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Research article

Existence of solutions for modified Kirchhoff-type equation without the Ambrosetti-Rabinowitz condition

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Abstract: This paper is devoted to studying a class of modified Kirchhoff-type equations

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+V(x)u-u\Delta(u^2)=f(x,u),\quad \text{in } \mathbb{R}^3,$$

where $a>0, b\geq 0$ are two constants and $V:\mathbb{R}^3\to\mathbb{R}$ is a potential function. The existence of nontrivial solution to the above problem is obtained by the perturbation methods. Moreover, when $u>0$ and $f(x,u)=f(u)$, under suitable hypotheses on $V(x)$ and $f(u)$, we obtain the existence of a positive ground state solution by using a monotonicity trick and a new version of global compactness lemma. The character of this work is that for $f(u)\sim|u|^{p-2}u$ we prove the existence of a positive ground state solution in the case where $p\in(2,3]$, which has few results for the modified Kirchhoff equation. Hence our results improve and extend the existence results in the related literatures.

Keywords: modified Kirchhoff-type equation; ground state solution; nehari manifold; pohozaev identity
Mathematics Subject Classification: 35J20, 35J60

1. Introduction

In the first part of this paper, we are dedicated to studying the following modified Kirchhoff-type problem with general nonlinearity:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u-u\Delta u^2+V(x)u=f(x,u),\ x\in\mathbb{R}^3, \tag{1.1}$$

where $a>0, b\geq 0$ are two constants and $V:\mathbb{R}^3\to\mathbb{R}$ is a potential function satisfying:
(V): $V(x)\in C(\mathbb{R}^3)$, $V_0:=\inf\limits_{x\in\mathbb{R}^3}V(x)>0$. Furthermore, for any $M>0$, there is $r>0$ such that $B_r(y)$

centered at y with radius r satisfying

$$\text{meas}\{x \in B_r(y) : V(x) \le M\} \to 0, \text{ as } |y| \to \infty. \tag{1.2}$$

In addition, we suppose that the function $f(x,t)$ verifies:
(f_1): $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $|f(x,t)| \le C_1 \left(1 + |t|^{p-1}\right)$ for some $C_1 > 0$ and $p \in (4, 12)$;
(f_2): $f(x,t) = o(t)$ uniformly in x as $t \to 0$;
(f_3): $F(x,t)/t^4 \to \infty$ uniformly in x as $|t| \to \infty$, where $F(x,t) = \int_0^t f(x,s)ds$;
(f_4): $t \to f(x,t)/t^3$ is positive for $t \neq 0$, strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$.
Clearly, (f_1) and (f_2) show that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon|t| + C_\varepsilon |t|^{p-1} \text{ for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^3. \tag{1.3}$$

And (f_2) and (f_4) tell that

$$f(x,t)t > 4F(x,t) > 0\,, \text{ for } t \neq 0, \tag{1.4}$$

which is weaker than the following Ambrosetti-Rabinowitz type condition:

$$0 < F(x,t) := \int_0^t f(x,s)ds \le \frac{1}{\gamma} t f(x,t), \text{ where } \gamma > 4. \tag{A-R}$$

As is well known, the (A–R) condition is very useful in verifying the Palais-Smale condition for the energy functional associated problem (1.1). This is very much crucial in the applications of critical point theory. However, although (A–R) is a quite natural condition, it is somewhat restrictive and eliminates many nonlinearities. For example, the function

$$f(x,t) = t^3 log(1 + |t|)$$

does not satisfy (A–R) condition for any $\gamma > 4$. But it satisfies our conditions (f_1) − (f_4). For this reason, there have been efforts to remove (A–R) condition. For an overview of the relevant literature in this direction, we refer to the pioneering papers [1–6].

Problem (1.1) is a nonlocal problem due to the presence of the term $\int_{\mathbb{R}^3} |\nabla u|^2 dx$, and this fact indicates that (1.1) is not a pointwise identity. Moreover, problem (1.1) involves the quasilinear term $u\Delta(u^2)$, whose natural energy functional is not well defined in $H^1(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3)$ and variational methods cannot be used directly. These cause some mathematical difficulties, and in the meantime make the study of such a problem more interesting.

Some interesting results by variational methods can be found in [7–9] for Kirchhoff-type problem. Especially, in recent paper [10], Li and Ye studied the following problem:

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, & \text{in } \mathbb{R}^3, \end{cases} \tag{K}$$

where $p \in (3, 6)$. And they proved problem (K) has a positive ground state solution by using a monotonicity trick and a new version of global compactness lemma.

Thereafter, Guo [11] generalized the result in [10] to the following Kirchhoff-type problem with general nonlinearity

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{K1}$$

Guo proved problem (K1) also has a positive ground state solution by using the similar way. But applying Guo's result to problem (K), the condition $3 < p < 6$ in [10] can be weakened to $2 < p < 6$.

And two years later, Tang and Chen in [12] have obtained a ground state solution of Nehari-Pohozaev type for problem (K1) by using a more direct approach than [10, 11]. Moreover, Tang and Chen in [12] found that it does not seems to be sufficient to prove the inequality $c_\lambda < m_\lambda$ for $\lambda \in [\delta, 1]$ in Lemma 3.3 of [11]. Then by referring to [12], we correct this problem in the following Lemma 5.11 of the present paper.

In more recent paper [13], under more general assumptions on $V(x)$ than [10–12], He, Qin and Tang have proved the existence of ground state solutions for problem (K1) by using variational method and some new analytical techniques. Moreover, under general assumptions on the nonlinearity $f(u)$, He, Qin and Wu in [14] have obtained the existence of positive solution for problem (K1) by using property of the Pohozaev identity and some delicate analysis.

When $a = 1$ and $b = 0$, (1.1) is reduced to the well known modified nonlinear Schrödinger equation

$$-\Delta u + V(x)u - u\Delta u^2 = h(x, u), \; x \in \mathbb{R}^N. \tag{1.5}$$

Solutions of equation (1.5) are standing waves of the following quasilinear Schrödinger equation of the form:

$$i\psi_t + \Delta\psi - V(x)\psi + k\Delta\left(\alpha\left(|\psi|^2\right)\right)\alpha'\left(|\psi|^2\right)\psi + g(x, \psi) = 0, \; x \in \mathbb{R}^N, \tag{1.6}$$

where $V(x)$ is a given potential, k is a real constant, α and g are real functions. The quasilinear Schrödinger Eq (1.6) is derived as models of several physical phenomena, such as [15–17]. In [18], Poppenberg firstly began with the studies for Eq (1.6) in mathematics. For Eq (1.5) , there are several common ways to prove existence results, such as, the existence of a positive ground state solution has been studied in [19, 20] by using a constrained minimization argument; the problem is transformed to a semilinear one in [21, 22] by a change of variables (dual approach); Nehari method is used to get the existence results of ground state solutions in [23]. Especially, in [24], the following problem:

$$\begin{cases} -\sum_{j=1}^N D_j\left(a_j(x, u)D_i u\right) + \frac{1}{2}\sum_{j=1}^N D_j a_{ij}(x, u)D_j u D_j u = h(x, u), \text{ in } \Omega, \\ u = 0, \qquad \text{on } \partial\Omega \end{cases}$$

was studied via a perturbation method, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain.

Very recently, Huang and Jia in [25] studied the following autonomous modified Kirchhoff-type equation:

$$-\left(1 + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + u - \frac{1}{2}u\Delta u^2 = |u|^{p-2}u, \; x \in \mathbb{R}^N, \tag{1.7}$$

where $b \geq 0$, $p > 1$. For $p \in (1, 2] \cup [12, \infty)$, depending on the deduction of some suitable Pohozaev identity, they obtained the nonexistence result for Eq (1.7). And for $p \in (3, 4]$, they proved that the existence of ground state solution for Eq (1.7) by using the Nehari-Pohozaev manifold. But for $p \in (2, 3]$, they didn't give the existence of ground state solution for Eq (1.7). We refine the result in this paper.

We point out that $f(x, t)$ is C^1 with respect to t and $f(x, t)$ satisfies the Ambrosetti-Rabinowitz condition are very crucial in some related literatures. Since $f(x, t)$ is not assumed to be differentiable in t, the Nehari manifold of the corresponding Euler-Lagrange functional is not a C^1 functional. And if $f(x, t)$ dose not satisfy the Ambrosetti-Rabinowitz condition, the boundedness of Palais-Smale sequence (or minimizing sequence) seems hard to prove. In this case, their arguments become invalid. The first part of this paper intends to deal with the existence of non-trivial solution to problem (1.1) by the perturbation methods when $f(x, t)$ is C^1 in t and (A–R) condition are not established.

Now, we give our first main theorem as follows:

Theorem 1.1. *If (V) and $(f_1) - (f_4)$ hold, then problem (1.1) has a nontrivial solution.*

Remark 1.1. The condition (V) was firstly introduced by Bartsch and Wang [26] to guarantee the compactness of embeddings of the work space. The condition (V) can be replaced by one of the following conditions:
(V_1): $V(x) \in C(\mathbb{R}^3)$, $\text{meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < \infty$ for any $M > 0$;
(V_2): $V(x) \in C(\mathbb{R}^3)$, $V(x)$ is coercive, i.e., $\lim\limits_{|x| \to \infty} V(x) = \infty$.

Remark 1.2. Even though the condition (V) is critical to the proof of the compactness of the minimizing sequence for the energy functional, the existence result can also be obtained when V is a periodic potential because of the concentration-compactness principle.

Suppose that problem (1.1) has a periodic potential V and V satisfies
(V'): $V(x) \in C(\mathbb{R}^3)$ is 1-periodic in x_i for $1 \leq i \leq 3$, $V_0 := \inf\limits_{x \in \mathbb{R}^3} V(x) > 0$,
and $f(x, t)$ satisfies
(f_1'): $f(x, t) \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $f(x, t)$ is 1-periodic in x_i for $i = 1, 2, 3$ and $|f(x, t)| \leq C_2 \left(1 + |t|^{p-1}\right)$ for some $C_2 > 0$ and $p \in (4, 12)$.

Our second main result is

Theorem 1.2. *Suppose (V'), (f_1') and $(f_2) - (f_4)$ hold. Then equation (1.1) has a nontrivial solution.*

In the last part of our paper, we are absorbed in the following modified Kirchhoff-type equations with general nonlinearity:

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u - u \Delta u^2 + V(x) u = f(u), & \text{in } \mathbb{R}^3, \\ u \in \widetilde{E}, \ \ u > 0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.8}$$

where $a > 0, b \geq 0$, $\widetilde{E}$ is defined at the beginning of Section 5 and $V(x)$ satisfies:

(V_1^*): $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and there exists a positive constant $A < a$ such that

$$|(\nabla V(x), x)| \leqslant \frac{A}{2|x|^2} \quad \text{for all} \quad x \in \mathbb{R}^3 \backslash \{0\},$$

where$(\cdot,\cdot)$ is the usual inner product in $\mathbb{R}^3$;
(V_2^*): there exists a positive constant V_∞ such that for all $x \in \mathbb{R}^3$,

$$0 < V(x) \leqslant \liminf_{|y|\to+\infty} V(y) := V_\infty < +\infty.$$

Moreover, we assume that the function $f(s) \in C^1(\mathbb{R}^+, \mathbb{R})$ verifies:
(f_1^*): $f(s) = o(s)$ as $s \to 0^+$;
(f_2^*): $\lim\limits_{s\to+\infty} \frac{f'(s)}{s^{10}} = 0$;
(f_3^*): $\lim\limits_{s\to+\infty} \frac{f(s)}{s} = +\infty$;
(f_4^*): $\frac{f(s)}{s}$ is strictly increasing in $(0, +\infty)$.
Since we are only interested in positive solutions, we define $f(s) \equiv 0$ for $s \leq 0$.

Remark 1.3. There are a number of functions which satisfy $(V_1^*) - (V_2^*)$. For example, $V(x) = V_\infty - \frac{A}{8(1+|x|^2)}$, where $0 < A < \min\{2a, 8V_\infty\}$ is a constant. Moreover, by Lemma 5.1 mentioned later, we know that $|f(s)| \leq \varepsilon(|s| + |s|^{11}) + C_\varepsilon |s|^{p-1}$ for every $\varepsilon > 0$ and $p \in (2, 12)$.

The last main result is given below:

Theorem 1.3. *If $(V_1^*)-(V_2^*)$ and $(f_1^*)-(f_4^*)$ hold, then problem (1.8) has a positive ground state solution.*

In order to prove Theorem 1.3, we need to overcome several difficulties. First, since the Ambrosetti-Rabinowitz condition or 4-superlinearity does not hold, for $2 < p < 12$, it is difficult to get the boundedness of any (PS) sequence even if a (PS) sequence has been obtained. To overcome this difficulty, inspired by [27, 28], we use an indirect approach developed by Jeanjean. Second, the usual Nehari manifold is not suitable because it is difficult to prove the boundedness of the minimizing sequence. So we follow [29] to take the minimum on a new manifold, which is obtained by combining the Nehari manifold and the corresponding Pohozaev type identity. Third, since the Sobolev embedding $H_V^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ for $q \in [2, 2^*)$ is not compact, it seems to be hard to get a critical point of the corresponding functional from the bounded (PS) sequence. To solve this difficulty, we need to establish a version of global compactness lemma [10].

Remark 1.4. In Theorem 1.3, we especially give the existence result for the case where $p \in (2, 3]$, which has few results for this modified Kirchhoff problems and can be viewed as a partial extension of a main result in [10, 30], which dealt with the cases of $p \in (3, 6)$ and $p \in (4, 2 \times 2^*)$, respectively.

This paper is organized as follows. In Section 2, we describe the related mathematical tools. Theorem 1.1 and Theorem 1.2 are proved in Section 3 and in Section 4, respectively. In Section 5 we give the proof of Theorem 1.3.
In the whole paper, C_i, C_ε and C'_ε always express distinct constants.

2. Preliminaries

Let $L^p(\mathbb{R}^3)$ be the usual Lebesgue space with the norm $\|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p dx\right)^{\frac{1}{p}}$. And $H^1(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_H = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx\right)^{1/2}$. Moreover, $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with the norm $\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{1/2}$.

In order to deal with the perturbation functional I_λ (see Eq (2.3)), the work space E is defined by

$$E = W^{1,4}(\mathbb{R}^3) \cap H^1_V(\mathbb{R}^3),$$

where

$$H^1_V(\mathbb{R}^3) := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_{H_V} = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 \right) dx \right)^{1/2}$$

and $W^{1,4}(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_W = \left(\int_{\mathbb{R}^3} (|\nabla u|^4 + u^4) dx \right)^{1/4}.$$

Moreover, when $V(x) \equiv 1$, we define

$$\|u\|_H = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) dx \right)^{1/2}.$$

The norm of E is denoted by

$$\|u\| = \left(\|u\|^2_W + \|u\|^2_{H_V} \right)^{1/2}.$$

Notice that the embedding from $H^1_V(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$ is compact ([26]). Thus, by applying the interpolation inequality, we get that the embedding from E into $L^s(\mathbb{R}^3)$ for $2 \le s < 12$ is compact.

A function $u \in E$ is called a weak solution of problem (1.1), if for all $\varphi \in E$, there holds

$$\begin{aligned} &\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} \nabla u \nabla \varphi dx + 2 \int_{\mathbb{R}^3} (|\nabla u|^2 u\varphi + u^2 \nabla u \nabla \varphi) dx \\ &\qquad + \int_{\mathbb{R}^3} V(x)u\varphi dx - \int_{\mathbb{R}^3} f(x,u)\varphi dx = 0, \end{aligned} \tag{2.1}$$

which is formally associated to the energy functional given by

$$I(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 dx + \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \int_{\mathbb{R}^3} F(x,u) dx, \tag{2.2}$$

for $u \in E$, where $F(x,u) = \int_0^u f(x,s)ds$.

Remind that $\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx$ is not convex and well-defined in $H^1_V(\mathbb{R}^3)$, we need to take a perturbation functional of (2.2) given by

$$I_\lambda(u) = \frac{\lambda}{4} \int_{\mathbb{R}^3} \left(|\nabla u|^4 + u^4 \right) dx + I(u). \tag{2.3}$$

From condition (V), (1.3) and (1.4), it is normal to verify that $I_\lambda \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \langle I'_\lambda(u), \varphi \rangle =& \lambda \int_{\mathbb{R}^3} \left(|\nabla u|^2 \nabla u \nabla \varphi + u^3 \varphi \right) dx + \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} \nabla u \nabla \varphi dx \\ &+ 2 \int_{\mathbb{R}^3} (|\nabla u|^2 u\varphi + u^2 \nabla u \nabla \varphi) dx + \int_{\mathbb{R}^3} V(x)u\varphi dx - \int_{\mathbb{R}^3} f(x,u)\varphi dx, \text{for all } \varphi \in E. \end{aligned} \tag{2.4}$$

3. Proof of Theorem 1.1

First of all, let us briefly describe the proof of Theorem 1.1. We first discuss the properties of the perturbed family of functionals I_λ on the Nehari manifold

$$\mathcal{N}_\lambda = \{u \in E \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\}.$$

Then we prove that $I_\lambda(u_\lambda) = \inf_{\mathcal{N}_\lambda} I_\lambda$ is achieved. Moreover, since the Nehari manifold $\mathcal{N}_\lambda$ is not a C^1-manifold, we use the general Nehari theory in [31] to prove that the minimizer u_λ is a critical point of I_λ. Finally, solutions of problem (1.1) can be obtained as limits of critical points of I_λ.

Lemma 3.1. *Assume* (V) *and* $(f_1)-(f_4)$ *hold. Then, for* $\lambda \in (0,1]$, *we get the following results:*
(1) For $u \in E \setminus \{0\}$, *there exists a unique* $t_u = t(u) > 0$ *such that* $m(u) := t_u u \in \mathcal{N}_\lambda$ *and*

$$I_\lambda(m(u)) = \max_{t\in\mathbb{R}^+} I_\lambda(tu);$$

(2) For all $u \in \mathcal{N}_\lambda$, *there exists* $\alpha_0 > 0$ *such that* $\|u\|_W \geq \alpha_0$*;*
(3) There exists $\rho > 0$ *such that* $c := \inf_{\mathcal{N}_\lambda} I_\lambda \geq \inf_{S_\rho} I_\lambda > 0$, *where* $S_\rho := \{u \in E : \|u\| = \rho\}$*;*
(4) If $\mathcal{V} \subset E \setminus \{0\}$ *is a compact subset, there exists* $R > 0$ *such that* $I_\lambda \leq 0$ *on* $\mathcal{W} \setminus B_R(0)$, *where* $\mathcal{W} = \{\mathbb{R}^+ u : u \in \mathcal{V}\}$.

Proof. (1) For any $u \in E \setminus \{0\}$, we define a function $h_u(t) = I_\lambda(tu)$ for $t \in (0,\infty)$, i.e.,

$$\begin{aligned} h_u(t) =& \frac{\lambda t^4}{4}\int_{\mathbb{R}^3}\left(|\nabla u|^4 + u^4\right)dx + \frac{t^2}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2 + V(x)u^2)dx + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2 dx\Big)^2 \\ &+ t^4\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - \int_{\mathbb{R}^3} F(x,tu)dx. \end{aligned} \tag{3.1}$$

And since the Sobolev embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ for $s \in [2,12]$ is continuous, combined with (1.3), for $t > 0$ and small $\varepsilon > 0$ one has

$$\begin{aligned} h_u(t) \geq& \frac{\lambda t^4}{4}\|u\|_W^4 + \min\{a,1\}\frac{t^2}{2}\|u\|_{H_V}^2 + t^4\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2 dx\Big)^2 \\ &- \frac{\varepsilon t^2}{2}\int_{\mathbb{R}^3}|u|^2 dx - \frac{C_\varepsilon t^p}{p}\int_{\mathbb{R}^3}|u|^p dx \\ \geq& \frac{\lambda t^4}{4}\|u\|_W^4 + \min\{a,1\}\frac{t^2}{4}\|u\|_{H_V}^2 - C_3 t^p\|u\|_p^p, \end{aligned}$$

where the constant C_3 is independent of t. Since $u \neq 0$ and $p > 4$, then for $t > 0$ small enough, we deduce $h_u(t) > 0$.

On the other hand, noticing that $|tu(x)| \to \infty$ if $u(x) \neq 0$ and $t \to \infty$, by (f_3) and Fatou's lemma, we get

$$\begin{aligned} h_u(t) \leq& \frac{\lambda t^4}{4}\|u\|_W^4 + \max\{a,1\}\frac{t^2}{2}\|u\|_{H_V}^2 + C_4 t^4\|u\|_W^4 + C_5 t^4\|u\|_{H_V}^4 \\ &- t^4\int_{\mathbb{R}^3}\frac{F(x,tu)}{|tu|^4}u^4 dx \to -\infty, \text{ as } t \to \infty. \end{aligned}$$

Hence, $h_u(t)$ has a positive maximum and there exists a $t_u = t(u) > 0$ such that $h'_u(t_u) = 0$ and $t_u u \in \mathcal{N}_\lambda$.

Next, we prove the uniqueness of t_u. To this aim, we may suppose that there exists $t_u^* > 0$ with $t_u^* \neq t_u$ such that $h'_u(t_u^*) = 0$. Then we obtain

$$\lambda\|u\|_W^4 + \frac{1}{(t_u^*)^2}\Big(a\int_{\mathbb{R}^3}|\nabla u|^2dx + \int_{\mathbb{R}^3}V(x)u^2dx\Big) + b\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 + 4\int_{\mathbb{R}^3}u^2|\nabla u|^2dx = \int_{\mathbb{R}^3}\frac{f(x,t_u^*u)}{(t_u^*u)^3}u^4dx.$$

This together with

$$\lambda\|u\|_W^4 + \frac{1}{(t_u)^2}\Big(a\int_{\mathbb{R}^3}|\nabla u|^2dx + \int_{\mathbb{R}^3}V(x)u^2dx\Big) + b\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 + 4\int_{\mathbb{R}^3}u^2|\nabla u|^2dx = \int_{\mathbb{R}^3}\frac{f(x,t_uu)}{(t_uu)^3}u^4dx$$

implies that

$$\left(\frac{1}{(t_u^*)^2} - \frac{1}{(t_u)^2}\right)\left(a\int_{\mathbb{R}^3}|\nabla u|^2dx + \int_{\mathbb{R}^3}V(x)u^2dx\right) = \int_{\mathbb{R}^3}\left(\frac{f(x,t_u^*u)}{(t_u^*u)^3} - \frac{f(x,t_uu)}{(t_uu)^3}\right)u^4dx,$$

which contradicts with (f_4).

(2) By $u \in \mathcal{N}_\lambda$ and (1.3), for $\varepsilon > 0$ small enough, one has

$$\begin{aligned}
0 &\geq \lambda\|u\|_W^4 + \min\{a,1\}\|u\|_{H_V}^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3}|u|^2dx - \frac{C_\varepsilon}{p}\int_{\mathbb{R}^3}|u|^pdx \\
&\geq \lambda\|u\|_W^4 + \frac{1}{2}\min\{a,1\}\|u\|_{H_V}^2 - C_6\|u\|_W^p \\
&\geq \lambda\|u\|_W^4 - C_6\|u\|_W^p,
\end{aligned}$$

which implies that there exists a constant $\alpha_0 > 0$ such that $\|u\|_W \geq \alpha_0 > 0$ for all $u \in \mathcal{N}_\lambda$.

(3) For some $\rho > 0$ and $u \in E \setminus \{0\}$ with $\|u\| \leq \rho$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^3}|u|^2|\nabla u|^2dx \leq C\rho^4.$$

By (V), (f_1), (f_2) and the Sobolev inequality, without loss of generality, we take $\rho < 1$ small enough and $\varepsilon = \frac{V_0}{4}\min\{a,1\}$, then

$$\begin{aligned}
I_\lambda(u) \geq& \frac{\lambda}{4}\|u\|_W^4 + \frac{1}{2}\min\{a,1\}\|u\|_{H_V}^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 + \int_{\mathbb{R}^3}u^2|\nabla u|^2dx \\
&- \varepsilon\int_{\mathbb{R}^3}|u|^2dx - C_\varepsilon\int_{\mathbb{R}^3}|u|^{12}dx \\
\geq& \frac{\lambda}{4}\|u\|_W^4 + \frac{1}{4}\min\{a,1\}\|u\|_{H_V}^2 + \int_{\mathbb{R}^3}u^2|\nabla u|^2dx - C_7\left(\int_{\mathbb{R}^3}u^2|\nabla u|^2dx\right)^3 \\
\geq& \frac{\lambda}{4}\|u\|_W^4 + \frac{1}{4}\min\{a,1\}\|u\|_{H_V}^2 \\
\geq& \frac{1}{8}\min\{\lambda,a,1\}\|u\|^4,
\end{aligned} \tag{3.2}$$

whenever $\|u\| \leq \rho$. For any $u \in \mathcal{N}_\lambda$, Lemma 3.1-(1) implies that

$$I_\lambda(u) = \max_{t\in\mathbb{R}^+} I_\lambda(tu). \tag{3.3}$$

Take $s > 0$ with $su \in S_\rho$. It follows from (3.2) and (3.3) that

$$I_\lambda(u) \geq I_\lambda(su) \geq \inf_{v \in S_\rho} I_\lambda(v) \geq \frac{1}{8}\min\{\lambda, a, 1\}\rho^4 > 0.$$

Therefore

$$c := \inf_{\mathcal{N}_\lambda} I_\lambda \geq \inf_{S_\rho} I_\lambda > 0.$$

(4) Arguing by contradiction, then there must exist $u_n \in \mathcal{V}$ and $v_n = t_n u_n$ such that $I_\lambda(v_n) \geq 0$ for all n and $t_n \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that $\|u_n\| = 1$ for every $u_n \in \mathcal{V}$. Up to a subsequence, there exists $u \in E$ with $\|u\| = 1$ such that $u_n \to u$ strongly in E. Since $|v_n(x)| \to \infty$ if $u(x) \neq 0$, by (f_3) and Fatou's lemma, then

$$\int_{\mathbb{R}^3} \frac{F(x, v_n)}{v_n^4} u_n^4 dx \to \infty, \text{ as } n \to \infty,$$

which implies that

$$\begin{aligned}
0 \leq & \frac{I_\lambda(v_n)}{\|v_n\|^4} \\
= & \frac{1}{\|v_n\|^4}\left(\frac{\lambda}{4}\|v_n\|_W^4 + \frac{a}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x) v_n^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\Big)^2 + \int_{\mathbb{R}^3} v_n^2 |\nabla v_n|^2 dx\right) \\
& - \int_{\mathbb{R}^3} \frac{F(x, v_n)}{v_n^4} u_n^4 dx \\
\leq & C_8 - \int_{\mathbb{R}^3} \frac{F(x, v_n)}{v_n^4} u_n^4 dx \to -\infty \text{ as } n \to \infty.
\end{aligned}$$

This is a contradiction. □

Now we are ready to study the minimizing sequence for I_λ on $\mathcal{N}_\lambda$.

Lemma 3.2. *For fixed $\lambda \in (0, 1]$, let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence for I_λ. Then $\{u_n\}$ is bounded in E. Moreover, passing to a subsequence there exists $u \in E (u \neq 0)$ such that $u_n \to u$ in E.*

Proof. Let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence of I_λ, i.e.,

$$I_\lambda(u_n) \to c := \inf_{\mathcal{N}_\lambda} I_\lambda \text{ and } \langle I'_\lambda(u_n), u_n\rangle = 0. \tag{3.4}$$

From (3.4), we have

$$\begin{aligned}
c + o(1) &= I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\
&= \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x) u_n^2 dx + \int_{\mathbb{R}^3}\left(\frac{1}{4} f(x, u_n) u_n - F(x, u_n)\right) dx \\
&\geq \frac{1}{4}\min\{a, 1\}\|u_n\|_{H_V}^2.
\end{aligned}$$

Thus, we deduce $\{\|u_n\|_{H_V}\}$ is bounded.

Next, we need to prove that $\{\|u_n\|_W\}$ is also bounded. By contradiction, if $\{u_n\}$ is unbounded in $W^{1,4}(\mathbb{R}^3)$, setting $\omega_n = \|u_n\|_W^{-1} u_n$, we have

$$\omega_n \rightharpoonup \omega \text{ weakly in } W^{1,4}(\mathbb{R}^3),\ \omega_n \to \omega \text{ strongly in } L^p(\mathbb{R}^3),\ \omega_n \to \omega\ a.e. \text{ on } x \in \mathbb{R}^3.$$

The proof is divided into two cases as follows:

Case 1: $\omega = 0$. From Lemma 3.1-(1), we see

$$I_\lambda(u_n) = \max_{t\in\mathbb{R}^+} I_\lambda(tu_n).$$

For any $m > 0$ and setting $v_n = (8m)^{1/4}\omega_n$, since $v_n \to 0$ strongly in $L^p(\mathbb{R}^3)$, we deduce from (1.3) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(x, v_n)dx = 0. \tag{3.5}$$

So for n large enough, $(8m)^{1/4}\|u_n\|_W^{-1} \in (0, 1)$, and

$$\begin{aligned} I_\lambda(u_n) \geq & I_\lambda(v_n) \\ = & 2\lambda m + (2m)^{1/2} \min\{a, 1\}\frac{\|u_n\|_{H_V}^2}{\|u_n\|_W^2} + 2bm\frac{\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2}{\|u_n\|_W^4} + 8m\frac{\int_{\mathbb{R}^3} u_n^2|\nabla u_n|^2 dx}{\|u_n\|_W^4} - \int_{\mathbb{R}^3} F(x, v_n)dx \\ \geq & \lambda m + o(1). \end{aligned}$$

That is, for fixed $\lambda > 0$, from the arbitrariness of m, we get $I_\lambda(u_n) \to \infty$. This contradicts with $I_\lambda(u_n) \to c > 0$.

Case 2: $\omega \neq 0$. Due to $\omega \neq 0$, the set $\Theta = \{x \in \mathbb{R}^3 : \omega(x) \neq 0\}$ has a positive Lebesgue measure. For $x \in \Theta$, we have $|u_n(x)| \to \infty$. This together with condition (f_3), implies

$$\frac{F(x, u_n(x))}{|u_n(x)|^4}|\omega_n(x)|^4 \to \infty \text{ as } n \to \infty.$$

It follows from $I_\lambda(u_n) \to c$, (f_3), Sobolev inequality and Fatou's Lemma that

$$\begin{aligned} \frac{c + o(1)}{\|u_n\|_W^4} = & \frac{\lambda}{4} + \frac{1}{2\|u_n\|_W^4}\left(a\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} V(x)u_n^2 dx\right) + \frac{b}{4\|u_n\|_W^4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 \\ & + \frac{1}{\|u_n\|_W^4}\int_{\mathbb{R}^3} u_n^2|\nabla u_n|^2 dx - \frac{1}{\|u_n\|_W^4}\int_{\mathbb{R}^3} F(x, u_n)dx \\ \leq & \frac{\lambda}{4} + C_9 - \left(\int_{\omega\neq0} + \int_{\omega=0}\right)\frac{F(x, u_n(x))}{|u_n(x)|^4}|\omega_n(x)|^4 dx \\ \leq & \frac{\lambda}{4} + C_9 - \int_{\omega\neq0} \frac{F(x, u_n(x))}{|u_n(x)|^4}|\omega_n(x)|^4 dx \to -\infty, \text{ as } n \to \infty, \end{aligned}$$

where C_9 is a constant independent on n. This is impossible.

In both cases, we all get a contradiction. Therefore, $\{u_n\}$ is bounded in $W^{1,4}(\mathbb{R}^3)$. It follows that $\{u_n\}$

is bounded in E, so $u_n \rightharpoonup u$ weakly in E after passing to a subsequence. If $u = 0$, for n large enough and $u_n \in \mathcal{N}_\lambda$, we see as in (3.5) that

$$c + 1 \geq I_\lambda(u_n) \geq I_\lambda(su_n) \geq C_{10}s^4 - \int_{\mathbb{R}^3} F(x, su_n)dx \to C_{10}s^4$$

for all $s > 0$, where $C_{10} = \frac{\lambda}{4}\left(\inf_{u\in\mathcal{N}_\lambda} \|u\|_W\right)^4 > 0$. It is a contradiction. Hence $u \neq 0$.

Since the embedding $H^1_V(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is compact for each $p \in [2, 12)$, similar to Lemma 2.2 in [30], it is well known that $u_n \to u$ strongly in E. □

Lemma 3.3. *For fixed $\lambda \in (0, 1]$, there exists $u \in \mathcal{N}_\lambda$ such that $I_\lambda(u) = \inf_{\mathcal{N}_\lambda} I_\lambda$.*

Proof. Let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence of I_λ, then $\{u_n\}$ is bounded in E by lemma 3.2. Thus, up to a subsequence there exists $u \in E(u \neq 0)$ such that $u_n \rightharpoonup u$ in E and $I'_\lambda(u) = 0$. It follows that $u \in \mathcal{N}_\lambda$. Thus, $I_\lambda(u) \geq c > 0$. In order to complete the proof, it suffices to show that $I_\lambda(u) \leq c$. Indeed, from (1.4), Fatou's lemma and the weakly lower semi-continuity of norm, we have

$$\begin{aligned}
c + o(1) &= I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\
&\geq \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x, u_n)u_n - F(x, u_n)\right)dx \\
&\geq \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x, u)u - F(x, u)\right)dx + o(1) \\
&= I_\lambda(u) + o(1).
\end{aligned}$$

The proof is completed. □

Let S be the unit sphere in E. Define a mapping $m(\omega) : S \to \mathcal{N}_\lambda$ and a functional $J_\lambda(\omega) : S \to \mathbb{R}$ by

$$m(\omega) = t_\omega\omega \text{ and } J_\lambda(\omega) := I_\lambda(m(\omega)),$$

where t_ω is as shown in Lemma 3.1-(1). As Proposition 2.9 and Corollary 2.10 in [31], the following proposition is a consequence of Lemma 3.1 and the above observation.

Proposition 3.1. *Assume (V) and $(f_1) - (f_4)$ hold. For fixed $\lambda \in (0, 1]$, then*
(1) $J_\lambda \in C^1(S, \mathbb{R})$, and

$$\langle J'_\lambda(\omega), z\rangle = \|m(\omega)\|\langle I'_\lambda(m(\omega)), z\rangle$$

for any $z \in T_\omega S = \{v \in E : \langle v, \omega\rangle = 0, \forall\omega \in S\}$;
(2) $\{\omega_n\}$ is a Palais-Smale sequence for J_λ if and only if $\{m(\omega_n)\}$ is a Palais-Smale sequence for I_λ;
(3) $\omega \in S$ is a critical point of J_λ if and only if $m(\omega) \in \mathcal{N}$ is a critical point of I_λ. Moreover, the corresponding critical values of J_λ, I_λ coincide and $c = \inf_S J_\lambda = \inf_{\mathcal{N}_\lambda} I_\lambda$.

Finally, for the proof of Theorem 1.1, we need to introduce the following result.

Lemma 3.4. *Assume the conditions* (V) *and* $(f_1)-(f_4)$ *hold. Let* $\{\lambda_n\} \subset (0,1]$ *be such that* $\lambda_n \to 0$. *Let* $\{u_n\} \subset E$ *be a sequence of critical points of* I_{λ_n} *with* $I_{\lambda_n}(u_n) \le C$ *for some constant C independent of n. Then, passing to a subsequence, we have* $u_n \to \widetilde{u}$ *in* $H^1_V(\mathbb{R}^3)$, $u_n\nabla u_n \to \widetilde{u}\nabla\widetilde{u}$ *in* $L^2(\mathbb{R}^3)$, $\lambda_n\int_{\mathbb{R}^3}\left(|\nabla u_n|^4 + u_n^4\right)dx \to 0$, $I_{\lambda_n}(u_n) \to I(\widetilde{u})$ *and* $\widetilde{u}$ *is a critical point of I.*

Proof. First, similar to Lemma 3.2, we can get $\{u_n\}$ is bounded in E. Then, this together with Theorem 3.1 in [30] can complete the proof. □

Proof of Theorem 1.1 Let $\{\omega_n\} \subset S$ be a minimizing sequence for J_λ. As is mentioned above, we may assume $J'_\lambda(\omega_n) \to 0$ and $J_\lambda(\omega_n) \to c$ by Ekeland's variational principle. From Proposition 3.1-(2), for $u_n = m(\omega_n)$ we have $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$. Therefore, $\{u_n\}$ is a minimizing sequence for I_λ on $\mathcal{N}_\lambda$ and from Lemma 3.3 there exists a minimizer u of $I_\lambda|_{\mathcal{N}_\lambda}$. Then $m^{-1}(u) \in S$ is a minimizer of J_λ and a critical point of J_λ, thus by Proposition 3.1-(3) u is a critical point of I_λ, as required.

Let $\lambda_i \in (0,1]$ with $\lambda_i \to 0$ as $i \to \infty$. Let $\{u_i\} \subset E$ be a sequence of critical points of I_{λ_i} with $I_{\lambda_i}(u_i) = c_{\lambda_i} \le C$. According to Lemma 3.4, there exists a critical point $\widetilde{u}$ of I such that $\widetilde{u} \in H^1_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. In the following, we will show that $\widetilde{u}$ is a non-trivial critical point of I. Considering $\langle I'_{\lambda_i}(u_i), u_i\rangle = 0$, it follows from Sobolev inequality, interpolation inequality, and Young's inequality that

$$\begin{aligned}
0 &= \lambda_i\|u_i\|_W^4 + a\int_{\mathbb{R}^3}|\nabla u_i|^2dx + \int_{\mathbb{R}^3}V(x)u_i^2dx + b\left(\int_{\mathbb{R}^3}|\nabla u_i|^2dx\right)^2 + 4\int_{\mathbb{R}^3}u_i^2|\nabla u_i|^2dx - \int_{\mathbb{R}^3}f(x,u_i)u_idx \\
&\ge \min\{a,1\}\|u_i\|_{H_V}^2 + 4\int_{\mathbb{R}^3}u_i^2|\nabla u_i|^2dx - \frac{\varepsilon}{2}\int_{\mathbb{R}^3}|u_i|^2dx - \frac{C_\varepsilon}{p}\int_{\mathbb{R}^3}|u_i|^pdx \\
&\ge \frac{1}{2}\min\{a,1\}\|u_i\|_{H_V}^2 + C_{11}\|u_i\|_p^4 - C_{12}\|u_i\|_p^p \\
&\ge C_{11}\|u_i\|_p^4 - C_{12}\|u_i\|_p^p,
\end{aligned}$$

which implies $\|u_i\|_p \ge (\frac{C_{11}}{C_{12}})^{1/(p-4)}$. Recall that $u_i \to \widetilde{u}$ strongly in $L^p(\mathbb{R}^3)$ for $4 \le p < 12$. Therefore, we see that $\widetilde{u} \neq 0$.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to that made in Section 3. From Lemmas 3.1 and 3.2, it is clear that the functional I_λ on $\mathcal{N}_\lambda$ has a bounded minimizing sequence $\{u_n\}$. But we cannot ensure this sequence to be convergent in $E^* := W^{1,4}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, which endowed with the norm

$$\|u\|_{E^*} = \left(\|u\|_W^2 + \|u\|_H^2\right)^{1/2}.$$

Thus, we need to study some compact properties of the minimizing sequence for I_λ on the Nehari manifold $\mathcal{N}_\lambda^*$, where

$$\mathcal{N}_\lambda^* = \{u \in E^* \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\}.$$

Firstly, we have the following result due to P.L. Lions ([32]):

Lemma 4.1. *Let* $r > 0$. *If* $\{u_n\}$ *is bounded in* E^* *and*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}|u_n|^2dx = 0,$$

we have $u_n \to 0$ *strongly in* $L^s(\mathbb{R}^3)$ *for any* $s \in (2,12)$.

Next, we are going to discuss the minimizing sequence for I_λ on $\mathcal{N}_\lambda^*$.

Lemma 4.2. *Let $\{u_n\} \subset \mathcal{N}_\lambda^*$ be a minimizing sequence for I_λ. Then $\{u_n\}$ is bounded in E^*. Moreover, after a suitable $\mathbb{Z}^3$-translation, passing to a subsequence there exists $u \in \mathcal{N}_\lambda^*$ such that $u_n \rightharpoonup u$ and $I_\lambda(u) = \inf\limits_{\mathcal{N}_\lambda^*} I_\lambda$.*

Proof. Set $c = \inf\limits_{\mathcal{N}_\lambda^*} I_\lambda$. Remind that $\{u_n\}$ is bounded by Lemma 3.2, $u_n \rightharpoonup u$ weakly in E^* after passing to a subsequence. If

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_n|^2 dx = 0,$$

then $u_n \to 0$ strongly in $L^s(\mathbb{R}^3)$ for any $s \in (2, 12)$ due to Lemma 4.1. Then, by (1.3) it is easy to see that

$$\int_{\mathbb{R}^3} f(x, u_n)u_n dx = o(\|u_n\|_W).$$

Therefore,

$$\begin{aligned}
o\left(\|u_n\|_{E^*}\right) &= \langle I'_\lambda(u_n), u_n\rangle \\
&= \lambda\|u_n\|_W^4 + a\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} V(x)u_n^2 dx + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 \\
&\quad + 4\int_{\mathbb{R}^3} u_n^2|\nabla u_n|^2 dx - \int_{\mathbb{R}^3} f(x, u_n)u_n dx \\
&\geq \lambda\|u_n\|_W^4 - o(\|u_n\|_W),
\end{aligned}$$

which implies $\|u_n\|_W \to 0$. This contradicts with Lemma 3.1-(2). Hence, there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 dx \geq \delta > 0,$$

where we may assume $y_n \in \mathbb{Z}^3$. Due to the invariance of I_λ on $\mathcal{N}_\lambda^*$ under translations, $\{y_n\}$ is bounded in $\mathbb{Z}^3$. Hence, passing to a subsequence we imply $u_n \rightharpoonup u \neq 0$ weakly in E^* and $I'_\lambda(u) = 0$. It follows that $u \in \mathcal{N}_\lambda^*$, and then $I_\lambda(u) \geq c > 0$.

From (1.4), Fatou's lemma and the weakly lower semi-continuity of norm, we have

$$\begin{aligned}
c + o(1) &= I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\
&\geq \frac{1}{4}\min\{a, 1\}\|u_n\|_{H_V}^2 + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(x, u_n)u_n - F(x, u_n)\right) dx \\
&\geq \frac{1}{4}\min\{a, 1\}\|u\|_{H_V}^2 + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(x, u)u - F(x, u)\right) dx + o(1) \\
&= I_\lambda(u) + o(1),
\end{aligned}$$

which implies $I_\lambda(u) \leq c$. This completes the proof. □

Proof of Theorem 1.2 Combining Lemma 4.2 and the methods in proving Theorem 1.1, we can prove that the conclusion of Theorem 1.2 is true.

5. Proof of Theorem 1.3

In this section, we firstly need to consider the associated "limit problem" of (1.8):

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u - u\Delta u^2 + V_\infty u = f(u), & \text{in } \mathbb{R}^3, \\ u \in \widetilde{E}, \ \ u>0, & \text{in } \mathbb{R}^3, \end{cases} \tag{5.1}$$

where $a>0, b\geq 0$, V_∞ is defined as shown in (V_2^*).

Since problem (5.1) involves the quasilinear term $u\Delta(u^2)$ and the nonlocal term, its natural energy functional is not well defined in $H_V^1(\mathbb{R}^3)$. To solve this difficulty, we set

$$\widetilde{E} = \left\{u \in H_V^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} u^2|\nabla u|^2dx < +\infty\right\} = \left\{u : u^2 \in H_V^1(\mathbb{R}^3)\right\}.$$

In addition, for convenience, we make use of the following notations:
- $H_r^1(\mathbb{R}^3) := \left\{u : u \in \widetilde{E}, u(x) = u(|x|)\right\}$;
- $P := \left\{u \in \widetilde{E} | u \geq 0\right\}$ denotes the positive cone of $\widetilde{E}$ and $P_+ = P\backslash\{0\}$;
- $u^+ := \max\{u, 0\}$ and $u^- = \min\{u, 0\}$;
- For any $u \in \widetilde{E}\backslash\{0\}, u_t$ is defined as

$$u_t(x) = \begin{cases} 0, & t=0, \\ \sqrt{t}u\left(\frac{x}{t}\right), & t>0. \end{cases} \tag{5.2}$$

Now we give some preliminary results as follows.

Lemma 5.1. *Assume $f \in C^1(\mathbb{R}^+, \mathbb{R})$ satisfies $(f_1^*) - (f_4^*)$, then*
(i) For every $\varepsilon > 0$ and $p \in (2, 12)$, there is $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon(|s| + |s|^{11}) + C_\varepsilon |s|^{p-1};$$

(ii) $F(s) > 0, sf(s) > 2F(s)$ and $sf'(s) > f(s)$ if $s > 0$.

Proof. It is easy to get the results by direct calculation, so we omit the proof. □

Lemma 5.2. *(Pohozaev identity, [33]) Assume that $(f_1^*) - (f_4^*)$ hold. If $u \in \widetilde{E}$ is a weak solution to equation (5.1), then the following Pohozaev identity holds:*

$$P(u) := \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}V_\infty|u|^2dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 + \int_{\mathbb{R}^3}u^2|\nabla u|^2dx - 3\int_{\mathbb{R}^3}F(u)dx = 0. \tag{5.3}$$

Proof. The proof is standard, so we omit it. □

Lemma 5.3. *Assume that (f_3^*) holds. Then the functional*

$$I_{V_\infty}(u) := \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V_\infty|u|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 + \int_{\mathbb{R}^3}u^2|\nabla u|^2dx - \int_{\mathbb{R}^3}F(u)dx$$

is not bounded from below.

Proof. For any $u \in P_+$, we obtain

$$\begin{aligned} I_{V_\infty}(u_t) =& \frac{a}{2}t^2 \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2}t^4 \int_{\mathbb{R}^3} V_\infty |u|^2 dx + \frac{b}{4}t^4 \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \\ &+ t^3 \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - t^4 \int_{\mathbb{R}^3} \frac{F(\sqrt{t}u)}{(\sqrt{t}u)^2} u^2 dx. \end{aligned} \tag{5.4}$$

By (f_3^*), it is clear that $I_{V_\infty}(u_t) \to -\infty$ as $t \to +\infty$. □

Lemma 5.4. *Let C_{13}, C_{14}, C_{15} be positive constants and $u \in P_+$. If $f \in C^1$ satisfies $(f_1^*) - (f_4^*)$, then the function*

$$\eta(t) = C_{13}t^2 + C_{14}t^3 + C_{15}t^4 - t^3 \int_{\mathbb{R}^3} F(\sqrt{t}u)dx \quad for\ t \geq 0$$

has a unique positive critical point which corresponds to its maximum.

Proof. The conclusion is easily obtained by elementary calculation. □

Now set

$$\mathcal{M} = \left\{ u \in \widetilde{E} \backslash \{0\} | u \in P_+, G(u) = \frac{1}{2} \langle I'_{V_\infty}(u), u \rangle + P(u) = 0 \right\},$$

where $P(u)$ is given by (5.3). Then, by direct calculation we have

$$\begin{aligned} G(u) =& a \int_{\mathbb{R}^3} |\nabla u|^2 dx + 2 \int_{\mathbb{R}^3} V_\infty |u|^2 dx + b \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + 3 \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx \\ &- 3 \int_{\mathbb{R}^3} F(u) dx - \frac{1}{2} \int_{\mathbb{R}^3} f(u) u dx \\ =& \frac{dI_{V_\infty}(u_t)}{dt}\Big|_{t=1}. \end{aligned}$$

Lemma 5.5. *For any $u \in P_+$, there exists a unique $\widetilde{t} > 0$ such that $u_{\widetilde{t}} \in \mathcal{M}$. Moreover, $I_{V_\infty}(u_{\widetilde{t}}) = \max_{t>0} I_{V_\infty}(u_t)$.*

Proof. For any $u \in P_+$ and $t > 0$, let $\gamma(t) := I_{V_\infty}(u_t)$. By Lemma 5.4, $\gamma(t)$ has a unique critical point $\widetilde{t} > 0$ corresponding to its maximum, i.e., $\gamma(\widetilde{t}) = \max_{t>0} \gamma(t)$ and $\gamma'(\widetilde{t}) = 0$. It follows that $G(u_{\widetilde{t}}) = \widetilde{t}\gamma'(\widetilde{t}) = 0$. Thus, $u_{\widetilde{t}} \in \mathcal{M}$. □

We define

$$z_1 = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} I_{V_\infty}(\eta(t)), \quad z_2 = \inf_{u \in P_+} \max_{t>0} I_{V_\infty}(u_t(x)),$$

and

$$z_3 = \inf_{u \in \mathcal{M}} I_{V_\infty}(u), \qquad z_4 = \inf_{u \in H_r^1(\mathbb{R}^3) \cap \mathcal{M}} I_{V_\infty}(u),$$

where $u_t(x)$ is given by (5.2) and

$$\Gamma = \left\{ \eta \in C([0,1], \widetilde{E}) | \eta(0) = 0, I_{V_\infty}(\eta(1)) \leq 0, \eta(1) \neq 0 \right\}.$$

Lemma 5.6. $z_1 = z_2 = z_3 = z_4 > 0$.

Proof. We divide the proof into the following three steps:

Step 1. $z_3 > 0$. For any $u \in \mathcal{M}$, by Lemma 5.1-(i), the continuous embedding $\widetilde{E} \hookrightarrow L^s(\mathbb{R}^3)$ for $s \in [2, 12)$ and Sobolev inequality, we get

$$\begin{aligned}
I_{V_\infty}(u) =& \max_{t>0} I_{V_\infty}(u_t)\\
\geq& \frac{a}{2}t^2 \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2}t^4 \int_{\mathbb{R}^3} V_\infty |u|^2 dx + \frac{b}{4}t^4 \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + t^3 \int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - t^3 \int_{\mathbb{R}^3} F(\sqrt{t}u) dx\\
\geq& \frac{a}{2}t^2 \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2}t^4 \int_{\mathbb{R}^3} V_\infty |u|^2 dx + t^3 \int_{\mathbb{R}^3} u^2|\nabla u|^2 dx\\
&- \frac{\varepsilon}{2}t^4 \int_{\mathbb{R}^3} |u|^2 dx - \frac{\varepsilon}{12}t^9 \int_{\mathbb{R}^3} |u|^{12} dx - C_\varepsilon t^{\frac{6+p}{2}} \int_{\mathbb{R}^3} |u|^p dx,
\end{aligned}$$

where $C_\varepsilon > 0$ is a constant depending on ε. Since $u \neq 0$ and $p > 2$, then for $\varepsilon, t > 0$ small enough, we deduce $I_{V_\infty}(u) > 0$. Furthermore, we get $z_3 > 0$.

Step 2. $z_1 = z_2 = z_3$. The proof is similar to the argument of Nehari manifold method in [34]. One can make obvious modification by Lemma 5.4 and 5.5.

Step 3. $z_3 = z_4$. Since equation (5.1) is autonomous, the proof is standard by Schwartz symmetric arrangement. □

In the following discussion, for convenience, we set $z = z_1 (= z_2 = z_3 = z_4)$.

Lemma 5.7. *If z is attained at some $u \in \mathcal{M}$, then u is a critical point of I_{V_∞} in $\widetilde{E}$.*

Proof. Since this proof is analogous to the proof of Lemma 2.7 in [11], we omit it. □

Lemma 5.8. *Assume $(f_1^*) - (f_4^*)$ hold. Then problem (5.1) has a positive ground state solution.*

Proof. From Lemma 5.6 and Lemma 5.7, we only need to prove that z is achieved for some $u \in H_r^1(\mathbb{R}^3) \cap \mathcal{M}$.

Letting $\{u_n\} \subset H_r^1(\mathbb{R}^3) \cap \mathcal{M}$ be a minimizing sequence of I_{V_∞}, then we have

$$\begin{aligned}
1 + z > I_{V_\infty}(u_n) =& I_{V_\infty}(u_n) - \frac{1}{4}G(u_n)\\
=& \frac{a}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} u_n^2|\nabla u_n|^2 dx - \frac{1}{8} \int_{\mathbb{R}^3} [2F(u_n) - f(u_n) u_n] dx,
\end{aligned}$$

for n large enough. Therefore, $\left\{\|\nabla u_n\|_2^2\right\}$ and $\left\{\|\nabla (u_n^2)\|_2^2\right\}$ are bounded. In the following we prove $\left\{\|u_n\|_2^2\right\}$ is also bounded. By $u_n \in \mathcal{M}$ and Lemma 5.1-(ii) we obtain

$$\begin{aligned}
&2 \int_{\mathbb{R}^3} V_\infty |u_n|^2 dx\\
=&3 \int_{\mathbb{R}^3} F(u_n) dx + \frac{1}{2} \int_{\mathbb{R}^3} f(u_n) u_n dx - a \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 - 3 \int_{\mathbb{R}^3} u_n^2|\nabla u_n|^2 dx\\
\leq& \varepsilon \left(\|u_n\|_2^2 + \|u_n\|_{12}^{12}\right) + C_\varepsilon \|u_n\|_q^q + C_{16},
\end{aligned}$$

where $q \in (2, 12)$. According to the interpolation and Sobolev inequalities, we have

$$\|u_n\|_q^q \leq \|u_n\|_2^{q\theta} \|u_n\|_{12}^{q(1-\theta)} \leq C_{17} \|u_n\|_2^{q\theta} \|\nabla (u_n^2)\|_2^{\frac{q(1-\theta)}{2}},$$

where $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{12}$. Noting $q\theta < 2$, by Young's inequality, we derive for some $C'_\varepsilon > 0$

$$C_\varepsilon \|u_n\|_q^q \leq \varepsilon \|u_n\|_2^2 + C'_\varepsilon \|\nabla (u_n^2)\|_2^{\frac{q(1-\theta)}{2-q\theta}}.$$

Hence we obtain $\left\{\|u_n\|_2^2\right\}$ is also bounded if we pick $\varepsilon = \frac{1}{2} V_\infty$. Therefore, $\{u_n\}$ is bounded in $\widetilde{E}$.

Recall the compact embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ for $p \in (2, 12)$. Thus, going if necessary to a subsequence, we may assume that there exists a function $u \in \widetilde{E}$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } H_r^1(\mathbb{R}^3), \\ u_n \to u & \text{in } L^s(\mathbb{R}^3), \ \forall s \in (2, 12), \\ u_n \to u & \text{a.e. on } \mathbb{R}^3. \end{cases}$$

It is easy to check $u^+ \neq 0$ and $G(u) \leq 0$. By Lemma 5.5, $u_{t_0} \in \mathcal{M}$ for some $0 < t_0 \leqslant 1$. If $t_0 \in (0, 1)$, one can easily verify $I_{V_\infty}(u_{t_0}) < z$. Hence $t_0 = 1$ and z is attained at some $u \in \mathcal{M}$.

The strong maximum principle and standard argument [35] imply that $u(x)$ is positive for all $x \in \mathbb{R}^3$. Therefore, u is a positive ground state solution of problem (5.1). □

So far, we have proved that the associated "limit problem" of (1.8) has a ground state solution. Next, on this basis, we are going to prove Theorem 1.3.

Since V is not a constant, that is to say, problem (1.8) is no longer autonomous, the method to prove Lemma 5.8 cannot be applied. Moreover, due to the lack of the variant Ambrosetti-Rabinowitz condition, we could not obtain the boundedness of any $(PS)_c$ sequence. In order to overcome this difficulty, we make use of the monotone method due to L. Jeanjean.

Proposition 5.1. *([36], Theorem 1.1) Let $\left(\widetilde{E}, \|\cdot\|_{H_V}\right)$ be a Banach space and $T \subset \mathbb{R}^+$ be an interval. Consider a family of C^1 functionals on $\widetilde{E}$ of the form*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in T,$$

with $B(u) \geqslant 0$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_{H_V} \to +\infty$. Assume that there are two points $v_1, v_2 \in \widetilde{E}$ such that

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) > \max \left\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\right\}, \quad \forall \lambda \in T,$$

where

$$\Gamma = \left\{\gamma \in C([0, 1], \widetilde{E}) | \gamma(0) = v_1, \gamma(1) = v_2\right\}.$$

Then, for almost every $\lambda \in T$, there is a bounded $(PS)_{c_\lambda}$ sequence in $\widetilde{E}$.

Letting $T = [\delta, 1]$, where $\delta \in (0, 1)$ is a positive constant, we investigate a family of functionals on $\widetilde{E}$ with the following form

$$I_{V,\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(x)|u|^2\right) dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^3} F(u) dx, \ \forall \lambda \in [\delta, 1].$$

Then let $I_{V,\lambda}(u) = A(u) - \lambda B(u)$, where

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 + V(x)|u|^2\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 + \int_{\mathbb{R}^3} u^2|\nabla u|^2 dx,$$

and

$$B(u) = \int_{\mathbb{R}^3} F(u)dx.$$

It is easy to see that $A(u) \to \infty$ as $\|u\|_{H_V} \to \infty$ and $B(u) \geq 0$.

Lemma 5.9. *Under the assumptions of Theorem 1.3 we have*
(i) there exists $v \in \widetilde{E}\backslash\{0\}$ *such that* $I_{V,\lambda}(v) \leqslant 0$ *for all* $\lambda \in [\delta, 1]$*;*
(ii) $c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{V,\lambda}(\gamma(t)) > \max\{I_{V,\lambda}(0), I_{V,\lambda}(v)\}$ *for all* $\lambda \in [\delta, 1]$, *where*

$$\Gamma = \left\{\gamma \in C([0,1], \widetilde{E})|\gamma(0) = 0, \gamma(1) = v\right\}.$$

Proof. (i) For any $\lambda \in [\delta, 1]$, $t > 0$ and $u \in P_+$, we get

$$\begin{aligned} I_{V,\lambda}(u_t) \leq I_{V_\infty,\delta}(u_t) = &\frac{at^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{t^4}{2}\int_{\mathbb{R}^3} V_\infty|u|^2 dx + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 \\ &+ t^3\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - \delta t^4\int_{\mathbb{R}^3}\frac{F(\sqrt{t}u)}{(\sqrt{t}u)^2}u^2 dx. \end{aligned}$$

Then by (f_3^*), we infer that there exists $t > 0$ such that $I_{V,\lambda}(u_t) \leq I_{V_\infty,\delta}(u_t) < 0$.

(ii) Depending on Lemma 5.1-(i), for $\varepsilon > 0$ small enough and $p \in (2, 12)$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} I_{V,\lambda}(u) &\geqslant \frac{1}{2}\min\{a,1\}\|u\|_{H_V}^2 + \int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - \int_{\mathbb{R}^3} F(u)dx \\ &\geq \frac{1}{2}\min\{a,1\}\|u\|_{H_V}^2 + \int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - \int_{\mathbb{R}^3}\left[\varepsilon\left(|u|^2 + |u|^{12}\right) + C_\varepsilon|u|^p\right]dx \\ &\geq \frac{1}{4}\min\{a,1\}\|u\|_{H_V}^2 - C_\varepsilon\int_{\mathbb{R}^3}|u|^p dx. \end{aligned}$$

Then by standard argument there exists $r > 0$ such that

$$b = \inf_{\|u\|_{H_V}=r} I_{V,\lambda}(u) > 0 = I_{V,\lambda}(0) > I_{V,\lambda}(v),$$

and hence $c_\lambda > \max\{I_{V,\lambda}(0), I_{V,\lambda}(u_t)\}$. Then the conclusion follows with $v = u_t$. □

Lemma 5.10. *([36], Lemma 2.3) Under the assumptions of Proposition 5.1, the map* $\lambda \to c_\lambda$ *is non-increasing and left continuous.*

By Lemma 5.8, we infer that for any $\lambda \in [\delta, 1]$, the "limit problem" of the following type:

$$\begin{cases} -\left(a + b\int|\nabla u|^2 dx\right)\Delta u + V_\infty u - \Delta(u^2)u = \lambda f(u), & \text{in } \mathbb{R}^3, \\ u \in \widetilde{E}, \ \ u > 0, & \text{in } \mathbb{R}^3 \end{cases} \tag{5.5}$$

has a positive ground state solution in $\widetilde{E}$. Thus we further derive that for any $\lambda \in [\delta, 1]$, there exists

$$u_\lambda \in \mathcal{M}_\lambda := \left\{u \in \widetilde{E} | u \neq 0, G_\lambda(u) = 0\right\}$$

such that $u_\lambda(x) > 0$ for all $x \in \mathbb{R}^3$, $I'_{V_\infty,\lambda}(u_\lambda) = 0$ and

$$I_{V_\infty,\lambda}(u_\lambda) = m_\lambda := \inf_{u \in \mathcal{M}_\lambda} I_{V_\infty,\lambda}(u), \tag{5.6}$$

where

$$\begin{aligned} G_\lambda(u) =& a \int_{\mathbb{R}^3} |\nabla u|^2 dx + 2 \int_{\mathbb{R}^3} V_\infty |u|^2 dx + b \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \\ &+ 3 \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - 3\lambda \int_{\mathbb{R}^3} F(u) dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} u f(u) dx. \end{aligned} \tag{5.7}$$

Lemma 5.11. *Suppose that* $(V_1^*) - (V_2^*)$, $(f_1^*) - (f_4^*)$ *hold and* $V(x) \not\equiv V_\infty$. *Then there exists* $\bar{\lambda} \in [\delta, 1)$ *such that* $c_\lambda < m_\lambda$ *for any* $\lambda \in [\bar{\lambda}, 1]$.

Proof. First of all, for convenience, we set $I_{V,\lambda}(u) = I_{V,1}(u)$, $m_\lambda = m_1$ and $c_\lambda = c_1$ when $\lambda = 1$. And let u_λ, u_1 be the minimizer of $I_{V,\lambda}, I_{V,1}$, respectively. By Lemma 5.3, we see that there exists $K > 0$ independent of λ such that $I_{V,\lambda}((u_1)_K) < 0$ for all $\lambda \in [\delta, 1]$. Moreover, It is easy to see that $I_{V,\lambda}((u_1)_t)$ is continuous on $t \in [0, \infty)$. Hence for any $\lambda \in [\delta, 1)$, we can choose $t_\lambda \in (0, K)$ such that $I_{V,\lambda}\left((u_1)_{t_\lambda}\right) = \max_{t \in [0,K]} I_{V,\lambda}((u_1)_t)$. Note that $I_{V,\delta}((u_1)_t) \to -\infty$ as $t \to \infty$, thus there exists $K_0 > 0$ such that

$$I_{V,\delta}((u_1)_t) \leq I_{V,1}(u_1) - 1, \quad \forall t \geq K_0.$$

By the definition of t_λ, one has

$$I_{V,1}(u_1) \leq I_{V,\lambda}(u_1) \leq I_{V,\lambda}\left((u_1)_{t_\lambda}\right) \leq I_{V,\delta}\left((u_1)_{t_\lambda}\right), \quad \forall \lambda \in [\delta, 1].$$

Then the above two inequalities implies $t_\lambda < K_0$ for $\lambda \in [\delta, 1]$. Let $\beta_0 = \inf_{\lambda \in [\delta,1]} t_\lambda$. If $\beta_0 = 0$, then by contradiction, there exists a sequence $\{\lambda_n\} \subset [\delta, 1]$ such that $\lambda_n \to \lambda_0 \in [\delta, 1]$ and $t_{\lambda_n} \to 0$. It follows that

$$0 < c_1 \leq c_{\lambda_n} \leq I_{\lambda_n}\left((u_1)_{t_{\lambda_n}}\right) = o(1),$$

which implies $\beta_0 > 0$. Thus

$$0 < \beta_0 \leq t_\lambda < K_0, \quad \forall \lambda \in [\delta, 1].$$

Let

$$\bar{\lambda} := \max\left\{\delta,\ 1 - \frac{\beta_0^4 \min_{\beta_0 \leq s \leq T_0} \int_{\mathbb{R}^3} [V_\infty - V(sx)] |u_1|^2 \,\mathrm{d}x}{2K_0^3 \int_{\mathbb{R}^3} F\left(K_0^{1/2} u_1\right) \mathrm{d}x}\right\}.$$

Then $\delta \leq \bar{\lambda} < 1$. From the definition of $\bar{\lambda}$ and $0 < \beta_0 \leq t_\lambda < K_0$ for $\forall \lambda \in [\delta, 1]$, we have

$$\begin{aligned} m_\lambda \geq m_1 &= I_{V_\infty,1}(u_1) \geq I_{V_\infty,1}\left((u_1)_{t_\lambda}\right) \\ &= I_{V,\lambda}\left((u_1)_{t_\lambda}\right) - (1-\lambda) t_\lambda^3 \int_{\mathbb{R}^3} F\left(t_\lambda^{1/2} u_1\right) \mathrm{d}x + \frac{t_\lambda^4}{2} \int_{\mathbb{R}^3} [V_\infty - V(t_\lambda x)] |u_1|^2 \,\mathrm{d}x \\ &> c_\lambda - (1-\lambda) K_0^3 \int_{\mathbb{R}^3} F\left(K_0^{1/2} u_1\right) \mathrm{d}x + \frac{\beta_0^4}{2} \min_{\beta_0 \leq s \leq T_0} \int_{\mathbb{R}^3} [V_\infty - V(sx)] |u_1|^2 \,\mathrm{d}x \\ &\geq c_\lambda, \quad \forall \lambda \in [\bar{\lambda}, 1]. \end{aligned}$$

□

Next, we will introduce the following global compactness lemma, which is used for proving that the functional $I_{V,\lambda}$ satisfies $(PS)_{c_\lambda}$ condition for all $\lambda \in [\bar{\lambda}, 1]$.

Lemma 5.12. *Suppose that* $(V_1^*) - (V_2^*)$ *and* $(f_1^*) - (f_4^*)$ *hold. For* $c > 0$ *and* $\lambda \in [\delta, 1]$, *let* $\{u_n\} \subset \widetilde{E}$ *be a bounded* $(PS)_c$ *sequence for* $I_{V,\lambda}$. *Then there exists* $v_0 \in \widetilde{E}$ *and* $A \in \mathbb{R}$ *such that* $J'_{V,\lambda}(v_0) = 0$, *where*

$$J_{V,\lambda}(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \left(V(x)|u|^2 + 2|u|^2|\nabla u|^2 \right) dx - \lambda \int_{\mathbb{R}^3} F(u) dx. \tag{5.8}$$

Moreover, there exists a finite (possibly empty) set $\{v_1, \ldots, v_l\} \subset \widetilde{E}$ *of nontrivial solutions for*

$$-\left(a + bA^2\right) \Delta u + V_\infty u - \Delta(u^2) u = \lambda f(u), \tag{5.9}$$

and $\left\{y_n^k\right\} \subset \mathbb{R}^3$ *for* $k = 1, \ldots, l$ *such that*

$$\begin{aligned}
&\left|y_n^k\right| \to \infty, \ \left|y_n^k - y_n^{k'}\right| \to \infty, \ k \neq k', n \to \infty, \\
&c + \frac{bA^4}{4} = J_{V,\lambda}(v_0) + \sum_{k=1}^{l} J_{V_\infty,\lambda}(v_k), \\
&\left\| u_n - v_0 - \sum_{k=1}^{l} v_k\left(\cdot - y_n^k\right) \right\|_{H_V} \to 0, \\
&A^2 = \|\nabla v_0\|_2^2 + \sum_{k=1}^{l} \|\nabla v_k\|_2^2.
\end{aligned}$$

Proof. The proof is analogous to Lemma 5.3 in [10]. Here we only point out the difference. Since f satisfies $(f_1^*) - (f_4^*)$, for $u_n \rightharpoonup u$ in $\widetilde{E}$, we have

$$f(u_n) - f(u_n - u) \to f(u) \quad \text{in } \widetilde{E}',$$

where $\widetilde{E}'$ is the conjugate space of $\widetilde{E}$. Moreover, by referring to Lemma 3.4-(12) in [23], we can get

$$\int_{\mathbb{R}^3} |u_n|^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^3} |u_n - u|^2 |\nabla u_n - \nabla u|^2 dx \to \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 dx.$$

Then the rest proof can be derived by obvious modification from line to line. □

Lemma 5.13. *Suppose that* $(V_1^*) - (V_2^*)$ *and* $(f_1^*) - (f_4^*)$ *hold. For* $\lambda \in [\bar{\lambda}, 1]$, *let* $\{u_n\} \subset \widetilde{E}$ *be a bounded* $(PS)_{c_\lambda}$ *sequence of* $I_{V,\lambda}$. *Then there exists a nontrivial* $u_\lambda \in \widetilde{E}$ *such that*

$$u_n \to u_\lambda \ \text{ in } \widetilde{E}.$$

Proof. According to Lemma 5.12 and referring to the proof of Lemma 3.5 in [10], we can easily complete this proof. So we omit the detailed proof. □

In order to prove that the problem (1.8) has a positive ground state solution, we define

$$m = \inf_{\mathcal{X}} I_V(u),$$

where $\mathcal{X} := \left\{ u \in \widetilde{E} \backslash \{0\} : I'_V(u) = 0 \right\}$.

Lemma 5.14. $\mathcal{X} \neq \emptyset$.

Proof. Depending on Lemma 5.9 and Proposition 5.1, we see for almost everywhere $\lambda \in [\bar{\lambda}, 1]$, there exists a bounded sequence $\{u_n\} \subset \widetilde{E}$ such that

$$I_{V,\lambda}(u_n) \to c_\lambda, \;\; I'_{V,\lambda}(u_n) \to 0.$$

It follows from Lemma 5.13 that $I_{V,\lambda}$ has a nontrivial critical point $u_\lambda \in \widetilde{E}$ and $I_{V,\lambda}(u_\lambda) = c_\lambda$.

Based on the above discussion, there exists a sequence $\{\lambda_n\} \subset [\bar{\lambda}, 1]$ with $\lambda_n \to 1^-$ and an associated sequence $\{u_{\lambda_n}\} \subset \widetilde{E}$ such that $I_{V,\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}, I'_{V,\lambda_n}(u_{\lambda_n}) = 0$.

Next, we prove that $\{u_{\lambda_n}\}$ is bounded in $\widetilde{E}$. By (V_1^*) and Hardy inequality, using the proof of Lemma 5.8, we can refer that $\{\|\nabla u_{\lambda_n}\|_2\}$ and $\{\|u_{\lambda_n}\|_2\}$ are bounded. Thus, $\{u_{\lambda_n}\}$ is bounded in $\widetilde{E}$.

Since $\lambda_n \to 1^-$, we claim that $\{u_{\lambda_n}\}$ is a $(PS)_{c_1}$ sequence of $I_V = I_{V,1}$. Indeed, by Lemma 5.10 we obtain that

$$\lim_{n\to\infty} I_{V,1}(u_{\lambda_n}) = \lim_{n\to\infty}\left(I_{V,\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1)\int_{\mathbb{R}^3} F(u_{\lambda_n})\,dx\right) = \lim_{n\to\infty} c_{\lambda_n} = c_1,$$

and for all $\varphi \in H^1(\mathbb{R}^3)\backslash\{0\}$,

$$\lim_{n\to\infty} \frac{\left|\left\langle I'_{V,1}(u_{\lambda_n}), \varphi\right\rangle\right|}{\|\varphi\|_H} \leq \lim_{n\to\infty} \frac{1}{\|\varphi\|_H} |\lambda_n - 1| \int_{\mathbb{R}^3}\left(\left|u_{\lambda_n}\right| + C_{18}\left|u_{\lambda_n}\right|^{11}\right) dx \|\varphi\|_H = 0.$$

Hence $\{u_{\lambda_n}\}$ is a bounded $(PS)_{c_1}$ sequence of I_V. Then by Lemma 5.13, I_V has a nontrivial critical point $u_0 \in \widetilde{E}$ and $I_V(u_0) = c_1$. Thus, $\mathcal{X} \neq \emptyset$. □

Proof of Theorem 1.3 Firstly, in order to get a nontrivial $(PS)_m$ sequence, we need to prove $m > 0$.

For all $u \in \mathcal{X}$, we have $\langle I'_V(u), u\rangle = 0$. Thus by standard argument we see $\|u\|_{H_V} \geq \xi$ for some positive constant ξ. On the other hand, the Pohozaev identity (5.3) holds, i.e., $P_V(u) = 0$. Now by Lemma 5.1-(ii) we can get

$$I_V(u) = I_V(u) - \frac{1}{8}\left[\langle I'_V(u), u\rangle + 2P_V(u)\right] \geq \frac{1}{4}a\int |\nabla u|^2 dx - \frac{1}{8}\int_{\mathbb{R}^3} (\nabla V(x), x)\, u^2 dx.$$

Then from (V_1^*) and Hardy inequality, we infer

$$I_V(u) \geq C_{19}\int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

Therefore, we obtain $m \geq 0$.

In the following let us rule out $m = 0$. By contradiction, let $\{u_n\}$ be a $(PS)_0$ sequence of I_V. Then it is easy to show that $\lim\limits_{n\to\infty} \|u_n\|_{H_V} = 0$, which contradicts with $\|u_n\|_{H_V} \geq \xi > 0$ for all $n \in \mathbb{N}$.

Next, we may assume that there exists a sequence $\{u_n\} \subset P_+$ satisfying $I'_V(u_n) = 0$ and $I_V(u_n) \to m$. Similar to the argument in the proofs of Lemma 5.14, we can conclude that $\{u_n\}$ is a bounded $(PS)_m$ sequence of I_V. Then by Lemma 5.13 and strong maximal principle, there exists a function $u \in \widetilde{E}$ such that

$$I_V(u) = m, \;\; I'_V(u) = 0 \;\text{ and }\; u(x) > 0 \;\text{ for all }\; x \in \mathbb{R}^3.$$

So u is a positive ground state solution for problem (1.8). The proof is completed.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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