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Research article

Dynamical transition for a 3-component Lotka-Volterra model with diffusion

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Abstract: The main objective of this article is to investigate the dynamical transition for a 3-component Lotka-Volterra model with diffusion. Based on the spectral analysis, the principle of exchange of stability conditions for eigenvalues is obtained. In addition, when $\delta_0 < \delta_1$, the first eigenvalues are complex, and we show that the system undergoes a continuous or jump transition. In the small oscillation frequency limit, the transition is always continuous and the time periodic rolls are stable after the transition. In the case where $\delta_0 > \delta_1$, the first eigenvalue is real. Generically, the first eigenvalue is simple and all three types of transition are possible. In particular, the transition is mixed if $\int_\Omega e_{k_0}^3 dx \neq 0$, and is continuous or jump in the case where $\int_\Omega e_{k_0}^3 dx = 0$. In this case we also show that the system bifurcates to two saddle points on $\delta < \delta_1$ as $\tilde{\theta} > 0$, and bifurcates to two stable singular points on $\delta > \delta_1$ as $\tilde{\theta} < 0$ where $\tilde{\theta}$ depends on the system parameters.

Keywords: center manifold reduction; diffusion; dynamic transition; the spectral analysis; 3-component Lotka-Volterra model
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1. Introduction

In this paper, we consider the following 3-component Lotka-Volterra model with diffusion

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_1 u - c_1 v - k_1 w), \\ v_t = d_2\Delta v + v(a_2 - c_2 u - b_2 v - k_2 w), \\ w_t = d_3\Delta w + w(-r + \alpha_1 k_1 u + \alpha_2 k_2 v), \end{cases} \tag{1.1}$$

where u, v are the population densities of two competing prey and w is the population density of its predator. The habitat $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. $r, a_i, b_i, c_i, \alpha_i, k_i (i = 1, 2)$

and $d_j(j = 1, 2, 3)$ are positive constants. a_1 and a_2 represent the intrinsic growth rate, b_i and $c_i(i = 1, 2)$ represent the intra-specific and inter-specific competition rates of u and v, k_1 and k_2 are the predation rate of w, α_1 and α_2 are the transformation rate of predation, and r is the death rate of w. d_1, d_2 and d_3 are represent the diffusion rates of u, v and w respectively.

Here, we focus on the system (1.1) supplemented with the following initial condition:

$$u(x, 0) = u^0, \quad v(x, 0) = v^0, \quad w(x, 0) = w^0, \tag{1.2}$$

and the Neumann boundary condition:

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0, \tag{1.3}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial\Omega$. The Neumann boundary condition in (1.3) was interpreted as the condition that the system is self-contained with zero population flux across the boundary.

For two species Lotka-Volterra (LV) systems, there has been largely discussed in the past several decades. Kuto and Tsujikawa [1] considered a general stationary Lotka-Volterra competition model with diffusion. They obtained the existence of nonconstant solutions by the Leray-Schauder degree theory and derived a limiting system as diffusion of one of the species tending to infinity. Eilbeck et al. [2] studied the two species Lotka-Volterra competition model and obtained the existence and non-uniqueness of coexistence solutions for a wide range of parameters. Other results related to the two species Lotka-Volterra competition model, we refer to other studies [3–6] and the references therein.

However, very little result is known about the three species Lotka-Volterra model. In general, three species systems are very complicated even in the ordinary differential equations case. In recent years, the three species Lotka-Volterra model with diffusion was studied by some investigators. Ni et al. [7] considered the role of cross-diffusion in the 3×3 Lotka-Volterra competition model, and obtained the existence of non-constant steady states created by cross-diffusion in 3×3 systems. In [8], Pang and Wang studied a strongly coupled system of a two-predator-one-prey ecosystem, they demonstrated the emergence of stationary patterns for this system, and showed that it is due to the cross diffusion that arises naturally in the model. Moreover, Ali et al. [9] studied the prey-predator-top-predator system, in addition, they point out that the system exhibits Bogdanov-Takens bifurcation, saddle-node bifurcation, Hopf bifurcation for suitable choice of the relevant parameters. There are other related works on three species model, see [10–17] and references therein.

Although considerable work has been done concerning three competition model and two-predator-one-prey ecosystem, it is worth noting that it is interesting to investigate two prey and one predator system. In some circumstances, predation may have a tendency to increase species diversity in competitive communities, which is called predation mediated coexistence. For instance, in [10], the authors considered the coexistence problem of two competing species mediated by the presence of predator, and speculate that the possibility. Furthermore, Kan-on and Mimura [11] proved the existence of stable spatially inhomogeneous positive stationary solutions of (1.1). In addition, Yukio Kan-on [12] studied the positive stationary solutions by using the singular perturbation method and the associated singular limit spectral analysis. Moreover, Wang [13] considered the strongly coupled version of (1.1) and established the existence and non-existence of non-constant positive solutions.

Motivated by the above papers, what we are concerned in this paper is to describe the dynamic stability and transition for the system (1.1). The technical method for the analysis is the dynamical transition theory, which has been developed by Ma and Wang [18–20] and has been used to solve many interesting mathematical and physical problems, see [21–26]. As is well known, for the system (1.1), due to non-selfadjoint linear operator, the transition can be caused by real or complex eigenvalues crossing the imaginary axis. Let δ_0 and δ_1 be critical values given by (3.17) and (3.20). δ_0 is derived from $AB - C = 0$, which determine the signs of the eigenvalues of the linearized eigenvalue equations (3.7). The same is δ_1, which is derived from $C_k = 0$. When $\delta_0 > \delta_1$, the first eigenvalue is real and simple, and all three types of transition are possible depending on a non-dimensional number exactly given in terms of the system parameters. In particular, the transition is mixed if $\int_\Omega e_{k_0}^3 dx \neq 0$, and in the case where $\int_\Omega e_{k_0}^3 dx = 0$ we show that the transition of the system is continuous as $\tilde{\theta} < 0$, and is jump as $\tilde{\theta} > 0$ where $\tilde{\theta}$ is defined in (4.43). when $\delta_0 < \delta_1$, the first eigenvalues are complex, and we show that the system undergoes a continuous or jump transition. In the small oscillation frequency limit, the transition is always continuous and the time periodic rolls are stable after the transition.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries on dynamical transition theory. Section 3 recapitulates (1) the nondimensional form and the nonnegative basic states of the steady-state equations for the system (1.1), (2) an abstract form for (1.1), and (3) linear theory and principle of exchange of stabilities(PES). The main theorems of this artical are stated and proved in Section 4.

2. Preliminaries

In this section, we introduce the dynamical transition theory for nonlinear dissipative systems developed by Ma and Wang [18–20], which provides the basic method for the following research of this paper.

Let H and H_1 be two Hilbert spaces, $H_1 \subset H$ be a compact and dense inclusion. Consider the following abstract nonlinear equation defined on H, given by

$$\begin{cases} \frac{d\chi}{dt} = L_\lambda \chi + G(\chi, \lambda), \\ \chi(0) = \chi_0, \end{cases} \tag{2.1}$$

where $\chi(t)$ is an unknown function, $L_\lambda : H_1 \to H$ is a linear operator, and $G : H_1 \to H$ is a nonlinear operator, λ is the system parameter.

Assume that $L_\lambda : H_1 \to H$ is a parameterized linear completely continuous field depending continuously on λ. which satisfies

$$\begin{cases} L_\lambda = -A + B & \text{is a sectorial operator}, \\ A : H_1 \to H & \text{is a linear homeomorphism}, \\ B : H_1 \to H & \text{is a compact operator}. \end{cases} \tag{2.2}$$

Furthermore, we assume that the nonlinear term $G : H_\sigma \to H (0 \leq \sigma < 1)$ is a C^r bounded operators ($r \geq 1$) and depending continuously on λ, where H_σ is the fractional order space, and

$$G(\chi, \lambda) = o(\|\chi\|_{H_\sigma}). \tag{2.3}$$

Hereafter, we always assume the conditions (2.2) and (2.3) hold true, which imply that the system (2.1) has a dissipative structure.

At first, we recall the mathematical definition of transition for the system (2.1).

Definition 2.1. *([20]) We say that the system* (2.1) *has a transition from* $(\chi, \lambda) = (0, \lambda_0)$ *at* λ_0 *if the following two conditions hold true:*

(1) *if* $\lambda < \lambda_0$, $\chi = 0$ *is locally asymptotically stable for* (2.1)*, and*

(2) *if* $\lambda > \lambda_0$*, there exists a neighborhood* $U \subset H$ *of* $\chi = 0$ *independent of* λ*, such that for any* $\chi_0 \in U \setminus \Gamma_\lambda$ *the solution* $\chi_\lambda(t, \chi_0)$ *of* (2.1) *satisfies that*

$$\begin{aligned} &\limsup_{t\to\infty} \|\chi_\lambda(t,\chi_0)\|_H \geq \delta(\lambda) > 0, \\ &\lim_{\lambda\to\lambda_0} \delta(\lambda) \geq 0, \end{aligned} \tag{2.4}$$

where Γ_λ *is the stable manifold of* $\chi = 0$*, with codim* $\Gamma_\lambda \geq 1$ *in* H *for* $\lambda > \lambda_0$.

Let the eigenvalues (counting multiplicity) of L_λ be given by $\{\beta_j(\lambda) \in \mathbb{C} | j = 1, 2, \cdots\}$, and let

$$Re\beta_i(\lambda) \begin{cases} > 0, & \text{if } \lambda > \lambda_0, \\ = 0, & \text{if } \lambda = \lambda_0, \quad \text{for any } \ 1 \leq i \leq m, \\ < 0, & \text{if } \lambda < \lambda_0, \end{cases} \tag{2.5}$$

$$Re\beta_j(\lambda_0) < 0, \qquad \text{for any } \ j \geq m+1. \tag{2.6}$$

The following theorem is a basic principle of transitions from equilibrium states, which provides sufficient conditions and a basic classification for transitions of nonlinear dissipative systems. The proof of this Lemma is given in Ma and Wang [19, 20].

Lemma 2.1. *Let the conditions* (2.5) *and* (2.6) *hold true. Then the system* (2.1) *must have a transition from* $(\chi, \lambda) = (0, \lambda_0)$*, and there is a neighborhood* $U \subset H$ *of* $\chi = 0$ *such that the transition is one of the following three types:*

(1) *Continuous transition: There exists an open and dense set* $\widetilde{U_\lambda} \subset U$ *such that for any* $\chi_0 \in \widetilde{U_\lambda}$*, the solution* $\chi_\lambda(t, \chi_0)$ *of (2.1)satisfies*

$$\lim_{\lambda\to\lambda_0} \limsup_{t\to\infty} \|\chi_\lambda(t,\chi_0)\|_H = 0.$$

(2) *Jump transition: For any* $\lambda_0 < \lambda < \lambda_0 + \varepsilon$ *with some* $\varepsilon > 0$*, there is an open and dense set* $U_\lambda \subset U$ *such that for any* $\chi_0 \in U_\lambda$*,*

$$\limsup_{t\to\infty} \|\chi_\lambda(t,\chi_0)\|_H \geq \delta > 0, \qquad \textit{for some } \delta > 0 \textit{ is independent of } \lambda.$$

(3) *Mixed transition: For any* $\lambda_0 < \lambda < \lambda_0 + \varepsilon$ *with some* $\varepsilon > 0$*,* U *can be decomposed into two open sets* U_1^λ *and* U_2^λ *(*U_i^λ *not necessarily connected):* $\bar{U} = \bar{U}_1^\lambda + \bar{U}_2^\lambda, U_1^\lambda \cap U_2^\lambda = \emptyset$*, such that*

$$\begin{aligned} &\lim_{\lambda\to\lambda_0} \limsup_{t\to\infty} \|\chi_\lambda(t,\chi_0)\|_H = 0, \qquad &\forall \chi_0 \in U_1^\lambda, \\ &\limsup_{t\to\infty} \|\chi_\lambda(t,\chi_0)\|_H \geq \delta > 0, \qquad &\forall \chi_0 \in U_2^\lambda, \end{aligned}$$

where U_1^λ *and* U_2^λ *are called metastable domain.*

3. Mathematical setting and linear problem

3.1. Mathematical setting

The forthcoming analysis is to formulate the evolution equations given in (1.1) using an abstract functional setting that is standard in the framework of dynamic transitions.

First, we introduce a set of the following to nondimensionalize the system (1.1):

$$\begin{aligned} &u = \frac{a_1}{b_1}u', \quad v = \frac{a_2}{b_2}v', \quad w = \frac{a_1}{k_1}w', \\ &x = lx', \qquad t = t_0 t'(t_0 = \frac{b_1}{\alpha_1 k_1 a_1}), \end{aligned}$$

where the prime denotes nondimensionalized variables. Substituting these nondimensional variables into (1.1) and neglecting the prime for all variables for convenience, we obtain

$$\begin{cases} u_t = \varepsilon_1 \Delta u + \bar{a}_1 u(1 - u - \alpha v - w), \\ v_t = \varepsilon_2 \Delta v + \bar{a}_2 v(1 - \gamma u - v - \kappa w), \\ w_t = \varepsilon_3 \Delta w + w(-\sigma + u + qv), \end{cases} \tag{3.1}$$

here

$$\begin{aligned} &\varepsilon_1 = \frac{d_1 t_0}{l^2}, \quad \bar{a}_1 = a_1 t_0, \quad \alpha = \frac{a_2 c_1}{a_1 b_2}, \\ &\varepsilon_2 = \frac{d_2 t_0}{l^2}, \quad \bar{a}_2 = a_2 t_0, \quad \gamma = \frac{a_1 c_2}{a_2 b_1}, \\ &\varepsilon_3 = \frac{d_3 t_0}{l^2}, \quad \sigma = r t_0, \quad \kappa = \frac{a_1 k_2}{a_2 k_1}, \quad q = \frac{a_2 b_1 \alpha_2 k_2}{a_1 b_2 \alpha_1 k_1}, \end{aligned}$$

and the unknown functions are $u, v, w \geq 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, the parameters are positive constants:

$$\varepsilon_i (1 \leq i \leq 3),\ \bar{a}_1,\ \bar{a}_2,\ \alpha,\ \gamma,\ \kappa,\ \sigma,\ q.$$

Furthermore, let

$$\begin{aligned} \mathbb{R}^m_+ &= \{(x_1, \cdots, x_m) \in \mathbb{R}^m_+ \big| x_i \geq 0, 1 \leq i \leq m\}, \\ \lambda &= (\varepsilon_1, \varepsilon_2, \varepsilon_3, \bar{a}_1, \bar{a}_2, \alpha, \gamma, \kappa, \sigma, q) \in \mathbb{R}^{10}_+. \end{aligned}$$

Then we define the following function spaces,

$$\begin{aligned} H &= L^2(\Omega)^3, \\ H_1 &= \left\{ \chi \in H^2(\Omega)^3 \middle| \frac{\partial \chi}{\partial n} = 0 \ on\ \partial\Omega \right\}, \end{aligned}$$

where $\chi = (u, v, w)$.

Define the operators $L_\lambda = A_\lambda + B_\lambda$ and $G_\lambda : H_1 \to H$ by

$$\begin{aligned} A_\lambda \chi &= (\varepsilon_1 \Delta u, \varepsilon_2 \Delta v, \varepsilon_3 \Delta w), \\ B_\lambda \chi &= (\bar{a}_1 u, \bar{a}_2 v, -\sigma w), \\ G(\chi, \lambda) &= (-\bar{a}_1 u^2 - \bar{a}_1 \alpha u v - \bar{a}_1 u w, \\ &\quad - \bar{a}_2 \gamma u v - \bar{a}_2 v^2 - \bar{a}_2 \kappa v w, uw + qvw). \end{aligned}$$

Obviously, $L_\lambda : H_1 \to H$ is a parameterized linear completely continuous field depending continuously on λ, and $G_\lambda : H_1 \to H$ represents the nonlinear terms of the Eq (3.1).

Thus, the Eqs (3.1) with (1.2) and (1.3), take the following operator form:

$$\begin{cases} \dfrac{d\chi}{dt} = L_\lambda \chi + G(\chi, \lambda), \\ \chi(0) = \varphi, \end{cases} \tag{3.2}$$

where $\lambda = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \bar{a}_1, \bar{a}_2, \alpha, \gamma, \kappa, \sigma, q) \in \mathbb{R}^{10}_+$.

On the other hand, we study the steady-state solutions for the system (3.1). It is easy to check that the system (3.1) admits seven biologically realistic constant steady-state solutions:

$$\begin{aligned} &\varphi_0 = (0,0,0)^T, \quad \varphi_1 = (0, \frac{\sigma}{q}, \frac{q-\sigma}{\kappa q})^T, \quad \varphi_2 = (\sigma, 0, 1-\sigma)^T, \\ &\varphi_3 = (\frac{1-\alpha}{1-\alpha\gamma}, \frac{1-\gamma}{1-\alpha\gamma}, 0)^T, \quad \varphi_4 = (1,0,0)^T, \quad \varphi_5 = (0,1,0)^T, \\ &\varphi_6 = (u_0, v_0, w_0)^T, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} u_0 &= \frac{-\sigma(1-\alpha\kappa) - q(\kappa-1)}{q(\gamma-\kappa) - (1-\alpha\kappa)}, \\ v_0 &= \frac{(\kappa-1) + \sigma(\gamma-\kappa)}{q(\gamma-\kappa) - (1-\alpha\kappa)}, \\ w_0 &= \frac{q\gamma - 1 + \sigma - q + \alpha - \sigma\alpha\gamma}{q(\gamma-\kappa) - (1-\alpha\kappa)}. \end{aligned}$$

Biologically, only positive solutions ($u_0 > 0, v_0 > 0, w_0 > 0$) are of interest in the competition of biological population. Hence, we make the natural assumption: $u_0 > 0, v_0 > 0, w_0 > 0$.

In this paper, we mainly focus on the bifurcation and transition problem of (3.1) at the more general positive steady-state solution φ_6 in (3.3).

For this purpose, we take the translation

$$u = u'' + u_0, \quad v = v'' + v_0, \quad w = w'' + w_0, \tag{3.4}$$

Omitting the primes, the system (3.1) with (1.2)–(1.3) becomes

$$\begin{cases} u_t = \varepsilon_1 \Delta u - \bar{a}_1 u_0 u - \bar{a}_1 \alpha u_0 v - \bar{a}_1 u_0 w \\ \qquad\qquad - \bar{a}_1 u^2 - \bar{a}_1 \alpha u v - \bar{a}_1 u w, \\ v_t = \varepsilon_2 \Delta v - \bar{a}_2 \gamma v_0 u - \bar{a}_2 v_0 v - \bar{a}_2 \kappa v_0 w \\ \qquad\qquad - \bar{a}_2 \gamma u v - \bar{a}_2 v^2 - \bar{a}_2 \kappa v w, \\ w_t = \varepsilon_3 \Delta w + w_0 u + q w_0 v + u w + q v w, \end{cases} \tag{3.5}$$

with the initial-boundary conditions

$$\begin{aligned} &u(x,0) = u^0 - u_0, \quad v(x,0) = v^0 - v_0, \quad w(x,0) = w^0 - w_0, \\ &\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = \frac{\partial w}{\partial n}\Big|_{\partial\Omega} = 0. \end{aligned} \tag{3.6}$$

Then it suffices to study the bifurcation solution of (3.5) at the steady-state solution $\chi = (0,0,0)^T$.

3.2. Linear theory and principle of exchange of stabilities (PES)

The linearized eigenvalue equations of (3.5) are given by

$$\begin{cases} \varepsilon_1\Delta u - \bar{a}_1 u_0 u - \bar{a}_1\alpha u_0 v - \bar{a}_1 u_0 w = \beta(\lambda)u, \\ \varepsilon_2\Delta v - \bar{a}_2\gamma v_0 u - \bar{a}_2 v_0 v - \bar{a}_2\kappa v_0 w = \beta(\lambda)v, \\ \varepsilon_3\Delta w + w_0 u + q w_0 v = \beta(\lambda)w. \end{cases} \tag{3.7}$$

Let ρ_k and e_k be the kth eigenvalue and eigenvector of the Laplace operator Δ with the Neumann boundary condition:

$$\begin{cases} \Delta e_k = -\rho_k e_k, \quad (\rho_k \geq 0), \\ \dfrac{\partial e_k}{\partial n}|_{\partial\Omega} = 0. \end{cases} \tag{3.8}$$

Let M_k be the matrix given by

$$M_k = \begin{pmatrix} -\varepsilon_1\rho_k - \bar{a}_1 u_0 & -\bar{a}_1\alpha u_0 & -\bar{a}_1 u_0 \\ -\bar{a}_2\gamma v_0 & -\varepsilon_2\rho_k - \bar{a}_2 v_0 & -\bar{a}_2\kappa v_0 \\ w_0 & q w_0 & -\varepsilon_3\rho_k \end{pmatrix}. \tag{3.9}$$

Thus, all eigenvalues $\beta(\lambda) = \beta_{ki}(\lambda)$ of (3.7) satisfy

$$M_k\eta_{ki} = \beta_{ki}(\lambda)\eta_{ki}, \quad 1 \leq i \leq 3, \ \ k = 1, 2, \cdots, \tag{3.10}$$

where $\eta_{ki} \in \mathbb{R}^3$ is the eigenvector of M_k corresponding to $\beta_{ki}(\lambda)$. Hence, the eigenvector ψ_{ki} of (3.7) corresponding to $\beta_{ki}(\lambda)$ is

$$\psi_{ki}(x) = \eta_{ki}e_k(x), \quad k = 1, 2, \cdots, \ \ 1 \leq i \leq 3, \tag{3.11}$$

where $e_k(x)$ is as in (3.8).

In particular, $\rho_1 = 0$ and e_1 is a constant, and

$$M_1 = \begin{pmatrix} -\bar{a}_1 u_0 & -\bar{a}_1\alpha u_0 & -\bar{a}_1 u_0 \\ -\bar{a}_2\gamma v_0 & -\bar{a}_2 v_0 & -\bar{a}_2\kappa v_0 \\ w_0 & q w_0 & 0 \end{pmatrix}. \tag{3.12}$$

By simple calculation, it is not difficult to find that the eigenvalues $\beta(\lambda) = \beta_{1i}(\lambda)(i = 1, 2, 3)$ satisfy the following equation:

$$\beta_{1i}(\lambda)^3 + A\beta_{1i}(\lambda)^2 + B\beta_{1i}(\lambda) + C = 0, \tag{3.13}$$

where

$$\begin{aligned} A &= \bar{a}_2 v_0 + \bar{a}_1 u_0, \\ B &= \bar{a}_1\bar{a}_2 u_0 v_0 + \bar{a}_1 u_0 w_0 - \bar{a}_1\bar{a}_2\alpha\gamma u_0 v_0 + \bar{a}_2\kappa q v_0 w_0, \\ C &= \bar{a}_1\bar{a}_2(-\kappa\alpha - \gamma q + 1 + \kappa q)u_0 v_0 w_0. \end{aligned} \tag{3.14}$$

It is known that all solutions of (3.13) have negative real parts if and only if

$$A > 0, \quad C > 0, \quad AB - C > 0. \tag{3.15}$$

If we suppose $q(\gamma - \kappa) - (1 - \kappa\alpha) < 0$ in (3.3), then, direct calculation indicates that these two parameters A and C in (3.14) are positive

$$A > 0, \quad C > 0. \tag{3.16}$$

Note that

$$\beta_{11}(\lambda)\beta_{12}(\lambda)\beta_{13}(\lambda) = -C < 0,$$

which implies that all real eigenvalues of (3.12) do not change their signs, and at least one of these real eigenvalues is negative.

In addition, let $\delta = \alpha\gamma = \frac{c_1c_2}{b_1b_2}$, and we can derive from $AB - C = 0$, the critical number

$$\delta_0 = 1 + \frac{\bar{a}_2^2\kappa q v_0^2 w_0 + \bar{a}_1^2 u_0^2 w_0 + \bar{a}_1\bar{a}_2(\kappa\alpha + \gamma q)u_0v_0w_0}{\bar{a}_1\bar{a}_2^2 u_0 v_0^2 + \bar{a}_1^2\bar{a}_2 u_0^2 v_0}. \tag{3.17}$$

It is then clear that

$$AB - C \begin{cases} < 0 & \text{if } \delta > \delta_0, \\ = 0 & \text{if } \delta = \delta_0, \\ > 0 & \text{if } \delta < \delta_0. \end{cases} \tag{3.18}$$

Next, we check the other eigenvalues $\beta_{kj}(\lambda)$ with $j = 1, 2, 3,\ k \geq 2$. By calculation, the eigenvalues $\beta_{kj}(\lambda)(j = 1, 2, 3, k \geq 2)$ of (3.9) satisfy

$$\beta_{kj}(\lambda)^3 + A_k\beta_{kj}(\lambda)^2 + B_k\beta_{kj}(\lambda) + C_k = 0, \tag{3.19}$$

where

$$\begin{aligned} A_k &= A + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_k, \\ B_k &= B + (\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_2)\rho_k^2 \\ &\quad + (\bar{a}_1u_0\varepsilon_3 + \bar{a}_2v_0\varepsilon_3 + \bar{a}_2v_0\varepsilon_1 + \bar{a}_1u_0\varepsilon_2)\rho_k, \\ C_k &= C + \varepsilon_1\varepsilon_2\varepsilon_3\rho_k^3 + (\varepsilon_1\varepsilon_3\bar{a}_2v_0 + \varepsilon_2\varepsilon_3\bar{a}_1u_0)\rho_k^2 \\ &\quad + \bar{a}_1\bar{a}_2(1 - \delta)u_0v_0\varepsilon_3\rho_k + \bar{a}_1u_0w_0\varepsilon_2\rho_k + \bar{a}_2\kappa q v_0w_0\varepsilon_1\rho_k. \end{aligned}$$

We introduce another critical number

$$\delta_1 = \min_{\rho_k \neq 0}\left[1 + \frac{C + \tau_1\rho_k^3 + \tau_2\rho_k^2 + \tau_3\rho_k}{\bar{a}_1\bar{a}_2u_0v_0\varepsilon_3\rho_k}\right], \tag{3.20}$$

where

$$\begin{aligned} \tau_1 &= \varepsilon_1\varepsilon_2\varepsilon_3, \\ \tau_2 &= (\varepsilon_1\varepsilon_3\bar{a}_2v_0 + \varepsilon_2\varepsilon_3\bar{a}_1u_0), \\ \tau_3 &= \bar{a}_1u_0w_0\varepsilon_2 + \bar{a}_2\kappa q v_0w_0\varepsilon_1. \end{aligned} \tag{3.21}$$

The following lemma characterizes the principle of exchange stability (PES) for the eigenvalue Eq (3.7).

Lemma 3.1. *Assume that $q(\gamma - \kappa) - (1 - \kappa\alpha) < 0$, let δ_0 and δ_1 be the numbers given by* (3.17) *and* (3.20)*, then the eigenvalues $\beta_{ki}(i = 1, 2, 3, k \geq 1)$ of L_λ satisfy the following properties:*

(1) *If $\delta_1 < \delta_0$ and k_0 be the integer that δ_1 in* (3.20) *reaches its minimum at ρ_{k_0}, then $\beta_{k_0 1}$ is a real eigenvalue of* (3.7)*, and*

$$\beta_{k_0 1}(\delta)\begin{cases} > 0 & \textit{if } \delta > \delta_1, \\ = 0 & \textit{if } \delta = \delta_1, \quad \textit{for } \rho_k = \rho_{k_0}, \\ < 0 & \textit{if } \delta < \delta_1, \end{cases}$$

$$Re\beta_{ij}(\delta_1) < 0 \ \ \forall (i, j) \neq (k_0, 1) \ \ \textit{with} \ \ \rho_k = \rho_{k_0}.$$

(2) *If $\delta_1 > \delta_0$, then $\beta_{11}(\delta) = \bar{\beta}_{12}(\delta)$ are a pair of complex eigenvalues of* (3.7)*, and*

$$Re\beta_{11}(\delta) = Re\beta_{12}(\delta)\begin{cases} > 0 & \textit{if } \delta > \delta_0, \\ = 0 & \textit{if } \delta = \delta_0, \\ < 0 & \textit{if } \delta < \delta_0, \end{cases}$$

$$Re\beta_{ij}(\delta_0) < 0 \ \ \forall (i, j) \neq (1, 1), (1, 2).$$

Proof. According to the assumption, C is positive. We see that

$$\begin{aligned} A &= \bar{a}_2 v_0 + \bar{a}_1 u_0, \\ B &= \bar{a}_1\bar{a}_2 u_0 v_0 + \bar{a}_1 u_0 w_0 - \bar{a}_1\bar{a}_2\alpha\gamma u_0 v_0 + \bar{a}_2\kappa q v_0 w_0. \end{aligned}$$

By the direct calculation, we can see that

$$A_k > 0, \quad A_k B_k - C_k > 0, \quad \forall k \geq 2.$$

As $\delta_1 < \delta_0$, we infer from (3.16) and (3.18) that

$$Re\beta_{1j}(\delta_1) < 0, \quad \forall 1 \leq j \leq 3.$$

In addition, it is clear that there must exist a k_0 satisfying (3.20), furthermore,

$$C_{k_0}(\delta)\begin{cases} < 0 & \text{if } \delta > \delta_1, \\ = 0 & \text{if } \delta = \delta_1, \\ > 0 & \text{if } \delta < \delta_1, \end{cases}$$

$$C_k(\delta_1) > 0 \quad \text{for all } \ k \neq k_0,$$

thus, assertion (1) is approved.

As $\delta_1 > \delta_0$, through the analysis of above, we know that $C_k > 0$ at $\delta = \delta_0$ for all $k \geq 2$. Since all real eigenvalues $\beta_{1j}(1 \leq j \leq 3)$ of (3.12) do not change their signs, and at least one of these real eigenvalues is negative, so the condition (3.18) implies that there exists a pair of complex eigenvalues $\beta_{11} = \bar{\beta}_{12}$ crossing the imaginary axis at $\delta = \delta_0$. Then assertion (2) follows. The Lemma is proved.

4. Main results and proofs

The following theorems will show the types of transition that the system (3.5) undergoes as the bifurcation parameter δ crosses the critical value δ_0 or δ_1 basing on Lemma 3.1. Hereafter, we will give different transition theorems basing on Lemma 3.1.

First, we consider the case that $\delta_0 < \delta_1$.

4.1. Transitions from complex eigenvalues

By Lemma 3.1, as $\delta_0 < \delta_1$, the first critical eigenvalues will be a pair of complex numbers β_{11} and β_{12}, the problem (3.5) undergoes a dynamic transition to a periodic solution from δ_0. To determine the types of transition, we introduce a parameter b_0 which is defined by (4.30) as follows:

$$
\begin{aligned}
b_0 = & \frac{1}{D^2 D_0}\Big[-\frac{\rho^2}{\bar{a}_1^2 u_0^3}(A(3F_1+F_3)E_2+\rho a\tilde{b}(F_1+3F_3)+\rho F_2E_2) \\
& +\frac{\rho w_0}{\bar{a}_1 u_0} b\tilde{b}\bar{a}_2(3E_5F_1+E_5F_3+E_7F_2)+a\tilde{b}(3F_1E_6+F_3E_6-F_2E_8 \\
& -\frac{\rho^2 A}{\bar{a}_1^2 u_0^3}F_2)+(3F_3+F_1)(E_3E_7+E_2E_8)+F_2(E_3E_5-E_2E_6)\Big] \\
& +\frac{2b\tilde{b}}{D^4\bar{a}_1u_0}\Big[\frac{\rho^2 w_0}{\bar{a}_1^2u_0^2}a(\bar{a}_2(a+\kappa w_0 b)+qa)(E_3\eta_2(\eta_2+\kappa\eta_3)-E_1E_2) \\
& -(w_0\bar{a}_2\xi_2(\gamma+\xi_2+\kappa\xi_3)+\frac{\rho^2}{\bar{a}_1u_0}a\xi_3)(E_3\xi_2(\gamma+\xi_2+\kappa\xi_3)+E_1E_2)\Big] \\
& +\frac{1}{D^4\bar{a}_1u_0}\bar{a}_2[\xi_2(\gamma+\xi_2+\kappa\xi_3)+\eta_2(\eta_2+\kappa\eta_3)] \\
& (-\frac{\rho^3}{\bar{a}_1^2u_0^3}\tilde{b}+\frac{\rho}{\bar{a}_1u_0}\bar{a}_2E_7+E_4\tilde{a})(a^2+2w_0ab\tilde{a}+w_0^2b^2(\tilde{a}^2+\tilde{b}^2)),
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
D_0 = & \frac{\bar{a}_1u_0\bar{a}_2v_0w_0(q-\alpha)(\kappa\alpha-1)+A^2(A\alpha-qw_0)-qw_0^2(\bar{a}_2v_0\kappa q+\bar{a}_1u_0)}{\bar{a}_1u_0(A\alpha-qw_0)^2}, \\
D_1 = & \Big[-\bar{a}_2\xi_2(\gamma+\xi_2+\kappa\xi_3)-\frac{\rho^2 qw_0}{\bar{a}_1^2u_0^2}ab\frac{\bar{a}_1u_0+\bar{a}_2v_0\alpha\kappa}{A\alpha-qw_0}\Big], \\
D_2 = & \frac{\rho}{\bar{a}_1u_0}\Big[\frac{\rho^2w_0}{\bar{a}_1u_0}b^2\frac{\bar{a}_1u_0+\bar{a}_2v_0\alpha\kappa}{A\alpha-qw_0}-\frac{\bar{a}_1u_0A+\bar{a}_2v_0w_0\kappa q}{u_0(A\alpha-qw_0)} \\
& -\bar{a}_2(\gamma a+(a+\kappa b)(\frac{\rho^2}{\bar{a}_1u_0q}b-\frac{1}{q}))\Big], \\
D_3 = & -\frac{\rho}{\bar{a}_1u_0}a\Big[\frac{q^2w_0}{\bar{a}_1u_0}a\frac{\bar{a}_1u_0+\bar{a}_2v_0\alpha\kappa}{A\alpha-qw_0}+\bar{a}_2(\xi_2+\kappa\xi_3)\Big],
\end{aligned}
$$

$$
\begin{aligned}
D_4 &= \frac{\bar{a}_1 u_0 + \bar{a}_2 v_0 \alpha\kappa}{A\alpha - q w_0} q\eta_2\eta_3 - \bar{a}_2\eta_2(\eta_2 + \kappa\eta_3),\\
D^2 &= \frac{\rho^2}{\bar{a}_1^2 u_0^2}(w_0^2 b^2 \tilde{b}^2 + (a + w_0 b\tilde{a})^2), \qquad E_1 = \frac{q\rho^3 w_0}{\bar{a}_1^3 u_0^3} ab,\\
E_2 &= a\tilde{a} + w_0 b(\tilde{a}^2 + \tilde{b}^2), \qquad E_3 = \frac{\rho}{\bar{a}_1 u_0}(a + w_0 b\tilde{a})\bar{a}_2,\\
E_4 &= \frac{\rho^2 w_0}{\bar{a}_1^3 u_0^3}(\rho^2 b^2 - q^2 a^2),\\
E_5 &= \frac{A v_0 \kappa}{\bar{a}_1 u_0^2} - (\gamma(\xi_2 + \zeta_2) + 2\xi_2\zeta_2 + \kappa(\xi_2\zeta_3 + \xi_3\zeta_2)),\\
E_6 &= \frac{\rho w_0}{\bar{a}_1^3 u_0^3}\frac{\bar{a}_1 u_0 \alpha + \bar{a}_2 v_0 q}{A\alpha - q w_0}(\rho^2 b + A q a),\\
E_7 &= \frac{\rho v_0 \kappa}{\bar{a}_1 u_0^2} - (\gamma\eta_2 + 2\eta_2\xi_2 + \kappa(\eta_2\xi_3 + \eta_3\xi_2)),\\
E_8 &= \frac{\rho^2 w_0}{\bar{a}_1^3 u_0^3}\frac{\bar{a}_1 u_0 \alpha + \bar{a}_2 v_0 q}{A\alpha - q w_0}(-qa + Ab),\\
F_1 &= \frac{D_1}{A} + \frac{2\rho^2(D_4 - D_1)}{A(A^2 + 4\rho^2)} - \frac{\rho(D_2 + D_3)}{A^2 + 4\rho^2},\\
F_2 &= \frac{D_2 + D_3}{A} - \frac{4\rho^2(D_2 + D_3)}{A(A^2 + 4\rho^2)} + \frac{2\rho(D_1 - D_4)}{A^2 + 4\rho^2},\\
F_3 &= \frac{D_4}{A} + \frac{2\rho^2(D_1 + D_4)}{A(A^2 + 4\rho^2)} + \frac{\rho(D_2 + D_3)}{A^2 + 4\rho^2}.
\end{aligned} \tag{4.2}
$$

Here A is as in (3.14), and $a, b, \tilde{a}, \tilde{b}$ are as in (4.6) and (4.8).

Then, we have the following dynamic transition theorem.

Theorem 4.1. *Consider b_0 which is given by* (4.1). *Let $\delta_0 < \delta_1$. Assume $q(\gamma - \kappa) - (1 - \kappa\alpha) < 0$ and assume that the critical index are $(k, j) = (1, 1)$ and $(k, j) = (1, 2)$, then the problem* (3.5) *undergoes a transition to periodic solutions at $\delta = \delta_0$, and the following assertions holds true:*

(1) *If $b_0 < 0$, then the transition of* (3.5) *is continuous, and the system bifurcates to a periodic solution on $\delta > \delta_0$, which is an attractor.*

(2) *If $b_0 > 0$, then the transition of* (3.5) *is jump, and the system bifurcates on $\delta < \delta_0$ to a unique unstable periodic orbit.*

Proof. We shall prove the theorem in the following two steps.

Step 1. Calculate the critical eigenvectors.

By Lemma 3.1, at δ_0 there is a pair of imaginary eigenvalues $\beta_{11} = \bar{\beta}_{12} = -i\rho$ of (3.7). Let $z = \xi + i\eta$ and $z^* = \xi^* + i\eta^*$ be the eigenvectors and conjugate eigenvectors of (3.7) corresponding to $-i\rho$, i.e. z and z^* satisfy that

$$
\begin{aligned}
(M_1 + i\rho)z = 0,\\
(M_1^* - i\rho)z^* = 0.
\end{aligned} \tag{4.3}
$$

For $z = (z_1, z_2, z_3)$, from the first equation of (4.3) we obtain

$$\begin{cases} \Big[(q-\alpha) - i\dfrac{\rho q}{\bar{a}_1 u_0}\Big] z_1 = \Big[-q + i\dfrac{\rho\alpha}{w_0}\Big] z_3, \\ z_1 + q z_2 = -i\dfrac{\rho}{w_0} z_3. \end{cases} \tag{4.4}$$

Thus, we derive from (4.4) the eigenvectors $z = \xi + i\eta$ as follows:

$$\begin{aligned} \xi &= (\xi_1, \xi_2, \xi_3) = (1,\ \frac{\rho^2}{\bar{a}_1 u_0 q} b - \frac{1}{q},\ -\frac{q w_0}{\bar{a}_1 u_0} a), \\ \eta &= (\eta_1, \eta_2, \eta_3) = (0,\ \frac{\rho}{\bar{a}_1 u_0} a,\ \frac{\rho w_0}{\bar{a}_1 u_0} b), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} a &= \frac{\bar{a}_1 u_0 w_0 (q-\alpha) + \rho^2 \alpha}{q^2 w_0^2 + \rho^2 \alpha^2}, \\ b &= \frac{q^2 w_0 - \bar{a}_1 u_0 \alpha (q-\alpha)}{q^2 w_0^2 + \rho^2 \alpha^2}. \end{aligned} \tag{4.6}$$

In the same fashion, we derive from the second equation of (4.3) the conjugate eigenvectors $z^* = \xi^* + i\eta^*$ as

$$\begin{aligned} \xi^* &= (\xi_1^*, \xi_2^*, \xi_3^*) = (\frac{\rho\tilde{b} - \bar{a}_2 v_0 \kappa}{\bar{a}_1 u_0},\ 1,\ \tilde{a}), \\ \eta^* &= (\eta_1^*, \eta_2^*, \eta_3^*) = (-\frac{\rho\tilde{a}}{\bar{a}_1 u_0},\ 0,\ \tilde{b}), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} \tilde{a} &= \frac{\bar{a}_2 v_0 (1-\alpha\kappa) q w_0 + \rho^2 \alpha}{q^2 w_0^2 + \rho^2 \alpha^2}, \\ \tilde{b} &= \frac{-\bar{a}_2 v_0 \rho\alpha (1-\alpha\kappa) + \rho q w_0}{q^2 w_0^2 + \rho^2 \alpha^2}. \end{aligned} \tag{4.8}$$

It is easy to show that

$$\begin{aligned} \langle \xi, \xi^* \rangle &= \langle \eta, \eta^* \rangle = \frac{\rho w_0}{\bar{a}_1 u_0} b\tilde{b}, \\ \langle \xi, \eta^* \rangle &= -\langle \eta, \xi^* \rangle = -\Big(\frac{\rho}{\bar{a}_1 u_0} a + \frac{\rho w_0}{\bar{a}_1 u_0} b\tilde{a}\Big). \end{aligned} \tag{4.9}$$

It is known that functions $\psi_1^* + i\psi_2^*$ given by

$$\begin{aligned} \psi_1^* &= \frac{1}{\langle \xi, \xi^* \rangle} [\langle \xi, \xi^* \rangle \xi^* + \langle \xi, \eta^* \rangle \eta^*], \\ \psi_2^* &= \frac{1}{\langle \eta, \eta^* \rangle} [\langle \eta, \xi^* \rangle \xi^* + \langle \eta, \eta^* \rangle \eta^*], \end{aligned} \tag{4.10}$$

also satisfy the second equation of (4.3) with

$$\begin{aligned}&\langle \xi, \psi_1^* \rangle = \langle \eta, \psi_2^* \rangle \neq 0,\\ &\langle \xi, \psi_2^* \rangle = \langle \eta, \psi_1^* \rangle = 0.\end{aligned} \tag{4.11}$$

On the other hand, we know that

$$\beta_{13} \cdot (i\rho) \cdot (-i\rho) = \rho^2 \beta_{13} = -C. \tag{4.12}$$

In addition, because $\pm i\rho$ are solutions of (3.13), and $AB - C = 0$ at δ_0, we deduce that

$$\rho^2 = B = \frac{C}{A}. \tag{4.13}$$

Then, we obtain

$$\beta_{13} = -A = -(\bar{a}_2 v_0 + \bar{a}_1 u_0). \tag{4.14}$$

From the equation

$$(M_1 - \beta_{13})\zeta = 0, \tag{4.15}$$

we derive the eigenvector

$$\begin{aligned}\zeta &= (\zeta_1, \zeta_2, \zeta_3)\\ &= \left(1,\ \frac{\bar{a}_1 u_0 w_0 + \bar{a}_2 v_0 A}{\bar{a}_1 u_0 (A\alpha - q w_0)},\ -\frac{w_0}{\bar{a}_1 u_0} \frac{\bar{a}_1 u_0 \alpha + \bar{a}_2 v_0 q}{A\alpha - q w_0}\right).\end{aligned} \tag{4.16}$$

In the same fashion, from

$$(M_1^* - \beta_{13})\zeta^* = 0, \tag{4.17}$$

we derive the conjugate eigenvector of β_{13} as follows:

$$\begin{aligned}\zeta^* &= (\zeta_1^*, \zeta_2^*, \zeta_3^*)\\ &= \left(\frac{\bar{a}_2 v_0 w_0 \kappa q + \bar{a}_1 u_0 A}{\bar{a}_1 u_0 (A\alpha - q w_0)},\ 1,\ \frac{\bar{a}_1 u_0 + \bar{a}_2 v_0 \alpha \kappa}{A\alpha - q w_0}\right).\end{aligned} \tag{4.18}$$

Step 2. Derivation of evolution equation.

Let $\chi \in H$ be a solution of (3.5) expressed as

$$\chi = x\xi + y\eta + \Phi(x, y), \quad x, y \in \mathbb{R}^1,$$

where $\Phi(x, y)$ is the center manifold function of (3.5) at δ_0.

Based on the center manifold reduction, the reduced equations of (3.5) on the center manifold are given by

$$\begin{aligned}\frac{dx}{dt} &= -\rho y + \frac{1}{\langle \xi, \psi_1^* \rangle} \langle G(x\xi + y\eta + \Phi), \psi_1^* \rangle,\\ \frac{dy}{dt} &= \rho x + \frac{1}{\langle \eta, \psi_2^* \rangle} \langle G(x\xi + y\eta + \Phi), \psi_2^* \rangle,\end{aligned} \tag{4.19}$$

where $G(\chi) = G(\chi, \chi)$ is the bilinear operator defined by

$$\begin{aligned} G(\chi, \chi_1) = (&-\bar{a}_1 u_1 u_2 - \bar{a}_1 \alpha u_1 v_2 - \bar{a}_1 u_1 w_2, \\ &- \bar{a}_2 \gamma u_1 v_2 - \bar{a}_2 v_1 v_2 - \bar{a}_2 \kappa v_1 w_2, u_1 w_2 + q v_1 w_2), \end{aligned} \tag{4.20}$$

for $\chi = (u_1, v_1, w_1), \chi_1 = (u_2, v_2, w_2) \in H_1$.

Now we are in position to solve the center manifold function $\Phi(x, y)$. By the approximation formula (B.4.1) in [20], the center manifold function Φ satisfy

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + o(2), \tag{4.21}$$

where

$$\begin{aligned} \ell\Phi_1 &= -x^2 P_2 G_{11} - xy(P_2 G_{12} + P_2 G_{21}) - y^2 P_2 G_{22}, \\ (\ell^2 + 4\rho^2)\ell\Phi_2 &= 2\rho^2(x^2 - y^2) P_2 G_{11} + 4\rho^2 xy(P_2 G_{12} + P_2 G_{21}) \\ &\quad + 2\rho^2(y^2 - x^2) P_2 G_{22}, \\ (\ell^2 + 4\rho^2)\Phi_3 &= \rho(y^2 - x^2)(P_2 G_{12} + P_2 G_{21}) \\ &\quad + 2\rho xy(P_2 G_{11} - P_2 G_{22}), \end{aligned} \tag{4.22}$$

here $P_2 : H \to E_2$ is the canonical projection, E_2 is the orthogonal complement of $E_1 = span\{\xi, \eta\}$, and ℓ is the linearized operator of (3.5).

Remark 4.1. *In* (4.21), $o(2)$ *represent* $o(|(x, y)|^2)$. *hereafter, we make the following convention*

$$o(m) = o(|\mathbf{x}|^m) \quad \textit{for } \mathbf{x} \in \mathbb{R}^n \textit{near 0}. \tag{4.23}$$

Direct calculation shows that

$$\begin{aligned} &\langle \zeta, \zeta^* \rangle = D_0, \\ &\langle G_{11}, \zeta^* \rangle = D_1, \quad \langle G_{12}, \zeta^* \rangle = D_2, \\ &\langle G_{21}, \zeta^* \rangle = D_3, \quad \langle G_{22}, \zeta^* \rangle = D_4 \end{aligned}$$

and D_0, D_1, D_2, D_3, D_4 are as in (4.2).

By (4.5), (4.18) and (4.20), it is clear that

$$\begin{aligned} P_2 G_{11} &= \langle G_{11}, \zeta^* \rangle \zeta = D_1 \zeta, \\ P_2 G_{12} &= \langle G_{12}, \zeta^* \rangle \zeta = D_2 \zeta, \\ P_2 G_{21} &= \langle G_{21}, \zeta^* \rangle \zeta = D_3 \zeta, \\ P_2 G_{22} &= \langle G_{22}, \zeta^* \rangle \zeta = D_4 \zeta. \end{aligned} \tag{4.24}$$

Hence, $\Phi_1, \Phi_2, \Phi_3 \in span\{\zeta\}$, which implies

$$\ell\Phi_j = M_1 \Phi_j = -A\Phi_j. \tag{4.25}$$

We infer from (4.21)–(4.25) the center manifold function as follows:

$$\begin{aligned}\Phi &= \frac{\zeta}{D_0}\Big[\Big(\frac{D_1}{A} + \frac{2\rho^2(D_4 - D_1)}{A(A^2 + 4\rho^2)} - \frac{\rho(D_2 + D_3)}{A^2 + 4\rho^2}\Big)x^2 \\ &\quad + \Big(\frac{D_2 + D_3}{A} - \frac{4\rho^2(D_2 + D_3)}{A(A^2 + 4\rho^2)} + \frac{2\rho(D_1 - D_4)}{A^2 + 4\rho^2}\Big)xy \\ &\quad + \Big(\frac{D_4}{A} + \frac{2\rho^2(D_1 + D_4)}{A(A^2 + 4\rho^2)} + \frac{\rho(D_2 + D_3)}{A^2 + 4\rho^2}\Big)y^2\Big] + o(2) \\ &= \frac{1}{D_0}(F_1x^2 + F_2xy + F_3y^2)\zeta + o(2),\end{aligned} \tag{4.26}$$

where F_1, F_2, F_3 are as in (4.2).

Inserting (4.26) into (4.19), by direct circulation, we have

$$\begin{aligned}\frac{dx}{dt} &= -\rho y + \frac{1}{D^2}\Big\{[\langle\xi,\xi^*\rangle\langle G_{11},\xi^*\rangle + \langle\xi,\eta^*\rangle\langle G_{11},\eta^*\rangle]x^2 \\ &\quad + [\langle\xi,\xi^*\rangle\langle G_{22},\xi^*\rangle + \langle\xi,\eta^*\rangle\langle G_{22},\eta^*\rangle]y^2 \\ &\quad + [\langle\xi,\xi^*\rangle\langle G_{12} + G_{21},\xi^*\rangle + \langle\xi,\eta^*\rangle\langle G_{12} + G_{21},\eta^*\rangle]xy \\ &\quad + \langle\xi,\xi^*\rangle\langle G(\xi,\zeta) + G(\zeta,\xi),\xi^*\rangle\frac{1}{D_0}(F_1x^3 + F_2x^2y + F_3xy^2) \\ &\quad + \langle\xi,\eta^*\rangle\langle G(\xi,\zeta) + G(\zeta,\xi),\eta^*\rangle\frac{1}{D_0}(F_1x^3 + F_2x^2y + F_3xy^2) \\ &\quad + \langle\xi,\xi^*\rangle\langle G(\eta,\zeta) + G(\zeta,\eta),\xi^*\rangle\frac{1}{D_0}(F_1x^2y + F_2xy^2 + F_3y^3) \\ &\quad + \langle\xi,\eta^*\rangle\langle G(\eta,\zeta) + G(\zeta,\eta),\eta^*\rangle\frac{1}{D_0}(F_1x^2y + F_2xy^2 + F_3y^3)\Big\} \\ &\quad + o(3),\end{aligned} \tag{4.27}$$

$$\begin{aligned}\frac{dy}{dt} &= \rho x + \frac{1}{D^2}\Big\{[-\langle\xi,\eta^*\rangle\langle G_{11},\xi^*\rangle + \langle\xi,\xi^*\rangle\langle G_{11},\eta^*\rangle]x^2 \\ &\quad + [-\langle\xi,\eta^*\rangle\langle G_{22},\xi^*\rangle + \langle\xi,\xi^*\rangle\langle G_{22},\eta^*\rangle]y^2 \\ &\quad + [-\langle\xi,\eta^*\rangle\langle G_{12} + G_{21},\xi^*\rangle + \langle\xi,\xi^*\rangle\langle G_{12} + G_{21},\eta^*\rangle]xy \\ &\quad - \langle\xi,\eta^*\rangle\langle G(\xi,\zeta) + G(\zeta,\xi),\xi^*\rangle\frac{1}{D_0}(F_1x^3 + F_2x^2y + F_3xy^2) \\ &\quad + \langle\xi,\xi^*\rangle\langle G(\xi,\zeta) + G(\zeta,\xi),\eta^*\rangle\frac{1}{D_0}(F_1x^3 + F_2x^2y + F_3xy^2) \\ &\quad - \langle\xi,\eta^*\rangle\langle G(\eta,\zeta) + G(\zeta,\eta),\xi^*\rangle\frac{1}{D_0}(F_1x^2y + F_2xy^2 + F_3y^3) \\ &\quad + \langle\xi,\xi^*\rangle\langle G(\eta,\zeta) + G(\zeta,\eta),\eta^*\rangle\frac{1}{D_0}(F_1x^2y + F_2xy^2 + F_3y^3)\Big\} \\ &\quad + o(3),\end{aligned} \tag{4.28}$$

where $D^2 = \langle\xi,\xi^*\rangle^2 + \langle\xi,\eta^*\rangle^2$.

In view of (4.9), Eqs (4.27) and (4.28) become

$$\begin{aligned}\frac{dx}{dt} &= -\rho y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ &\quad + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + o(3), \\ \frac{dy}{dt} &= \rho x + b_{20}x^2 + b_{11}xy + b_{02}y^2 \\ &\quad + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + o(3),\end{aligned} \tag{4.29}$$

where

$$\begin{aligned}
a_{20} &= \frac{1}{D^2}\frac{\rho}{\bar a_1 u_0} b\tilde b[-w_0\bar a_2\xi_2(\gamma+\xi_2+\kappa\xi_3) - \frac{\rho^2}{\bar a_1 u_0}a\xi_3],\\
a_{02} &= -\frac{1}{D^2}\frac{\rho^3 w_0}{\bar a_1^3 u_0^3} ab\tilde b[\bar a_2(a+\kappa w_0 b) + qa],\\
a_{11} &= -\frac{1}{D^2}[-E_5E_2 + \frac{\rho^2 w_0}{\bar a_1^2 u_0^2} b\tilde b\bar a_2 E_4 - E_6 a\tilde b],\\
a_{30} &= \frac{F_1}{D^2 D_0}[-\frac{\rho^2 A}{\bar a_1^2 u_0^3}E_2 + \frac{\rho w_0}{\bar a_1 u_0} b\tilde b\bar a_2 E_7 + E_8 a\tilde b],\\
a_{03} &= \frac{F_3}{D^2 D_0}[-\frac{\rho^3}{\bar a_1^2 u_0^3}E_2 + \frac{\rho w_0}{\bar a_1 u_0} b\tilde b\bar a_2 E_9 - E_{10} a\tilde b],\\
b_{20} &= \frac{1}{D^2}[-E_3\xi_2(\gamma+\xi_2+\kappa\xi_3) - E_1E_2],\\
b_{02} &= \frac{1}{D^2}[-E_3\eta_2(\eta_2+\kappa\eta_3) + E_1E_2],\\
b_{11} &= \frac{1}{D^2}[-E_5 a\tilde b + \frac{\rho}{\bar a_1 u_0}E_3E_4 + E_2E_6],\\
b_{30} &= \frac{F_1}{D^2 D_0}[-\frac{\rho^2 A}{\bar a_1^2 u_0^3}a\tilde b + E_3E_7 - E_2E_8],\\
b_{03} &= \frac{F_3}{D^2 D_0}[-\frac{\rho^3}{\bar a_1^2 u_0^3}a\tilde b + E_3E_9 + E_2E_{10}],\\
a_{21} &= \frac{F_2}{F_1}a_{30} + \frac{F_1}{F_3}a_{03}, \qquad a_{12} = \frac{F_3}{F_1}a_{30} + \frac{F_2}{F_3}a_{03},\\
b_{21} &= \frac{F_2}{F_1}b_{30} + \frac{F_1}{F_3}b_{03}, \qquad b_{12} = \frac{F_3}{F_1}b_{30} + \frac{F_2}{F_3}b_{03}.
\end{aligned}$$

The transition of (3.5)–(3.6) is determined by the sign of the following number; see [20],

$$\begin{aligned}b_0 &= 3(a_{30}+b_{03}) + (a_{12}+b_{21}) + \frac{2}{\rho}(a_{02}b_{02} - a_{20}b_{20}) \\ &\quad + \frac{1}{\rho}(a_{11}a_{20} + a_{11}a_{02} - b_{11}b_{20} - b_{11}b_{02}),\end{aligned} \tag{4.30}$$

which is the same as that given by (4.1). From (4.1) and (4.2), it is easy to show that $b_0 < 0$ in the limit of small ρ. Thus the proof is complete.

Second, we consider the case that $\delta_0 > \delta_1$.

4.2. Transitions from real eigenvalues

Thanks to Lemma 3.1, for $\delta_0 > \delta_1$, the transition of the system (3.5) occurs at δ_1, which is from real eigenvalues. The following theorems will show the types of transition that the system (3.5) undergoes as the bifurcation parameter δ crosses the critical value δ_1

$$\delta_1 = \min_{\rho_k \neq 0} [1 + \frac{C + \tau_1\rho_k^3 + \tau_2\rho_k^2 + \tau_3\rho_k}{\bar{a}_1\bar{a}_2 u_0 v_0 \varepsilon_3 \rho_k}].$$

Let δ_1 achieve it minimum at ρ_{k_0}, ρ_{k_0} be the eigenvalues of (3.8), e_{k_0} be the eigenvector of (3.8) corresponding to ρ_{k_0}. Assume that $\beta_{k_0 1}$ is simple near δ_1. Hereafter, we will give different transition theorems basing on the two cases about e_{k_0}.

First, we consider the case where

$$\int_\Omega e_{k_0}^3 dx \neq 0. \tag{4.31}$$

For simplicity, let

$$\theta = \frac{P_k \int_\Omega e_{k_0}^3 dx}{Q_k \int_\Omega e_{k_0}^2 dx}, \tag{4.32}$$

where

$$\begin{aligned} P_k =& \bar{a}_1^2 u_0 w_0 a^2 c^2 [\bar{a}_1 u_0 w_0 ac + \alpha \bar{a}_2 v_0 w_0 bd + w_0^2 ab] \\ &+ \bar{a}_2^2 v_0 w_0 b^2 cd [\gamma \bar{a}_1 u_0 w_0 ac + \bar{a}_2 v_0 w_0 bd + \kappa w_0^2 ab] \\ &+ w_0^2 ab^2 e [\bar{a}_1 u_0 w_0 ac + q \bar{a}_2 v_0 w_0 bd], \\ Q_k =& [-\bar{a}_1 u_0 w_0 a^2 c^2 - \bar{a}_2 v_0 w_0 b^2 cd + w_0^2 ab^2 e], \end{aligned} \tag{4.33}$$

and a, b, c, d, e in (4.33) are given by

$$\begin{aligned} a &= (-\varepsilon_2\rho_{k_0} - \bar{a}_2 v_0) + \bar{a}_2 v_0 \beta q, \\ b &= (-\varepsilon_1\rho_{k_0} - \bar{a}_1 u_0) q + \bar{a}_1 u_0 \alpha, \\ c &= \varepsilon_3 \rho_{k_0} \alpha + q w_0, \\ d &= \varepsilon_3 \rho_{k_0} \beta + \kappa w_0, \\ e &= (-\varepsilon_2\rho_{k_0} - \bar{a}_2 v_0) + \bar{a}_2 v_0 \kappa \alpha. \end{aligned} \tag{4.34}$$

Then, under the condition (4.31), we have the second dynamic transition theorem.

Theorem 4.2. *Assume that* $q(\gamma - \kappa) - (1 - \kappa\alpha) < 0$*. Let* $\delta_0 > \delta_1$*. If* $\theta \neq 0$*, then we have the following assertions:*

(1) *The transition of* (3.5) *at* $\delta = \delta_1$ *is mixed. More precisely, there exists a neighborhood* $U \subset H_1$ *of* $\chi = 0$ *such that* U *is separated into two disjoint open sets* U_1 *and* U_2 *by the stable manifold* Γ *of* $\chi = 0$ *satisfying*

  (*a*) $U = U_1 + U_2 + \Gamma$,

(*b*) *the transition in* U_1 *is jump, and in* U_2 *is continuous.*

(2) *The system* (3.5) *bifurcates in* U_2 *to a unique singular point* $\bar{\chi}$ *on* $\delta > \delta_1$*, which is an attractor such that for any initial value* $\varphi \in U_2$*,*

$$\lim_{t \longrightarrow \infty} \|\chi(t,\varphi) - \bar{\chi}\|_{H_1} = 0.$$

(3) *The system* (3.5) *bifurcates on* $\delta < \delta_1$ *to a unique saddle point* $\bar{\chi}$*.*

(4) *The bifurcated singular point* $\bar{\chi}$ *can be expressed as*

$$\bar{\chi} = -\frac{\beta_{k_0 1}}{\theta}\xi e_{k_0} + o(|\beta_{k_0 1}|), \tag{4.35}$$

where θ *is as in* (4.32) *and* $\xi = (\xi_1, \xi_2, \xi_3)$ *are given by*

$$\begin{aligned}
\xi_1 &= \bar{a}_1 u_0 w_0[(-\varepsilon_2\rho_{k_0} - \bar{a}_2 v_0) + \bar{a}_2 v_0 \gamma q][\varepsilon_3 \rho_{k_0}\alpha + q w_0],\\
\xi_2 &= \bar{a}_2 v_0 w_0[(-\varepsilon_1\rho_{k_0} - \bar{a}_1 u_0) q + \bar{a}_1 u_0 \alpha][\varepsilon_3 \rho_{k_0}\gamma + \kappa w_0],\\
\xi_3 &= w_0^2[(-\varepsilon_2\rho_{k_0} - \bar{a}_2 v_0) + \bar{a}_2 v_0 \gamma q][(-\varepsilon_1\rho_{k_0} - \bar{a}_1 u_0) q + \bar{a}_1 u_0 \alpha].
\end{aligned}$$

Proof. We shall prove the theorem in the following two steps.
Step 1. Calculate the critical eigenvectors and decompose space.

Let ξ and $\xi^* \in \mathbb{R}^3$ be the eigenvectors of M_{k_0} and $M^*_{k_0}$ corresponding to $\beta_{k_0 1}(\delta_1) = 0$, i.e.

$$M_{k_0}\xi = 0, \qquad M^*_{k_0}\xi^* = 0,$$

where M_{k_0} is the matrix (3.9) with $k = k_0$. It is easy to see that ξ is as in (4.35), and

$$\begin{aligned}
\xi^* &= (\xi_1^*, \xi_2^*, \xi_3^*),\\
\xi_1^* &= -[(-\varepsilon_2\rho_{k_0} - \bar{a}_2 v_0) + \bar{a}_2 v_0 \gamma q][\varepsilon_3 \rho_{k_0}\alpha + q w_0],\\
\xi_2^* &= -[(-\varepsilon_1\rho_{k_0} - \bar{a}_1 u_0) q + \bar{a}_1 u_0 \alpha][\varepsilon_3 \rho_{k_0}\alpha + q w_0],\\
\xi_3^* &= [(-\varepsilon_2\rho_{k_0} - \bar{a}_2 v_0) + \bar{a}_2 v_0 \kappa \alpha][(-\varepsilon_1\rho_{k_0} - \bar{a}_1 u_0) q + \bar{a}_1 u_0 \alpha].
\end{aligned} \tag{4.36}$$

On the other hand, based on the spectral theory of linear completely continuous field, the spaces H and H_1 can be decomposed into the following form:

$$H = E_1 \oplus E_2, \qquad H_1 = E_1 \oplus \bar{E}_2,$$

where $E_1 = span\{\xi e_{k_0}\}$, $E_2 = E_1^{\perp}$. Then near δ_1, the solution of the Eq (3.5) can be expressed as

$$\chi = x\xi e_{k_0} + z, \qquad z = \sum_{j=2}^{3} x_{k_0 j}\psi_{k_0 j} + \sum_{k \neq k_0, j=1}^{3} x_{kj}\psi_{kj}, \tag{4.37}$$

where $x\xi e_{k_0} \in E_1$, $z \in E_2$. $\psi_{kj}(j = 1,2,3, k \neq k_0)$ is the eigenvector corresponding to the eigenvalue β_{kj}.

Thus, in the space E_1, the Eq (3.5) can be reduced to

$$\begin{aligned}\langle \xi e_{k_0}, \xi^* e_{k_0}\rangle \frac{dx}{dt} &= \langle L_\delta(\chi), \xi^* e_{k_0}\rangle + \langle G(\chi), \xi^* e_{k_0}\rangle \\ &= \beta_{k_0 1}\langle \xi e_{k_0}, \xi^* e_{k_0}\rangle x + \langle G(\chi), \xi^* e_{k_0}\rangle.\end{aligned} \tag{4.38}$$

Note that $\langle\cdot,\cdot\rangle$ denotes the inner product in H.

Step 2. Derivation of evolution equation.

According to the condition (4.31), we do not need to consider the influence of the center manifold function. That is to say, we let $\chi = x\xi e_{k_0}$ in (4.38).

Hence, we derive the following reduced bifurcation equation

$$\frac{dx}{dt} = \beta_{k_0 1} x + \frac{\langle G(x\xi e_{k_0}), \xi^* e_{k_0}\rangle}{\langle \xi e_{k_0}, \xi^* e_{k_0}\rangle}. \tag{4.39}$$

For the operator G, we can derive that

$$\frac{\langle G(x\xi e_{k_0}), \xi^* e_{k_0}\rangle}{\langle \xi e_{k_0}, \xi^* e_{k_0}\rangle} = \theta x^2 + o(2).$$

where θ is as in (4.32).

Thus, the Eq (4.39) can be rewritten as

$$\frac{dx}{dt} = \beta_{k_0 1} x + \theta x^2 + o(2). \tag{4.40}$$

It is known that the transition of the Eq (3.5) and its local topological structure are determined completely by (4.40). If $\theta \neq 0$, it is clear that (4.40) has exactly a bifurcated solution as follows:

$$\bar{x} = -\frac{\beta_{k_0 1}}{\theta} + o(|\beta_{k_0 1}|). \tag{4.41}$$

Therefore,

$$\bar{\chi} = -\frac{\beta_{k_0 1}}{\theta}\xi e_{k_0} + o(|\beta_{k_0 1}|)$$

is the bifurcated singular point of (3.5).

Obviously, $\bar{\chi}$ is a locally asymptotically stable singular point on $\delta > \delta_1$, which implies the problem (3.5)–(3.6) has a continuous transition in U_2. Meanwhile, the original equilibrium state loses its stability and the problem (3.5)–(3.6) has a jump transition in U_1. And $\bar{\chi}$ is an unstable saddle point on $\delta < \delta_1$ (see Figure 1). Thus, the Theorem is proved.

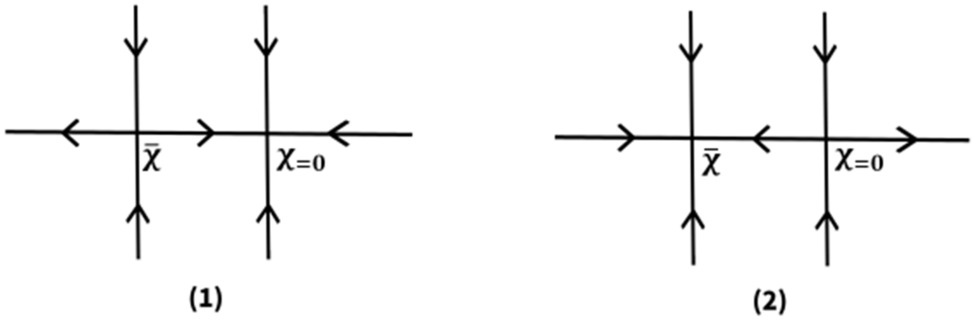

Figure 1. If $\int_\Omega e_{k_0}^3 dx \neq 0$, then the local topological structure of the transitions of (3.5) is : (1) when $\delta < \delta_1$, the system bifurcates from stable equilibrium point $\chi = 0$ to an unstable saddle point $\bar{\chi}$; (2) when $\delta > \delta_1$, the system bifurcates from $\chi = 0$ to a attractor $\bar{\chi}$.

In the following, we consider the case that

$$\int_\Omega e_{k_0}^3 dx = 0. \tag{4.42}$$

Let

$$\tilde{\theta} = \frac{\tilde{P}_k}{Q_k \int_\Omega e_{k_0}^2 dx}, \tag{4.43}$$

here

$$\begin{aligned}
\tilde{P}_k = \Big\{&[2\bar{a}_1^2 u_0 w_0 a^2 c^2 + \bar{a}_1\bar{a}_2\alpha v_0 w_0 abcd + \bar{a}_1 w_0^2 a^2 bc + \bar{a}_2^2\gamma v_0 w_0 b^2 cd \\
&+ w_0^2 ab^2 e]\int_\Omega \phi_1 e_{k_0}^2 dx + [\bar{a}_1^2\alpha u_0 w_0 a^2 c^2 + \bar{a}_1\bar{a}_2\gamma u_0 w_0 abc^2 \\
&+ 2\bar{a}_2^2 v_0 w_0 b^2 cd + \bar{a}_2\kappa w_0^2 ab^2 c + q w_0^2 ab^2 e]\int_\Omega \phi_2 e_{k_0}^2 dx \\
&+ [\bar{a}_1^2 u_0 w_0 a^2 c^2 + \bar{a}_2^2\kappa v_0 w_0 b^2 cd + \bar{a}_1 u_0 w_0 abce \\
&+ q\bar{a}_2 v_0 w_0 b^2 de]\int_\Omega \phi_3 e_{k_0}^2 dx\Big\},
\end{aligned} \tag{4.44}$$

Q_k is as in (4.33), $\phi = (\phi_1, \phi_2, \phi_3)$ satisfies

$$L\phi = -G(\xi)e_{k_0}^2, \tag{4.45}$$

and the operators L and G are defined by

$$L\phi = \begin{cases} \varepsilon_1\Delta u - \bar{a}_1 u_0 u - \bar{a}_1\alpha u_0 v - \bar{a}_1 u_0 w, \\ \varepsilon_2\Delta v - \bar{a}_2\gamma v_0 u - \bar{a}_2 v_0 v - \bar{a}_2\kappa v_0 w, \\ \varepsilon_3\Delta w + w_0 u + q w_0 v, \end{cases}$$

$$G(\xi)=\begin{cases}-\bar{a}_1^2u_0w_0ac(\bar{a}_1u_0w_0ac+\bar{a}_2\alpha v_0w_0bd+w_0^2ab),\\ -\bar{a}_2^2v_0w_0bd(\bar{a}_1\gamma u_0w_0ac+\bar{a}_2v_0w_0bd+\kappa w_0^2ab),\\ w_0^2ab(\bar{a}_1u_0w_0ac+\bar{a}_2qv_0w_0bd).\end{cases}$$

Then we have the following theorem,

Theorem 4.3. *Assume that* $q(\gamma-\kappa)-(1-\kappa\alpha)<0$. *Let* $\tilde{\theta}\neq 0$ *be the number given by* (4.43), *and* $\delta_0>\delta_1$. *Then the transition of* (3.5) *at* δ_1 *is continuous as* $\tilde{\theta}<0$, *and is jump as* $\tilde{\theta}>0$. *Moreover, we have the following assertions:*

(1) *When* $\tilde{\theta}>0$, *the system* (3.5) *bifurcates from* $(\chi,\delta)=(0,\delta_1)$ *to two steady-state solutions* χ_+ *and* χ_- *on* $\delta<\delta_1$, *which are saddles, and no bifurcation solutions on* $\delta>\delta_1$.

(2) *When* $\tilde{\theta}<0$, *the system* (3.5) *bifurcates from* $(\chi,\delta)=(0,\delta_1)$ *to two steady-state solutions* χ_+ *and* χ_- *on* $\delta>\delta_1$, *which are attractors, and no bifurcation solutions on* $\delta<\delta_1$.

(3) *The bifurcated solutions* $\chi_\pm$ *can be expressed as*

$$\chi_\pm=\pm\Big[-\frac{\beta_{k_01}}{\tilde{\theta}}\Big]^{\frac{1}{2}}\xi e_{k_0}+o(|\beta_{k_01}|^{\frac{1}{2}}),$$

where ξ *is as in* (4.35), β_{k_01} *as in Lemma* 3.1.

Proof. We shall prove the theorem in the following two steps.

Step 1.We deduce the evolution equation.

Space decomposition is the same as the step 1 in the proof of Theorem 4.2, so we omit it.

Analogously, in the space E_1, the Eq (3.5) can be reduced to

$$\begin{aligned}\langle\xi e_{k_0},\xi^*e_{k_0}\rangle\frac{dx}{dt}&=\langle L_\delta(\chi),\xi^*e_{k_0}\rangle+\langle G(\chi),\xi^*e_{k_0}\rangle\\&=\beta_{k_01}\langle\xi e_{k_0},\xi^*e_{k_0}\rangle x+\langle G(\chi),\xi^*e_{k_0}\rangle.\end{aligned}\tag{4.46}$$

Let

$$\chi=x\xi e_{k_0}+\Phi(x),\quad x\in\mathbb{R}^1,\tag{4.47}$$

and $\Phi(x)$ is the center manifold function. To evaluate the last term in (4.46), we need to know the center manifold function $\Phi:E_1\to E_2$. Let $\Phi=x^2\phi=o(2)$, then by the approximation formula of center manifolds (see (A.10) in [27]), ϕ satisfies

$$L\phi=-G(\xi e_{k_0})=-G(\xi)e_{k_0}^2.\tag{4.48}$$

Direct circulation to get

$$
\begin{aligned}
&\langle G(x\xi e_{k_0}+\Phi),\xi^* e_{k_0}\rangle \\
&=\int_\Omega\Big[-\bar{a}_1(x\xi_1 e_{k_0}+\Phi_1)^2\xi_1^* e_{k_0}-\bar{a}_1\alpha(x\xi_1 e_{k_0}+\Phi_1)(x\xi_2 e_{k_0}+\Phi_2)\xi_1^* e_{k_0} \\
&\quad-\bar{a}_1(x\xi_1 e_{k_0}+\Phi_1)(x\xi_3 e_{k_0}+\Phi_3)\xi_1^* e_{k_0}-\bar{a}_2\gamma(x\xi_1 e_{k_0}+\Phi_1)(x\xi_2 e_{k_0}+\Phi_2)\xi_2^* e_{k_0} \\
&\quad-\bar{a}_2(x\xi_2 e_{k_0}+\Phi_2)^2\xi_2^* e_{k_0}-\bar{a}_2\kappa(x\xi_2 e_{k_0}+\Phi_2)(x\xi_3 e_{k_0}+\Phi_3)\xi_2^* e_{k_0} \\
&\quad+(x\xi_1 e_{k_0}+\Phi_1)(x\xi_3 e_{k_0}+\Phi_3)\xi_3^* e_{k_0}+q(x\xi_2 e_{k_0}+\Phi_2)(x\xi_3 e_{k_0}+\Phi_3)\xi_2^* e_{k_0}\Big]dx \\
&=x^3\Big\{[-2\bar{a}_1\xi_1\xi_1^*-\bar{a}_1\alpha\xi_2\xi_1^*-\bar{a}_1\xi_3\xi_1^*-\bar{a}_2\gamma\xi_2\xi_2^*+\xi_3\xi_3^*]\int_\Omega\phi_1 e_{k_0}^2dx \\
&\qquad+[-\bar{a}_1\alpha\xi_1\xi_1^*-\bar{a}_2\gamma\xi_1\xi_2^*-2\bar{a}_2\xi_2\xi_2^*-\bar{a}_2\kappa\xi_3\xi_2^*+q\xi_3\xi_3^*]\int_\Omega\phi_2 e_{k_0}^2dx \\
&\qquad+[-\bar{a}_1\xi_1\xi_1^*-\bar{a}_2\kappa\xi_2\xi_2^*+\xi_1\xi_3^*+q\xi_2\xi_3^*]\int_\Omega\phi_3 e_{k_0}^2dx\Big\}+o(3).
\end{aligned}
$$

Hence, we have

$$\frac{\langle G(x\xi e_{k_0}+\Phi),\xi^* e_{k_0}\rangle}{\langle \xi e_{k_0},\xi^* e_{k_0}\rangle}=\tilde{\theta}x^3+o(3), \tag{4.49}$$

where $\tilde{\theta}$ is defined by (4.43).

Combining (4.46) and (4.49), we deduce that the following reduced bifurcation equation:

$$\frac{dx}{dt}=\beta_{k_0 1}x+\tilde{\theta}x^3+o(3). \tag{4.50}$$

Step 2. Bifurcation analysis.

Obviously, when $\tilde{\theta}>0$, the Eq (4.50) bifurcates two saddle points on $\delta<\delta_1$, and when $\tilde{\theta}<0$, the Eq (4.50) bifurcates two stable singular points on $\delta>\delta_1$. The bifurcated solutions can be expressed as

$$x_\pm=\pm\Big[-\frac{\beta_{k_0 1}}{\tilde{\theta}}\Big]^{\frac{1}{2}}+o(|\beta_{k_0 1}|^{\frac{1}{2}}).$$

It is known that the transition and local topological structure of Eq (3.5) are determined completely by (4.50). Therefore,

$$\chi_+=+\Big[-\frac{\beta_{k_0 1}}{\tilde{\theta}}\Big]^{\frac{1}{2}}\xi e_{k_0}+o(|\beta_{k_0 1}|^{\frac{1}{2}})$$

and

$$\chi_-=-\Big[-\frac{\beta_{k_0 1}}{\tilde{\theta}}\Big]^{\frac{1}{2}}\xi e_{k_0}+o(|\beta_{k_0 1}|^{\frac{1}{2}})$$

are the bifurcated singular points of (3.5). The stability of χ_+ and χ_- are the same as that of $x_\pm$, see Figure 2. The Theorem is proved.

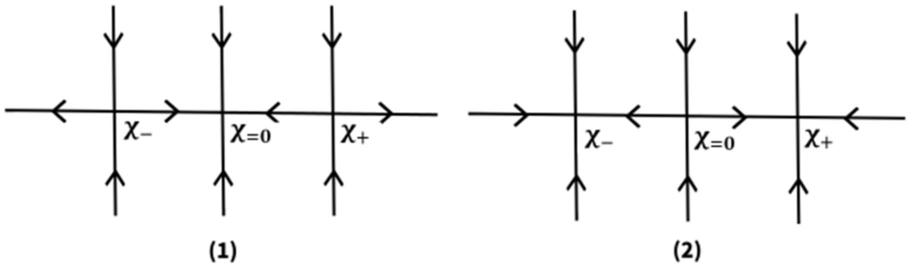

Figure 2. If $\int_\Omega e_{k_0}^3 dx = 0$, then the local topological structure of (3.5) is: (1) when $\tilde{\theta} > 0$, $\delta < \delta_1$, the system bifurcates from an stable equilibrium point $\chi = 0$ to two saddle points χ_+ and χ_-; (2)when $\tilde{\theta} < 0$, $\delta > \delta_1$, the system bifurcates from an equilibrium point $\chi = 0$ to two stable singular points χ_+ and χ_-.

5. Conclusions

In this work, we study the dynamical transition for a 3-component Lotka-Volterra model with diffusion from the perspective of dynamic transition recently developed by Ma and Wang [18–20]. By using the Principle of Exchange of Stabilities condition for 3-component Lotka-Volterra model, we note that the model produces stationary periodic in space patterns for $\delta < \min\{\delta_0, \delta_1\}$, where δ_0 and δ_1 are determined by parameters $\lambda = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \bar{a}_1, \bar{a}_2, \alpha, \gamma, \kappa, \sigma, q) \in \mathbb{R}_+^{10}$. However, when $\delta > \min\{\delta_0, \delta_1\}$, i.e., the ratio of the inter-specific competition rates and the intra-specific rates are greater than the specified value, the stability is broken. Then we obtained a few main characteristics are as follows.

First, when $\delta_0 < \delta_1$, the first eigenvalues are complex, and we show that the system undergoes a continuous or jump transition. In the small oscillation frequency limit, the transition is always continuous and the time periodic rolls are stable after the transition. That provide some sufficient conditions for the stable coexistence equilibrium and periodic solution.

Second, when $\delta_0 > \delta_1$, it was demonstrated that chaotic coexistence bifurcates from the periodic when $\int_\Omega e_{k_0}^3 dx \neq 0$. If $\int_\Omega e_{k_0}^3 dx = 0$, we show that the permanent coexistence was existed for the two-prey-one-predator model with intra-specific competition for predator.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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