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Research article

On modified convex interval valued functions and related inclusions via the interval valued generalized fractional integrals in extended interval space

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Abstract: In this paper, we propose a new family of interval valued (IV) convex functions termed as generalized modified (p, h)-convex IV functions. We obtain the counterpart of Hermite-Hadamard $H \cdot H$ inequality by extending the IV fractional integral to the IV ψ_k-Riemann-Liouville $(\psi_k - RL)$ fractional integrals. Also, several inequalities using extended operations on the newly defined class of convex IV functions are given.

Keywords: $H \cdot H$ inequality; $\psi_k - RL$ fractional integrals; IV convex function
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1. Introduction

The function $H : \mathcal{T} \to \mathbb{R}$ ($\mathcal{T}$ is a finite interval of $\mathbb{R}$) will be convex if the inequality:

$$H(\mu a_1 + (1-\mu)a_2) \leq \mu H(a_1) + (1-\mu)H(a_2), \tag{1.1}$$

holds for all $a_1, a_2 \in \mathcal{T}, \mu \in [0,1]$. The beauty of mentioned functions was revealed when, Hermite and Hadamard [1] discovered the inequality

$$H\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} H(t)dt \leq \frac{H(a_1)+H(a_2)}{2}, \tag{1.2}$$

called $H{\cdot}H$ inequality, which ranked among the best-established inequalities. Due to a nice geometrical interpretation, various applications has been found, see for instance [2, 3]. For a detailed study about the generalization of this inequality, we refer to [4–13].

Interval analysis is a special case of set-valued analysis. The subject emerged as an attempt to handle the interval uncertainty which appears in different computational or mathematical models of some deterministic real-world problems. One of the oldest example of an interval enclosure is Archimedes method which is related to computation of the circumference of a circle. In 1966, Moore [14] wrote the first manuscript on to the interval analysis. After his book, scientists began to explore the theory and application of interval arithmetic.

The interval analysis has its applications in robotics, computer graphics, chemical and structured engineering, economics, behavioral ecology, constraint satisfaction, signal processing and global optimization, neural network output optimization and many others [15–17].

Due to the mentioned applications, researchers have investigated several important inequalities including Jensen's inequality, $H \cdot H$ and Ostrowski inequalities etc. Chalco-Cano et al. [18, 19] obtained Ostrowski type inequalities via the Hukuhara derivative for IV functions. Romn-Flores et al. [20], established Minkowski and Beckenbachs inequalities. For some other inequalities, see [20–22]. Sadowska [23], gave the $H \cdot H$ inequality. For other studies, see [23, 24]. In [25], Zhao et al. proved $H \cdot H$ inequality for IV approximately h-convex functions via generalized fractional integrals. Recently, Kamran et al. [26], established the $H \cdot H$ inequality by introducing the notion of IV generalized p-convex functions.

Motivated by these papers, we develop some new $H \cdot H$ inequalities by introducing a new class of IV functions termed as modified generalized (p, h)-convex IV function. Furthermore we explore the counter part of $H \cdot H$ inequality for the new and larger class of IV convex functions. We also extend the concept from interval arithmetic to extended interval arithmetic for the deduction of the including sub-classes of modified generalized (p, h)-convex IV functions. We establish the $H \cdot H$ inequality via IV $\psi_k - RL$ fractional integrals, which is the extension of IV Riemann–Liouville fractional integral introduced in [27].

2. Preliminaries

In this section, we recall some useful definitions and results which will be helpful for the further study. Most of the literature has been taken from [28–30].

2.1. Interval arithmetic; $\mathcal{S} = (\mathcal{S}, +, \times, \subseteq)$

In this subsection we recall some basic concepts of interval-arithmetic.

An interval $[u, v], u \le v$ is a compact subset of $\mathbb{R}$, the real line, defined by $[u, v] = \{x | u \le x \le v\}$. The set $\{[u, v] | u, v \in \mathbb{R}\}$ of all intervals is denoted by $\mathcal{TR}$. The left end-point of any interval $U \in \mathcal{TR}$ is denoted by u^- or U^-, while the right end-point by u^+ or U^+, so that $U = [u^-, u^+] = [U^-, U^+]$. So, for $U \in \mathcal{TR}$ the symbol u^r (or U^r), with $r \in \{+, -\}$, represents the left or right end-point of U, depending on the value of r. Define the product rs for $r, s \in \{+, -\}$ by setting $-- = + = ++, +- = -+ = -$, so that $u^{--} = u^{++} = u^+$ etc. Let us denote the set of intervals containing zero by $\mathcal{Z} = \{U \in \mathcal{TR} |\ 0 \in U\} = \{U \in \mathcal{TR} |\ u^- \le 0, u^+ \ge 0\}$ and the set of intervals which do not contain zero is $\mathcal{TR} \setminus \mathcal{Z} = \{U \in \mathcal{TR} |\ 0 \notin U\}$. Define $\varphi : \mathcal{TR} \setminus \mathcal{Z} \to \{+, -\}$ by means of

$$\varphi(U) = \begin{cases} +, & \text{if} \quad 0 < u^-; \\ -, & \text{if} \quad u^+ < 0. \end{cases} \tag{2.1}$$

The interval arithmetic $\mathcal{G} = (\mathcal{TR}, +, ., /, \subseteq)$ consists of the set $\mathcal{TR}$ together with a relation for inclusion $\subseteq$ and the basic operations addition $+ : \mathcal{TR} \otimes \mathcal{TR} \to \mathcal{TR}$, multiplication $. : \mathcal{TR} \otimes \mathcal{TR} \to \mathcal{TR}$ and inversion (reciprocal value) $/ : \mathcal{TR} \setminus \mathcal{Z} \to \mathcal{TR}$ defined by:

$$U \subseteq V \Leftrightarrow v^- \le u^- \text{ and } u^+ \le v^+ \text{ for } U, V \in \mathcal{TR}. \tag{2.2}$$

$$U + V = [u^- + v^-, u^+ + v^+]. \tag{2.3}$$

$$U.V = \begin{cases} [u^{-\varphi(V)} v^{-\varphi(U)}, u^{\varphi(V)} v^{\varphi(U)}], & \text{for} \quad U, V \in \mathcal{TR} \setminus \mathcal{Z}; \\ [u^{\delta} v^{-\delta}, u^{\delta} v^{\delta}], \delta = \varphi(U) & \text{for} \quad U \in \mathcal{TR} \setminus \mathcal{Z}, V \in \mathcal{Z}; \\ [u^{-\delta} v^{\delta}, u^{\delta} v^{\delta}], \delta = \varphi(V) & \text{for} \quad U \in \mathcal{Z}, V \in \mathcal{TR} \setminus \mathcal{Z}. \end{cases} \tag{2.4}$$

$$U.V = \left[min\left\{u^- v^+, u^+ v^-\right\}, max\left\{u^- v^-, u^+ v^+\right\}\right], \quad \text{for all } U, V \in \mathcal{Z}. \tag{2.5}$$

$$1/V = \left[\frac{1}{v^+}, \frac{1}{v^-}\right], V \in \mathcal{TR} \setminus \mathcal{Z}. \tag{2.6}$$

In particular, if U is a degenerate interval of the form $U = [u, u] = u$, then

$$U.V = u.V = \begin{cases} [uv^{-\varphi(u)}, uv^{\varphi(u)}] = [uv^-, uv^+], & \text{for} \quad u \ge 0; \\ [uv^{-\varphi(u)}, uv^{\varphi(u)}] = [uv^-, uv^+], & \text{for} \quad u < 0. \end{cases} \tag{2.7}$$

For $u = -1$, we have $(-1).V = -V = -[v^-, v^+] = [-v^+, -v^-]$. The operation of subtraction $U - V$ and division U/V are defined in $\mathcal{G}$ as composite operations respectively by

$$U - V = U + (-1).V = U + (-V) = [u^- - v^+, u^+ - v^-], \quad \text{for } U, V \in \mathcal{TR}, \tag{2.8}$$

$$U/V = U.(1/V) = \begin{cases} [u^{-\varphi(V)}/v^{\varphi(U)}, u^{\varphi(V)}/v^{-\varphi(U)}], & \text{for} \quad U, V \in \mathcal{TR} \setminus \mathcal{Z}; \\ [u^{-\delta}/v^{-\delta}, u^{\delta}/v^{-\delta}], \delta = \varphi(V) & \text{for} \quad U \in \mathcal{Z}, V \in \mathcal{TR} \setminus \mathcal{Z}. \end{cases} \tag{2.9}$$

Note that the operation inversion $1/V$ in $\mathcal{G}$ can not be composed by means of " + " and "." and thus should be assumed as basic. One can observe that the operations $+, -, ., /$ in $\mathcal{G}$ defined by (2.3)–(2.8) satisfy: $U \circledast V = \{a \circledast b | a \in U, v \in V\}, \circledast \in \{+, -, ., /\}$. The properties of $\mathcal{G} = (\mathcal{TR}, +, ., /, \subseteq)$ are well established. Also note that:

G$_1$. The operations " + " and "." satisfy the following associative laws:

$$(U + V) + W = U + (V + W), \ (U.V).W = U.(V.W).$$

G$_2$. The operations (2.3)–(2.8) in $\mathcal{G}$ are isotone w. r. t. $\subseteq$:
$U \subseteq U_1, V \subseteq V_1 \Rightarrow U \circledast V \subseteq U_1 \circledast V_1, \circledast \in \{+, -, ., /\}$. Furthermore, the distributive law holds in special case. For example, if $U = [2, 4], V = [3, 6]$ and $W = [-3, -1]$, then

$$U.(V + W) = [0, 20]$$

and

$$U.V + U.W = [-6, 22].$$

The distributive law is valid if $V.W > 0$ (see [14]).

2.2. Building the interval; $\mathcal{H} = (\mathcal{H}, +, ., \subseteq)$

In this subsection we reload the concept of extended interval arithmetic. Recall that an interval $U = [u^-, u^+] \in \mathcal{H}$ is a proper or regular interval if $u^- \leq u^+$, and improper interval if $u^- \geq u^+$. Let us denote the space of all proper intervals by $\mathcal{TR}$ and improper intervals by $\overline{\mathcal{TR}}$. Then $\mathcal{H} = \{[a, b]| \ a, b \in \mathbb{R}\} \cong \mathbb{R}^2$ of all ordered couples of real numbers is the extended space. The basic operations are extended by replacing $\mathcal{TR}$ by $\mathcal{H}$ in the extended intervals space $\mathcal{H}$. For any $U \in \mathcal{TR}$, denote an improper interval $[-u^-, -u^+]$ by $-_hU$. Obviously, $U + (-_hU) = 0$. Thus, U and $-_hU$ are additive inverses of each other. One should not confuse with opposite element $-U = [-u^+, -u^-]$ and the additive inverse $-_hU = [-u^-, -u^+]$ of U. Thus $\mathcal{H}$ is a group. For detail see [28–30]. The inclusion isotonicity property $\mathbf{G}_2$ and the association property $\mathbf{G}_1$ remain true in $\mathcal{H}$ in the same forms as developed in $\mathcal{G}$. Moreover, $U.(V + W) = U.V + U.W$ if $\varphi(U) = \varphi(V) = \varphi(U + V)$, for $U, V, W \in \mathcal{H}$, see [30, Example 3].

3. Interval integration

This section deals with reloading the notion of integral for IV functions. Throughout the study, we will denote set of positive proper intervals by $\mathcal{TR}^+$, negative $\mathcal{TR}^-$ and all proper intervals by $\mathcal{TR}$ and the set of extended interval set by $\mathcal{H}$, positive extended intervals by $\mathcal{H}^+$ and negative by $\mathcal{H}^-$.

An IV function F of t on $[a_1, a_2]$ assigns a nonempty interval to each $t \in [a_1, a_2]$ $F(t) = [\underline{F}(t), \overline{F}(t)]$.

Let P be a partition of $[a_1, a_2]$ having the form $P : a_1 = t_0 < t_1 < \ldots < t_n = a_2$ and $mesh(P) = max\left\{t_j t_{j-1} : j = 1, 2, \ldots, n\right\}$ be the mesh of the partition P. Also, the set of all partitions of $[a_1, a_2]$ is denote by $P([a_1, a_2])$. Let $P(\delta, [a_1, a_2])$ be the set of all P in $P([a_1, a_2])$ such that $mesh(P) < \delta$. Let us choose an arbitrary point ζ_j in $[t_{j-1}, t_j]$, $(i = 1, 2, \ldots, n)$ and let us define the sum $S(F, P, \delta) = \sum\limits_{j=1}^{\infty} F(\zeta_j)[t_j t_{j-1}]$, where $F : [a_1, a_2] \to \mathbb{R}_I$. Then, we say that $S(F, P, \delta)$ is a Riemann sum of F corresponding to $P \in P(\delta, [a_1, a_2])$.

Definition 3.1 ([31, 32]). We say that a function $F : [a_1, a_2] \to \mathcal{TR}$ is an interval Riemann integrable (($I\mathfrak{R}$)-integrable) on $[a_1, a_2]$, if there exists $M \in \mathcal{TR}$ such that for any $\epsilon > 0$, there exists $\delta > 0$ with

$$d(S(F, P, \delta), M) < \epsilon. \tag{3.1}$$

For any Riemann sum S of F corresponding to $P \in P(\delta, [a_1, a_2])$ and independent from choice of $\zeta_j \in [t_{j-1}, t_j]$ for all $1 \le j \le n$, we call M as the $(I\mathfrak{R})$-integral of F on $[a_1, a_2]$ and denoted by

$$M = \int_{a_1}^{a_2} F(v)dv. \tag{3.2}$$

Note that, $I\mathfrak{R}([a_1, a_2])$ is the collection of all functions that are $(I\mathfrak{R})$-integrable on $[a_1, a_2]$ and $\mathfrak{R}([a_1, a_2])$ is the collection of Riemann integrable function on $[a_1, a_2]$. In what follows we obtain a relation between $(I\mathfrak{R})$-integrable and Riemann integrable ($\mathfrak{R}$-integrable).

Theorem 3.1. *[33] Assume that $H : [a_1, a_2] \to \mathcal{TR}$ is an* IV *function with $H(v) = [H^-(v), H^+(v)]$. Then, $H \in I\mathfrak{R}([a_1, a_2])$ if and only if $H^-(v), H^+(v) \in \mathfrak{R}([a_1, a_2])$ and*

$$(I\mathfrak{R}) \int_{a_1}^{a_2} H(v)dv = \left[\mathfrak{R} \int_{a_1}^{a_2} H^-(v)dv, \mathfrak{R} \int_{a_1}^{a_2} H^+(v)dv \right]. \tag{3.3}$$

In recent past, researchers have focused interval analysis and established different inequalities, for detail see [18–26, 34, 35]. In [36], Varošanec introduced the class of SX functions as follows.

Definition 3.2. For a non-negative function $h : [a_1, a_2] \to \mathbb{R}$, $(0, 1) \subseteq [a_1, a_2]$ and $h \neq 0$, we say that $H : [a_1, a_2] \to \mathbb{R}$ is an h-convex function, or that h belongs to the class $SX(h, [a_1, a_2], \mathbb{R})$, if h is non-negative and

$$H(vx + (1 - v)y) \le h(v)H(x) + h(1 - v)H(y), \quad x, y \in [a_1, a_2], \ v \in (0, 1). \tag{3.4}$$

In [37], Zhao et al. introduced the notion of h-convex IV function as follows.

Definition 3.3. For any non-negative function $h : [a_1, a_2] \to \mathbb{R}$, $(0, 1) \subseteq [a_1, a_2]$ and $h(v) \neq 0$, we say that $H : [a_1, a_2] \to \mathcal{TR}^+$ is an h-convex IV function, if

$$h(v)H(x) + h(1 - v)H(y) \subseteq H(vx + (1 - v)y), \quad x, y \in [a_1, a_2], \ v \in (0, 1). \tag{3.5}$$

Remark 3.1. *The h-convex* IV *function* (3.5) *with $h(v) = v$ gives the convex* IV *function (see [38]).*

Theorem 3.2 ([37])**.** *Let $H : [a_1, a_2] \to \mathcal{TR}^+$ be an* IV *function with $H(v) = [H^-(v), H^+(v)]$ such that $H \in I\mathfrak{R}([a_1, a_2])$, $h : [a_1, a_2] \to \mathbb{R}$ be a non-negative function and $h\left(\frac{1}{2}\right) \neq 0$. If $H \in SX(h, [a_1, a_2], \mathcal{TR}^+)$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} H\left(\frac{a_1 + a_2}{2}\right) \supseteq \frac{1}{a_2 - a_1}(I\mathfrak{R}) \int_{a_1}^{a_2} H(v)dv \supseteq [H(a_1) + H(a_2)] \int_0^1 h(v)dv. \tag{3.6}$$

Remark 3.2. *For $h(v) = v$, the inequality* (3.6) *reduces to*

$$H\left(\frac{a_1 + a_2}{2}\right) \supseteq \frac{1}{a_2 - a_1}(I\mathfrak{R}) \int_{a_1}^{a_2} H(v)dv \supseteq \frac{H(a_1) + H(a_2)}{2}. \tag{3.7}$$

Lupulescu [27] derived the following IV definition for the left-sided RL fractional integral.

Definition 3.4. Assume that $H : [a_1, a_2] \to \mathcal{T}R^+$ is an IV function with $H(t) = [H^-(t), H^+(t)]$ and $H \in I\mathfrak{R}_{([a_1,a_2])}$. Then, the IV left-sided RL fractional integral of function H is given by

$$^{RL}\mathcal{J}^{\nu}_{a_1^+} H(y) = \frac{1}{\Gamma(\nu)}(I\mathfrak{R}) \int_{a_1}^{y} (y-\mu)^{\nu-1} H(\mu) d\mu, \quad y > a_1, \nu > 0. \tag{3.8}$$

Budak et al. [39] defined the IV right-sided RL fractional integrals in the similar passion as following.

$$^{RL}\mathcal{J}^{\nu}_{a_2^-} G(y) = \frac{1}{\Gamma(\nu)}(I\mathfrak{R}) \int_{y}^{a_2} (\mu-y)^{\nu-1} G(\mu) d\mu, \quad y < a_2, \nu > 0. \tag{3.9}$$

Budak et al. [39] establish the following $H \cdot H$ inequalities for convex IV functions.

Theorem 3.3. *If $H : [a_1, a_2] \to \mathcal{T}R^+$ is a convex* IV *function with $H(\nu) = [\underline{H}(\nu), \overline{H}(\nu)], \nu > 0$, then*

$$H\left(\frac{a_1+a_2}{2}\right) \supseteq \frac{\Gamma(\nu+1)}{2(a_2-a_1)^{\nu}} \left[{}^{RL}\mathcal{J}^{\nu}_{a_1^+} H(a_2) + {}^{RL}\mathcal{J}^{\nu}_{a_2^-} H(a_1)\right] \supseteq \frac{H(a_1)+H(a_2)}{2}. \tag{3.10}$$

Theorem 3.4. *If $H, K : [a_1, a_2] \to \mathcal{T}R^+$ are two convex* IV *functions such that $H(t) = [H^-(t), H^+(t)]$, and $K(t) = [K^-(t), K^+(t)]$, then for $\nu > 0$, we have*

$$\frac{\Gamma(\nu+1)}{2(a_2-a_1)^{\nu}} \left[{}^{RL}\mathcal{J}^{\nu}_{a_1^+} H(a_2) + {}^{RL}\mathcal{J}^{\nu}_{a_2^-} H(a_1)\right]$$
$$\supseteq \left(\frac{1}{2} - \frac{\nu}{(\nu+1)(\nu+2)}\right) M(a_1, a_2) + \frac{\nu}{(\nu+1)(\nu+2)} N(a_1, a_2), \tag{3.11}$$

and

$$2H\left(\frac{a_1+a_2}{2}\right) K\left(\frac{a_1+a_2}{2}\right) \supseteq \frac{\Gamma(\nu+1)}{2(a_2-a_1)^{\nu}} \left[{}^{RL}\mathcal{J}^{\nu}_{a_1^+} H(a_2) + {}^{RL}\mathcal{J}^{\nu}_{a_2^-} H(a_1)\right]$$
$$+ \frac{\nu}{(\nu+1)(\nu+2)} M(a_1, a_2) + \left(\frac{1}{2} - \frac{\nu}{(\nu+1)(\nu+2)}\right) N(a_1, a_2), \tag{3.12}$$

where $M(a_1, a_2) = H(a_1)K(a_1) + H(a_2)K(a_2)$ and $N(a_1, a_2) = H(a_1)K(a_2) + H(a_2)K(a_1)$.

Interested reader may see also the recent papers on inclusions of $H \cdot H$ type [40–42]. The purpose of this paper is to derive some new counterparts of $H \cdot H$ inequality via generalized (p, h)-convex IV functions in terms of $\psi_k - RL$ fractional integrals in interval form. In order to obtain the main results of the paper we first introduce the notion of (p, h)-convex IV functions on $\mathcal{S}$ and $\psi_k - RL$ fractional integrals in interval form. Then we obtain some subclasses of (p, h)-convex IV functions by considering the interval family $\mathcal{K}$. We obtain a number of inequalities for including classes for the extended class. Finally, the validity of our findings is checked by taking some particular examples.

Before we start off our main results, let us recall some known fractional integrals.

Definition 3.5 ([43]). Let $\psi : [a_1, a_2] \to \mathbb{R}$ be an increasing and positive monotone function on $[a_1, a_2]$, and let $\psi'(x)$ be continuous on (a_1, a_2). The left and right-sided fractional integrals of H with according to ψ on $[a_1, a_2]$ of order $\nu > 0$ are respectively given as follows:

$$ {}^{RL}\mathcal{J}_{a_1^+}^{\nu,\psi} H(y) = \frac{1}{\Gamma(\nu)} \int_{a_1}^{y} \frac{\psi'(\tau)H(\tau)}{[\psi(y) - \psi(\tau)]^{1-\nu}} d\tau, \quad y > a_1, \tag{3.13} $$

$$ {}^{RL}\mathcal{J}_{a_2^-}^{\nu,\psi} H(y) = \frac{1}{\Gamma(\nu)} \int_{y}^{a_2} \frac{\psi'(\tau)H(\tau)}{[\psi(\tau) - \psi(y)]^{1-\nu}} d\tau, \quad y < a_2. \tag{3.14} $$

Kwun et al. [44] introduced the k-fractional version of the integrals (3.13) and (3.14) as follows:

Definition 3.6. Let $\psi : [a_1, a_2] \to \mathbb{R}$ be an increasing and positive monotone function on $[a_1, a_2]$, and let $\psi'(x)$ be continuous on (a_1, a_2). The left and right-sided fractional integrals of H with according to ψ on $[a_1, a_2]$ of order $\nu, k > 0$ are respectively given as follows: defined by:

$$ {}^{RL}\mathcal{J}_{a_1^+}^{\nu,k,\psi} H(y) = \frac{1}{k\Gamma_k(\nu)} \int_{a_1}^{y} \frac{\psi'(\tau)H(\tau)}{[\psi(y) - \psi(\tau)]^{1-\frac{\nu}{k}}} d\tau, \quad y > a_1, \tag{3.15} $$

$$ {}^{RL}\mathcal{J}_{a_1^+}^{\nu,k,\psi} H(y) = \frac{1}{k\Gamma_k(\nu)} \int_{y}^{a_2} \frac{\psi'(\tau)H(\tau)}{[\psi(\tau) - \psi(y)]^{1-\frac{\nu}{k}}} d\tau, \quad y < a_2, \tag{3.16} $$

where $\Gamma_k(\cdot)$ is the k-gamma function. We call integrals (3.15) and (3.16) as $\psi_k - RL$ fractional integrals.

Remark 3.3. *The operators given by* (3.15) *and* (3.16) *have following special cases.*

(i) If we take $\psi(\tau) = \ln\tau, k = 1$*, then we achieve Hadamard fractional integral as follows [43]:*

$$ \begin{aligned} {}^{H}\mathcal{J}_{a_1^+}^{\nu} H(x) &= \frac{1}{\Gamma(\nu)} \int_{a_1}^{x} \left(\ln\frac{x}{\tau}\right)^{\nu-1} \frac{H(\tau)}{\tau} d\tau, \quad x > a_1, \\ {}^{H}\mathcal{J}_{a_2^-}^{\nu} H(x) &= \frac{1}{\Gamma(\nu)} \int_{x}^{a_2} \left(\ln\frac{\tau}{x}\right)^{\nu-1} \frac{H(\tau)}{\tau} d\tau, \quad x < a_2. \end{aligned} \tag{3.17} $$

(ii) If $\psi(\tau) = \frac{\tau^{r+\beta}}{r+\beta}, r+\beta \neq 0$*, then we have generalized conformable fractional integral as follows [45]:*

$$ \begin{aligned} {}^{GC}\mathcal{J}_{a_1^+}^{\nu,\beta,r} H(x) &= \frac{(r+\beta)^{1-\nu}}{\Gamma(\nu)} \int_{a_1}^{x} \frac{\tau^{r} H(\tau)}{(x^{r+\beta} - \tau^{r+\beta})^{1-\nu}} d_{\beta}\tau, \quad x > a_1, \\ {}^{GC}\mathcal{J}_{a_2^-}^{\nu,\beta,r} H(x) &= \frac{(r+\beta)^{1-\nu}}{\Gamma(\nu)} \int_{x}^{a_2} \frac{\tau^{r} H(\tau)}{(\tau^{r+\beta} - x^{r+\beta})^{1-\nu}} d_{\beta}\tau, \quad x < a_2. \end{aligned} \tag{3.18} $$

(iii) For $\psi(\tau) = \frac{\tau^{\rho}}{\rho}$, $\rho > 0$ *and* $k = 1$*, then we obtain Katugampola fractional integral [46].*

$$ \begin{aligned} {}^{KG}\mathcal{J}_{a_1^+}^{\nu,\rho} H(x) &= \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_{a_1}^{x} \frac{\tau^{\rho-1} H(\tau)}{(x^{\rho} - \tau^{\rho})^{1-\nu}} d\tau, \quad x > a_1, \\ {}^{KG}\mathcal{J}_{a_2^-}^{\nu,\rho} H(x) &= \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_{x}^{a_2} \frac{\tau^{\rho-1} H(\tau)}{(\tau^{\rho} - x^{\rho})^{1-\nu}} d\tau, \quad x < a_2. \end{aligned} \tag{3.19} $$

(iv) If we take $\psi(\tau) = \frac{\tau^{1+s}}{1+s}$, then we have (k, s)-fractional integral as follows [47]:

$$\begin{aligned} {}^{KS}\mathcal{J}_{a_1^+}^{\nu,k,s} H(x) &= \frac{(s+1)^{1-\frac{\nu}{k}}}{k\Gamma_k(\nu)} \int_{a_1}^{x} \frac{\tau^s H(\tau)}{\left(x^{s+1} - \tau^{s+1}\right)^{1-\frac{\nu}{k}}} d\tau, \quad x > a_1, \\ {}^{KS}\mathcal{J}_{a_2^-}^{\nu,k,s} H(x) &= \frac{(s+1)^{1-\frac{\nu}{k}}}{k\Gamma_k(\nu)} \int_{x}^{a_2} \frac{\tau^s H(\tau)}{\left(\tau^{s+1} - x^{s+1}\right)^{1-\frac{\nu}{k}}} d\tau, \quad x < a_2. \end{aligned} \tag{3.20}$$

(v) For $\psi(\tau) = \tau$, the k version of RL fractional integral can be obtained as follows [48]:

$$\begin{aligned} {}^{KRL}\mathcal{J}_{a_1^+}^{\nu,k} H(x) &= \frac{1}{\Gamma(\nu)} \int_{a_1}^{x} (x-\tau)^{\frac{\nu}{k}-1} H(\tau)\, d\tau, \quad x > a_1, \\ {}^{KRL}\mathcal{J}_{a_2^-}^{\nu,k} H(x) &= \frac{1}{\Gamma(\nu)} \int_{x}^{a_2} (\tau-x)^{\frac{\nu}{k}-1} H(\tau)\, d\tau, \quad x < a_2. \end{aligned} \tag{3.21}$$

(vi) For $\psi(\tau) = \tau, k = 1$, we can re-capture the RL fractional integral.

Definition 3.7 ([49])**.** Let $\psi : [a_1, a_2] \to \mathbb{R}$ be an increasing and positive function on $[a_1, a_2]$, and let ψ' be continuous on (a_1, a_2). Assume that $H : [a_1, a_2] \to \mathcal{T}R^+$ is an IV function with $H(\mu) = [H^-(\mu), H^+(\mu)]$ and $H \in I\mathfrak{R}_{([a,b])}$. The IV left and right-sided RL fractional integrals of H according to ψ on $[a_1, a_2]$ of order $\nu > 0$ are respectively given by:

$$ {}^{IRL}\mathcal{J}_{a_1^+}^{\nu,\psi} H(x) = \frac{1}{\Gamma(\nu)} (I\mathfrak{R}) \int_{a_1}^{x} \frac{\psi'(\tau) H(\tau)}{\left[\psi(x) - \psi(\tau)\right]^{1-\nu}} d\tau, \quad x > a_1, \tag{3.22}$$

$$ {}^{IRL}\mathcal{J}_{a_2^-}^{\nu,\psi} H(x) = \frac{1}{\Gamma(\nu)} (I\mathfrak{R}) \int_{x}^{a_2} \frac{\psi'(\tau) H(\tau)}{\left[\psi(\tau) - \psi(x)\right]^{1-\nu}} d\tau, \quad x < a_2. \tag{3.23}$$

Now, we can extend the above definition to the IV $\psi_k - RL$ fractional integrals.

Definition 3.8. Let $\psi : [a_1, a_2] \to \mathbb{R}$ be an increasing and positive function on $[a_1, a_2]$, and let ψ' be continuous on (a_1, a_2). Assume that $H : [a_1, a_2] \to \mathcal{T}R^+$ is an IV function with $H(\mu) = [H^-(\mu), H^+(\mu)]$ and $H \in I\mathfrak{R}_{([a,b])}$. The IV left and right-sided RL fractional integrals of H according to ψ on $[a_1, a_2]$ of order $\nu, k > 0$ are respectively given by:

$$ {}^{IRL}\mathcal{J}_{a_1^+}^{\nu,k,\psi} H(x) = \frac{1}{k\Gamma_k(\nu)} (I\mathfrak{R}) \int_{a_1}^{x} \frac{\psi'(\tau) H(\tau)}{\left[\psi(x) - \psi(\tau)\right]^{1-\frac{\nu}{k}}} d\tau, \quad x > a_1, \tag{3.24}$$

$$ {}^{IRL}\mathcal{J}_{a_2^-}^{\nu,k,\psi} H(x) = \frac{1}{k\Gamma_k(\nu)} (I\mathfrak{R}) \int_{x}^{a_2} \frac{\psi'(\tau) H(\tau)}{\left[\psi(\tau) - \psi(x)\right]^{1-\frac{\nu}{k}}} d\tau, \quad x < a_2, \tag{3.25}$$

where $\Gamma_k(\cdot)$ is the k-gamma function.

Theorem 3.5. *If $H : [a_1, a_2] \to \mathcal{T}R$ is an* IV *function with $H(t) = [H^-(t), H^+(t)]$ with $H^-(t), H^+(t) \in \mathfrak{R}_{([a_1,a_2])}$, then we have*

$$ {}^{IRL}\mathcal{J}_{a_1^+}^{\nu,k,\psi} H(t) = \left[{}^{RL}\mathcal{J}_{a_1^+}^{\nu,k,\psi} H^-(t), {}^{RL}\mathcal{J}_{a_1^+}^{\nu,k,\psi} H^+(t) \right], \tag{3.26}$$

and

$$ {}^{IRL}\mathcal{J}_{a_2^-}^{\nu,k,\psi} H(t) = \left[{}^{RL}\mathcal{J}_{a_2^-}^{\nu,k,\psi} H^-(t), {}^{RL}\mathcal{J}_{a_2^-}^{\nu,k,\psi} H^+(t) \right]. \tag{3.27}$$

Proof. The desired result can be obtained by making use of Theorem 3.1. □

Interval analysis have uncertainty while solving equations of the form $U + [1, 2] = [4, 6]$ are not solvable by interval arithmetic. To remove this kind of uncertainty, extended interval arithmetic is introduced. The main objective of this paper is to discover $H \cdot H$ for general classes where extended arithmetic can be use successfully. For the desirable results, we introduce the notion of generalized modified (p, h)-convex IV functions. We have develop several inequalities using extended operations on the newly defined class of convex IV functions.

4. Main results

This section deals with introducing the notion of generalized modified (p, h)-convex IV function.

Definition 4.1 ([50]). A set $\mathcal{K}_1$ is p-convex set, if for any $x_1, x_2 \in \mathcal{K}_1, \mu \in [0, 1]$, we have

$$[ua_1^p + (1-u)a_2^p]^{\frac{1}{p}} \in \mathcal{K}_1, \tag{4.1}$$

where $p = 2r_1 + 1$ or $p = \frac{m_1}{n_1}, m_1 = 2s_1, n_1 = 2k_1 + 1$ and $r_1, s_1, k_1 \in \mathbb{N}$.

Definition 4.2 ([50]). A mapping H defined from a p-convex set $\mathcal{K}_1$ to $\mathbb{R}$ is said to be p-convex function, if

$$H([ua_1^p + (1-u)a_2^p]^{\frac{1}{p}}) \leq uH(a_1) + (1-u)H(a_2) \tag{4.2}$$

for all $a_1, a_2 \in \mathcal{K}_1, u \in [0, 1]$.

Definition 4.3 ([51]). The mapping H defined from $\mathcal{K}_1$ to $\mathbb{R}$ is said to be η-convex if

$$H(\mu a_1 + (1-\mu)a_2) \leq H(a_2) + \mu\eta(H(a_1), H(a_2)) \tag{4.3}$$

with respect to $\eta : B_2 \times B_2 \to B_3$ for appropriate $B_2, B_3 \subseteq \mathbb{R}$ for each $a_1, a_2 \in \mathcal{K}_1, \mu \in [0, 1]$.

Definition 4.4 ([52]). The mapping H defined from $\mathcal{K}_1$ to $\mathbb{R}$ is said to be generalized p-convex, if

$$H([ua_1^p + (1-u)a_2^p]^{\frac{1}{p}}) \leq H(a_2) + \mu\eta(H(a_1), H(a_2)) \tag{4.4}$$

for $\eta : B_2 \times B_2 \to B_3$ a bi-function for appropriate $B_2, B_3 \subseteq \mathbb{R}$ and for each $a_1, a_2 \in \mathcal{K}_1, \mu \in [0, 1]$.

Definition 4.5 ([53]). Let $h : [0, 1] \to \mathbb{R}_0^+ := [0, \infty)$ and $H : \Xi \subseteq \mathbb{R} \to \mathbb{R}_0^+$. Then, H is called modified h-convex function if we have

$$H(\mu\gamma + (1-\mu)\delta) \leq h(\mu)H(\gamma) + (1 - h(\mu))H(\delta), \tag{4.5}$$

for all $\gamma, \delta \in \Xi$ and $\mu \in (0, 1)$. It is important to notice that we respectively have convex function and $(\alpha, 1)$-convex functions provided that $h(\mu) = \mu, h(\mu) = \mu^\alpha, \alpha \in (0, 1)$.

Throughout the study, we consider an IV bi-function $\eta : \Delta_{\mathcal{T}} \times \Delta_{\mathcal{T}} \to \Delta_{\mathcal{T}}$ for suitable $\Delta_{\mathcal{T}} \subseteq \mathcal{H}$. Furthermore, if $U = [3, 5]$ is a proper interval and $V = [-2, -3]$ an improper interval, then $U + V = [1, 2]$ is a proper positive interval. By this motivation, we may restrict η, so that $U + \eta(V, W) \in \mathcal{T}R^+$.

Now we introduce the notion of generalized modified (p, h)-convex IV function.

Definition 4.6. Let $h : [a_1, a_2] \to \mathbb{R}$ be a non-negative function, $(0, 1) \subset [a_1, a_2]$. Let $\eta : \Delta_{\mathcal{T}} \times \Delta_{\mathcal{T}} \to \Delta_{\mathcal{T}}$ where $\Delta_{\mathcal{T}} \subseteq \mathcal{H}$. A mapping $H : \mathcal{K}_1 \to \mathcal{T}R^+$ is said to be generalized modified (p, h)-convex IV function with respect to η, if

$$H(a_2) + h(\mu)\eta\left(H(a_1), H(a_2)\right) \subseteq H\left([\mu a_1^p + (1-\mu)a_2^p]^{\frac{1}{p}}\right), \tag{4.6}$$

for all $a_1, a_2 \in \mathcal{K}_1, \mu \in [0, 1]$.

Now we present some special cases by taking into account the algebraic properties of intervals.

Remark 4.1. *The generalized modified* (p, h)*-convex* IV *function has following special cases.*

(a) If $h(\mu) = \mu$, then H is generalized p-convex IV *function [26].*
(b) If $p = 1$, then H is called generalized modified h-convex IV *function.*
(c) If $p = 1$ and $\eta(H(a_1), H(a_2)) = H(a_1) - H(a_2)$, then (4.6) *gives*

$$h(\mu)H(a_1) + (1 - h(\mu))H(a_2) \subseteq H\left([\mu a_1 + (1-\mu)a_2]\right). \tag{4.7}$$

We call it modified h-convex IV *function. Furthermore, if $h(\mu) = \mu^s$, $s \in (0, 1)$, then H is called s-convex* IV *function of first kind.*

(d) If $\eta(H(a_1), H(a_2)) = H(a_1) - H(a_2)$, then

$$h(\mu)H(a_1) + (1 - h(\mu))H(a_2) \subseteq H\left([\mu a_1^p + (1-\mu)a_2^p]^{\frac{1}{p}}\right). \tag{4.8}$$

In this case H is modified (p, h)-convex IV *function. Moreover, if $h(\mu) = \mu^s$, $s \in (0, 1)$, then H is a (p, s)-convex* IV *function of first kind. Furthermore, if h is identity function, then we obtain p-convex* IV *function.*

(e) If $h(\mu) = \mu$, $p = 1$ and $\eta(H(a_1), H(a_2)) = H(a_1) - H(a_2)$,

$$\mu H(a_1) + (1-\mu)H(a_2) \subseteq H\left([\mu a_1 + (1-\mu)a_2]\right), \tag{4.9}$$

which is convex IV *function, see [39].*

(f) If $p = -1$, then (4.6) *gives*

$$H(a_2) + h(\mu)\eta\left(H(a_1), H(a_2)\right) \subseteq H\left(\frac{a_1 a_2}{(1-\mu)a_1 + \mu a_2}\right), \tag{4.10}$$

we say H is a generalized modified harmonically h-convex IV *function. We have following cases for harmonic convexity subject to the different choices of h and η as following.*

(a) If $\eta(H(a_1), H(a_2)) = H(a_1) - H(a_2)$, then (4.10) *reduces to*

$$h(\mu)H(a_1) + (1 - h(\mu))H(a_2) \subseteq H\left(\frac{a_1 a_2}{(1-\mu)a_1 + \mu a_2}\right), \tag{4.11}$$

we call it harmonically modified h-convex IV *function.*

(b) If $\eta(H(a_1), H(a_2)) = H(a_1) - H(a_2)$ and $h(\mu) = \mu^s$, $s \in (0, 1)$, then (4.10) *reduces to*

$$\mu^s H(a_1) + (1 - \mu^s)H(a_2) \subseteq H\left(\frac{a_1 a_2}{(1-\mu)a_1 + \mu a_2}\right), \tag{4.12}$$

we call it harmonically s-convex IV *function of first kind.*

(c) If $\eta(H(a_1), H(a_2)) = H(a_1) - H(a_2)$ and $h(\mu) = \mu$, then (4.10) *reduces to*

$$\mu H(a_1) + (1-\mu)H(a_2) \subseteq H\left(\frac{a_1 a_2}{(1-\mu)a_1 + \mu a_2}\right), \tag{4.13}$$

we call it harmonically convex IV *function.*

We start with the following double inequality of $H \cdot H$ type.

Theorem 4.1. *Let $\eta : \Delta_{\mathcal{T}} \times \Delta_{\mathcal{T}} \to \Delta_{\mathcal{T}}$ where $\Delta_{\mathcal{T}} \subseteq \mathcal{H}$. Let $H : [a_1, a_2] \to \mathcal{T}R^+$ be a generalized modified (p, h)-convex* IV *function with respect to η such that $h(t) = [H^-(t), H^+(t)]$, then for $\nu, k > 0$, we have*

$$\begin{aligned} &H\left(\left[\frac{a_1^p + a_2^p}{2}\right]^{\frac{1}{p}}\right) - Q \\ &\qquad \supseteq \frac{\Gamma_k(\nu+k)}{2(a_2^p - a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_1^p)^+}(H\circ\Psi)(\psi^{-1}(a_2^p)) + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_2^p)^-}(H\circ\Psi)(\psi^{-1}(a_1^p))\right] \\ &\qquad\qquad \supseteq \frac{H(a_1)+H(a_2)}{2} + \frac{\nu}{2k}\left[\eta(H(a_1), H(a_2)) + \eta(H(a_2), H(a_1))\right]\int_0^1 \mu^{\frac{\nu}{k}-1}h(\mu)d\mu, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} Q = \frac{h\left(\frac{1}{2}\right)}{2}(I\mathfrak{R})\int_0^1 \mu^{\frac{\nu}{k}-1}\Big\{&\eta\left(H\left([\mu a_1^p + (1-\mu)a_2^p]^{\frac{1}{p}}\right), H\left([\mu a_2^p + (1-\mu)a_1^p]^{\frac{1}{p}}\right)\right) \\ &+ \eta\left(H\left([\mu a_2^p + (1-\mu)a_1^p]^{\frac{1}{p}}\right), H\left([\mu a_1^p + (1-\mu)a_2^p]^{\frac{1}{p}}\right)\right)\Big\}d\mu, \end{aligned} \tag{4.15}$$

and $\Psi(t) = (\psi(t))^{\frac{1}{p}}$.

Proof. By the notion of generalized modified (p, h)-convexity of IV function H, we have

$$H\left(\left[\frac{x_1^p + x_2^p}{2}\right]^{\frac{1}{p}}\right) \supseteq H(x_2) + h\left(\frac{1}{2}\right)\eta\left(H(x_1), H(x_2)\right) \tag{4.16}$$

for all $x_1, x_2 \in [a_1, a_2]$. By taking $x_1^p = \mu a_1^p + (1-\mu)a_2^p$ and $x_2^p = \mu a_2^p + (1-\mu)a_1^p$, $\mu \in [0, 1]$ in (4.16), we get

$$\begin{aligned} H\left(\left[\frac{a_1^p + a_2^p}{2}\right]^{\frac{1}{p}}\right) &\supseteq H\left([\mu a_2^p + (1-\mu)a_1^p]^{\frac{1}{p}}\right) \\ &\quad + h\left(\frac{1}{2}\right)\eta\left(H\left([\mu a_1^p + (1-\mu)a_2^p]^{\frac{1}{p}}\right), H\left([\mu a_2^p + (1-\mu)a_1^p]^{\frac{1}{p}}\right)\right). \end{aligned} \tag{4.17}$$

By choosing $x_1^p = \mu a_2^p + (1-\mu)a_1^p$ and $x_2^p = \mu a_1^p + (1-\mu)a_2^p$ in (4.16), we have

$$H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)\supseteq H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)$$
$$+h\left(\frac{1}{2}\right)\eta\left(H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right),H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)\right). \quad (4.18)$$

Summing up (4.17) and (4.18) and multiplying by $\mu^{\frac{\nu}{k}-1}$, we obtain

$$\mu^{\frac{\nu}{k}-1}H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)-\frac{h\left(\frac{1}{2}\right)\mu^{\frac{\nu}{k}-1}}{2}\left\{\eta\left(H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right),H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right)\right.$$
$$\left.+\,\eta\left(H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right),H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)\right)\right\}$$
$$\supseteq\frac{\mu^{\frac{\nu}{k}-1}}{2}\left[H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)+H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right]. \quad (4.19)$$

Now, by interval integration, we obtain

$$(I\Re)\int_0^1\mu^{\frac{\nu}{k}-1}H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)d\mu$$
$$-\frac{h\left(\frac{1}{2}\right)}{2}(I\Re)\int_0^1\mu^{\frac{\nu}{k}-1}\left\{\eta\left(H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right),H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right)\right.$$
$$\left.+\,\eta\left(H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right),H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)\right)\right\}d\mu$$
$$\supseteq(I\Re)\int_0^1\mu^{\frac{\nu}{k}-1}\left[H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)+H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right]d\mu. \quad (4.20)$$

By the application of interval integration,

$$(I\Re)\int_0^1\mu^{\frac{\nu}{k}-1}H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)d\mu$$
$$=\left[(\Re)\int_0^1\mu^{\frac{\nu}{k}-1}H^-\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)d\mu,(\Re)\int_0^1\mu^{\frac{\nu}{k}-1}H^+\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)d\mu\right]$$
$$=\left[H^-\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)(\Re)\int_0^1\mu^{\frac{\nu}{k}-1}d\mu,H^+\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)(\Re)\int_0^1\mu^{\frac{\nu}{k}-1}d\mu\right]$$
$$=\left[\frac{k}{\nu}H^-\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right),\frac{k}{\nu}H^+\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)\right]=\frac{k}{\nu}H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right). \quad (4.21)$$

Furthermore,

$$(I\Re)\int_0^1 \mu^{\frac{\nu}{k}-1}\left(H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)+H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right)d\mu$$

$$=\Bigg[(\Re)\int_0^1 \mu^{\frac{\nu}{k}-1}\left(H^-\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)+H^-\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right)d\mu\,,$$

$$(\Re)\int_0^1 \mu^{\frac{\nu}{k}-1}\left(H^+\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)+H^+\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right)d\mu\Bigg]. \quad (4.22)$$

Since,

$$(\Re)\int_0^1 \mu^{\frac{\nu}{k}-1}\left(H^-\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)+H^-\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right)d\mu$$

$$=(\Re)\int_0^1 \mu^{\frac{\nu}{k}-1}\left(H^-[\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)d\mu+(\Re)\int_0^1 \lambda^{\frac{\nu}{k}-1}H^-\left([\lambda a_2^p+(1-\lambda)a_1^p]^{\frac{1}{p}}\right)d\lambda. \quad (4.23)$$

By selecting $\frac{\psi(\tau)-a_1^p}{a_2^p-a_1^p}=\lambda$ and $\frac{a_2^p-\psi(\tau)}{a_2^p-a_1^p}=\mu$, then

$$(\Re)\int_0^1 \mu^{\frac{\nu}{k}-1}\left(H^-[\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)d\mu+(\Re)\int_0^1 \lambda^{\frac{\nu}{k}-1}H^-\left([\lambda a_2^p+(1-\lambda)a_1^p]^{\frac{1}{p}}\right)d\lambda$$

$$=\frac{1}{(a_2^p-a_1^p)^{\frac{\nu}{k}}}\Bigg\{(\Re)\int_{\psi^{-1}(a_1^p)}^{\psi^{-1}(a_2^p)}\frac{\psi'(\tau)}{(a_2^p-\psi(\tau))^{1-\frac{\nu}{k}}}H^-\left([\psi(\tau)]^{\frac{1}{p}}\right)d\tau$$

$$+(\Re)\int_{\psi^{-1}(a_1^p)}^{\psi^{-1}(a_2^p)}\frac{\psi'(\tau)}{(\psi(\tau)-a_1^p)^{1-\frac{\nu}{k}}}H^-\left([\psi(\tau)]^{\frac{1}{p}}\right)d\tau\Bigg\}$$

$$=\frac{k\Gamma_k(\nu)}{(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{RL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_1^p)^+}(H^-\circ\Psi)(\psi^{-1}(a_2^p))+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_2^p)^-}(H^-\circ\Psi)(\psi^{-1}(a_1^p))\right], \quad (4.24)$$

where $\Psi(\tau)=[\psi(\tau)]^{\frac{1}{p}}$. Similarly,

$$(\Re)\int_0^1 \mu^{\frac{\nu}{k}-1}\left(H^+\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)+H^+\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right)d\mu$$

$$=\frac{k\Gamma_k(\nu)}{(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{RL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_1^p)^+}(H^+\circ\Psi)(\psi^{-1}(a_2^p))+{}^{RL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_2^p)^-}(H^+\circ\Psi)(\psi^{-1}(a_1^p))\right]. \quad (4.25)$$

Hence,

$$(I\Re)\int_0^1 \frac{\mu^{\frac{\nu}{k}-1}}{2}\left(H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)+H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right)d\mu$$

$$=\frac{k\Gamma_k(\nu)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_1^p)^+}(H\circ\Psi)(\psi^{-1}(a_2^p))+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_2^p)^-}(H\circ\Psi)(\psi^{-1}(a_1^p))\right]. \tag{4.26}$$

By using (4.21) and (4.26) in (4.20), we have

$$H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)-Q$$

$$\supseteq\frac{\Gamma_k(\nu+k)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_1^p)^+}(H\circ\Psi)(\psi^{-1}(a_2^p))+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_2^p)^-}(H\circ\Psi)(\psi^{-1}(a_1^p))\right]. \tag{4.27}$$

Again by the generalized modified (p,h)-convexity,

$$\frac{\mu^{\frac{\nu}{k}-1}}{2}\left[H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)+H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right]$$

$$\supseteq\frac{\mu^{\frac{\nu}{k}-1}}{2}\left[H(a_1)+H(a_2)+h(\mu)\left(\eta(H(a_1),H(a_2))+\eta(H(a_2),H(a_1))\right)\right]. \tag{4.28}$$

After interval integration, we get the second inequality, which then completes the proof. □

Example 4.1. Let $H(t)=[t^p,4-e^{t^p}]$, $\eta(U,V)=U-V$. Also let $a_1=-1, a_2=1$. Furthermore, ψ is identity and $\nu=k=1$

$$H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)-Q=[0,3], \tag{4.29}$$

and

$$(I\Re)\int_0^1\left(H\left([\mu a_1^p+(1-\mu)a_2^p]^{\frac{1}{p}}\right)+H\left([\mu a_2^p+(1-\mu)a_1^p]^{\frac{1}{p}}\right)\right)d\mu$$

$$=(I\Re)\int_0^1\left(H(1-2\mu)+H(2\mu-1)\right)d\mu$$

$$=\left[0,\Re\int_0^1\left(8-e^{1-2\mu}+e^{2\mu-1}\right)d\mu\right]=\left[0,8+\frac{e^{-1}-e^1}{2}\right]. \tag{4.30}$$

Also, we have

$$\frac{H(a_1)+H(a_2)}{2}=\frac{1}{2}[0,8-e^{-1}-e]. \tag{4.31}$$

It is clear that this example support the result presented in (4.14).

Remark 4.2. *If $h(\mu) = \mu$, then Theorem 4.1 gives the fractional counterpart of [26, Theorem 3].*

Corollary 4.1. *Let the conditions of Theorem 4.1 be satisfied, then it is easy to see that*

- *If H is modified h-convex* IV *function, then we have*

$$H\left(\frac{a_1+a_2}{2}\right) \supseteq \frac{\Gamma_k(\nu+k)}{2(a_2-a_1)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_1)^+}(H\circ\Psi)(\psi^{-1}(a_2)) + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_2)^-}(H\circ\Psi)(\psi^{-1}(a_1))\right] \supseteq \frac{H(a_1)+H(a_2)}{2}. \quad (4.32)$$

- *If H is modified (p,h)-convex* IV *function, then*

$$H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right) \supseteq \frac{\Gamma_k(\nu+k)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_1^p)^+}(H\circ\Psi)(\psi^{-1}(a_2^p)) + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}(a_2^p)^-}(H\circ\Psi)(\psi^{-1}(a_1^p))\right] \supseteq \frac{H(a_1)+H(a_2)}{2}. \quad (4.33)$$

- *If H is a generalized modified harmonically h-convex* IV *function, then*

$$\begin{aligned} H\left(\frac{2a_1a_2}{a_1+a_2}\right) - Q_1 &\supseteq \frac{\Gamma_k(\nu+k)}{2}\left(\frac{a_1a_2}{a_2-a_1}\right)^{\frac{\nu}{k}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}\left(\frac{1}{a_1}\right)^-}(H\circ\Psi)\left(\psi^{-1}\left(\frac{1}{a_2}\right)\right)\right. \\ &\qquad \left. + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}\left(\frac{1}{a_2}\right)^+}(H\circ\Psi)\left(\psi^{-1}\left(\frac{1}{a_1}\right)\right)\right] \\ &\supseteq \frac{H(a_1)+H(a_2)}{2} + \frac{\nu}{2k}\left[\eta(H(a_1),H(a_2)) + \eta(H(a_2),H(a_1))\right]\int_0^1 \mu^{\frac{\nu}{k}-1}h(\mu)d\mu. \quad (4.34) \end{aligned}$$

- *If H is harmonically modified h-convex function, then* (4.34) *reduces to*

$$\begin{aligned} H\left(\frac{2a_1a_2}{a_1+a_2}\right) &\supseteq \frac{\Gamma_k(\nu+k)}{2}\left(\frac{a_1a_2}{a_2-a_1}\right)^{\frac{\nu}{k}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}\left(\frac{1}{a_1}\right)^-}(H\circ\Psi)\left(\psi^{-1}\left(\frac{1}{a_2}\right)\right)\right. \\ &\qquad \left. + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^{-1}\left(\frac{1}{a_2}\right)^+}(H\circ\Psi)\left(\psi^{-1}\left(\frac{1}{a_1}\right)\right)\right] \\ &\supseteq \frac{H(a_1)+H(a_2)}{2}. \quad (4.35) \end{aligned}$$

Remark 4.3. *If H is harmonically convex* IV *function, then* (4.34) *again produces* (4.35).

Theorem 4.2. *Let $\eta : \Delta_{\mathcal{T}} \times \Delta_{\mathcal{T}} \to \Delta_{\mathcal{T}}$ where $\Delta_{\mathcal{T}} \subseteq \mathcal{T}R^+$. Let $H, K : [a_1, a_2] \to \mathcal{T}R^+$ be generalized modified (p,h)-convex functions with respect to η such that $H(t) = [H^-(t), H^+(t)]$ and $K(t) = [K^-(t), K^+(t)]$, then we have:*

$$\frac{\Gamma_k(\nu+k)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1^p)^+}(HK\circ\Psi)(\psi^-(a_2^p))+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2^p)^-}(HK\circ\Psi)(\psi^-(a_1^p))\right]$$
$$\supseteq (I\mathfrak{R})\int_0^1 \mu^{\frac{\nu}{k}-1}[\mathcal{M}(\mu)+\mathcal{N}(\mu)]d\mu, \quad (4.36)$$

where

$$\mathcal{M}(\mu)=[H(a_2)+h(\mu)\eta(H(a_1),H(a_2))][K(a_2)+h(\mu)\eta(K(a_1),K(a_2))],$$
$$\mathcal{N}(\mu)=[H(a_1)+h(\mu)\eta(H(a_2),H(a_1))][K(a_1)+h(\mu)\eta(K(a_2),K(a_1))].$$

Proof. By the indicated generalized (p,h)-convexity of H and K, we have

$$H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)\supseteq H(a_2)+h(\mu)\eta(H(a_1),H(a_2)), \quad (4.37)$$

and

$$K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)\supseteq K(a_2)+h(\mu)\eta(K(a_1),K(a_2)). \quad (4.38)$$

Multiplying (4.37) and (4.38), we have

$$H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)$$
$$\supseteq [H(a_2)+h(\mu)\eta(H(a_1),H(a_2))][K(a_2)+h(\mu)\eta(K(a_1),K(a_2))]. \quad (4.39)$$

Similarly,

$$H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)$$
$$\supseteq [H(a_1)+h(\mu)\eta(H(a_2),H(a_1))][K(a_1)+h(\mu)\eta(K(a_2),K(a_1))]. \quad (4.40)$$

On summing (4.39) and (4.40), we get

$$H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)+H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)$$
$$\supseteq \mathcal{M}(\mu)+\mathcal{N}(\mu). \quad (4.41)$$

Multiplying the above inequality by $\frac{\mu^{\frac{\nu}{k}-1}}{2}$, and applying interval integration,

$$(I\mathfrak{R})\int_0^1 \frac{\mu^{\frac{\nu}{k}-1}}{2}H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)d\mu$$
$$+(I\mathfrak{R})\int_0^1 \frac{\mu^{\frac{\nu}{k}-1}}{2}H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)d\mu$$

$$\supseteq (I\mathfrak{R})\int_{0}^{1}\frac{\mu^{\frac{\nu}{k}-1}}{2}[\mathcal{M}(\mu)+\mathcal{N}(\mu)]d\mu. \quad (4.42)$$

By applying Theorem 4.1, we get

$$(I\mathfrak{R})\int_{0}^{1}\frac{\mu^{\frac{\nu}{k}-1}}{2}H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)d\mu$$

$$=\frac{k\Gamma_k(\nu)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\,{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1^p)^+}\left(HK\circ\Psi\right)\left(\psi^-(a_2^p)\right), \quad (4.43)$$

and

$$(I\mathfrak{R})\int_{0}^{1}\frac{\mu^{\frac{\nu}{k}-1}}{2}H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)d\mu$$

$$=\frac{k\Gamma_k(\nu)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\,{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2^p)^-}\left(HK\circ\Psi\right)\left(\psi^-(a_1^p)\right). \quad (4.44)$$

Consequently, we can obtain the desired result (4.36) by substituting (4.43) and (4.44) in (4.42). □

Corollary 4.2. *If H and K are modified (p,h)-convex* IV *functions, then*

$$\frac{\Gamma_k(\nu+k)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1^p)^+}\left(HK\circ\Psi\right)\left(\psi^-(a_2^p)\right)+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2^p)^-}\left(HK\circ\Psi\right)\left(\psi^-(a_1^p)\right)\right]$$

$$\supseteq\frac{\nu M(a_1,a_2)}{2k}\int_{0}^{1}\mu^{\frac{\nu}{k}-1}[(h(\mu))^2+(1-h(\mu))^2]d\mu+\frac{\nu N(a_1,a_2)}{k}\int_{0}^{1}\mu^{\frac{\nu}{k}-1}h(\mu)(1-h(\mu))d\mu, \quad (4.45)$$

where $M(a_1,a_2)$ and $N(a_1,a_2)$ are same as defined in Theorem 3.4.

Proof. Let $\eta(U,V)=U-V$, then

$$\begin{aligned}\mathcal{M}(\mu)+\mathcal{N}(\mu)&=[h(\mu)H(a_1)+(1-h(\mu))H(a_2)][h(\mu)K(a_1)+(1-h(\mu))K(a_2)]\\&=[H(a_1)K(a_1)+H(a_2)K(a_2)][(h(\mu))^2+(1-h(\mu))^2]\\&\quad+2[H(a_1)K(a_2)+H(a_2)K(a_1)]h(\mu)(1-h(\mu)).\end{aligned} \quad (4.46)$$

Using (4.46) in (4.36), we get the desired inequality. □

Corollary 4.3. *Let the conditions of Theorem 4.2 be satisfied, then one can see that*

- *If H and K are p-convex* IV *functions, then*

$$\frac{\Gamma_k(\nu+k)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1^p)^+}\left(HK\circ\Psi\right)\left(\psi^-(a_2^p)\right)+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2^p)^-}\left(HK\circ\Psi\right)\left(\psi^-(a_1^p)\right)\right]$$

$$\supseteq\left[\frac{1}{2}-\frac{k\nu}{(\nu+k)(\nu+2k)}\right]M(a_1,a_2)+\frac{k\nu}{(\nu+k)(\nu+2k)}N(a_1,a_2), \quad (4.47)$$

where $M(a_1,a_2)$ and $N(a_1,a_2)$ are same as defined in Theorem 3.4.

- *If H and K are modified h-convex* IV *functions, then*

$$\frac{\Gamma_k(\nu+k)}{2(a_2-a_1)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1)^+}(HK\circ\Psi)(\psi^-(a_2))+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2)^-}(HK\circ\Psi)(\psi^-(a_1))\right]$$
$$\supseteq\frac{\nu M(a_1,a_2)}{2k}\int_0^1\mu^{\frac{\nu}{k}-1}[(h(\mu))^2+(1-h(\mu))^2]d\mu+\frac{\nu N(a_1,a_2)}{k}\int_0^1\mu^{\frac{\nu}{k}-1}h(\mu)(1-h(\mu))d\mu. \quad (4.48)$$

- *If H is convex* IV *function, then*

$$\frac{\Gamma_k(\nu+k)}{2(a_2-a_1)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1)^+}(HK\circ\Psi)(\psi^-(a_2))+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2)^-}(HK\circ\Psi)(\psi^-(a_1))\right]$$
$$\supseteq\left[\frac{1}{2}-\frac{k\nu}{(\nu+k)(\nu+2k)}\right]M(a_1,a_2)+\frac{k\nu}{(\nu+k)(\nu+2k)}N(a_1,a_2). \quad (4.49)$$

- *If H and K are (p,s)-convex* IV *functions of first kind, then*

$$\frac{\Gamma_k(\nu+k)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1^p)^+}(HK\circ\Psi)(\psi^-(a_2^p))+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2^p)^-}(HK\circ\Psi)(\psi^-(a_1^p))\right]$$
$$\supseteq\left[\frac{1}{2}-\frac{k\nu s}{(\nu+ks)(\nu+2ks)}\right]M(a_1,a_2)+\frac{k\nu s}{(\nu+ks)(\nu+2ks)}N(a_1,a_2). \quad (4.50)$$

- *If H and K are harmonically convex* IV *functions, then*

$$\frac{\Gamma_k(\nu+k)}{2}\left(\frac{a_1a_2}{a_2-a_1}\right)^{\frac{\nu}{k}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-\left(\frac{1}{a_2}\right)^+}(HK\circ\Psi)\left(\psi^-\left(\frac{1}{a_1}\right)\right)\right.$$
$$\left.+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-\left(\frac{1}{a_1}\right)^-}(HK\circ\Psi)\left(\psi^-\left(\frac{1}{a_2}\right)\right)\right]$$
$$\supseteq\left[\frac{1}{2}-\frac{k\nu}{(\nu+k)(\nu+2k)}\right]M(a_1,a_2)+\frac{k\nu}{(\nu+k)(\nu+2k)}N(a_1,a_2). \quad (4.51)$$

- *If H and K are modified harmonically h-convex* IV *functions, then*

$$\frac{\Gamma_k(\nu+k)}{2}\left(\frac{a_1a_2}{a_2-a_1}\right)^{\frac{\nu}{k}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-\left(\frac{1}{a_2}\right)^+}(HK\circ\Psi)\left(\psi^-\left(\frac{1}{a_1}\right)\right)\right.$$
$$\left.+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-\left(\frac{1}{a_1}\right)^-}(HK\circ\Psi)\left(\psi^-\left(\frac{1}{a_2}\right)\right)\right]$$
$$\supseteq\frac{\nu M(a_1,a_2)}{2k}\int_0^1\mu^{\frac{\nu}{k}-1}[(h(\mu))^2+(1-h(\mu))^2]d\mu+\frac{\nu N(a_1,a_2)}{k}\int_0^1\mu^{\frac{\nu}{k}-1}h(\mu)(1-h(\mu))d\mu. \quad (4.52)$$

- *If H and K are harmonically s-convex* IV *functions of first kind, then*

$$\frac{\Gamma_k(\nu+k)}{2}\left(\frac{a_1a_2}{a_2-a_1}\right)^{\frac{\nu}{k}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-\left(\frac{1}{a_2}\right)^+}(HK\circ\Psi)\left(\psi^-\left(\frac{1}{a_1}\right)\right)\right.$$
$$\left.+\,{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-\left(\frac{1}{a_1}\right)^-}(HK\circ\Psi)\left(\psi^-\left(\frac{1}{a_2}\right)\right)\right]$$
$$\supseteq\left[\frac{1}{2}-\frac{k\nu s}{(\nu+ks)(\nu+2ks)}\right]M(a_1,a_2)+\frac{k\nu s}{(\nu+ks)(\nu+2ks)}N(a_1,a_2). \quad (4.53)$$

Theorem 4.3. *Let $\eta:\Delta_I\times\Delta_I\to\Delta_I$, where $\Delta_I\subset\mathcal{T}R^+$. Let $H,K:[a_1,a_2]\to\mathcal{T}R^+$ be generalized modified (p,h)-convex functions with respect to η such that $H(t)=[H^-(t),H^+(t)]$ and $K(t)=[K^-(t),K^+(t)]$. then we have:*

$$2H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)K\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)$$
$$\supseteq\frac{\Gamma_k(\nu+k)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1^p)^+}(HK\circ\Psi)(\psi^-(a_2^p))+{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2^p)^-}(HK\circ\Psi)(\psi^-(a_1^p))\right]$$
$$+\frac{1}{2}[N(a_1,a_2)+\frac{\nu}{k}\mathcal{P}(a_1,a_2)\int_0^1\mu^{\frac{\nu}{k}-1}h(\mu)d\mu+\frac{\nu}{k}\mathcal{S}\int_0^1\mu^{\frac{\nu}{k}-1}h^2(\mu)d\mu]$$
$$+\frac{h\left(\frac{1}{2}\right)}{2}(IR)\int_0^1\mu^{\frac{\nu}{k}-1}\mathcal{A}d\mu+\frac{h^2\left(\frac{1}{2}\right)}{2}(IR)\int_0^1\mu^{\frac{\nu}{k}-1}Cd\mu. \quad (4.54)$$

where

$$\mathcal{A}=\left(H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)+H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right)$$
$$\times\left\{\eta\left(K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right),K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right)\right.$$
$$\left.+\,\eta\left(K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right),K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)\right)\right\}$$
$$+\left(K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)+K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right)$$
$$\times\left\{\eta\left(H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right),H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right)\right.$$
$$\left.+\,\eta\left(H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right),H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)\right)\right\}, \quad (4.55)$$

$$C=\left(\eta\left(H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right),H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right)\right.$$
$$\left.+\,\eta\left(H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right),H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)\right)\right)$$
$$\times\left(\eta\left(K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right),K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right)\right.$$

$$+\eta\left(K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right),K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)\right)\right), \quad (4.56)$$

$$\mathcal{P}=H(a_1)\eta(K(a_1),K(a_2))+H(a_2)\eta(K(a_2),K(a_1))+K(a_1)\eta(H(a_1),H(a_2)) \\ +K(a_2)\eta(H(a_2),H(a_1)), \quad (4.57)$$

$$\mathcal{S}=\eta(H(a_1),H(a_2))\eta(K(a_2),K(a_1))+\eta(H(a_2),H(a_1))\eta(K(a_1),K(a_2)), \quad (4.58)$$

and $N(a_1,a_2)$ is defined in Theorem 3.4.

Proof. By the IV generalized modified (p,h)-convexity of H and K, we have

$$\begin{aligned} &H\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right)K\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right)\\ &\supseteq\left(\frac{1}{2}[H(x)+H(y)]+\frac{h\left(\frac{1}{2}\right)}{2}\{\eta(H(x),H(y))+\eta(H(y),H(x))\}\right)\\ &\times\left(\frac{1}{2}[K(x)+K(y)]+\frac{h\left(\frac{1}{2}\right)}{2}\{\eta(K(x),K(y))+\eta(K(y),K(x))\}\right)\end{aligned}$$

$$\begin{aligned} &=\frac{1}{4}[H(x)K(x)+H(y)K(y)]+\frac{1}{4}[H(x)K(y)+H(y)K(x)]\\ &+\frac{h\left(\frac{1}{2}\right)}{4}\{[H(x)+H(y)][\eta(K(x),K(y))+\eta(K(y),K(x))]\}\\ &+\frac{h\left(\frac{1}{2}\right)}{4}\{[K(x)+K(y)][\eta(H(x),H(y))+\eta(H(y),H(x))]\}\\ &+\frac{h^2\left(\frac{1}{2}\right)}{4}\{\eta(H(x),H(y))+\eta(H(y),H(x))\}\{\eta(K(x),K(y))+\eta(K(y),K(x))\}. \end{aligned} \quad (4.59)$$

Using $x=\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}$ and $y=\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}$ and using indicated convexity, we have

$$\begin{aligned} H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)K\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)d\mu &\supseteq\frac{1}{4}\Big[H\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_1^p+(1-\mu)a_2^p\right]^{\frac{1}{p}}\right)\\ &+H\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_2^p+(1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\Big]\\ &+\frac{1}{4}[N(a_1,a_2)+\mathcal{P}(a_1,a_2)h(\mu)+\mathcal{S}(a_1,a_2)h^2(\mu)]+\frac{h\left(\frac{1}{2}\right)}{4}\mathcal{A}+\frac{h^2\left(\frac{1}{2}\right)}{4}C. \end{aligned} \quad (4.60)$$

After multiplying the last inequality with $\mu^{\frac{\nu}{k}-1}$ and then applying the interval integration over $[0,1]$, the desired inequality (4.54) yields. $\square$

Remark 4.4. *It is important to notice that, if B_I is a suitable subset of $\mathcal{H}$ the extended interval space, then we could not use the distributive property. In that case the inequality of Theorem 4.3 will be:*

$$2H\left(\left[\frac{a_1^p + a_2^p}{2}\right]^{\frac{1}{p}}\right)K\left(\left[\frac{a_1^p + a_2^p}{2}\right]^{\frac{1}{p}}\right) \supseteq \frac{\nu}{k}(I\mathfrak{R})\int_0^1 \mu^{\frac{\nu}{k}-1}ABd\mu, \tag{4.61}$$

where

$$\begin{aligned} A = \frac{1}{2}&\left[H\left(\left[\mu a_1^p + (1-\mu)a_2^p\right]^{\frac{1}{p}}\right) + H\left(\left[\mu a_2^p + (1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right] \\ &+ \frac{h\left(\frac{1}{2}\right)}{2}\left\{\eta\left(H\left(\left[\mu a_1^p + (1-\mu)a_2^p\right]^{\frac{1}{p}}\right), H\left(\left[\mu a_2^p + (1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right)\right. \\ &\left.+ \eta\left(H\left(\left[\mu a_2^p + (1-\mu)a_1^p\right]^{\frac{1}{p}}\right), H\left(\left[\mu a_1^p + (1-\mu)a_2^p\right]^{\frac{1}{p}}\right)\right)\right\}, \end{aligned} \tag{4.62}$$

and

$$\begin{aligned} B = \frac{1}{2}&\left[K\left(\left[\mu a_1^p + (1-\mu)a_2^p\right]^{\frac{1}{p}}\right) + K\left(\left[\mu a_2^p + (1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right] \\ &+ \frac{h\left(\frac{1}{2}\right)}{2}\left\{\eta\left(K\left(\left[\mu a_1^p + (1-\mu)a_2^p\right]^{\frac{1}{p}}\right), K\left(\left[\mu a_2^p + (1-\mu)a_1^p\right]^{\frac{1}{p}}\right)\right)\right. \\ &\left.+ \eta\left(K\left(\left[\mu a_2^p + (1-\mu)a_1^p\right]^{\frac{1}{p}}\right), K\left(\left[\mu a_1^p + (1-\mu)a_2^p\right]^{\frac{1}{p}}\right)\right)\right\}. \end{aligned} \tag{4.63}$$

Corollary 4.4. *If H and K are modified (p, h)-convex* IV *functions in Theorem 4.3, then*

$$\begin{aligned} &2H\left(\left[\frac{a_1^p + a_2^p}{2}\right]^{\frac{1}{p}}\right)K\left(\left[\frac{a_1^p + a_2^p}{2}\right]^{\frac{1}{p}}\right) \\ &\quad\supseteq \frac{\Gamma_k(\nu+k)}{2(a_2^p - a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1^p)^+}\left(HK\circ\Psi\right)\left(\psi^-(a_2^p)\right) + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2^p)^-}\left(HK\circ\Psi\right)\left(\psi^-(a_1^p)\right)\right] \\ &\qquad+ \frac{\nu}{2k}N(a_1,a_2)\int_0^1((h(\mu))^2 + (1-h(\mu))^2)d\mu + \frac{\nu}{k}M(a_1,a_2)\int_0^1 h(\mu)(1-h(\mu))d\mu, \end{aligned} \tag{4.64}$$

where $M(a_1, a_2)$ and $N(a_1, a_2)$ are defined in Theorem 3.4.

Proof. For the desired inequality, we choose $\eta(U, V) = U - V$, then by applying the mentioned convexity of H and K, we have from (4.62) and (4.63)

$$\begin{aligned} 4AB \supseteq\ &H\left(\left[\mu a_1^p + (1-\mu)a_2^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_1^p + (1-\mu)a_2^p\right]^{\frac{1}{p}}\right) \\ &+ H\left(\left[\mu a_2^p + (1-\mu)a_1^p\right]^{\frac{1}{p}}\right)K\left(\left[\mu a_2^p + (1-\mu)a_1^p\right]^{\frac{1}{p}}\right) \\ &+ N(a_1,a_2)[(h(\mu))^2 + (1-h(\mu))^2] + 2M(a_1,a_2)h(\mu)(1-h(\mu)). \end{aligned} \tag{4.65}$$

The required inequality yields by using (4.65) in (4.61). □

Corollary 4.5. *Let the conditions of Theorem 4.3 be satisfied, then it is easy to see that*

- *If H and K are p-convex* IV *functions, then*

$$2H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)K\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right) \supseteq \frac{\Gamma_k(\nu+k)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1^p)^+}(HK\circ\Psi)(\psi^-(a_2^p)) + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2^p)^-}(HK\circ\Psi)(\psi^-(a_1^p))\right] + \left[\frac{1}{2}-\frac{k\nu}{(\nu+k)(\nu+2k)}\right]N(a_1,a_2)+\frac{k\nu}{(\nu+k)(\nu+2k)}M(a_1,a_2). \quad (4.66)$$

- *If H and K are convex* IV *functions, then*

$$2H\left(\frac{a_1+a_2}{2}\right)K\left(\frac{a_1+a_2}{2}\right) \supseteq \frac{\Gamma_k(\nu+k)}{2(a_2-a_1)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1)^+}(HK\circ\Psi)(\psi^-(a_2)) + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2)^-}(HK\circ\Psi)(\psi^-(a_1))\right] + \left[\frac{1}{2}-\frac{k\nu}{(\nu+k)(\nu+2k)}\right]N(a_1,a_2)+\frac{k\nu}{(\nu+k)(\nu+2k)}M(a_1,a_2). \quad (4.67)$$

- *If H and K are (p,s)-convex* IV *functions of first kind, then*

$$2H\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right)K\left(\left[\frac{a_1^p+a_2^p}{2}\right]^{\frac{1}{p}}\right) \supseteq \frac{\Gamma_k(\nu+k)}{2(a_2^p-a_1^p)^{\frac{\nu}{k}}}\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_1^p)^+}(HK\circ\Psi)(\psi^-(a_2^p)) + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-(a_2^p)^-}(HK\circ\Psi)(\psi^-(a_1^p))\right] + \left[\frac{1}{2}-\frac{k\nu s}{(\nu+ks)(\nu+2ks)}\right]N(a_1,a_2)+\frac{k\nu s}{(\nu+ks)(\nu+2ks)}M(a_1,a_2). \quad (4.68)$$

- *If H and K are harmonically convex* IV *functions, then*

$$2H\left(\frac{2a_1a_2}{a_1+a_2}\right)K\left(\frac{2a_1a_2}{a_1+a_2}\right) \supseteq \frac{\Gamma_k(\nu+k)}{2}\left(\frac{a_1a_2}{a_2-a_1}\right)^{\frac{\nu}{k}} \times\left[{}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-\left(\frac{1}{a_2}\right)^+}(HK\circ\Psi)\left(\psi^-\left(\frac{1}{a_1}\right)\right) + {}^{IRL}\mathcal{J}^{\nu,k,\psi}_{\psi^-\left(\frac{1}{a_1}\right)^-}(HK\circ\Psi)\left(\psi^-\left(\frac{1}{a_2}\right)\right)\right] + \left[\frac{1}{2}-\frac{k\nu}{(\nu+k)(\nu+2k)}\right]N(a_1,a_2)+\frac{k\nu}{(\nu+k)(\nu+2k)}M(a_1,a_2). \quad (4.69)$$

5. Conclusions

In this study, we have obtain a several of $H\cdot H$ integral type for different classes of IV convex functions. We have introduced a general class which includes many classes of IV convex functions. The new class is termed as generalized modified (p,h)-convex functions. We have also coin the notion of $\psi_k - RL$ fractional integrals for interval calculus. We have used the new concept for extended interval arithmetic. This is the first effort to obtain the results for classical and fractional integral inequalities for extended interval space. We hope that current work will attract the attention of researchers working in mathematical analysis, fractional calculus and other related fields.

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Conflict of interest

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

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