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Research article

A study on weak hyperfilters of ordered semihypergroups

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Abstract: In this paper, the notion of weak hyperfilters of an ordered semihypergroup is introduced, and several related properties and applications are given. In particular, we discuss the relationship between the weak hyperfilters and the prime hyperideals in ordered semihypergroups. Furthermore, we define and investigate the equivalence relation $\mathcal{W}$ on an ordered semihypergroup by weak hyperfilters. We establish the relation of the equivalence relation $\mathcal{W}$ and Green's relations of an ordered semihypergroup. Finally, characterizations of intra-regular (duo) ordered semihypergroups are given by the properties of weak hyperfilters.

Keywords: ordered semihypergroup; weak hyperfilter; Green's relation; intra-regular (duo) ordered semihypergroup
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1. Introduction

It is well known that an ordered groupoid is a groupoid $(S, \cdot)$ endowed with an order relation " $\leq$ " in which the multiplication is compatible with the ordering; and it is denoted by $(S, \cdot, \leq)$. In particular, if the multiplication on S is associative, then $(S, \cdot, \leq)$ is called an ordered semigroup. The theory of ordered semigroups has important applications in the fields of formal languages, artificial intelligence and computer science. Similar to the study of semigroup (or ring) theory, filters of ordered semigroups have a great important contribution to characterizing the algebraic structures of ordered semigroups. A subsemigroup F of an ordered semigroup $(S, \cdot, \leq)$ is called a *filter* of S if it satisfies (1) $a \in F$ and $a \leq b \in S$ imply $b \in F$, and (2) $(\forall a, b \in S)$ $ab \in F$ implies $a, b \in F$ (see [1]). Since then, Xie and Wu [2] defined and studied the semilattice congruences $\mathcal{N}$ on ordered semigroups in terms of filters. In particular, they proved $\mathcal{N}$ is not the least semilattice congruence on an ordered semigroup S, but it is the least regular semilattice congruence on S.

The hyper structure theory (called also multialgebra) was first introduced in 1934 by Marty (see [3]). Later on, hyperstructures have a lot of applications in mathematics, automata, cryptography, codes, artificial intelligence and other fields, for example, see [4, 5]. In [6], Heidari and Davvaz defined and studied the ordered semihypergroups, and discussed several related properties. And then a lot of papers on ordered semihypergroups have been written, for instance, see [7–14]. In [7], Tang *et al.* defined the hyperfilters of ordered semihypergroups, and characterized completely prime hyperideals of an ordered semihypergroup in terms of hyperfilters. In order to characterize prime hyperideals of ordered semihypergroups in a similar way, in this paper the hyperfilters of ordered semihypergroups are studied in depth, and some related properties and results are generalized. To begin with, we define and discuss the weak hyperfilters of ordered semihypergroups. Especially, some concepts and results of ordered semigroups are generalized to ordered semihypergroups. We establish the relation of weak hyperfilters and prime hyperideals of an ordered semihypergroup. Moreover, we define and discuss the equivalence relation $\mathcal{W}$ on an ordered semihypergroup S by weak hyperfilters. Finally, we give some applications of weak hyperfilters. Especially, characterizations of intra-regular (duo) ordered semihypergroups are given by properties of weak hyperfilters. As an application of the present paper, corresponding notions and results on semihypergroup can be obtained, and this is because every semihypergroup endowed with the equality relation " = " is an ordered semihypergroup.

2. Preliminaries and some notations

In this section, we first present some definitions and results which will be used throughout this paper.

Let S be a nonempty set and $P^*(S)$ the set of all nonempty subsets of S. A mapping $\circ : S \times S \to P^*(S)$ is called a *hyperoperation* or *hypercomposition* on S. The couple $(S, \circ)$ is called a *hyperstructure*. In the above definition, if $x \in S$ and A, B are nonempty subsets of S, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hyperstructure $(S, \circ)$ is a *semihypergroup* [4] if for all $x, y, z \in S$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

Let T be a nonempty subset of a semihypergroup $(S, \circ)$. Then T is called a *subsemihypergroup* if $T \circ T \subseteq T$.

Let S be a nonempty set. The triplet $(S, \circ, \leq)$ is called an *ordered semihypergroup* [6] if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that

$$x \leq y \Rightarrow a \circ x \preceq a \circ y \text{ and } x \circ a \preceq y \circ a$$

for all $x, y, a \in S$. Here, the preorder " $\preceq$ " on $P^*(S)$ is defined by

$$(\forall A, B \in P^*(S))\ A \preceq B \text{ means that } (\forall a \in A)\ (\exists b \in B)\ a \leq b.$$

It is not difficult to understand that every ordered semigroup can be said to be an ordered semihypergroup. Also see [7]. In this paper, S stands for an ordered semihypergroup unless stated otherwise.

Let S be an ordered semihypergroup and $A \in P^*(S)$. A is called a *right* (resp. *left*) *hyperideal* of S if (1) $A \circ S \subseteq A$ (resp. $S \circ A \subseteq A$) and (2) $S \ni b \leq a, a \in A \Rightarrow b \in A$. A is called a (*two-sided*) *hyperideal* of S if it is both a right and a left hyperideal of S (see [6]). For $H \in P^*(S)$, we denote by

$(H]$ the subset of S defined by

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

If no confusion is possible, we write in short $(a]$ instead of $(\{a\}]$. We denote by $R(A)$ (resp. $L(A), I(A)$) the right (resp. left, two-sided) hyperideal of S generated by A ($A \in P^*(S)$). It can be easily shown that $R(A) = (A \cup A \circ S]$, $L(A) = (A \cup S \circ A]$ and $I(A) = (A \cup S \circ A \cup A \circ S \cup S \circ A \circ S]$. In particular, if $A = \{a\}$, then we write $L(a), R(a), I(a)$ instead of $L(\{a\}), R(\{a\}), I(\{a\})$, respectively.

Let I be a hyperideal of an ordered semihypergroup S. I is called *completely prime* if $(\forall a, b \in S)$ $(a \circ b) \cap I \neq \emptyset \Rightarrow a \in I$ or $b \in I$ (see [11]). I is called *prime* if $(\forall a, b \in S)$ $a \circ b \subseteq I \Rightarrow a \in I$ or $b \in I$ (see [7]). I is called *semiprime* if for every $a \in S$ such that $a \circ a \subseteq I$, we have $a \in I$. S is called *right* (resp. *left*) *duo* if every right (resp. left) hyperideal of S is a left (resp. right) hyperideal of S. S is called *duo* if it is both right duo and left duo.

Lemma 2.1 ([8]) *Let S be an ordered semihypergroup and $A, B \in P^*(S)$. Then the following conditions hold:*

(1) $A \subseteq (A], ((A]] = (A]$.
(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$.
(3) $(A] \circ (B] \subseteq (A \circ B]$ *and* $((A] \circ (B]] = (A \circ B]$.
(4) *If T is a hyperideal of S, then we have $(T] = T$.*
(5) *If A, B are hyperideals of S, then $(A \circ B]$ is a hyperideal of S.*
(6) *For any $a \in S$, $(S \circ a]$ and $(S \circ a \circ S]$ are a left hyperideal and a hyperideal of S, respectively.*
(7) *If I is a hyperideal of S, then $A \preceq B \subseteq I$ implies $A \subseteq I$.*
(8) $A \preceq B \Rightarrow C \circ A \preceq C \circ B$ *and* $A \circ C \preceq B \circ C$, $\forall C \in P^*(S)$.

Lemma 2.2 ([11]) *Let $\{A_i \mid i \in I\}$ be a family of hyperideals of S. Then*

(1) $\bigcup_{i \in I} A_i$ *is a hyperideal of S.*
(2) $\bigcap_{i \in I} A_i$ *is a hyperideal of S if* $\bigcap_{i \in I} A_i \neq \emptyset$.

A subsemihypergroup F of S is called a *hyperfilter* [7] of S if
(1) $(\forall\, a, b \in S)$ $(a \circ b) \cap F \neq \emptyset \Rightarrow a \in F$ and $b \in F$.
(2) If $a \in F$ and $a \leq b \in S$, then $b \in F$.

Let $x \in S$ and $N(x)$ denotes the hyperfilter of S generated by x. The equivalence relation "$\mathcal{N}$" on S is defined by

$$\mathcal{N} := \{(x, y) \in S \times S \mid N(x) = N(y)\}.$$

Let S be an ordered semihypergroup and $I \in P^*(S)$. The relation "δ_I" on S is defined by

$$\delta_I := \{(x, y) \in S \times S \mid x, y \in I \text{ or } x, y \notin I\}.$$

It can be easily shown that δ_I is an equivalence relation on S, also see [9]. Moreover, we have the following lemma:

Lemma 2.3 ([14]) *Let S be an ordered semihypergroup. Then*

$$\mathcal{N} = \cap\{\delta_I \mid I \in CP(S)\},$$

where $CP(S)$ is the set of all completely prime hyperideals of S.

For further information related to ordered semihypergroups, we refer to [5, 15].

3. Weak hyperfilters of ordered semihypergroups

In the current section, we shall define and investigate left weak hyperfilters, right weak hyperfilters and weak hyperfilters of an ordered semihypergroup. In particular, the relationship between the weak hyperfilters and the prime hyperideals in ordered semihypergroups is discussed.

Definition 3.1 Let S be an ordered semihypergroup. A nonempty subset W of S is called a *left* (resp. *right*) *weak hyperfilter* of S if

(1) $a, b \in W \Rightarrow (a \circ b) \cap W \neq \emptyset$.

(2) $(\forall\, a, b \in S)\ (a \circ b) \cap W \neq \emptyset \Rightarrow b \in W$ (resp. $a \in W$).

(3) $a \in W, a \leq b \in S \Rightarrow b \in W$.

If W is both a left weak hyperfilter and a right weak hyperfilter of S, then W is called a *weak hyperfilter* of S.

Obviously, every hyperfilter of an ordered semihypergroup S is a weak hyperfilter of S. However, the converse is not true, in general, as shown in the following counterexample.

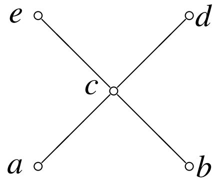
Example 3.2 Let $S := \{a, b, c, d, e\}$ with the hyperoperation " $\circ$ " and the order " $\leq$ " below:

$\circ$	a	b	c	d	e
a	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{c\}$
b	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{c\}$
c	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{c\}$
d	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{d, e\}$	$\{e\}$
e	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{e\}$	$\{e\}$

$$\leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (e, e)\}.$$

The covering relation "$\prec$" and the figure of S are given as follows:

$$\prec = \{(a, c), (b, c), (c, d), (c, e)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup. It is a routine matter to verify that $W = \{d\}$ is a weak hyperfilter of S, but not a hyperfilter of S. In fact, since $d \in W$, while $d \circ d = \{d, e\} \nsubseteq W$, i.e., W is not a subsemihypergroup of S.

Suppose $\{W_i \mid i \in I\}$ is a family of weak hyperfilters of S. Is it true that the union $\bigcup_{i \in I} W_i$ of W_i $(i \in I)$ is a weak hyperfilter of S? The following example gives a negative answer to this question.

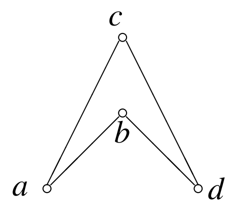
Example 3.3 Let $S := \{a, b, c, d\}$ with the hyperoperation " $\circ$ " and the order " $\leq$ " below:

$\circ$	a	b	c	d
a	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a\}$
b	$\{a, d\}$	$\{b\}$	$\{a, d\}$	$\{a, d\}$
c	$\{a, d\}$	$\{a, d\}$	$\{c\}$	$\{a, d\}$
d	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$

$$\leq := \{(a,a),(a,b),(a,c),(b,b),(c,c),(d,b),(d,c),(d,d)\}.$$

The covering relation "$\prec$" and the figure of S are given as follows:

$$\prec = \{(a,b),(a,c),(d,b),(d,c)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup (see [7]). It can be easily shown that $W_1 = \{b\}$, $W_2 = \{c\}$ are both weak hyperfilters of S. But $W_1 \cup W_2 = \{b, c\}$ is not a weak hyperfilter of S. In fact, since $b, c \in W_1 \cup W_2$, but $(b \circ c) \cap (W_1 \cup W_2) = \emptyset$.

Theorem 3.4 *Let W_1, W_2 be weak hyperfilters of an ordered semihypergroup S. Then $W_1 \cup W_2$ is a weak hyperfilter of S if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.*

Proof $\Leftarrow$. Clearly.

$\Rightarrow$. let $W_1 \cup W_2$ be a weak hyperfilter of S such that $W_1 \nsubseteq W_2$. We claim that $W_2 \subseteq W_1$. To prove our claim, let $a \in W_1, a \notin W_2$ and $b \in W_2$. Then $a, b \in W_1 \cup W_2$. Since $W_1 \cup W_2$ is a weak hyperfilter of S, we have $(a \circ b) \cap (W_1 \cup W_2) \neq \emptyset$. This means that there exists $c \in a \circ b$ such that $c \in W_1 \cup W_2$, which implies that $c \in W_1$ or $c \in W_2$. Assume that $c \in W_2$. Since W_2 is a weak hyperfilter of S, we have $a \in W_2$, which is a contradiction. It thus follows that $c \in W_1$. Also, since W_1 is a weak hyperfilter of S, it can be obtained that $b \in W_1$. Hence $W_2 \subseteq W_1$. □

Theorem 3.5 *Let S be an ordered semihypergroup. If $\{W_k \mid k \in I\}$ is a family of weak hyperfilters of S such that $W_i \subseteq W_j$ or $W_j \subseteq W_i$ for all $i, j \in I$, then $\bigcup_{k \in I} W_k$ is a weak hyperfilter of S, where $|I| \geq 2$.*

Proof. The proof is similar to that of Theorem 3.7 in [7] with suitable modification. □

As we know, filters of ordered semigroups can be characterized by prime ideals (see [15]). Similarly, the following theorem establishes the relation of weak hyperfilters and prime hyperideals of an ordered semihypergroup S.

Theorem 3.6 *Let S be an ordered semihypergroup and W a nonempty subset of S. Then W is a weak hyperfilter of S if and only if $S \backslash W = \emptyset$ or $S \backslash W$ is a prime hyperideal of S, where $S \backslash W$ is the complement of W in S.*

Proof. $\Rightarrow$. Assume that W is a weak hyperfilter of S and $S \backslash F \neq \emptyset$. We first claim that $S \backslash W$ is a hyperideal of S. To show our claim, let $x \in S \backslash W, y \in S$. If $x \circ y \nsubseteq S \backslash W$, then there exists $z \in x \circ y$ such that $z \in W$, which means that $(x \circ y) \cap W \neq \emptyset$. Since W is a weak hyperfilter of S, it can be concluded that $x \in W$, which is a contradiction. Hence $x \circ y \subseteq S \backslash W$, and we have $(S \backslash W) \circ S \subseteq S \backslash W$. Similarly, $S \circ (S \backslash W) \subseteq S \backslash W$. Let $y \in S \backslash W, S \ni x \leq y$. If $x \in W$, then, since W is a weak hyperfilter of S and $W \ni x \leq y \in S$, we have $y \in W$. It contradicts the fact that $y \in S \backslash W$. Therefore, $x \in S \backslash W$. Furthermore, we claim that $S \backslash W$ is prime. Indeed, let $x, y \in S$ be such that $x \circ y \subseteq S \backslash W$. If $x \in W$ and $y \in W$, then, since W is a weak hyperfilter of S, $(x \circ y) \cap W \neq \emptyset$. Impossible. Thus $x \in S \backslash W$ or $y \in S \backslash W$. Therefore, $S \backslash W$ is a prime hyperideal of S.

$\Leftarrow$. Let $S \backslash W = \emptyset$. Then $W = S$, and W is clearly a weak hyperfilter of S. If $S \backslash W$ is a prime hyperideal of S, then $(x \circ y) \cap W \neq \emptyset$ for any $x, y \in W$. In fact, if $(x \circ y) \cap W = \emptyset$, then we have $x \circ y \subseteq S \backslash W$. Since $S \backslash W$ is prime, it can be shown that $x \in S \backslash W$ or $y \in S \backslash W$, which is impossible. Next let $x, y \in S$ be such that $(x \circ y) \cap W \neq \emptyset$. Then $x \in W$ and $y \in W$. Indeed, if $x \in S \backslash W$, then, since $S \backslash W$ is a hyperideal of S, we have $x \circ y \subseteq S \backslash W$. It implies that $(x \circ y) \cap W = \emptyset$, which is a contradiction. Hence $x \in W$. In a similar way, we can show that $y \in W$. Furthermore, let $x \in W$ and $x \leq y \in S$. Then $y \in W$. In fact, if $y \in S \backslash W$, then, since $S \backslash W$ is a hyperideal of S and $S \ni x \leq y \in S \backslash W$, we have $x \in S \backslash W$. This is impossible. Thus $y \in W$. Therefore, W is a weak hyperfilter of S. □

Theorem 3.7 *Let $\{T_i \mid i \in I\}$ be a family of prime hyperideals of an ordered semihypergroup S. Then $\bigcap_{i \in I} T_i$ is a semiprime hyperideal of S if $\bigcap_{i \in I} T_i \neq \emptyset$.*

Proof. Let $\{T_i \mid i \in I\}$ be a family of prime hyperideals of S such that $\bigcap_{i \in I} T_i \neq \emptyset$. Then $\bigcap_{i \in I} T_i$ is a hyperideal of S by Lemma 2.2. Moreover, we claim that $\bigcap_{i \in I} T_i$ is semiprime. To show our claim, let $a \in S$ be such that $a \circ a \subseteq \bigcap_{i \in I} T_i$. Then $a \circ a \subseteq T_i$ for every $i \in I$. Thus, by hypothesis, $a \in T_i$ for any $i \in I$. It thus concludes that $a \in \bigcap_{i \in I} T_i$. Consequently, $\bigcap_{i \in I} T_i$ is a semiprime hyperideal of S. □

By Theorem 3.7, every nonempty intersection of prime hyperideals of S is a semiprime hyperideal of S. However, the nonempty intersection of prime hyperideals of S is not necessarily prime. It can be illustrated by the following counterexample.

Example 3.8 Let $(S, \circ, \leq)$ be the ordered semihypergroup given in Example 3.3. We have shown that $W_1 = \{b\}$ and $W_2 = \{c\}$ are weak hyperfilters of S. Thus, by Theorem 3.6, $T_1 = S \backslash W_1 = \{a, c, d\}$ and $T_2 = S \backslash W_2 = \{a, b, d\}$ are both prime hyperideals of S. But $T_1 \cap T_2 = \{a, d\}$ is not a prime hyperideal of S. Indeed, $b \circ c = \{a, d\} \subseteq T_1 \cap T_2$, but $b \notin T_1 \cap T_2$ and $c \notin T_1 \cap T_2$.

In the following theorem, we give a condition for every nonempty intersection of prime hyperideals of S to be a prime hyperideal of S.

Theorem 3.9 *Let S be an ordered semihypergroup. Then every nonempty intersection of prime hyperideals of S is a prime hyperideal of S if and only if the set of prime hyperideals of S is a chain under inclusion.*

Proof. Let I_1 and I_2 be any two prime hyperideals of S. It can be obtained that $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. In fact, if there exist $x, y \in S$ such that $x \in I_1 \backslash I_2$ and $y \in I_2 \backslash I_1$, then, since I_1 and I_2 are both hyperideals of S, $x \circ y \subseteq I_1 \cap I_2$. By hypothesis, $I_1 \cap I_2$ is a prime hyperideal of S. Hence we have $x \in I_1 \cap I_2$ or $y \in I_1 \cap I_2$, which is a contradiction. We have thus shown that $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. In other words, the set of prime hyperideals of S is indeed a chain under inclusion.

Conversely, assume that I_α ($\alpha \in \Lambda$) is a prime hyperideal of S and Λ is an index set. Let $I = \bigcap_{\alpha \in \Lambda} I_\alpha \neq \emptyset$. Then I is a hyperideal of S by Lemma 2.2. To prove that I is prime, by Theorem 3.6 we only need to show that $S \backslash I = \emptyset$ or $S \backslash I$ is a weak hyperfilter of S. Suppose that $S \backslash I \neq \emptyset$. It can be easily observed that

$$S \backslash I = S \backslash \bigcap_{\alpha \in \Lambda} I_\alpha = \bigcup_{\alpha \in \Lambda} (S \backslash I_\alpha).$$

Since I_α is a prime hyperideal of S for any $i \in \Lambda$, $S \backslash I_\alpha$ ($\alpha \in \Lambda$) is a weak hyperfilter of S if $S \backslash I_\alpha \neq \emptyset$ by Theorem 3.6. By hypothesis, $\{I_\alpha\}_{\alpha \in \Lambda}$ is a chain under inclusion, and thus, $\{S \backslash I_\alpha\}_{\alpha \in \Lambda}$ is also a chain under inclusion. Hence, by Theorem 3.5, $\bigcup\limits_{\alpha\in\Lambda}(S\setminus I_\alpha)$ is a weak hyperfilter of S. It implies that $S\setminus I$ is a weak hyperfilter of S. Thus, by Theorem 3.6, I is a prime hyperideal of S. □

4. Characterizations of intra-regular (duo) ordered semihypergroups

In this section we discuss the properties of weak hyperfilters of ordered semihypergroups in depth, and give some characterizations of intra-regular (duo) ordered semihypergroups.

Let S be an ordered semihypergroup and $a \in S$. We denote by $W(x)$ the weakly hyperfilter of S generated by a, and define a relation $\mathcal{W} := \{(x, y) \in S \times S \mid W(x) = W(y)\}$. It can be easily shown that $\mathcal{W}$ is an equivalence relation on S.

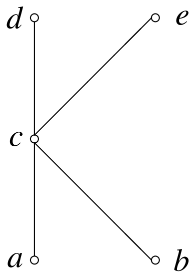
Example 4.1 We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":

$\circ$	a	b	c	d	e
a	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
c	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$
d	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$	$\{e\}$
e	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$

$$\leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (e, e)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, c), (b, c), (c, d), (c, e)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup (see [8]). It is easy to check that $W(a) = W(b) = S$, $W(c) = W(e) = \{c, d, e\}$, $W(d) = \{d\}$. Thus the equivalence relation $\mathcal{W}$ on S is as follows:

$$\mathcal{W} := \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, e), (d, d), (e, c), (e, e)\}.$$

Theorem 4.2 *Let S be an ordered semihypergroup. Then $\mathcal{W} = \bigcap\{\delta_I \mid I \in P(S)\}$, where $P(S)$ is the set of all prime hyperideals of S.*

Proof. Let $(x, y) \in \mathcal{W}$. Then we prove that $(x, y) \in \delta_I$ for any $I \in P(S)$. Indeed, if $(x, y) \notin \delta_I$ for some $I \in P(S)$, then ($x \notin I$ and $y \in I$) or ($x \in I$ and $y \notin I$). Let $x \notin I$ and $y \in I$. Then $\emptyset \neq S\setminus I \subseteq S$. Since $S\setminus(S\setminus I)$ $(= I)$ is a prime hyperideal of S, by Theorem 3.6, $S\setminus I$ is a weak hyperfilter of S. Since $x \in S\setminus I$, we have $W(x) \subseteq S\setminus I$, and thus $W(y) \subseteq S\setminus I$, i.e., $y \in S\setminus I$. It contradicts the fact that $y \in I$. Similarly, if $x \in I$ and $y \notin I$, we can also get a contradiction. This proves that $\mathcal{W} \subseteq \bigcap\{\delta_I \mid I \in P(S)\}$. To show the inverse inclusion, let $(x, y) \in \delta_I$ for any $I \in P(S)$. Assume that $(x, y) \notin \mathcal{W}$. Then it can be easily shown that $x \notin W(y)$ or $y \notin W(x)$. Let $x \notin W(y)$. Then $x \in S\setminus W(y)$. For the simplicity's sake, we denote $S\setminus W(y)$ by I'. Then $I' \neq \emptyset$. Since $W(y)$ is a weak hyperfilter of S, by Theorem 3.6, I' is a prime hyperideal of S. Thus we have $I' \in P(S), x \in I'$ and $y \notin I'$ (since $y \in W(y)$), and $(x, y) \notin \delta_{I'}$, which contradicts the hypothesis. From $y \notin W(x)$, similarly, we get a contradiction. Hence $(x, y) \in \mathcal{W}$. We have thus shown that $\bigcap\{\delta_I \mid I \in P(S)\} \subseteq \mathcal{W}$. The proof is completed. □

Let S be an ordered semihypergroup. The *Green's relations* of S are the equivalence relations $\mathcal{R}, \mathcal{L}, \mathcal{J}$ and $\mathcal{H}$ of S defined as follows:

$\mathcal{R} := \{(x, y) \mid R(x) = R(y)\}$.

$\mathcal{L} := \{(x, y) \mid L(x) = L(y)\}$.

$\mathcal{J} := \{(x, y) \mid I(x) = I(y)\}$.

$\mathcal{H} := \mathcal{R} \cap \mathcal{L}$.

We denote by $(x)_{\mathcal{R}}$ (resp. $(x)_{\mathcal{L}}$, $(x)_{\mathcal{J}}$) the $\mathcal{R}$-class (resp. $\mathcal{L}$-class, $\mathcal{J}$-class) containing x ($x \in S$) (see [16]).

Theorem 4.3 *Let S be an ordered semihypergroup. Then the following statements hold:*

(1) *If $\mathcal{A}$ is the set of all right hyperideals, $\mathcal{B}$ the set of all left hyperideals and $\mathcal{M}$ the set of all hyperideals of S, then*

$$\mathcal{R} = \bigcap\{\delta_I \mid I \in \mathcal{A}\}, \mathcal{L} = \bigcap\{\delta_I \mid I \in \mathcal{B}\}, \mathcal{J} = \bigcap\{\delta_I \mid I \in \mathcal{M}\}.$$

(2) $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{J} \subseteq \mathcal{W} \subseteq \mathcal{N}, \mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{J} \subseteq \mathcal{W} \subseteq \mathcal{N}$.

(3) *If A is a right hyperideal, B a left hyperideal and I a hyperideal of S, then*

$$A = \bigcup\{(x)_{\mathcal{R}} \mid x \in A\}, B = \bigcup\{(x)_{\mathcal{L}} \mid x \in B\}, I = \bigcup\{(x)_{\mathcal{J}} \mid x \in I\}.$$

Proof. The proofs of (1) and (3) come from Theorem 1 in [16].

(2) By Theorem 1 in [16], $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{J}$. Moreover, we can show that $\mathcal{J} \subseteq \mathcal{W}$. Indeed, by Theorem 4.2, $\mathcal{W} = \bigcap\{\delta_I \mid I \in P(S)\}$, where $P(S)$ is the set of all prime hyperideal of S. By (1), $\mathcal{J} = \bigcap\{\delta_I \mid I \in \mathcal{M}\}$, Since $P(S) \subseteq \mathcal{M}$, we have

$$\mathcal{J} = \bigcap\{\delta_I \mid I \in \mathcal{M}\} \subseteq \bigcap\{\delta_I \mid I \in P(S)\} = \mathcal{W}.$$

Furthermore, since every completely prime hyperideal of S is a prime hyperideal of S, by Lemma 2.3 and Theorem 4.2, we have $\mathcal{W} \subseteq \mathcal{N}$. Therefore, $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{J} \subseteq \mathcal{W} \subseteq \mathcal{N}$. Similarly, we can obtain that $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{J} \subseteq \mathcal{W} \subseteq \mathcal{N}$. □

An ordered semihypergroup $(S, \circ, \leq)$ is called *intra-regular* if, for every element a of S, there exist $x, y \in S$ such that $a \preceq x \circ a \circ a \circ y$. Equivalently, $a \in (S \circ a \circ a \circ S], \forall a \in S$. The following theorem provides a characterization of intra-regular ordered semihypergroups by the weak hyperfilters.

Theorem 4.4 *An ordered semihypergroup $(S, \circ, \leq)$ is intra-regular if and only if $W(x) = \{y \in S \mid x \in (S \circ y \circ S]\}$ for any $x \in S$.*

Proof. Assume that S is an intra-regular ordered semihypergroup and $x \in S$. Let $T := \{y \in S \mid x \in (S \circ y \circ S]\}$. Then we prove that T is the weak hyperfilter of S generated by x. In fact:

(1) Since S is an intra-regular ordered semihypergroup, we have $x \in (S \circ x \circ x \circ S] \subseteq (S \circ x \circ S]$, and thus $x \in T$.

(2) Let $y, z \in T$. Then, by Lemma 2.1, we have

$$\begin{aligned} x &\in (S \circ x \circ x \circ S] \subseteq (S \circ (S \circ z \circ S] \circ (S \circ y \circ S] \circ S] \subseteq (S \circ z \circ S \circ y \circ S] \\ &\subseteq (S \circ (S \circ (z \circ S \circ y) \circ (z \circ S \circ y) \circ S] \circ S] \subseteq (S \circ z \circ S \circ y \circ z \circ S \circ y \circ S] \\ &\subseteq (S \circ y \circ z \circ S]. \end{aligned}$$

Consequently, there exists $a \in y \circ z$ such that $x \in (S \circ a \circ S]$. It implies that $(y \circ z) \cap T \neq \emptyset$.

(3) Let $y, z \in S$ be such that $(y \circ z) \cap T \neq \emptyset$. Then there exists $a \in S$ such that $a \in y \circ z$ and $a \in T$, and thus we have

$$x \in (S \circ a \circ S] \subseteq (S \circ y \circ z \circ S] \subseteq (S \circ y \circ S], (S \circ z \circ S].$$

It thus follows that $y \in T, z \in T$.

(4) Let $y \in T, z \in S$ and $y \leq z$. Then $x \in (S \circ y \circ S] \subseteq (S \circ z \circ S]$, which implies that $z \in T$.

(5) Assume that W is a weak hyperfilter of S containing x. Then $T \subseteq W$. Indeed, let $y \in T$. Then $x \in (S \circ y \circ S]$, and there exist $s_1, s_2 \in S$ such that $x \leq s_1 \circ y \circ s_2$. Thus there exists $a \in s_1 \circ y \circ s_2$ such that $x \leq a$, and, since W is a weak hyperfilter of S containing x, we have $a \in W$. It implies that $(s_1 \circ y \circ s_2) \cap W \neq \emptyset$. Hence there exists $b \in s_1 \circ y$ such that $(b \circ s_2) \cap W \neq \emptyset$. Also, since W is a weak hyperfilter of S, we have $b \in W$, which means that $(s_1 \circ y) \cap W \neq \emptyset$. Consequently, $y \in W$.

Therefore, T is the weak hyperfilter of S generated by x. In other words, $W(x) = \{y \in S \mid x \in (S \circ y \circ S]\}$.

Conversely, let $x \in S$. Since $x \in W(x)$, we have $(x \circ x) \cap W(x) \neq \emptyset$. It implies that there exists $y \in x \circ x$ such that $y \in W(x)$. Thus we have $x \in (S \circ y \circ S] \subseteq (S \circ x \circ x \circ S]$. Hence S is intra-regular. □

Corollary 4.5 *An ordered semihypergroup $(S, \circ, \leq)$ is intra-regular if and only if $\mathcal{W} = \mathcal{J}$.*

Proof. Suppose that S is an intra-regular ordered semihypergroup. Let $(x, y) \in \mathcal{W}$. Then $x \in W(y)$, and thus, by Theorem 4.4, $y \in (S \circ x \circ S] \subseteq I(x)$. Similarly, it can be proved that $x \in I(y)$. Hence $I(x) = I(y)$, i.e., $(x, y) \in \mathcal{J}$. On the other hand, by Theorem 4.3(2), $\mathcal{J} \subseteq \mathcal{W}$. Therefore, $\mathcal{W} = \mathcal{J}$.

Conversely, assume that $\mathcal{W} = \mathcal{J}$ and $x \in S$. We first claim that $(x, y) \in \mathcal{W}$ for some $y \in x \circ x$. To prove our claim, it is enough to prove that $W(x) = W(y)$ for some $y \in x \circ x$. In fact, since $x \in W(x)$ and $W(x)$ is a weak hyperfilter of S, we have $(x \circ x) \cap W(x) \neq \emptyset$, and there exists $y \in x \circ x$ such that $y \in W(x)$, i.e., $W(y) \subseteq W(x)$. Similarly, since $y \in W(y)$ and $W(y)$ is also a weak hyperfilter of S, it can be shown that $W(x) \subseteq W(y)$. Therefore, $(x, y) \in \mathcal{W}$ for some $y \in x \circ x$. Furthermore, by the above proof, it can be also obtained that $(y, z) \in \mathcal{W}$ for some $z \in y \circ y$. By hypothesis, $(x, y) \in \mathcal{J}, (y, z) \in \mathcal{J}$. Thus $(x, z) \in \mathcal{J}$, and we have $x \in I(x) = I(z)$, where $z \in x \circ x \circ x \circ x$. Hence, by Lemma 2.1, we have

$$\begin{aligned} x &\in (z \cup S \circ z \cup z \circ S \cup S \circ z \circ S] \\ &\subseteq (x \circ x \circ x \circ x \cup S \circ x \circ x \circ x \circ x \cup x \circ x \circ x \circ x \circ S \cup S \circ x \circ x \circ x \circ x \circ S] \\ &\subseteq (S \circ x \circ x \circ S], \end{aligned}$$

which means that S is intra-regular. □

Theorem 4.6 *Let S be an ordered semihypergroup. Then every left hyperideal of S is semiprime and S is left duo if and only if $W(x) = \{y \in S \mid x \in (S \circ y]\}$ for any $x \in S$.*

Proof. $\Rightarrow$ Assume that $x \in S$. Let $T := \{y \in S \mid x \in (S \circ y]\}$. In order to prove that T is the weak hyperfilter of S generated by x, we now consider the following five steps:

(1) Since $x \circ x \subseteq (S \circ x]$ and $(S \circ x]$ is a left hyperideal of S, by hypothesis, we have $x \in (S \circ x]$, which means that $x \in T$.

(2) Let $y, z \in T$. Then $x \in (S \circ y], x \in (S \circ z]$. Since S is left duo, by Lemma 2.1, we have

$$\begin{aligned} x \circ x &\subseteq (S \circ y] \circ (S \circ z] \subseteq ((S \circ y] \circ (S \circ z]] \subseteq ((S \circ y] \circ S \circ z] \\ &\subseteq ((S \circ y] \circ z] = (S \circ y \circ z]. \end{aligned}$$

Thus $x \in (S \circ y \circ z]$, and there exists $a \in y \circ z$ such that $x \in (S \circ a]$, i.e., $a \in T$. This implies that $(y \circ z) \cap T \neq \emptyset$.

(3) Let $y, z \in S$ be such that $(y \circ z) \cap T \neq \emptyset$. Then there exists $a \in y \circ z$ and $a \in T$, and thus we have

$$x \in (S \circ a] \subseteq (S \circ y \circ z] \subseteq (S \circ z].$$

It implies that $z \in T$. On the other hand, since S is left duo, we have $x \in (S \circ y \circ z] \subseteq ((S \circ y] \circ z] \subseteq (S \circ y]$. Therefore, $y \in T$.

(4) Let $y \in T, z \in S$ and $y \leq z$. Then $x \in (S \circ y] \subseteq (S \circ z]$, which implies that $z \in T$.

(5) Suppose that W is a weak hyperfilter of S containing x. Then we claim that $T \subseteq W$. To prove our claim, let $y \in T$. Then $x \in (S \circ y]$, and there exists $s_1 \in S$ such that $x \preceq s_1 \circ y$. Hence there exists $a \in s_1 \circ y$ such that $x \leq a$, and, by hypothesis we have $a \in W$. It implies that $(s_1 \circ y) \cap W \neq \emptyset$. Thus there exists $b \in s_1 \circ y$ such that $b \in W$, which means that $(s_1 \circ y) \cap W \neq \emptyset$. Therefore, $y \in W$.

Hence T is the weak hyperfilter of S generated by x, and $W(x) = \{y \in S \mid x \in (S \circ y]\}$.

$\Leftarrow$. Assume that L is a left hyperideal of S. Let $x \in S$ be such that $x \circ x \subseteq L$. Similar to the proof of Corollary 4.5, there exists $y \in x \circ x$ such that $(x, y) \in \mathcal{W}$, and thus $y \in W(x)$. Then we have

$$x \in (S \circ y] \subseteq (S \circ x \circ x] \subseteq (S \circ L] \subseteq (L] = L.$$

Therefore, L is semiprime. Furthermore, we claim that L is also a right hyperideal of S. To show our claim, let $y \in L, z \in S$. Then, for any $a \in y \circ z$, since $a \in W(a)$ and $W(a)$ is a weak hyperfilter of S, we have $y, z \in W(a)$. Hence $(z \circ y) \cap W(a) \neq \emptyset$, and there exists $b \in z \circ y$ such that $b \in W(a)$. This implies that $b \in W(b) \subseteq W(a)$. Thus we have

$$a \in (S \circ b] \subseteq (S \circ z \circ y] \subseteq (S \circ y] \subseteq (S \circ L] \subseteq (L] = L.$$

Consequently, $L \circ S \subseteq S$, and S is left duo. □

Corollary 4.7 *Let S be a left duo ordered semihypergroup. Then every left hyperideal of S is semiprime if and only if $\mathcal{W} = \mathcal{L}$.*

Proof. Assume that every left hyperideal of S is semiprime. Let $(x, y) \in \mathcal{W}$. Then $x \in W(y), y \in W(x)$, and thus, by Theorem 4.6, we have

$$y \in (S \circ x] \subseteq L(x), x \in (S \circ y] \subseteq L(y).$$

Hence $L(x) = L(y)$, i.e., $(x, y) \in \mathcal{L}$. On the other hand, by Theorem 4.3(2), $\mathcal{L} \subseteq \mathcal{W}$. Therefore, $\mathcal{W} = \mathcal{L}$.

Conversely, suppose that L is a left hyperideal of S and $\mathcal{W} = \mathcal{L}$. Let $x \in S$ be such that $x \circ x \subseteq L$. Then, by the proof of Corollary 4.5, there exists $y \in x \circ x$ such that $(x, y) \in \mathcal{W}$, and thus $(x, y) \in \mathcal{L}$. Hence, by Lemma 2.1, we have

$$x \in L(y) = (y \cup S \circ y] \subseteq (x \circ x \cup S \circ x \circ x] \subseteq (L \cup S \circ L] \subseteq (L] = L.$$

We have thus shown that L is semiprime. □

Similarly, we have the following theorem:

Theorem 4.8 *Let S be a right duo ordered semihypergroup. Then the following conditions are equivalent:*

(1) *Every right hyperideal of S is semiprime.*

(2) $\mathcal{W} = \mathcal{R}$.

(3) $W(x) = \{y \in S \mid x \in (y \circ S]\}$ *for any* $x \in S$.

Theorem 4.9 *Let S be an ordered semihypergroup. Then every hyperideal of S is semiprime and S is duo if and only if $W(x) = \{y \in S \mid x \in (y \circ S \circ y]\}$ for any $x \in S$.*

Proof. The proof is similar to that of Theorem 4.6 with suitable modification, we omit it. □

By Corollary 4.7, Theorems 4.8 and 4.9, the following corollary can be immediately obtained:

Corollary 4.10 *Let S be a duo ordered semihypergroup. Then every hyperideal of S is semiprime if and only if $\mathcal{W} = \mathcal{H}$.*

5. Conclusions

Similar to the theory of ordered semigroups, hyperfilters and weak hyperfilters of ordered semihypergroups play an important role in studying the structures of ordered semihypergroups. In this paper, we introduced the concept of weak hyperfilters of ordered semihypergroups, which is a generalization of the concept of hyperfilters of ordered semihypergroups. Several properties of weak hyperfilters of ordered semihypergroups were studied. Especially, some concepts and results of ordered semigroups are generalized to ordered semihypergroups. We established the relation of weak hyperfilters and prime hyperideals of an ordered semihypergroup. Moreover, we defined and investigated the equivalence relation $\mathcal{W}$ on an ordered semihypergroup S by weak hyperfilters. Finally, we provided some applications of weak hyperfilters. Especially, characterizations of intra-regular (duo) ordered semihypergroups were given by properties of weak hyperfilters. We hope that this work would offer foundation for further study of the theory on ordered semihypergroups. We will continue our research along this direction and hopefully we will investigate some new results in future. Our results can be further applied to other algebraic hyperstructure.

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Conflict of interest

The authors declare no conflict of interest.

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