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Research article

The optimal problems for torsional rigidity

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Abstract: In this paper, we consider the optimization problems associated with the nonhomogeneous and homogeneous Orlicz mixed torsional rigidities by investigating the properties of the corresponding mixed torsional rigidity. As the main results, the existence and the continuity of the solutions to these problems are proved.

Keywords: geominimal surface area; Petty bodies; torsional rigidity; torsional measure; mixed torsional rigidity
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1. Introduction

The setting of this paper is in the n-dimensional Euclidean space $\mathbb{R}^n$ with inner product $\langle \cdot, \cdot \rangle$. A set of points K in $\mathbb{R}^n$ is convex if for all $x, y \in K$ satisfying $[x, y] \subseteq K$. If C, D are compact convex sets in $\mathbb{R}^n$ and $\lambda \geq 0$, the Minkowski sum of C and D is

$$C + D = \{x + y : x \in C, y \in D\},$$

and the scalar product λC is $\lambda C = \{\lambda x : x \in C\}$. Let $\mathcal{K}$ and $\mathcal{K}_0$ be the class of convex bodies (compact convex set with nonempty interior) and the class of convex bodies which contain the origin o in their interiors, respectively.

The variation of volume of Minkowski sum of $K \in \mathcal{K}$ and the unit ball $B_2^n \subset \mathbb{R}^n$ is the classical Borel measure, that is, the surface area of the convex body K can be formulated as:

$$S(K) = \lim_{\varepsilon \to 0} \frac{|K + \varepsilon B_2^n| - |K|}{\varepsilon},$$

where $|K|$ is the volume of K. More generally, for a fixed convex body Q, the relative surface area of a convex body K (relative to a convex body Q) can be given by

$$S(K, Q) = \lim_{\varepsilon \to 0} \frac{|K + \varepsilon Q| - |K|}{\varepsilon}. \tag{1.1}$$

However, if Q with $|Q^\circ| = |B_2^n|$ is taking over on $\mathcal{K}$, then Petty (see [28]) considered the following optimization problem and provided the solution as follows: there exists a convex body M with $|M^\circ| = |B_2^n|$ such that

$$S(K, M) = \inf\{S(K, Q) : Q \in \mathcal{K} \text{ with } |Q^\circ| = |B_2^n|\}, \tag{1.2}$$

where Q° is the polar body of Q defined by $Q^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1, \text{for all } y \in Q\}$. The minimum $S(K, M)$ is called the geominimal surface area of K, denoted by $G(K) = S(K, M)$ for geometric meaning, by Petty (see [28]).

On the other hand, it has been proved that the variational formula (1.1) can be viewed as the integral formula of the mixed volume of convex bodies K and Q:

$$V_1(K, Q) = \frac{1}{n}\int_{S^{n-1}} h_Q(u)dS(K, u), \tag{1.3}$$

where h_Q is the support function of Q on the unit sphere S^{n-1} in $\mathbb{R}^n$, i.e., $h_Q(u) = \max\{\langle x, u\rangle : x \in Q\}$ for any $u \in S^{n-1}$, and $S(K, \cdot)$ is the surface area measure of K (see e.g., [1, 7]). Combining (1.1), (1.2) and (1.3), one can conclude that the geominimal surface area $G(K)$ of a convex body K can be written as

$$G(K) = \inf\{nV_1(K, Q) : Q \in \mathcal{K} \text{ with } |Q^\circ| = |B_2^n|\}. \tag{1.4}$$

Based on (1.4), Petty proved some affine isoperimetric inequalities for the geominimal surface area $G(K)$ of a convex body K (see [28, 29]). Along the development of the L_p Brunn-Minkowski theory (see e.g., [3, 8, 14, 20–22, 26, 30, 31, 33]), the classical geominimal surface area was extended to L_p form by Lutwak (see [21] for $p \geq 1$) and Ye (see [35] for $p \in \mathbb{R}$). One can also find more references for L_p geominimal surface area (see e.g., [17, 38, 41, 44, 45]). Recently, the L_p Brunn-Minkowski theory was extended to the Orlicz-Brunn-Minkowski theory by Lutwak, Yang and Zhang (see [23, 24]), the Orlicz addition was also introduced by Gardner, Hug and Weil [9] and Xi, Jin and Leng [34], separately. This new theory is widely extended (see e.g., [2, 4, 9–13, 18, 27, 36, 43, 46, 47]). Simultaneously, the geominimal surface area is developed to Orlicz geominimal surface area (see e.g., [25, 32, 37, 39, 40]). Quite recently, the geominimal surface area associated with the capacity is also considered (see e.g., [15, 19, 42]).

In [16], Li and Zhu proved the Orlicz version of the Hadamard variational formula for torsional rigidity, and introduced the Orlicz L_φ mixed torsional rigidity: for $K, L \in \mathcal{K}_0$ and a continuous function $\varphi : (0, \infty) \to (0, \infty)$, define the Orlicz L_φ mixed torsional rigidity as follows

$$\tau_{1,\varphi}(K, L) = \frac{1}{n+2}\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)d\mu_\tau(K, u), \tag{1.5}$$

where μ_τ is the torsional measure given in (2.5). Obviously, $\tau_{1,\varphi}(\cdot, \cdot)$ is nonhomogeneous in its variables. Therefore, we introduce the definition of the homogeneous Orlicz mixed torsional rigidity in Section 3 as follows: for $K, L \in \mathcal{K}_0$ and a continuous function $\varphi : (0, \infty) \to (0, \infty)$, we define the homogeneous Orlicz mixed torsional rigidity, denoted by $\widehat{\tau}_{1,\varphi}(K, L)$, of K and L as

$$\int_{S^{n-1}} \varphi\left(\frac{\tau(K) \cdot h_L(u)}{\widehat{\tau}_{1,\varphi}(K, L) \cdot h_K(u)}\right) d\mu_\tau^*(K, u) = 1,$$

where μ_τ^* is a probability measure given by (2.7). In section 3, we will discuss some good properties for the nonhomogeneous and homogeneous Orlicz mixed torsional rigidities, such as the continuity of $\tau_{1,\varphi}(\cdot,\cdot)$ and $\widehat{\tau}_{1,\varphi}(\cdot,\cdot)$.

In Section 4, we consider the following optimization problems associated with $\tau_{1,\varphi}(\cdot,\cdot)$ and $\widehat{\tau}_{1,\varphi}(\cdot,\cdot)$: under what conditions on φ, the following problems have solutions

$$\sup/\inf\{\tau_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \quad \text{and} \quad |L^\circ| = |B_2^n|\};$$

$$\sup/\inf\{\widehat{\tau}_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \quad \text{and} \quad |L^\circ| = |B_2^n|\}.$$

On the help of the properties of $\tau_{1,\varphi}(\cdot,\cdot)$ and $\widehat{\tau}_{1,\varphi}(\cdot,\cdot)$, we prove that the above problems may be solvable. For example:

Theorem 1.1. *Suppose $K \in \mathcal{K}_0$ and $\varphi : (0,\infty) \to (0,\infty)$ is a strictly increasing function with $\lim_{t\to 0^+} \varphi(t) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$ and $\varphi(1) = 1$. The following statements hold:*
(i) *There exists a convex body $M \in \mathcal{K}_0$ such that $|M^\circ| = |B_2^n|$ and*

$$\tau_{1,\varphi}(K,M) = \inf\{\tau_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \ \ \textit{and} \ \ |L^\circ| = |B_2^n|\}.$$

(ii) *There exists a convex body $\widehat{M} \in \mathcal{K}_0$ such that $|\widehat{M}^\circ| = |B_2^n|$ and*

$$\widehat{\tau}_{1,\varphi}(K,\widehat{M}) = \inf\{\widehat{\tau}_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \ \ \textit{and} \ \ |L^\circ| = |B_2^n|\}.$$

In addition, both of M and $\widehat{M}$ are unique if φ is convex.

The solutions M and $\widehat{M}$ in Theorem 1.1 are called the *Orlicz − Petty bodies for torsional rigidity*. We use the set $P_{1,\varphi}(K)$ to denote the collection of convex bodies M, and the set $\widehat{P}_{1,\varphi}(K)$ to denote the collection of convex bodies $\widehat{M}$. For simplicity, we write

$$Q_{1,\varphi}(K) = \tau_{1,\varphi}(K,M) \ \text{ and } \ \widehat{Q}_{1,\varphi}(K) = \widehat{\tau}_{1,\varphi}(K,\widehat{M}).$$

Since the solutions M and $\widehat{M}$ are unique if φ is convex by Theorem 1.1, then the sets $P_{1,\varphi}(K)$ and $\widehat{P}_{1,\varphi}(K)$ contain only one element and thus define two operators, we still use $P_{1,\varphi}(K)$ and $\widehat{P}_{1,\varphi}(K)$ to denote these two operators. Thus the continuity of $Q_{1,\varphi}(K)$, $\widehat{Q}_{1,\varphi}(K)$, $\widehat{P}_{1,\varphi}(K)$ and $P_{1,\varphi}(K)$ can be obtained.

Theorem 1.2. *Let $\varphi : (0,\infty) \to (0,\infty)$ be strictly increasing function with $\lim_{t\to 0^+} \varphi(t) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$ and $\varphi(1) = 1$. Let $\{K_i\}_{i=1}^\infty \subseteq \mathcal{K}_0$ be a sequence that converges to $K \in \mathcal{K}_0$. Then, the following statements hold:*
(i) *$Q_{1,\varphi}(K_i) \to Q_{1,\varphi}(K)$ and $\widehat{Q}_{1,\varphi}(K_i) \to \widehat{Q}_{1,\varphi}(K)$ as $i \to \infty$.*
(ii) *If φ is convex, then $P_{1,\varphi}(K_i) \to P_{1,\varphi}(K)$ and $\widehat{P}_{1,\varphi}(K_i) \to \widehat{P}_{1,\varphi}(K)$ as $i \to \infty$.*

2. Background and preliminaries

A subset $K \subseteq \mathbb{R}^n$ is called convex if $\lambda x + (1-\lambda)y \in K$ for any $\lambda \in [0,1]$ and $x, y \in K$. A convex body is a convex compact subset of $\mathbb{R}^n$ with nonempty interior. Let $\mathcal{K}$ and $\mathcal{K}_0$ be the class of convex bodies and the class of convex bodies with the origin in their interiors, respectively. For $K, L \in \mathcal{K}$, denoted by $K + L$, the Minkowski sum, is defined as $K + L = \{x + y : x \in K, y \in L\}$. The scalar product of $\alpha \in \mathbb{R}$

and $K \in \mathcal{K}$, denote by αK, is defined as $\alpha K = \{\alpha x : x \in K\}$. For $K \in \mathcal{K}$, $|K|$ denotes to the volume of K and $|B_2^n| = \omega_n$ denotes the volume of the unit ball B_2^n in $\mathbb{R}^n$. For $K \in \mathcal{K}$, the volume radius of K, is defined as

$$\mathrm{vrad}(K) = \left(\frac{|K|}{\omega_n}\right)^{\frac{1}{n}}.$$

For any $K \in \mathcal{K}_0$, the surface area measure $S(K, \cdot)$ of K (see [1]), is defined as follows:

$$S(K, A) = \int_{\nu_K^{-1}(A)} d\mathcal{H}^{n-1}, \text{ for any measurable subset } A \subseteq S^{n-1}, \tag{2.1}$$

where $\nu_K^{-1} : S^{n-1} \to \partial K$ (where ∂ denotes the boundary) is the inverse Gauss map and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on ∂K.

Let $C(S^{n-1})$ be the class of all continuous functions on S^{n-1}. The following two Lemmas will be useful:

Lemma 2.1. (see [19, Lemma 2.1]) *If a sequence of measures $\{\mu_i\}_{i=1}^{\infty}$ on S^{n-1} converges weakly to a finite measure μ on S^{n-1} and a sequence of functions $\{f_i\}_{i=1}^{\infty} \subseteq C(S^{n-1})$ converges uniformly to a function $f \in C(S^{n-1})$, then*

$$\lim_{i\to\infty} \int_{S^{n-1}} f_i d\mu_i = \int_{S^{n-1}} f d\mu.$$

Lemma 2.2. (see [19, Lemma 2.2]) *Let $\{K_i\}_{i=1}^{\infty} \subseteq \mathcal{K}_0$ be a uniformly bounded sequence such that the sequence $\{|K_i^\circ|\}_{i=1}^{\infty}$ is bounded. Then, there exists a subsequence $\{K_{i_j}\}_{j=1}^{\infty}$ of $\{K_i\}_{i=1}^{\infty}$ and a convex body $K \in \mathcal{K}_0$ such that $K_{i_j} \to K$. Moreover, if $|K_i^\circ| = \omega_n$ for all $i = 1, 2, \ldots$, then $|K^\circ| = \omega_n$.*

Next we will introduce some basic concepts about the torsional rigidity which can be found in [6, 16]. Suppose $C_c^\infty(\mathbb{R}^n)$ is the class of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports. The torsional rigidity of a convex body K, denote by $\tau(K)$, is defined as (see [5]):

$$\frac{1}{\tau(K)} = \inf\left\{\frac{\int_K |\nabla u(x)|^2 dx}{(\int_K |u(x)| dx)^2} : u \in W_0^{1,2}(\mathrm{int}K) \text{ and } \int_K |u(x)| dx > 0\right\},$$

where ∇u is the gradient of u and $W^{1,2}(\mathrm{int}K)$ (where $\mathrm{int}K$ is the interior of K) is appropriate for the Sobolev space of the functions in $L^2(\mathrm{int}K)$ whose first-order weak derivatives belong to $L^2(\mathrm{int}K)$, and $W_0^{1,2}(\mathrm{int}K)$ denotes the closure of $C_c^\infty(\mathrm{int}K)$ in the Sobolev space $W^{1,2}(\mathrm{int}K)$. Clearly, $\tau(\cdot)$ is monotone by the definition. Namely, for $K, L \in \mathcal{K}$, one has

$$\tau(K) \le \tau(L) \text{ if } K \subseteq L. \tag{2.2}$$

Let $K \in \mathcal{K}$, if u is the unique solution of the boundary-value problem

$$\begin{cases} \Delta u = -2 & \text{in } K \\ u = 0 & \text{on } \partial K, \end{cases} \tag{2.3}$$

then

$$\tau(K) = \int_K |\nabla u(x)|^2 dx.$$

The torsional rigidity is positively homogeneity of degree $n+2$, that is, $\tau(aK) = a^{n+2}\tau(K)$, for any $K \in \mathcal{K}_0$ and $a > 0$. The torsional measure $\mu_\tau(K,\cdot)$ is a nonnegative Borel measure on S^{n-1} which can be defined as (see [6]): for any measurable subset $A \subseteq S^{n-1}$,

$$\mu_\tau(K,A) = \int_{\nu_K^{-1}(A)} |\nabla u(x)|^2 d\mathcal{H}^{n-1}(x). \tag{2.4}$$

For any $a > 0$, it is easy to check that

$$\mu_\tau(aK,\cdot) = a^{n+1}\mu_\tau(K,\cdot) \text{ on } S^{n-1}.$$

In addition, $\mu_\tau(K,\cdot)$ is not concentrated on any closed hemisphere of S^{n-1}, that is,

$$\int_{S^{n-1}} \langle v,u\rangle_+ d\mu_\tau(K,u) > 0 \text{ for any } v \in S^{n-1},$$

where $\langle v,u\rangle_+ = \max\{\langle v,u\rangle, 0\}$.

From the previous definition (2.1) and (2.4), we have the following relation between $\mu_\tau(K,\cdot)$ and $S(K,\cdot)$ as follows:

$$d\mu_\tau(K,v) = |\nabla u(\nu_K^{-1}(v))|^2 dS(K,v) \text{ for any } v \in S^{n-1}. \tag{2.5}$$

By using the previous Borel measure, the integral formula of torsional rigidity τ was provided by Colesanti and Fimiani (see [6]) as follows: suppose $K \in \mathcal{K}$ with h_K being the support function, then

$$\begin{aligned}\tau(K) &= \frac{1}{n+2}\int_{\partial K} h_K(\nu(x))|\nabla u(x)|^2 d\mathcal{H}^{n-1}(x)\\ &= \frac{1}{n+2}\int_{S^{n-1}} h_K(v)d\mu_\tau(K,v),\end{aligned} \tag{2.6}$$

where u is the solution of (2.3). For any $K \in \mathcal{K}_0$, by (2.6), denote $\mu_\tau^*(K,\cdot)$ by a probability measure on S^{n-1}

$$\mu_\tau^*(K,\cdot) = \frac{1}{n+2}\cdot\frac{h_K(\cdot)\mu_\tau(K,\cdot)}{\tau(K)} \text{ on } S^{n-1}. \tag{2.7}$$

3. The nonhomogeneous and homogeneous Orlicz L_φ mixed torsional rigidities

Let $\mathcal{I}$ be the set of continuous functions $\varphi : (0,\infty) \to (0,\infty)$, such that φ is strictly increasing with $\lim_{t\to0^+}\varphi(t) = 0$, $\lim_{t\to\infty}\varphi(t) = \infty$ and $\varphi(1) = 1$. Let $\mathcal{D}$ be the set of continuous functions $\varphi : (0,\infty) \to (0,\infty)$, such that φ is strictly decreasing with $\lim_{t\to0^+}\varphi(t) = \infty$, $\lim_{t\to\infty}\varphi(t) = 0$ and $\varphi(1) = 1$.

The definition of the nonhomogeneous Orlicz L_φ mixed torsional rigidity was provided in [16] as follows.

Definition 3.1. *Let $\varphi \in \mathcal{I}\cup\mathcal{D}$ and $K,L \in \mathcal{K}_0$. The Orlicz L_φ mixed torsional rigidity of K and L, denoted by $\tau_{1,\varphi}(K,L)$, is defined as*

$$\tau_{1,\varphi}(K,L) = \frac{1}{n+2}\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)d\mu_\tau(K,u).$$

Clearly, $\tau_{1,\varphi}(K,K)=\tau(K)$ for any $\varphi\in\mathcal{I}\cup\mathcal{D}$. In addition, for the particular example of the previous definition, it is easy to verify that

$$\begin{aligned}\tau_{1,\varphi}(B_2^n,cB_2^n)&=\varphi(c)\tau(B_2^n),\\ \tau_{1,\varphi}(cB_2^n,B_2^n)&=c^{n+2}\varphi(c^{-1})\tau(B_2^n)\end{aligned}$$

for any $c>0$. Thus $\tau_{1,\varphi}(\cdot,\cdot)$ is nonhomogeneous if φ is nonhomogeneous. In this section, we introduce the homogeneous Orlicz L_φ mixed torsional rigidity as follows.

Definition 3.2. *Suppose $\varphi\in\mathcal{I}\cup\mathcal{D}$ and $K,L\in\mathcal{K}_0$. The homogeneous Orlicz L_φ mixed torsional rigidity of K and L, denoted by $\widehat{\tau}_{1,\varphi}(K,L)$, is defined as*

$$\int_{S^{n-1}}\varphi\left(\frac{\tau(K)h_L(u)}{\widehat{\tau}_{1,\varphi}(K,L)h_K(u)}\right)d\mu_\tau^*(K,u)=1. \tag{3.1}$$

Since $\mu_\tau^*(K,\cdot)$ is a probability measure on S^{n-1} and $\varphi(1)=1$, then $\widehat{\tau}_{1,\varphi}(K,K)=\tau(K)$ for $K\in\mathcal{K}_0$. By (3.1), it can be easily checked that the functional $\widehat{\tau}_{1,\varphi}(\cdot,\cdot)$ is homogeneous as follows.

Corollary 3.1. *Let $K,L\in\mathcal{K}_0$, and $s,t>0$. If $\varphi\in\mathcal{I}\cup\mathcal{D}$, then*

$$\widehat{\tau}_{1,\varphi}(sK,tL)=s^{n+1}\cdot t\cdot\widehat{\tau}_{1,\varphi}(K,L).$$

Next we will prove that $\tau_{1,\varphi}(\cdot,\cdot)$ and $\widehat{\tau}_{1,\varphi}(\cdot,\cdot)$ are continuous on $\mathcal{K}_0\times\mathcal{K}_0$.

Theorem 3.1. *Suppose $\{K_i\}_{i=1}^\infty,\{L_i\}_{i=1}^\infty\subseteq\mathcal{K}_0$ are two sequences of convex bodies, $K,L\in\mathcal{K}_0$ and $\varphi\in\mathcal{I}\cup\mathcal{D}$. If $K_i\to K$, $L_i\to L$ as $i\to\infty$, then*

$$\begin{aligned}\tau_{1,\varphi}(K_i,L_i)&\to\tau_{1,\varphi}(K,L),\\ \widehat{\tau}_{1,\varphi}(K_i,L_i)&\to\widehat{\tau}_{1,\varphi}(K,L),\end{aligned}$$

as $i\to\infty$.

Proof. Since $K_i,L_i\in\mathcal{K}_0$, then $K_i\to K$ and $L_i\to L$ imply that $h(K_i,\cdot)\to h(K,\cdot)$ uniformly and $h(L_i,\cdot)\to h(L,\cdot)$ uniformly on S^{n-1}. These further imply that there exist $r,R>0$ with $r\le R$ such that

$$rB_2^n\subseteq K_i,L_i\subseteq RB_2^n \text{ for } i\ge 1, \tag{3.2}$$

and

$$\frac{h_{L_i}(u)}{h_{K_i}(u)}\in\left[\frac{r}{R},\frac{R}{r}\right] \text{ for } u\in S^{n-1} \text{ and } i\ge 1. \tag{3.3}$$

Together with the continuity of φ, we have

$$\varphi\left(\frac{h_{L_i}(u)}{h_{K_i}(u)}\right)h(K_i,u)\to\varphi\left(\frac{h_L(u)}{h_K(u)}\right)h(K,u) \text{ uniformly on } S^{n-1}.$$

The convergence $K_i\to K$, also yields that

$$\mu_\tau(K_i,\cdot)\to\mu_\tau(K,\cdot) \text{ weakly on } S^{n-1}.$$

Combining with Lemma 2.1, we have

$$\frac{1}{n+2}\int_{S^{n-1}}\varphi\left(\frac{h_{L_i}(u)}{h_{K_i}(u)}\right)h_{K_i}(u)d\mu_\tau(K_i,u)\to\frac{1}{n+2}\int_{S^{n-1}}\varphi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)d\mu_\tau(K,u).$$

Therefore $\tau_{1,\varphi}(K_i,L_i)\to\tau_{1,\varphi}(K,L)$ as $i\to\infty$.

Next, we first prove $\widehat{\tau}_{1,\varphi}(K_i,L_i)\to\widehat{\tau}_{1,\varphi}(K,L)$ under the case $\varphi\in\mathcal{D}$. Since $\varphi(1)=1$, then $\tau(rB_2^n)\le\tau(K_i)\le\tau(RB_2^n)$ by (2.2). Together with (3.2) and (3.3), we have

$$\varphi\left(\frac{\tau(RB_2^n)R}{\widehat{\tau}_{1,\varphi}(K_i,L_i)r}\right)\le\varphi\left(\frac{\tau(K_i)h_{L_i}(u)}{\widehat{\tau}_{1,\varphi}(K_i,L_i)h_K(u)}\right)\le\varphi\left(\frac{\tau(rB_2^n)r}{\widehat{\tau}_{1,\varphi}(K_i,L_i)R}\right).$$

Thus

$$\varphi\left(\frac{\tau(RB_2^n)R}{\widehat{\tau}_{1,\varphi}(K_i,L_i)r}\right)\le\int_{S^{n-1}}\varphi\left(\frac{\tau(K_i)h_{L_i}(u)}{\widehat{\tau}_{1,\varphi}(K_i,L_i)h_{K_i}(u)}\right)d\mu_\tau^*(K_i,u)=1\le\varphi\left(\frac{\tau(rB_2^n)r}{\widehat{\tau}_{1,\varphi}(K_i,L_i)R}\right).$$

Since $\varphi\in\mathcal{D}$ and $\varphi(1)=1$, thus for $i\ge1$,

$$0<\frac{\tau(rB_2^n)\cdot r}{R}\le\widehat{\tau}_{1,\varphi}(K_i,L_i)\le\frac{\tau(RB_2^n)\cdot R}{r}<\infty,$$

i.e., $\widehat{\tau}_{1,\varphi}(K_i,L_i)$ is bounded from above and below. Let

$$X=\liminf_{i\to\infty}\widehat{\tau}_{1,\varphi}(K_i,L_i)>0,$$
$$Y=\limsup_{i\to\infty}\widehat{\tau}_{1,\varphi}(K_i,L_i)<\infty.$$

So, there exist two subsequences $\{\widehat{\tau}_{1,\varphi}(K_{i_m},L_{i_m})\}_{m=1}^\infty$ and $\{\widehat{\tau}_{1,\varphi}(K_{i_n},L_{i_n})\}_{n=1}^\infty$ of $\widehat{\tau}_{1,\varphi}(K_i,L_i)$ such that

$$\widehat{\tau}_{1,\varphi}(K_{i_n},L_{i_n})<\frac{n+1}{n}X \text{ with } \lim_{n\to\infty}\widehat{\tau}_{1,\varphi}(K_{i_n},L_{i_n})=X$$

and

$$\widehat{\tau}_{1,\varphi}(K_{i_m},L_{i_m})>\frac{m}{m+1}Y \text{ with } \lim_{m\to\infty}\widehat{\tau}_{1,\varphi}(K_{i_m},L_{i_m})=Y$$

for $m,n\ge1$. Since $\varphi\in\mathcal{D}$, the Lemma 2.1 yields that,

$$\begin{aligned}1&=\lim_{m\to\infty}\int_{S^{n-1}}\varphi\left(\frac{\tau(K_{i_m})h_{L_{i_m}}(u)}{\widehat{\tau}_{1,\varphi}(K_{i_m},L_{i_m})h_{K_{i_m}}(u)}\right)d\mu_\tau^*(K_{i_m},u)\\&\ge\lim_{m\to\infty}\int_{S^{n-1}}\varphi\left(\frac{(m+1)\tau(K_{i_m})h_{L_{i_m}}(u)}{mYh_{K_{i_m}}(u)}\right)d\mu_\tau^*(K_{i_m},u)\\&=\int_{S^{n-1}}\varphi\left(\frac{\tau(K)h_L(u)}{Yh_K(u)}\right)d\mu_\tau^*(K,u).\end{aligned}\tag{3.4}$$

In the same manner, one can check that

$$1\le\int_{S^{n-1}}\varphi\left(\frac{\tau(K)\cdot h_L(u)}{X\cdot h_K(u)}\right)d\mu_\tau^*(K,u).\tag{3.5}$$

Combing (3.4) and (3.5), we have

$$\limsup_{i\to\infty}\widehat{\tau}_{1,\varphi}(K_i,L_i)\le\widehat{\tau}_{1,\varphi}(K,L)\le\liminf_{i\to\infty}\widehat{\tau}_{1,\varphi}(K_i,L_i).$$

This, together with the fact that " $\liminf\le\limsup$ ", yields that $\widehat{\tau}_{1,\varphi}(K_i,L_i)\to\widehat{\tau}_{1,\varphi}(K,L)$ as $i\to\infty$. As for $\varphi\in\mathcal{I}$, it can be obtained in the same way. □

Theorem 3.2. *Suppose $\varphi\in\mathcal{I}$, $\{K_i\}_{i=1}^{\infty}\subseteq\mathcal{K}_0$ and $K_i\to K\in\mathcal{K}_0$ as $i\to\infty$. If $\{M_i\}_{i=1}^{\infty}\subseteq\mathcal{K}_0$ such that $\{\tau_{1,\varphi}(K_i,M_i)\}_{i=1}^{\infty}$ or $\{\widehat{\tau}_{1,\varphi}(K_i,M_i)\}_{i=1}^{\infty}$ is bounded, then $\{M_i\}_{i=1}^{\infty}$ is uniformly bounded.*

Proof. Since $K_i\to K$, then $h_{K_i}(\cdot)\to h_K(\cdot)$ uniformly on S^{n-1}, and $\mu_\tau(K_i,\cdot)\to\mu_\tau(K,\cdot)$ weakly on S^{n-1}, we have $\tau(K_i)\to\tau(K)$ as $i\to\infty$. Since $\mu_\tau(K,\cdot)$ is not contained in a closed hemisphere of S^{n-1}, so $\int_{S^{n-1}}\langle u,v\rangle_+d\mu_\tau(K,u)>0$, for $v\in S^{n-1}$. This implies that there exist $n_0\in\mathbb{N}$ and a constant $c_0>0$ such that

$$\int_\Omega\langle u,v\rangle_+d\mu_\tau(K,u)\ge c_0,$$

where $\Omega=\{u\in S^{n-1}:\langle u,v\rangle_+\ge\frac{1}{n_0}\}$.
In addition, there exist two numbers $r_0,R_0>0$ with $r_0\le R_0$ such that

$$r_0\le h_{K_i}(u),h_K(u)\le R_0$$

for $i\ge1$ and $u\in S^{n-1}$.

Since $M_i\in\mathcal{K}_0$, let $R_i=\max\{\rho_{M_i}(u):u\in S^{n-1}\}$ for $i\ge1$. Suppose that $v_i\in S^{n-1}$ with $R_i=\rho(M_i,v_i)$ for some $i\ge1$. Then $[0,R_iv_i]\subset M_i$, thus $R_i\langle u,v_i\rangle_+\le h_{M_i}(u)$ for $u\in S^{n-1}$. Assume $\{M_i\}_{i=1}^{\infty}$ is not uniformly bounded, i.e., $\sup_{i\ge1}R_i=\infty$. As $\{\tau_{1,\varphi}(K_i,M_i)\}_{i=1}^{\infty}$ is bounded, then there exists a positive constant c such that

$$c\ge\tau_{1,\varphi}(K_i,M_i)$$

for $i\ge1$.

Let $v_i\to v\in S^{n-1}$ as $i\to\infty$ by the compactness of S^{n-1}. Since $\varphi\in\mathcal{I}$ is increasing, Definition 3.1 and Lemma 2.1, we have that for any constant $T>0$,

$$\begin{aligned}
c&\ge\liminf_{i\to\infty}\frac{1}{n+2}\int_{S^{n-1}}\varphi\left(\frac{h_{M_i}(u)}{h_{K_i}(u)}\right)h_{K_i}(u)d\mu_\tau(K_i,u)\\
&\ge\liminf_{i\to\infty}\frac{1}{n+2}\int_{S^{n-1}}\varphi\left(\frac{R_i\langle u,v_i\rangle_+}{R_0}\right)h_{K_i}(u)d\mu_\tau(K_i,u)\\
&\ge\liminf_{i\to\infty}\frac{1}{n+2}\int_{S^{n-1}}\varphi\left(\frac{T\cdot\langle u,v_i\rangle_+}{R_0}\right)h_{K_i}(u)d\mu_\tau(K_i,u)\\
&\ge\frac{1}{n+2}\int_{S^{n-1}}\liminf_{i\to\infty}\varphi\left(\frac{T\cdot\langle u,v_i\rangle_+}{R_0}\right)h_K(u)d\mu_\tau(K,u)\\
&=\frac{1}{n+2}\int_{S^{n-1}}\varphi\left(\frac{T\cdot\langle u,v\rangle_+}{R_0}\right)h_K(u)d\mu_\tau(K,u)\\
&\ge\frac{r_0}{n+2}\int_{S^{n-1}}\varphi\left(\frac{T\cdot\langle u,v\rangle_+}{R_0}\right)d\mu_\tau(K,u)
\end{aligned}$$

$$\geq \frac{r_0}{n+2}\varphi\left(\frac{T}{R_0 n_0}\right)\int_{\Omega}\langle u,v\rangle_+ d\mu_\tau(K,u)$$
$$\geq \frac{c_0 r_0}{n+2}\varphi\left(\frac{T}{R_0 n_0}\right).$$

Letting $T \to \infty$, then $c \geq \infty$. This is a contradiction, which shows that $\{M_i\}_{i=1}^{\infty}$ is uniformly bounded.

Along the same line, one can check that $\{M_i\}_{i=1}^{\infty}$ is uniformly bounded when $\{\widehat{\tau}_{1,\varphi}(K_i, M_i)\}_{i=1}^{\infty}$ is bounded. □

4. The Orlicz-Petty bodies for torsional rigidity

In this section, we will prove the existence, uniqueness and continuity of the Orlicz-Petty bodies for torsional rigidity. To do so, we study the following optimization problems for nonhomogeneous and homogeneous Orlicz L_φ mixed torsional rigidities:

$$\sup / \inf\{\tau_{1,\varphi}(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}; \tag{4.1}$$

$$\sup / \inf\{\widehat{\tau}_{1,\varphi}(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}. \tag{4.2}$$

The next theorem gives the existence of the solutions to the problems in (4.1) and (4.2).

Theorem 4.1. *Suppose that $K \in \mathcal{K}_0$ and $\varphi \in \mathcal{I}$. The following statements hold:*
(i) There exists a convex body $M \in \mathcal{K}_0$ with $|M^\circ| = \omega_n$ and

$$\tau_{1,\varphi}(K, M) = \inf\{\tau_{1,\varphi}(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}.$$

(ii) There exists a convex body $\widehat{M} \in \mathcal{K}_0$ with $|\widehat{M}^\circ| = \omega_n$ and

$$\widehat{\tau}_{1,\varphi}(K, \widehat{M}) = \inf\{\widehat{\tau}_{1,\varphi}(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}.$$

Moreover, both of M and $\widehat{M}$ are unique if $\varphi \in \mathcal{I}$ is convex.

Proof. For simplicity, we write

$$Q_{1,\varphi}(K) = \inf\{\tau_{1,\varphi}(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}; \tag{4.3}$$

$$\widehat{Q}_{1,\varphi}(K) = \inf\{\widehat{\tau}_{1,\varphi}(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}. \tag{4.4}$$

(i) By (4.3) and Definition 3.1, we have

$$Q_{1,\varphi}(K) \leq \tau_{1,\varphi}(K, B_2^n) < \infty.$$

Assume that $\{M_i\}_{i=1}^{\infty} \subseteq \mathcal{K}_0$ is an optimal sequence of (4.3), namely, $\tau_{1,\varphi}(K, M_i) \to Q_{1,\varphi}(K)$ as $i \to \infty$ and $|M_i^\circ| = \omega_n$ for $i \geq 1$. Then $\{M_i\}_{i=1}^{\infty}$ is uniformly bounded by Theorem 3.2. This together with Lemma 2.2, we have a subsequence $\{M_{i_k}\}_{k=1}^{\infty}$ of $\{M_i\}_{i=1}^{\infty}$ and $M \in \mathcal{K}_0$ such that $M_{i_k} \to M$ as $k \to \infty$ and $|M^\circ| = \omega_n$. By Theorem 3.1, we have

$$Q_{1,\varphi}(K) = \lim_{i\to\infty} \tau_{1,\varphi}(K, M_i) = \lim_{k\to\infty} \tau_{1,\varphi}(K, M_{i_k}) = \tau_{1,\varphi}(K, M).$$

Hence M is a solution to (4.1).
(ii) By (4.4) and Definition 3.2, we have

$$\widehat{Q}_{1,\varphi}(K) \le \widehat{\tau}_{1,\varphi}(K, B_2^n) < \infty.$$

Let $\{\widehat{M}_i\}_{i=1}^{\infty} \subseteq \mathcal{K}_0$ such that $\widehat{\tau}_{1,\varphi}(K, \widehat{M}_i) \to \widehat{Q}_{1,\varphi}(K)$ as $i \to \infty$ and $|\widehat{M}_i^{\circ}| = \omega_n$ for $i \ge 1$. Then $\{\widehat{M}_i\}_{i=1}^{\infty}$ is uniformly bounded by Theorem 3.2. This together with Lemma 2.2, we have a subsequence $\{\widehat{M}_{i_k}\}_{k=1}^{\infty}$ of $\{\widehat{M}_i\}_{i=1}^{\infty}$ and $\widehat{M} \in \mathcal{K}_0$ such that $\widehat{M}_{i_k} \to \widehat{M}$ as $k \to \infty$ and $|\widehat{M}^{\circ}| = \omega_n$. Thus, Theorem 3.1 yields

$$\widehat{Q}_{1,\varphi}(K) = \lim_{i\to\infty} \widehat{\tau}_{1,\varphi}(K, \widehat{M}_i) = \lim_{k\to\infty} \widehat{\tau}_{1,\varphi}(K, \widehat{M}_{i_k}) = \widehat{\tau}_{1,\varphi}(K, \widehat{M}).$$

The proofs of the uniqueness of M and $\widehat{M}$ are similar, so we only provide the proof for M. Assume that $M_1, M_2 \in \mathcal{K}_0$ and M_1, M_2 satisfy

$$|M_1^{\circ}| = |M_2^{\circ}| = \omega_n, \ \tau_{1,\varphi}(K, M_1) = Q_{1,\varphi}(K) = \tau_{1,\varphi}(K, M_2).$$

Let $N = (M_1 + M_2)/2$, by the Brunn-Minkowski inequality, $\mathrm{vrad}(N^{\circ}) \le 1$ with equality if and only if $M_1 = M_2$. By the monotonicity and convexity of φ, one has

$$\begin{aligned}
Q_{1,\varphi}(K) &\le \tau_{1,\varphi}(K, \mathrm{vrad}(N^{\circ}) \cdot N) \\
&= \frac{1}{n+2} \int_{S^{n-1}} \varphi\left(\frac{\mathrm{vrad}(N^{\circ}) \cdot h_N(u)}{h_K(u)}\right) h_K(u) d\mu_{\tau}(K, u) \\
&\le \frac{1}{n+2} \int_{S^{n-1}} \varphi\left(\frac{h_N(u)}{h_K(u)}\right) h_K(u) d\mu_{\tau}(K, u) \\
&\le \frac{1}{n+2} \int_{S^{n-1}} \left[\frac{1}{2}\varphi\left(\frac{h_{M_1}(u)}{h_K(u)}\right) h_K(u) + \frac{1}{2}\varphi\left(\frac{h_{M_2}(u)}{h_K(u)}\right) h_K(u)\right] d\mu_{\tau}(K, u) \\
&= \frac{\tau_{1,\varphi}(K, M_1) + \tau_{1,\varphi}(K, M_2)}{2} \\
&= Q_{1,\varphi}(K).
\end{aligned}$$

This shows that $\mathrm{vrad}(N^{\circ}) = 1$, or equivalently $M_1 = M_2$. □

We call the solutions M and $\widehat{M}$ *Orlicz − Petty bodies for torsional rigidity*. Following the idea of Petty, we call the minimums $Q_{1,\varphi}(K) = \tau_{1,\varphi}(K, M)$ and $\widehat{Q}_{1,\varphi}(K) = \widehat{\tau}_{1,\varphi}(K, \widehat{M})$ the corresponding *geominimal surface area for torsional rigidity*. We use $P_{1,\varphi}(\cdot)$ and $\widehat{P}_{1,\varphi}(\cdot)$ to denote the sets of M and $\widehat{M}$, respectively.

Definition 4.1. *Suppose that $K \in \mathcal{K}_0$ and $\varphi \in \mathcal{I}$. Define the set*

$$P_{1,\varphi}(K) = \{M \in \mathcal{K}_0 : |M^{\circ}| = \omega_n \text{ and } \tau_{1,\varphi}(K, M) = Q_{1,\varphi}(K)\}.$$

Analogously, define the set

$$\widehat{P}_{1,\varphi}(K) = \{\widehat{M} \in \mathcal{K}_0 : |\widehat{M}^{\circ}| = \omega_n \text{ and } \widehat{\tau}_{1,\varphi}(K, \widehat{M}) = \widehat{Q}_{1,\varphi}(K)\}.$$

Obviously, the sets $P_{1,\varphi}(K)$ and $\widehat{P}_{1,\varphi}(K)$ are nonempty which follow from Theorem 4.1 if $\varphi \in \mathcal{I}$. Since $P_{1,\varphi}(K)$ and $\widehat{P}_{1,\varphi}(K)$ contain one element if $\varphi \in \mathcal{I}$ is convex, $P_{1,\varphi} : \mathcal{K}_0 \to \mathcal{K}_0$ and $\widehat{P}_{1,\varphi} : \mathcal{K}_0 \to \mathcal{K}_0$ define two operators on $\mathcal{K}_0$. The next theorem shows the continuity of $Q_{1,\varphi}(\cdot)$, $\widehat{Q}_{1,\varphi}(\cdot)$, $P_{1,\varphi}(\cdot)$ and $\widehat{P}_{1,\varphi}(\cdot)$.

Theorem 4.2. *Let $\varphi \in \mathcal{I}$ and $\{K_i\}_{i=1}^{\infty} \subseteq \mathcal{K}_0$ and $K \in \mathcal{K}_0$ be such that $K_i \to K$ as $i \to \infty$. The following statements hold:*
(i) $Q_{1,\varphi}(K_i) \to Q_{1,\varphi}(K)$ and $\widehat{Q}_{1,\varphi}(K_i) \to \widehat{Q}_{1,\varphi}(K)$ as $i \to \infty$.
(ii) If $\varphi \in \mathcal{I}$ is convex, then $P_{1,\varphi}(K_i) \to P_{1,\varphi}(K)$ and $\widehat{P}_{1,\varphi}(K_i) \to \widehat{P}_{1,\varphi}(K)$ as $i \to \infty$.

Proof. (i) First of all, we will show that $Q_{1,\varphi}(K_i) \to Q_{1,\varphi}(K)$ as $i \to \infty$. If $M \in P_{1,\varphi}(K)$ and $M_i \in P_{1,\varphi}(K_i)$ for $i \geq 1$, then Theorem 3.1 and (4.3) yields that

$$Q_{1,\varphi}(K) = \tau_{1,\varphi}(K, M) = \lim_{i\to\infty} \tau_{1,\varphi}(K_i, M) = \limsup_{i\to\infty} \tau_{1,\varphi}(K_i, M) \geq \limsup_{i\to\infty} Q_{1,\varphi}(K_i). \tag{4.5}$$

Thus, $\{Q_{1,\varphi}(K_i)\}_{i=1}^{\infty}$ is bounded. Since $Q_{1,\varphi}(K_i) = \tau_{1,\varphi}(K_i, M_i)$ for $i \geq 1$, then $\{M_i\}_{i=1}^{\infty}$ is uniformly bounded by Theorem 3.2. Let $\{M_{i_k}\}_{k=1}^{\infty} \subseteq \{M_i\}_{i=1}^{\infty}$ be a bounded subsequence such that

$$\lim_{k\to\infty} Q_{1,\varphi}(K_{i_k}) = \liminf_{i\to\infty} Q_{1,\varphi}(K_i).$$

Since $\{M_{i_k}\}_{k=1}^{\infty}$ is uniformly bounded, and by Lemma 2.2, there exist a subsequence $\{M_{i_{k_j}}\}_{j=1}^{\infty} \subseteq \{M_{i_k}\}_{k=1}^{\infty}$ and $M_0 \in \mathcal{K}_0$ such that $M_{i_{k_j}} \to M_0$ as $j \to \infty$ and $|M_0^{\circ}| = \omega_n$. Hence, Theorem 3.1 leads to

$$\begin{aligned} \liminf_{i\to\infty} Q_{1,\varphi}(K_i) = \lim_{j\to\infty} Q_{1,\varphi}(K_{i_{k_j}}) &= \lim_{j\to\infty} \tau_{1,\varphi}(K_{i_{k_j}}, M_{i_{k_j}}) \\ &= \tau_{1,\varphi}(K, M_0) \geq Q_{1,\varphi}(K). \end{aligned} \tag{4.6}$$

Combining (4.5) with (4.6), we have

$$Q_{1,\varphi}(K) = \lim_{i\to\infty} Q_{1,\varphi}(K_i). \tag{4.7}$$

Next, we prove that $\widehat{Q}_{1,\varphi}(K_i) \to \widehat{Q}_{1,\varphi}(K)$ as $i \to \infty$. Let $\widehat{M} \in \widehat{P}_{1,\varphi}(K)$ and $\widehat{M}_i \in \widehat{P}_{1,\varphi}(K_i)$ for $i \geq 1$. By Theorem 3.1 and (4.4), we have

$$\begin{aligned} \widehat{Q}_{1,\varphi}(K) &= \widehat{\tau}_{1,\varphi}(K, \widehat{M}) \\ &= \lim_{i\to\infty} \widehat{\tau}_{1,\varphi}(K_i, \widehat{M}) \\ &= \limsup_{i\to\infty} \widehat{\tau}_{1,\varphi}(K_i, \widehat{M}) \\ &\geq \limsup_{i\to\infty} \widehat{Q}_{1,\varphi}(K_i). \end{aligned} \tag{4.8}$$

This leads to $\{\widehat{Q}_{1,\varphi}(K_i)\}_{i=1}^{\infty}$ is bounded. It follows from Theorem 3.2 and $\widehat{Q}_{1,\varphi}(K_i) = \widehat{\tau}_{1,\varphi}(K_i, \widehat{M}_i)$ for $i \geq 1$ that $\{\widehat{M}_i\}_{i=1}^{\infty}$ is uniformly bounded. Let $\{K_{i_l}\}_{l=1}^{\infty} \subseteq \{K_i\}_{i=1}^{\infty}$ be a subsequence such that

$$\lim_{l\to\infty} \widehat{Q}_{1,\varphi}(K_{i_l}) = \liminf_{i\to\infty} \widehat{Q}_{1,\varphi}(K_i).$$

Since $\{\widehat{M}_{i_l}\}_{l=1}^{\infty}$ is uniformly bounded, and by Lemma 2.2, there exists a subsequence $\{\widehat{M}_{i_{l_j}}\}_{j=1}^{\infty}$ of $\{\widehat{M}_{i_l}\}_{l=1}^{\infty}$ and $\widehat{M}_0 \in \mathcal{K}_0$ such that $\widehat{M}_{i_{l_j}} \to \widehat{M}_0$ as $j \to \infty$ and $|\widehat{M_0}^{\circ}| = \omega_n$. Thus

$$\begin{aligned}\liminf_{i\to\infty} \widehat{Q}_{1,\varphi}(K_i) &= \lim_{j\to\infty} \widehat{Q}_{1,\varphi}(K_{i_{l_j}})\\ &= \lim_{j\to\infty} \widehat{\tau}_{1,\varphi}(K_{i_{l_j}}, \widehat{M}_{i_{l_j}})\\ &= \widehat{\tau}_{1,\varphi}(K, \widehat{M}_0)\\ &\geq \widehat{Q}_{1,\varphi}(K).\end{aligned} \tag{4.9}$$

From (4.8) and (4.9), one concludes that

$$\widehat{Q}_{1,\varphi}(K) = \lim_{i\to\infty} \widehat{Q}_{1,\varphi}(K_i). \tag{4.10}$$

(ii) Assume that $\varphi \in \mathcal{I}$ is convex. By Theorem 4.1, $P_{1,\varphi}(K)$, $P_{1,\varphi}(K_i)$, $\widehat{P}_{1,\varphi}(K)$ and $\widehat{P}_{1,\varphi}(K_i)$ contain one element which will be denoted by M, M_i, $\widehat{M}$ and $\widehat{M}_i$ for $i \geq 1$, respectively. Let $\{M_{i_k}\}_{k=1}^{\infty} \subseteq \{M_i\}_{i=1}^{\infty}$ and $\{\widehat{M}_{i_l}\}_{l=1}^{\infty} \subseteq \{\widehat{M}_i\}_{i=1}^{\infty}$. By (4.7) and (4.10)

$$Q_{1,\varphi}(K) = \lim_{k\to\infty} Q_{1,\varphi}(K_{i_k}) = \lim_{k\to\infty} \tau_{1,\varphi}(K_{i_k}, M_{i_k}); \tag{4.11}$$

$$\widehat{Q}_{1,\varphi}(K) = \lim_{l\to\infty} \widehat{Q}_{1,\varphi}(K_{i_l}) = \lim_{k\to\infty} \widehat{\tau}_{1,\varphi}(K_{i_l}, \widehat{M}_{i_l}). \tag{4.12}$$

It follows that $\{\tau_{1,\varphi}(K_{i_k}, M_{i_k})\}_{k=1}^{\infty}$ and $\{\widehat{\tau}_{1,\varphi}(K_{i_l}, \widehat{M}_{i_l})\}_{l=1}^{\infty}$ are uniformly bounded. Thus, by Theorem 3.2, $\{M_{i_k}\}_{k=1}^{\infty}$ and $\{\widehat{M}_{i_l}\}_{l=1}^{\infty}$ are bounded. By Lemma 2.2, there exist subsequences $\{M_{i_{k_{j_1}}}\}_{j_1=1}^{\infty} \subseteq \{M_{i_k}\}_{k=1}^{\infty}$ and $\{\widehat{M}_{i_{l_{j_2}}}\}_{j_2=1}^{\infty} \subseteq \{\widehat{M}_{i_l}\}_{l=1}^{\infty}$, respectively, and $S, I \in \mathcal{K}_0$ such that $M_{i_{k_{j_1}}} \to S$, $\widehat{M}_{i_{l_{j_2}}} \to I$, and $|S^{\circ}| = |I^{\circ}| = \omega_n$. By Theorem 3.1, (4.11) and (4.12), we have

$$Q_{1,\varphi}(K) = \lim_{j_1\to\infty} Q_{1,\varphi}(K_{i_{k_{j_1}}}) = \lim_{j_1\to\infty} \tau_{1,\varphi}(K_{i_{k_{j_1}}}, M_{i_{k_{j_1}}}) = \tau_{1,\varphi}(K, S);$$

$$\widehat{Q}_{1,\varphi}(K) = \lim_{j_2\to\infty} \widehat{Q}_{1,\varphi}(K_{i_{l_{j_2}}}) = \lim_{j_2\to\infty} \widehat{\tau}_{1,\varphi}(K_{i_{l_{j_2}}}, \widehat{M}_{i_{l_{j_2}}}) = \widehat{\tau}_{1,\varphi}(K, I).$$

It follows that $M = S$ and $\widehat{M} = I$. That is, $M_i \to M$ and $\widehat{M_i} \to \widehat{M}$ as $i \to \infty$. □

The following proposition shows that the Orlicz-Petty bodies for torsional rigidity of polytopes are still polytopes.

Proposition 4.1. *If $\varphi \in \mathcal{I}$ and $K \in \mathcal{K}_0$ is a polytope, then the elements in $P_{1,\varphi}(K)$ and $\widehat{P}_{1,\varphi}(K)$ are polytopes with faces parallel to those of K.*

Proof. Since K is a polytope, then $S(K, \cdot)$ must be concentrated on a finite subset $\{u_1, u_2, \ldots, u_m\} \subseteq S^{n-1}$. By (2.5), the torsional measure $\mu_\tau(K, \cdot)$ is also concentrated on $\{u_1, u_2, \ldots, u_m\}$. If $M \in P_{1,\varphi}(K)$, then let P_1 be a polytope with $\{u_1, u_2, \ldots, u_m\}$ as the unit normal vectors of its faces such that $P_1 = \bigcap_{1\leq i\leq m}\{x \in \mathbb{R}^n : \langle x, u_i\rangle \leq h_M(u_i)\}$. Therefore, we have $h_{P_1}(u_i) = h_M(u_i)$ $(1 \leq i \leq m)$. Then

$$\tau_{1,\varphi}(K, P_1) = \frac{1}{n+2}\int_{S^{n-1}} \varphi\left(\frac{h_{P_1}(u)}{h_K(u)}\right) h_K(u)d\mu_\tau(K, u)$$

$$
\begin{aligned}
&= \frac{1}{n+2}\sum_{i=1}^{m}\varphi\left(\frac{h_{P_1}(u_i)}{h_K(u_i)}\right)h_K(u_i)\cdot\mu_\tau(K,\{u_i\})\\
&= \frac{1}{n+2}\sum_{i=1}^{m}\varphi\left(\frac{h_M(u_i)}{h_K(u_i)}\right)h_K(u_i)\cdot\mu_\tau(K,\{u_i\})\\
&= \frac{1}{n+2}\int_{S^{n-1}}\varphi\left(\frac{h_M(u)}{h_K(u)}\right)h_K(u)d\mu_\tau(K,u)\\
&= \tau_{1,\varphi}(K,M).
\end{aligned}
$$

By (4.3), we have

$$\tau_{1,\varphi}(K,P_1) = \tau_{1,\varphi}(K,M) = Q_{1,\varphi}(K) \le \tau_{1,\varphi}(K,\mathrm{vrad}(P_1^\circ)P_1).$$

Since φ is strictly increasing, then $\mathrm{vrad}(P_1^\circ) \ge 1$. The inclusion $P_1^\circ \subseteq M^\circ$ shows that $\mathrm{vrad}(P_1^\circ) \le \mathrm{vrad}(M^\circ) = 1$. Hence, $\left|P_1^\circ\right| = |M^\circ|$. Then $M = P_1$, that is, each $M \in P_{1,\varphi}(K)$ is a polytope with faces parallel to those of K.

Using the same method, one can prove that each $\widehat{M} \in \widehat{P}_{1,\varphi}(K)$ is a polytope with faces parallel to those of K. □

Finally, we list some counterexamples to show that the problems (4.1) and (4.2) may not be solvable in general case.

Proposition 4.2. *Let $K \in \mathcal{K}_0$ be a polytope with surface area measure $S(K,\cdot)$ being concentrated on a finite subset $\{u_1,u_2,\dots,u_m\} \subseteq S^{n-1}$.*
(i) If $\varphi \in \mathcal{D}$ and the jth coordinates of $u_1,u_2,\dots,u_m$ are nonzero, then

$$
\begin{aligned}
\inf\{\tau_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\} = 0;\\
\sup\{\widehat{\tau}_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\} = \infty.
\end{aligned}
$$

(ii) If $\varphi \in \mathcal{I}$, then

$$
\begin{aligned}
&\sup\{\tau_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\}\\
&= \sup\{\widehat{\tau}_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\} = \infty.
\end{aligned}
$$

Proof. (i) Let $b_j = \min_{1\le i\le m}\{\left|(u_i)_j\right|\}$ be the *jth* coordinate of u_i ($1 \le i \le m$ and $1 \le j \le n$), by assumption, $b_j > 0$. Then there exists a constant $b > 0$ such that $b_j \ge b$ for all $1 \le j \le n$. Since K is a polytope with $u_1,u_2,\dots,u_m$ as the unit normal vectors of its faces, we know that K is bounded, then there exists a constant $c > 0$ such that $h_K(u_i) \le c$ for $1 \le i \le m$. For any $d > 0$, we write

$$T_d = \mathrm{diag}(d,\dots,d,1,d,\dots,d) \text{ and } L_d = d^{\frac{1-n}{n}}T_dB_2^n,$$

where 1 is in the *jth* column of the matrix T_d. Then, $L_d^\circ = d^{\frac{n-1}{n}}(T_d^t)^{-1}B_2^n$ and $|L_d^\circ| = \omega_n$. It is easily check that,

$$|T_du_i| = \sqrt{d^2(u_i)_1^2 + \cdots + (u_i)_j^2 + \cdots + d^2(u_i)_n^2} \ge |(u_i)_j| \ge b$$

for $1 \le i \le m$. Thus,

$$h_{L_d}(u_i) = \max_{v_1 \in L_d} \langle v_1, u_i \rangle = \max_{v_2 \in B_2^n} \langle T_d v_2 d^{\frac{1-n}{n}}, u_i \rangle$$
$$= d^{\frac{1-n}{n}} \max_{v_2 \in B_2^n} \langle v_2, T_d u_i \rangle = d^{\frac{1-n}{n}} |T_d u_i| \ge \frac{b}{d^{\frac{n-1}{n}}}.$$

Due to $\varphi \in \mathcal{D}$ is decreasing, so

$$\frac{1}{n+2} \int_{S^{n-1}} \varphi\left(\frac{h_{L_d}(u)}{h_K(u)}\right) h_K(u) d\mu_\tau(K, u) = \frac{1}{n+2} \sum_{i=1}^m \varphi\left(\frac{h_{L_d}(u_i)}{h_K(u_i)}\right) h_K(u_i)\mu_\tau(K, \{u_i\})$$
$$\le \frac{1}{n+2} \sum_{i=1}^m \varphi\left(\frac{1}{c} \frac{b}{d^{\frac{n-1}{n}}}\right) c\mu_\tau(K, \{u_i\})$$
$$= \frac{c}{n+2} \varphi\left(\frac{b}{cd^{\frac{n-1}{n}}}\right) \mu_\tau(K, S^{n-1}).$$

Since $\varphi\left(b/cd^{\frac{n-1}{n}}\right) \to 0$ as $d \to 0$ by the monotonicity of φ, then

$$\inf\{\tau_{1,\varphi}(K, L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\} \le \frac{c}{n+2} \varphi\left(\frac{b}{cd^{\frac{n-1}{n}}}\right) \mu_\tau(K, S^{n-1}) \to 0 \text{ as } d \to 0.$$

Similarly, we can check that $\sup\{\widehat{\tau}_{1,\varphi}(K, L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\} = \infty$ if $\varphi \in \mathcal{D}$.
(ii) Firstly, suppose that $\mu_\tau(K, \{u_1\}) > 0$. Since $K \in \mathcal{K}_0$, then there exists a positive number c_1 such that $h_K(u_i) \ge c_1 > 0$ as $1 \le i \le m$. Since K is a polytope with $u_1, u_2, \ldots, u_m$ as the unit normal vectors of its faces, then K is bounded, namely, there exists a constant $c_0 > 0$ such that $h_K(u_i) \le c_0$ for $1 \le i \le m$. By the Schmidt orthogonalization, it can be found an orthogonal matrix $T \in O(n)$ with u_1 as its first column vector. For any $d > 0$, let

$$T_d = T \operatorname{diag}(d^{-1}, d, 1, 1, \ldots, 1)T^t \text{ and } L_d = T_d B_2^n.$$

It follows that, $|L_d^\circ| = \omega_n$ and

$$h_{L_d}(u_1) = \max_{v_1 \in L_d} \langle v_1, u_1 \rangle = \max_{v_2 \in B_2^n} \langle T_d v_2, u_1 \rangle = \max_{v_2 \in B_2^n} \langle v_2, T_d u_1 \rangle$$
$$= \max_{v_2 \in B_2^n} \langle v_2, d^{-1} u_1 \rangle = \frac{1}{d}.$$

Then

$$\frac{1}{n+2} \int_{S^{n-1}} \varphi\left(\frac{h_{L_d}(u)}{h_K(u)}\right) h_K(u) d\mu_\tau(K, u) = \frac{1}{n+2} \sum_{i=1}^m \varphi\left(\frac{h_{L_d}(u_i)}{h_K(u_i)}\right) h_K(u_i)\mu_\tau(K, \{u_i\})$$
$$\ge \frac{1}{n+2} \varphi\left(\frac{h_{L_d}(u_1)}{h_K(u_1)}\right) h_K(u_1)\mu_\tau(K, \{u_1\})$$
$$\ge \frac{1}{n+2} \varphi\left(\frac{1}{c_0} \frac{1}{d}\right) h_K(u_1)\mu_\tau(K, \{u_1\})$$

$$\geq \frac{c_1}{n+2}\varphi\left(\frac{1}{c_0}\frac{1}{d}\right)\mu_\tau(K,\{u_1\}).$$

Sinse φ is increasing, then

$$\sup\{\tau_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\} = \infty \text{ as } d \to 0.$$

Similarly, one can check that $\sup\{\widehat{\tau}_{1,\varphi}(K,L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\} = \infty$ under the condition that $\varphi \in \mathcal{I}$. □

5. Conclusions

In this paper, we introduce the definition of the homogeneous Orlicz mixed torsional rigidities and obtain some properties of the nonhomogeneous and homogeneous Orlicz mixed torsional rigidities. Then we consider the optimization problems about the corresponding mixed torsional rigidity. As the main results, we prove the existence and the continuity of the solutions to these problems.

Conflict of interest

We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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