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Research article

Blow up of solutions for a system of two singular nonlocal viscoelastic equations with damping, general source terms and a wide class of relaxation functions

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Abstract: This work studies the blow up result of the solution of a coupled nonlocal singular viscoelastic equation with damping and general source terms under some suitable conditions.

Keywords: viscoelastic equation; blow up; source term
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1. Introduction

Mixed non local problems for hyperbolic and parabolic PDEs, have been studied intensively in recent decades. Such equations or systems with constraints modelize many time-dependant physical phenomena. These constraints can be a data measured directly on the boundary or giving integral boundary conditions (se for example [5–19]).

The aim of this work, is to show the blow up of solutions of the viscoelastic one-dimensional system

$$
\begin{cases}
u_{tt} - \frac{(xu_x)_x}{x} + \int_0^t \frac{1}{x} g_1(t-s)(xu_x(x,s))_x ds + \mu_1 u_t(x,t) = f_1(u,v), \text{ in } D_T, \\
v_{tt} - \frac{(xv_x)_x}{x} + \int_0^t \frac{1}{x} g_2(t-s)(xv_x(x,s))_x ds + \mu_2 v_t(x,t) = f_2(u,v), \text{ in } D_T, \\
v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ x \in (0,L), \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in (0,L), \\
\int_0^L xu(x,t)dx = \int_0^L xv(x,t)dx = 0,\ \ u(L,t) = v(L,t) = 0.
\end{cases} \tag{1.1}
$$

$f_1(.,.), f_2(.,.) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are given by

$$
\begin{cases}
f_1(u,v) = a_1|u+v|^{2(r+1)}(u+v) + b_1|u|^r.u.|v|^{r+2}, \\
f_2(u,v) = a_1|u+v|^{2(r+1)}(u+v) + b_1|v|^r.v.|u|^{r+2},
\end{cases} \tag{1.2}
$$

with $r \geq -1,\ a_1, b_1 \in \mathbb{R}$.
$D_T = (0,L) \times (0,T)$, L, T, μ_1 and μ_2 are positive constants, $g_1, g_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are functions to be specified later.

This work is motivated by [11], where S. Mesloub studied a problem modelizing the movement of a two-dimensional viscoelastic object on a disc :

$$
\begin{cases}
u_t - \frac{1}{x}(xu_x)_x - \frac{1}{x}(xu_x)_{xt} = f(x,t,u,u_x),\ in\ D_T, \\
u(x,0) = u_0(x), \\
u_x(l,t) = 0,\ \ (u_0)_x(l) = 0, \\
\int_0^l xu(x,t)dx = 0,\ \ \int_0^l xu_0(x)dx = 0,
\end{cases} \tag{1.3}
$$

where $D_T = \{(x,t) \in \mathbb{R}^2,\ 0 < x < l,\ 0 < t < T\}$, and the source term f verifies some Lipschitz conditions.

Using an iterative process, he proved the existence and uniqueness of the solution of the nonlinear problem (1.3).

Later in [14], S.A. Messaoudi showed the existence of solutions with positive initial energy that blow up in finite time of the following nonlinear viscoelastic hyperbolic problem

$$
\begin{cases}
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + u_t|u_t|^{m-2} = |u|^{p-2}u,\ \ \text{in } \Omega \times (0,+\infty), \\
u(x,t) = 0,\ \ x \in \partial\Omega,\ t \geq 0, \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega.
\end{cases}
$$

Ω is a bounded domain of $\mathbb{R}^n$, $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $p > 2$, $m \geq 1$.

More work followed up on similar nonlocal singular viscoelastic equations and systems in [4, 7, 8, 15, 20, 21].

Recently in [3], we prove global existence and decay for system 1.1, by constructing a Lyapunov function combined with a perturbed energy.

In [1] with absence of the damping term ($\mu_1 = \mu_2 = 0$), the authors established a general decay result to the system 1.1 .

In this work, we continue our study on system 1.1. We start by giving the fundamental definitions and theorems on function spaces that we need, then we state the local existence theorem. Finally, we state and prove with suitable conditions the blow up in finite time of solutions for system 1.1.

2. Preliminaries

We start by defining the weighted Banach space $L_x^p = L_x^p((0, L))$ equipped with the norm

$$\|u\|_{L_x^p} = \left(\int_0^L x\,|u|^p\,dx\right)^{1/p}. \tag{2.1}$$

For $p = 2$, we obtain the Hilbert space $H = L_x^2((0, L))$ equipped the finite norm

$$\|u\|_H = \left(\int_0^L xu^2 dx\right)^{1/2}. \tag{2.2}$$

Consider also the Hilbert space $V := V_x^1((0, L))$ with finite norm

$$\|u\|_V = \left(\|u_x\|_H^2 + \|u\|_H^2\right)^{1/2}, \tag{2.3}$$

and

$$V_0 = \{u \in V,\ \ u(L) = 0\}. \tag{2.4}$$

As in Sobolev spaces, one can prove the following lemma

Lemma 1. *There exists $C > 0$ such that*

$$\int_0^L xv^2(x)dx \leq C \int_0^L x(v_x(x))^2 dx, \ \ \textit{for all } v \textit{ in } V_0. \tag{2.5}$$

Remark 1. *Observe that in V_0, $\|u\|_V$ is equivalent to $\|u_x\|_H$.*

Theorem 1. *([2]) There exists constant $C_p > 0$ depending only on L and p, such that for $2 < p < 4$, and any v in V_0 we have*

$$\int_0^L x|v(x)|^p dx \leq C_p\,\|v_x\|_H^p. \tag{2.6}$$

Before stating the local existence of solutions result, we consider the following assumptions
(**A1**) $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are two differentiable and decreasing functions with

$$g_1(t) \geq 0 \quad , \quad l_1 := 1 - \int_0^\infty g_1(s)\, ds > 0,$$
$$g_2(t) \geq 0 \quad , \quad l_2 := 1 - \int_0^\infty g_2(s)\, ds > 0. \tag{2.7}$$

(**A2**)There exist non-increasing differentiable functions $\vartheta_1, \vartheta_2 : [0, +\infty) \to (0, +\infty)$ and C^1 functions $\Phi_1, \Phi_2 : [0, +\infty) \to [0, +\infty)$ which are linear or strictly increasing and strictly convex C^2 functions on $(0, \varepsilon], \varepsilon \leq g_i(0)$, with $\Phi_i(0) = \Phi_i'(0) = 0$ such that

$$g_1'(t) \leq -\vartheta_1(t)\Phi_1(g_1(t)) \quad , \quad t \geq 0,$$
$$g_2'(t) \leq -\vartheta_2(t)\Phi_2(g_2(t)) \quad , \quad t \geq 0. \tag{2.8}$$

(**A3**) $r > -1$.
First we have to introduce the definition of a weak solution to (1.1).

Definition 1. *We say that dualism* (u, v) *is a weak solution to the system (1.1) on* $[0, T]$ *if*

$$u, v \in C\left([0, T]; V_0(0, L) \cap L_x^{2(r+2)}(0, L)\right),$$
$$u_t, v_t \in C([0, T]; H).$$

In addition, (u, v) *satisfies*

$$\begin{aligned}
& \int_0^L xu'(t)\phi dx - \int_0^L xu_1(t)\phi dx + \mu_1 \int_0^L xu(t)\phi dx - \mu_1 \int_0^L xu_0(t)\phi dx \\
& + \int_0^t \int_0^L xu_x\phi_x dx d\tau + \int_0^t \int_0^L \int_0^\tau g_1(\tau - s)xu_x(x, s)\phi_x ds dx d\tau \\
= & \int_0^t \int_0^L xf_1(u(\tau), v(\tau))\phi dx d\tau, \\
& \int_0^L xv'(t)\varphi dx - \int_0^L xv_1(t)\varphi dx + \mu_2 \int_0^L xv(t)\varphi dx - \mu_2 \int_0^L xv_0(t)\varphi dx \\
& + \int_0^t \int_0^L xv_x\varphi_x dx d\tau + \int_0^t \int_0^L \int_0^\tau g_2(\tau - s)xv_x(x, s)\varphi_x ds dx d\tau \\
= & \int_0^t \int_0^L xf_2(u(\tau), v(\tau))\varphi dx d\tau,
\end{aligned}$$

for all test function $\phi, \varphi \in V_0(0, L)$*, and for a.e.* $t \in [0, T]$

Theorem 2. *Assume (**A1**)–(**A3**) hold.*

Then, there exists a small enough positive number T^* *such that system (1.1) admits a unique local solution* $(u, v) \in \left[C((0, T^*); V_0) \cap C^1((0, T^*); H)\right]^2$,
$\forall (u_0, v_0) \in V_0^2, \;\; \forall (u_1, v_1) \in H^2$.

Remark 2. *The proof of this theorem can be established exactly as in [22], and [3] where we also proved a global existence result for problem (1.1).*

Lemma 2. *Let $F(u,v)$ be a function defined as follows*

$$F(u,v) = \frac{1}{2(r+2)}\left[a_1|u+v|^{2(r+2)} + 2b_1|uv|^{r+2}\right] \geq 0.$$

Then

$$\frac{1}{2(r+2)}\left[uf_1(u,v) + vf_2(u,v)\right] = F(u,v)$$

and

$$\frac{\partial F}{\partial u} = f_1(u,v), \quad \frac{\partial F}{\partial v} = f_2(u,v).$$

Take $a_1 = b_1 = 1$ for convenience.

Lemma 3. *[17] There exist c_1, c_2 positive constants such that*

$$\frac{c_1}{2(r+2)}\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \leq F(u,v) \leq \frac{c_2}{2(r+2)}\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \tag{2.9}$$

Lemma 4. *Assume (**A1**), (**A2**) and (**A3**) hold, and (u,v) be a solution of (1.1), then the energy functional*

$$\begin{aligned} E(t) \; := \; & \frac{1}{2}\|u_t\|_H^2 + \frac{1}{2}\|v_t\|_H^2 + \frac{1}{2}l_1\|u_x\|_H^2 + \frac{1}{2}l_2\|v_x\|_H^2 \\ & + \frac{1}{2}(g_1 o u_x) + \frac{1}{2}(g_2 o v_x) - \int_0^L xF(u,v)dx, \end{aligned} \tag{2.10}$$

is non-increasing and it satisfies

$$\begin{aligned} E'(t) \; = \; & -\mu_1\|u_t\|_H^2 - \mu_2\|v_t\|_H^2 + \frac{1}{2}g_1' \circ u_x + \frac{1}{2}g_2' \circ v_x \\ & -\|u_x\|_H^2 \int_0^t g_1(s)ds - \|v_x\|_H^2 \int_0^t g_2(s)ds \\ \leq \; & 0, \end{aligned} \tag{2.11}$$

where

$$(g \circ u_x)(t) = \int_0^L \int_0^t xg(t-s)\left|u_x(x,t) - u_x(x,s)\right|^2 dsdx, \tag{2.12}$$

and

$$\int_0^L xF(u,v)dx = \frac{1}{2(r+2)}\left(\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{(r+2)}\right). \tag{2.13}$$

Proof. By integration by parts, we obtain

$$\int_0^L xu_{tt}u_t dx = \frac{1}{2}\frac{d}{dt}\left[\int_0^L xu_t^2 dx\right], \tag{2.14}$$

$$\int_0^L xv_{tt}v_t dx = \frac{1}{2}\frac{d}{dt}\left[\int_0^L xv_t^2 dx\right], \tag{2.15}$$

$$-\int_0^L (xu_x)_x u_t dx = \frac{1}{2}\frac{d}{dt}\left[\int_0^L xu_x^2 dx\right], \tag{2.16}$$

$$-\int_0^L (xv_x)_x v_t dx = \frac{1}{2}\frac{d}{dt}\left[\int_0^L xv_x^2 dx\right], \tag{2.17}$$

$$\begin{aligned}\int_0^L \int_0^t g_1(t-s)(xu_x(s))_x ds u_t(t) dx = &\frac{1}{2}\frac{d}{dt}\left[(g_1 \circ u_x)(t) - \int_0^t g_1(s)ds \int_0^L xu_x^2 dx\right] \\ &- \frac{1}{2}(g_1' \circ u_x)(t) + \frac{1}{2}g_1(t)\int_0^L xu_x^2 dx,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\int_0^L \int_0^t g_2(t-s)(xv_x(s))_x ds v_t(t) dx = &\frac{1}{2}\frac{d}{dt}\left[(g_2 \circ v_x)(t) - \int_0^t g_2(t)ds \int_0^L xv_x^2 dx\right] \\ &- \frac{1}{2}(g_2' \circ v_x)(t) + \frac{1}{2}g_2(t)\int_0^L xv_x^2 dx.\end{aligned} \tag{2.19}$$

By multiplying the first and second equations of system (1.1) by xu_t, xv_t respectively, and integrating over $(0, L)$, then using (2.14)–(2.19), we get

$$\begin{aligned}\frac{d}{dt} \quad &\left\{\frac{1}{2}\|u_t\|_H^2 + \frac{1}{2}\|v_t\|_H^2 + \frac{1}{2}l_1\|u_x\|_H^2 + \frac{1}{2}l_2\|v_x\|_H^2 + \frac{1}{2}(g_1 \circ u_x)\right. \\ &\left. + \frac{1}{2}(g_2 \circ v_x) - \int_0^L xF(u,v)dx\right\} \\ = \quad &-\mu_1\|u_t\|_H^2 - \mu_2\|v_t\|_H^2 + \frac{1}{2}g_1' \circ u_x + \frac{1}{2}g_2' \circ v_x \\ &-\|u_x\|_H^2 \int_0^t g_1(s)ds - \|v_x\|_H^2 \int_0^t g_2(s)ds.\end{aligned} \tag{2.20}$$

From (2.7), (2.8) and (2.20), we obtain (2.11). □

Lemma 5. *[16] There exist positive constants d and t_0 such that, for any $t \in [0, t_0]$, we have*

$$g_i'(t) \le -dg_i(t), \quad i = 1, 2. \tag{2.21}$$

Lemma 6. *If (2.7) hold. Then, for any $\phi \in V_0$, $0 < \alpha < 1$ and $i = 1, 2$, we have*

$$\int_0^L x\left(\int_0^t g_i(t-s)(\phi(t) - \phi(s))ds\right)^2 dx \le C_{\alpha,i}(h_i \circ \phi)(t), \quad i = 1, 2 \tag{2.22}$$

where $C_{\alpha,i} := \int_0^\infty \frac{g_i^2(s)}{\alpha g_i(s) - g_i'(s)} ds$ and $h_i := \alpha g_i - g_i'$.

3. Blow up

Now, we give the main result of this paper

Theorem 3. *Assume (**A1**)–(**A3**) hold, and $E(0) < 0$. Then the solution of problem (1.1) blows up in finite time.*

Proof. Let us define the functional I by

$$\begin{aligned} I(t) = -E(t) &= -\frac{1}{2}\|u_t\|_H^2 - \frac{1}{2}\|v_t\|_H^2 - \frac{1}{2}l_1\|u_x\|_H^2 - \frac{1}{2}l_2\|v_x\|_H^2 \\ &\quad -\frac{1}{2}(g_1 o u_x) - \frac{1}{2}(g_2 o v_x) \\ &\quad +\frac{1}{2(r+2)}\Big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{r+2}}^{r+2}\Big]. \end{aligned} \tag{3.1}$$

From (2.11) and the assumption $E(0) < 0$, we get

$$E(t) < 0, \tag{3.2}$$

and

$$I'(t) \geq 0. \tag{3.3}$$

By (2.13), (2.9) and (3.3) we have

$$\begin{aligned} 0 \leq I(0) \leq I(t) &\leq \frac{1}{2(r+2)}\Big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{r+2}\Big] \\ &\leq \frac{c_2}{2(r+2)}\Big[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\Big]. \end{aligned} \tag{3.4}$$

Set

$$\mathcal{J}(t) = I^{1-\delta} + \varepsilon\int_0^L x(uu_t + vv_t)dx + \frac{\varepsilon}{2}(\mu_1\|u\|_H^2 + \mu_2\|v\|_H^2), \tag{3.5}$$

with

$$0 < \delta < \frac{2r+2}{4(r+2)} < 1. \tag{3.6}$$

By multiplying the first and the second equations of system (1.1) by xu, xv, we can verify that the derivative of (3.5) is given by

$$\begin{aligned} \mathcal{J}'(t) &= (1-\delta)I^{-\delta}(t)I'(t) + \varepsilon(\|u_t\|_H^2 + \|v_t\|_H^2) - \varepsilon(\|u_x\|_H^2 + \|v_x\|_H^2) \\ &\quad +\varepsilon\int_0^L u_x\int_0^t g_1(t-s)xu_x(s)dsdx + \varepsilon\int_0^L v_x\int_0^t g_2(t-s)xv_x(s)dsdx \\ &\quad +\varepsilon\Big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{r+2}\Big]. \end{aligned} \tag{3.7}$$

we have

$$\begin{aligned}
& \varepsilon\int_0^t g_1(t-s)ds\int_0^L u_x.xu_x(s)dxds \\
= \ & \varepsilon\int_0^t g_1(t-s)ds\int_0^L u_x.(xu_x(s)-xu_x(t))dxds+\varepsilon(\int_0^t g_1(s)ds)\|u_x\|_H^2 \\
\geq \ & \varepsilon\Big(\frac{1}{2}\int_0^t g_1(s)ds\Big)\|u_x\|_H^2-\frac{\varepsilon}{2}C_{\alpha,1}(h_1\circ u_x), \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
& \varepsilon\int_0^t g_2(t-s)ds\int_0^L v_x.xv_x(s)dxds \\
= \ & \varepsilon\int_0^t g_2(t-s)ds\int_0^L v_x.(xv_x(s)-xv_x(t))dxds+\varepsilon(\int_0^t g_2(s)ds)\|v_x\|_H^2 \\
\geq \ & \varepsilon\Big(\frac{1}{2}\int_0^t g_2(s)ds\Big)\|v_x\|_H^2-\frac{\varepsilon}{2}C_{\alpha,2}(h_2\circ u_x). \qquad (3.9)
\end{aligned}$$

So, by (3.7)

$$\begin{aligned}
\mathcal{J}'(t) \ \geq \ & (1-\delta)I^{-\delta}(t)I'(t)+\varepsilon(\|u_t\|_H^2+\|v_t\|_H^2) \\
& -\varepsilon\Big\{\Big(1-\frac{1}{2}\int_0^t g_1(s)ds\Big)\|u_x\|_H^2+\Big(1-\frac{1}{2}\int_0^t g_2(s)ds\Big)\|v_x\|_H^2\Big\} \\
& -\frac{\varepsilon}{2}C_{\alpha,1}(h_1\circ u_x)-\frac{\varepsilon}{2}C_{\alpha,2}(h_2\circ v_x) \\
& +\varepsilon\Big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)}+2\|uv\|_{L_x^{(r+2)}}^{r+2}\Big]. \qquad (3.10)
\end{aligned}$$

For $0<\lambda<1$, from (3.1)

$$\begin{aligned}
\varepsilon\Big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)}+2\|uv\|_{L_x^{(r+2)}}^{r+2}\Big] = \ & \varepsilon\lambda\Big[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)}+2\|uv\|_{L_x^{(r+2)}}^{r+2}\Big] \\
& +2\varepsilon(r+2)(1-\lambda)I(t) \\
& +\varepsilon(r+2)(1-\lambda)(\|u_t\|_H^2+\|v_t\|_H^2) \\
& +\varepsilon(r+2)(1-\lambda)(g_1\circ u_x) \\
& +\varepsilon(r+2)(1-\lambda)(g_2\circ v_x) \\
& +\varepsilon(r+2)(1-\lambda)l_1\|u_x\|_H^2 \\
& +\varepsilon(p+2)(1-\lambda)l_2\|v_x\|_H^2. \qquad (3.11)
\end{aligned}$$

We obtain by substituting in (3.10)

$$\begin{aligned}
\mathcal{J}'(t) \ \geq \ & (1-\delta)I^{-\delta}(t)I'(t)+\varepsilon[(r+2)(1-\lambda)+1](\|u_t\|_H^2+\|v_t\|_H^2) \\
& +\varepsilon\Big[(r+2)(1-\lambda)\Big(1-\int_0^t g_1(s)ds\Big)-\Big(1-\frac{1}{2}\int_0^t g_2(s)ds\Big)\Big]\|u_x\|_H^2 \\
& +\varepsilon\Big[(r+2)(1-\lambda)\Big(1-\int_0^t g_2(s)ds\Big)-\Big(1-\frac{1}{2}\int_0^t g_2(s)ds\Big)\Big]\|v_x\|_H^2
\end{aligned}$$

$$+\varepsilon\Big[(r+2)(1-\lambda)\Big](g_1ou_x+g_2ov_x)+2\varepsilon(r+2)(1-\lambda)I(t)$$
$$-\frac{\varepsilon}{2}\Big(C_{\alpha,1}(h_1\circ u_x)(t)+C_{\alpha,1}(h_1\circ u_x)(t)\Big)$$
$$+\varepsilon\lambda\Big[\|u+v\|^{2(r+2)}_{L_x^{2(r+2)}}+2\|uv\|^{r+2}_{L_x^{(r+2)}}\Big], \tag{3.12}$$

by (2.22), we find

$$\begin{aligned}\mathcal{J}'(t) \ \geq \ & (1-\delta)I^{-\delta}(t)I'(t)+\varepsilon[(r+2)(1-\lambda)+1](\|u_t\|^2_H+\|v_t\|^2_H)\\
&+\varepsilon\Big[(r+2)(1-\lambda)\Big(1-\int_0^t g_1(s)ds\Big)-\Big(1-\frac{1}{2}\int_0^t g_2(s)ds\Big)\Big]\|u_x\|^2_H\\
&+\varepsilon\Big[(r+2)(1-\lambda)\Big(1-\int_0^t g_2(s)ds\Big)-\Big(1-\frac{1}{2}\int_0^t g_2(s)ds\Big)\Big]\|v_x\|^2_H\\
&+\varepsilon\Big[(r+2)(1-\lambda)-\frac{1}{2}\alpha C_\alpha\Big](g_1ou_x+g_2ov_x)+2\varepsilon(r+2)(1-\lambda)I(t)\\
&+\frac{\varepsilon}{2}\Big(C_{\alpha,1}(g_1'\circ u_x)(t)+C_{\alpha,2}(g_2'\circ u_x)(t)\Big)\\
&+\varepsilon\lambda\Big[\|u+v\|^{2(r+2)}_{L_x^{2(r+2)}}+2\|uv\|^{r+2}_{L_x^{(r+2)}}\Big],\end{aligned} \tag{3.13}$$

where $C_\alpha=\min\{C_{\alpha,1},C_{\alpha,2}\}$.

Now, one cause the Lebesgue dominated convergence theorem with the fact that $\frac{g_i^2(s)}{\alpha g_i(s)-g_i'(s)}<g_i(s)$, for $i=1,2$, to prove that

$$\lim_{\alpha\to 0^+}\alpha C_\alpha=0.$$

Therefore, there exists $\alpha_0\in(0,1)$ such that if $\alpha<\alpha_0$, then, by letting $\alpha=\varepsilon(r+2)(1-\lambda)<\alpha_0$, we get

$$\alpha C_\alpha<(r+2)(1-\lambda).$$

We put $\lambda>0$ small enough, we have

$$\beta_1=(r+2)(1-\lambda)-1>0,$$

at this point, we take

$$\max\Big\{\int_0^\infty g_1(s)ds,\int_0^\infty g_2(s)ds\Big\}<\frac{(r+2)(1-\lambda)-1}{\Big((r+2)(1-\lambda)-\frac{1}{2}\Big)}=\frac{2\beta_1}{2\beta_1+1}, \tag{3.14}$$

then, we have

$$\begin{aligned}\beta_2&=\Big[\Big((r+2)(1-\lambda)-1\Big)-\int_0^t g_1(s)ds\Big((r+2)(1-\lambda)-\frac{1}{2}\Big)\Big]\\
&>\beta_1-\frac{2\beta_1}{2\beta_1+1}\Big(\beta_1+\frac{1}{2}\Big)\end{aligned}$$

i.e $\beta_2 > 0$.

Similarly

$$\beta_3 = \left[\left((r+2)(1-\lambda)-1\right) - \int_0^t g_2(s)ds\left((r+2)(1-\lambda)-\frac{1}{2}\right)\right] > 0.$$

Choosing ε small enough, thus we have

$$I(0) + \varepsilon\int_0^L x(u_0u_1 + v_0v_1)dx + \frac{\varepsilon}{2}\left(\mu_1\|u_0\|_H^2 + \mu_2\|v_0\|_H^2\right) > 0.$$

Then, for some $\gamma_1, \gamma_2 > 0$, estimate (3.12) becomes

$$\begin{aligned}\mathcal{J}'(t) \geq\ & \gamma_1\Big\{I(t) + \|u_t\|_H^2 + \|v_t\|_H^2 + \|u_x\|_H^2 + \|v_x\|_H^2 \\ & +(g_1ou_x) + (g_2ov_x) + \left[\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{r+2}\right]\Big\} \\ & +\gamma_2\Big\{(g_1'ou_x) + (g_2'ov_x)\Big\}. \end{aligned} \tag{3.15}$$

By (2.9) and (2.11), for some $\gamma_3 > 0$, we get

$$\begin{aligned}\mathcal{J}'(t) \geq\ & \gamma_3\Big\{I(t) + \|u_t\|_H^2 + \|v_t\|_H^2 + \|u_x\|_H^2 + \|v_x\|_H^2 \\ & +(g_1ou_x) + (g_2ov_x) + \left[\|u\|_{L_x^{2(r+2)}}^{2(p+2)} + \|u\|_{L_x^{2(r+2)}}^{2(r+2)}\right]\Big\} \\ & +2\gamma_2E'(t). \end{aligned} \tag{3.16}$$

We obtain,

$$\begin{aligned}\mathcal{J}_1'(t) \geq\ & \gamma_3\Big\{I(t) + \|u_t\|_H^2 + \|v_t\|_H^2 + \|u_x\|_H^2 + \|v_x\|_H^2 \\ & +(g_1ou_x) + (g_2ov_x) + \left[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|u\|_{L_x^{2(r+2)}}^{2(r+2)}\right]\Big\}, \end{aligned} \tag{3.17}$$

where $\mathcal{J}_1(t) := \mathcal{J}(t) + \gamma_2 I(t)$,
and

$$\mathcal{J}_1(t) \geq \mathcal{J}_1(0) > 0, \quad t > 0. \tag{3.18}$$

On the other hand, we have by Holder's and Young's inequalities,

$$\left|\int_0^L x(uu_t + vv_t)dx\right|^{\frac{1}{1-\delta}} \leq C\left[\|u\|_{L_x^{2(r+2)}}^{\frac{\mu}{1-\delta}} + \|u_t\|_H^{\frac{\nu}{1-\delta}} + \|v\|_{L_x^{2(r+2)}}^{\frac{\mu}{1-\delta}} + \|v_t\|_H^{\frac{\nu}{1-\delta}}\right], \tag{3.19}$$

where $\frac{1}{\nu} + \frac{1}{\mu} = 1$.
Taking $\mu = 2(1-\delta)$, we get

$$\frac{\nu}{1-\delta} = \frac{2}{1-2\delta} \leq 2(r+2).$$

Then, for $s = \frac{2}{(1-2\delta)}$ and from (3.1), we obtain

$$\|u\|_{L_x^{2(r+2)}}^{\frac{2}{1-2\delta}} \leq c(\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + I(t))$$

$$\|v\|_{L_x^{2(r+2)}}^{\frac{2}{1-2\delta}} \leq c(\|v\|_{L_x^{2(r+2)}}^{2(r+2)} + I(t)), \quad \forall t \geq 0.$$

So, for some $k_2 > 0$

$$|\int_0^L x(uu_t + vv_t)dx|^{\frac{1}{1-\delta}} \leq k_2\Big[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)} + \|u_t\|_H^2 + \|v_t\|_H^2 + I(t)\Big]. \tag{3.20}$$

Therefore, there exist $k_3 > 0$ such that

$$\begin{aligned}
\mathcal{J}_1^{\frac{1}{1-\delta}}(t) &= (\mathcal{J}(t) + \gamma_2 I(t))^{\frac{1}{1-\delta}} \\
&= \Big(I^{1-\delta} + \varepsilon \int_0^L x(uu_t + vv_t)dx + \frac{\varepsilon}{2}(\mu_1\|u\|_H^2 + \mu_2\|v\|_H^2) + \gamma_2 I(t)\Big)^{\frac{1}{1-\delta}} \\
&\leq k_3\Big[I(t) + |\int_0^L x(uu_t + vv_t)dx|^{\frac{1}{1-\delta}} + (\|u\|_H^2 + \|v\|_H^2)^{\frac{1}{1-\delta}} \\
&\quad + \Big(\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\Big)^{\frac{1}{1-\delta}}\Big] \\
&\leq k_3\Big[I(t) + \|u_t\|_H^2 + \|v_t\|_H^2 + \|u_x\|_H^2 + \|v_x\|_H^2 + (g_1 o u_x) \\
&\quad + (g_2 o v_x) + \|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\Big].
\end{aligned} \tag{3.21}$$

From (3.15) and (3.21), we finally get the required inequality

$$\mathcal{J}_1'(t) \geq k_4 \mathcal{J}_1^{\frac{1}{1-\delta}}(t), \tag{3.22}$$

where $k_4 > 0$ depending only on k_1 and k_3.
By integration of (3.22), we find

$$\mathcal{J}_1^{\frac{\delta}{1-\delta}}(t) \geq \frac{1}{\mathcal{J}_1^{\frac{-\alpha}{1-\delta}}(0) - k_4\frac{\delta}{(1-\delta)}t}.$$

Hence, $\mathcal{J}_1(t)$ blows up at most at the finite time

$$T^* = \frac{1-\delta}{k_4\delta\mathcal{J}_1^{\delta/(1-\delta)}(0)}.$$

□

4. Conclusions

Motivated by last recent mentioned papers (see [1, 3, 4]) and under some sufficient conditions, we have stated and proved the blow up in finite time of solutions for system (1.1). In the next work, we have been extend our recent work to the high dimension. Also some numerical examples have been explained in order to ensure the theory study.

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Conflict of interest

This work does not have any conflicts of interest.

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