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Research article

Positive solutions to a semipositone superlinear elastic beam equation

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Abstract: A semipositone fourth-order two-point boundary value problem is considered. In mechanics, the problem describes the deflection of an elastic beam rigidly fastened on the left and simply supported on the right. Under some conditions concerning the first eigenvalue corresponding to the relevant linear operator, the existence of nontrivial solutions and positive solutions to this boundary value problem is obtained. The main results are obtained by using the topological method and the fixed point theory of superlinear operators.

Keywords: topological degree; fixed point; nontrivial solutions and positive solutions; elastic beam equations
Mathematics Subject Classification: 34K10, 37C25

1. Introduction

The purpose of this paper is to investigate the existence of nontrivial solutions and positive solutions to the following nonlinear fourth-order two-point boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)),\ 0 \le t \le 1, \\ u(0) = u'(0) = u(1) = u''(1) = 0, \end{cases} \tag{P}$$

where λ is a positive parameter, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. One function $u \in C[0, 1]$ is called a positive solution of problem (P) if u is a solution of (P) and $u(t) > 0, 0 < t < 1$.

Fourth-order two-point boundary value problems appear in beam analysis (See [1–15]). The deflection of an elastic beam rigidly fastened on the left and simply supported on the right leads to problem (P). The existence and multiplicity of positive solutions for the elastic beam equations have been studied extensively when the nonlinear term satisfies

$$f(t, u) \ge 0,\ \forall u \ge 0, \tag{1}$$

see for example [1–3,6–9,11] and references therein. Agarwal and Chow [1] investigated the problem (P) by using contraction mapping and iterative methods. Bai [3] applied upper and lower solution method and Yao [11] used Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type. However there are only a few papers concerned with the non-positone or semipositone elastic beam equations. Yao [12] considered the existence of positive solutions of semipositone elastic beam equations by using a special cone and the fixed point theorem of cone expansion-compression type. In this paper, we assume that there exists a constant $b > 0$ such that $f(t,u) \geq -b$ for all $t \in [0,1]$ and $u(t) \in \mathbb{R}$, which implies that problem (P) is semipositone. We obtain the existence of nontrivial solutions and positive solutions to the semipositone boundary value problem (P) under some conditions concerning the first eigenvalue corresponding to the relevant linear operator by the topological method and the fixed point theory of superlinear operators.

2. Preliminaries

In this section, we give some preliminaries required in our subsequent discussions.

Let E be an ordered Banach space in which the partial ordering $\leq$ is induced by a cone $P \subset E$, θ the zero element of E. For the concepts and properties about the cone we refer to [16, 17, 18].

Lemma 1. (see [19]). *Let $\Omega \subset E$ be a bounded open set, $\theta \in \Omega$, and $A : \overline{\Omega} \to E$ a completely continuous operator. If*

$$Ax \neq \mu x, \ \forall x \in \partial\Omega, \mu \geq 1,$$

then $\deg(I - A, \Omega, \theta) = 1$.

Lemma 2. (see [19]). *Let $\Omega \subset E$ be a bounded open set and $A : \overline{\Omega} \to E$ a completely continuous operator. If there exists $u_0 \in E\backslash\{\theta\}$ such that*

$$u - Au \neq \mu u_0, \ \forall u \in \partial\Omega, \mu \geq 0,$$

then $\deg(I - A, \Omega, \theta) = 0$.

The following is the famous Leray-Schauder theorem.

Lemma 3. (see [18]). *Let $\Omega \subset E$ be a bounded open set, $\theta \in \Omega$ and $A : \overline{\Omega} \to E$ a completely continuous operator with $A\theta = \theta$. Suppose that the Frechet derivative A'_θ of A at θ exists and 1 is not an eigenvalue of A'_θ. Then there exists $r_0 > 0$ such that for any $0 < r < r_0$,*

$$\deg(I - A, T_r, \theta) = \deg(I - A'_\theta, T_r, \theta) = (-1)^\kappa,$$

where κ is the sum of algebraic multiplicities for all eigenvalues of A'_θ lying in the interval (0, 1) and $T_r = \{x \in E \mid \|x\| < r\}$.

Let $G(t,s)$ be the Green's function of homogeneous linear problem $u^{(4)}(t) = 0, u(0) = u'(0) = u(1) = u''(1) = 0$, which can be explicitly given by

$$G(t,s) = \begin{cases} \dfrac{1}{12}(1-t)s^2[3(1-s) - (1-t)^2(3-s)], & 0 \leq s \leq t \leq 1, \\ \dfrac{1}{12}(1-s)t^2[3(1-t) - (1-s)^2(3-t)], & 0 \leq t \leq s \leq 1. \end{cases} \tag{2}$$

By (2) and Yao [13] we have
(G_1) $G(t,s) = G(s,t),\ 0 \le t,s \le 1$;
(G_2) $\frac{1}{6}s^2(1-s)t^2(1-t) \le G(t,s) \le \frac{1}{4}t^2(1-t),\ 0 \le t,s \le 1$;
By (G_1)(G_2),

$$\begin{aligned} G(t,s) &\ge \tfrac{1}{6}s^2(1-s)t^2(1-t) \\ &\ge \tfrac{1}{4}s^2(1-s)\tfrac{2}{3}t^2(1-t) \\ &\ge \tfrac{1}{4}s^2(1-s)\tfrac{3}{4}t^3(1-t)^2 \\ &\ge \tfrac{3}{4}t^3(1-t)^2G(s,\tau) \\ &= \tfrac{3}{4}t^3(1-t)^2G(\tau,s), \end{aligned}$$

and we have
(G_3) $G(t,s) \ge \frac{3}{4}t^3(1-t)^2G(\tau,s),\ 0 \le t,s,\tau \le 1$.

It is well known that the problem (P) is equivalent to the integral equation

$$u(t) = \lambda \int_0^1 G(t,s)f(s,u(s))ds.$$

Let

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s))ds, \tag{3}$$

$$(Bu)(t) = \int_0^1 G(t,s)u(s)ds. \tag{4}$$

By the famous Krein-Rutmann theorem (See [16]) and similar to Lemma 3 in [20], we have

Lemma 4. *Suppose that the linear operator B is defined by (4). Then the spectral radius $r(B) \neq 0$ and B has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = r^{-1}(B)$.*

3. Nontrivial solutions

In the sequel we always take $E = C[0,1]$ with the norm $\|u\| = \max_{0\le t\le 1}|u(t)|$ and the cone $P = \{u \in C[0,1] \mid u(t) \ge 0, 0 \le t \le 1\}$.

Now let us list the following conditions which will be used in this paper:

(H_1) There exists a constant $\alpha > 0$ satisfying

$$\liminf_{u\to+\infty} \frac{f(t,u)}{u} \ge \alpha,\ \forall t \in [0,1]. \tag{5}$$

(H_2) There exists a constant β with $0 < \beta < \alpha$ satisfying

$$\limsup_{u\to 0} \left|\frac{f(t,u)}{u}\right| \le \beta,\ \forall t \in [0,1]. \tag{6}$$

(H_3) There exists a constant $b > 0$ such that

$$f(t,u) \ge -b,\ \forall t \in [0,1], u(t) \in \mathbb{R}. \tag{7}$$

(H$_4$) $\lim\limits_{u\to+\infty}\dfrac{f(t,u)}{u}=+\infty$.

Theorem 5. *Suppose that (H$_1$)-(H$_3$) hold. Then for any $\lambda\in(\frac{\lambda_1}{\alpha},\frac{\lambda_1}{\beta})$, problem (P) has at least one nontrivial solution, where $\lambda_1=r^{-1}(B)$ and B is given by (4).*

Proof. Let $(Fu)(t)=f(t,u(t))$ for all $u\in E$, then by (4), $A=BF$. Applying the Arzela-Ascoli theorem and a standard argument, we can prove that $(\lambda A):E\to E$ is a completely continuous operator. It is known to all that the nonzero fixed points of the operator λA are the nontrivial solutions of the boundary value problem (P).

Now we show that there exists $R_0>0$, such that for any $R>R_0$,

$$u-\lambda Au\neq\mu u^*,\forall\mu\geq 0,u\in\partial T_R, \tag{8}$$

where u^* is the positive eigenfunction of B corresponding to its first eigenvalue $\lambda_1=r^{-1}(B)$, $T_R=\{u\in C[0,1]\mid\|u\|<R\}$ is a bounded open subset of E.

If (8) is not true, then there exist $\mu_0>0$ (if $\mu_0=0$, then Theorem 5 holds) and $u_0\in\partial T_R$ such that

$$u_0-\lambda Au_0=\mu_0u^*. \tag{9}$$

By (5) and (6), there exist $l>0,r_0>0$ and $0<\varepsilon<\min\{\frac{\alpha\lambda-\lambda_1}{\lambda},\frac{\lambda_1-\beta\lambda}{\lambda}\}$ such that

$$f(t,u)\geq(\alpha-\varepsilon)u-l,\ \forall t\in[0,1],u\geq 0, \tag{10}$$

$$|f(t,u)|\leq(\beta+\varepsilon)|u|,\ \forall t\in[0,1],|u|\leq r_0. \tag{11}$$

It follows from (7) that

$$f(t,u)\geq -b\geq(\alpha-\varepsilon)u-b,\forall t\in[0,1],u\leq 0. \tag{12}$$

Take $\omega=\max\{b,l\}$, then by (10) and (12) we have $f(t,u)\geq(\alpha-\varepsilon)u-\omega,\forall t\in[0,1],u\in R$. Let $\alpha_1=\alpha-\varepsilon$, then

$$\lambda Au=\lambda BFu\geq\lambda\alpha_1Bu-u_1,\ \forall u\in E, \tag{13}$$

where $u_1=\int_0^1G(t,s)\lambda\omega ds$. Take

$$\delta=\frac{2}{3}\lambda_1\int_0^1 s^2(1-s)u^*(s)ds>0.$$

By (4) and (G$_2$) we have

$$\begin{aligned}u^*(t)&=\lambda_1Bu^*(t)=\lambda_1\int_0^1G(t,s)u^*(s)ds\\&\geq\lambda_1\int_0^1\frac{1}{6}s^2(1-s)t^2(1-t)u^*(s)ds\\&\geq\frac{1}{6}\lambda_1t^2(1-t)\int_0^1 s^2(1-s)u^*(s)ds\\&\geq\frac{2}{3}\lambda_1G(t,s)\int_0^1 s^2(1-s)u^*(s)ds\\&=\delta G(t,s).\end{aligned}$$

Let

$$P_1 = \left\{ u \in P \mid \int_0^1 u^*(t)u(t)dt \geq \lambda_1^{-1}\delta\|u\| \right\}.$$

For any $u \in P$, we have

$$\begin{aligned}
\int_0^1 u^*(t)(\lambda Bu)(t)dt &= \int_0^1 u^*(t)[\int_0^1 G(t,s)\lambda u(s)ds]dt \\
&= \int_0^1 \lambda u(s)[\int_0^1 G(s,t)u^*(t)dt]ds \\
&= \int_0^1 \lambda u(s)Bu^*(s)ds \\
&\geq \lambda_1^{-1}\delta \int_0^1 G(t,s)\lambda u(s)ds \\
&= \lambda_1^{-1}\delta(\lambda Bu)(t).
\end{aligned}$$

And so $\int_0^1 u^*(t)(\lambda Bu)(t)dt \geq \lambda_1^{-1}\delta\|\lambda Bu\|$, i.e., $(\lambda B)(P) \subset P_1$. Since $Fu_0+\omega \in P$, then $(\lambda B)(Fu_0+\omega) \in P_1$ and $\mu_0 u^* = \mu_0\lambda_1 Bu^* \in P_1$. By (9) we have $u_0 + \lambda B\omega = \lambda Au_0 + \mu_0 u^* + \lambda B\omega = \lambda B(Fu_0+\omega) + \mu_0 u^* \in P_1$. Thus

$$\int_0^1 u^*(t)(u_0 + \lambda B\omega)(t)dt \geq \lambda_1^{-1}\delta\|u_0 + \lambda B\omega\| \geq \lambda_1^{-1}\delta\|u_0\| - \lambda_1^{-1}\delta\|\lambda B\omega\|. \tag{14}$$

Take $\varepsilon_0 = \alpha_1\lambda r(B) - 1 > 0$. By (13) we have

$$\begin{aligned}
\int_0^1 u^*(t)(\lambda Au_0)(t)dt &\geq \int_0^1 u^*(t)\alpha_1(\lambda Bu_0)(t)dt - \int_0^1 u^*(t)u_1(t)dt \\
&= \alpha_1\lambda r(B)\int_0^1 u^*(t)u_0(t)dt - \int_0^1 u^*(t)u_1(t)dt \\
&= \int_0^1 u^*(t)u_0(t)dt + \varepsilon_0\int_0^1 u^*(t)(u_0 + \lambda B\omega)(t)dt \\
&\quad -\varepsilon_0\int_0^1 u^*(t)(\lambda B\omega)(t)dt - \int_0^1 u^*(t)u_1(t)dt.
\end{aligned}$$

Take $R_0 = \frac{\lambda_1}{\varepsilon_0\delta}[\varepsilon_0\delta\lambda_1^{-1}\|\lambda B\omega\| + \varepsilon_0\int_0^1 u^*(t)(\lambda B\omega)(t)dt + \int_0^1 u^*(t)u_1(t)dt]$. For any $\|u_0\| = R > R_0$, by (14) we have

$$\begin{aligned}
\int_0^1 u^*(t)(\lambda Au_0 - u_0)(t)dt &\geq \varepsilon_0[\lambda_1^{-1}\delta\|u_0\| - \lambda_1^{-1}\delta\|\lambda B\omega\|] \\
&\quad -\varepsilon_0\int_0^1 u^*(t)(\lambda B\omega)(t)dt - \int_0^1 u^*(t)u_1(t)dt \\
&> \varepsilon_0\delta\lambda_1^{-1}R_0 - \varepsilon_0\delta\lambda_1^{-1}\|\lambda B\omega\| \\
&\quad -\varepsilon_0\int_0^1 u^*(t)(\lambda B\omega)(t)dt - \int_0^1 u^*(t)u_1(t)dt \\
&= 0.
\end{aligned}$$

But we see from (9) that

$$\int_0^1 u^*(t)(u_0 - \lambda Au_0)(t)dt = \int_0^1 u^*(t)\mu_0 u^*(t)dt \geq 0,$$

which is a contradiction. So (8) is true. By Lemma 2 we have

$$\deg(I-\lambda A, T_R, \theta) = 0. \tag{15}$$

Next we show that

$$(\lambda A)u \neq \mu u,\ \forall u \in \partial T_r, \mu \geq 1, \tag{16}$$

where $0 < r < \min\{r_0, R_0\}$. Assume on the contrary that there exist $u_0 \in \partial T_r$ and $\mu_0 \geq 1$ such that $(\lambda A)u_0 = \mu_0 u_0$. Since λA has no fixed point on ∂T_r, we have $\mu_0 > 1$. Let $B_1 = \lambda(\beta+\varepsilon)B$, then $r(B_1) < 1$. By (11) we have $|\lambda A u_0| \leq (\beta+\varepsilon)\lambda B|u_0| = B_1|u_0|$, then $\mu_0|u_0| \leq B_1|u_0|$, and therefore

$$\mu_0^n|u_0| \leq B_1^n|u_0|. \tag{17}$$

Let $D = \{v \mid v \geq |u_0|\}$. It follows from (17) that $\{\mu_0^{-n}B_1^n|u_0| \mid n = 1, 2, \cdots\} \subset D$. And $u_0 \in \partial T_r$ and $\theta \in T_r$ imply that $d = d(\theta, D) > 0$. Then one can have that

$$\|B_1^n\| \geq \frac{1}{\|u_0\|}\|B_1^n u_0\| \geq \frac{d}{\|u_0\|}\mu_0^n, \qquad n = 1, 2, \cdots,$$

which shows

$$r(B_1) = \lim_{n\to\infty}(\|B_1^n\|)^{1/n} \geq \lim_{n\to\infty}\left(\frac{d}{\|u_0\|}\mu_0^n\right)^{1/n} = \mu_0 > 1.$$

This contradicts $r(B_1) < 1$. So (16) holds. By Lemma 1, we have

$$\deg(I-\lambda A, T_r, \theta) = 1. \tag{18}$$

By (15) and (18), we have

$$\deg(I-\lambda A, T_R\backslash\overline{T_r}, \theta) = \deg(I-\lambda A, T_R, \theta) - \deg(I-\lambda A, T_r, \theta) = -1.$$

Then λA has at least one fixed point in $T_R\backslash\overline{T_r}$. This means that problem (P) has at least one nontrivial solution. □

Remark 6. For any $\alpha > 0$ and $\beta = 0$, the proof of Theorem 5 is valid. Then for any $\lambda > 0$, problem (P) has at least one nontrivial solution.

Corollary 7. *Suppose that (H_1) and (H_2) hold. Assume there exists a constant $b^* > 0$ such that*

$$f(t,u) \geq -\frac{b^*}{K},\ \forall t \in [0,1], u \geq -\frac{\lambda_1 b^*}{\beta},$$

where $K = \max_{t\in[0,1]}\int_0^1 G(t,s)ds$. Then for any $\lambda \in (\frac{\lambda_1}{\alpha}, \frac{\lambda_1}{\beta})$ problem (P) has at least one nontrivial solution.

Proof. Let

$$f_1(t,u) = \begin{cases} f(t,u), & \forall u \geq -\frac{\lambda_1 b^*}{\beta},\quad t \in [0,1], \\ f(t, -\frac{\lambda_1 b^*}{\beta}), & \forall u < -\frac{\lambda_1 b^*}{\beta},\quad t \in [0,1], \end{cases}$$

$$(A_1u)(t) = \int_0^1 G(t,s)f_1(s,u(s))ds.$$

Then all conditions of Theorem 5 hold for f_1. By Theorem 5, λA_1 has at least one nonzero fixed point $v^*(t)$, and

$$v^*(t) = \lambda \int_0^1 G(t,s)f_1(s,v^*(s))ds \geq -\lambda \frac{b^*}{K}\int_0^1 G(t,s)ds \geq -\frac{\lambda_1 b^*}{\beta}.$$

Thus

$$v^*(t) = \lambda \int_0^1 G(t,s)f_1(s,v^*(s))ds = \lambda \int_0^1 G(t,s)f(s,v^*(s))ds = \lambda Av^*(t).$$

This indicates $v^*(t)$ is a nontrivial solution of problem (P). □

Theorem 8. *Suppose that* (H_1) *and* (H_3) *hold. Let* $f(t,0) \equiv 0, \forall t \in [0,1]$ *and*

$$\lim_{u\to 0}\frac{f(t,u)}{u} = \rho. \tag{19}$$

Then for any $\lambda \in (\frac{\lambda_1}{\alpha}, +\infty)$ *and* $\lambda \neq \frac{\lambda_1}{\rho}$, *problem (P) has at least one nontrivial solution.*

Proof. From the proof of Theorem 5, if (H_1) and (H_3) hold, then there exists $R_0 > 0$ such that for any $R > R_0$ and $\lambda > \frac{\lambda_1}{\alpha}$ (15) holds.

Since $f(t,0) \equiv 0, \forall t \in [0,1]$, then $A\theta = \theta$. By (19) we have that the Frechet derivative A'_θ of A at θ exists and $(A'_\theta u)(t) = \int_0^1 G(t,s)\rho u(s)ds$. Notice that $\lambda \neq \frac{\lambda_1}{\rho}$, then 1 is not an eigenvalue of $\lambda A'_\theta$. By Lemma 3 there exists $r_0 > 0$, for any $0 < r < \min\{r_0, R_0\}$,

$$\deg(I - \lambda A, T_r, \theta) = \deg(I - \lambda A'_\theta, T_r, \theta) = (-1)^\kappa \neq 0, \tag{20}$$

where κ is the sum of algebraic multiplicities for all eigenvalues of $\lambda A'_\theta$ lying in the interval (0, 1).

By (15) and (20) λA has at least one nonzero fixed point. Thus problem (P) has at least one nontrivial solution. □

4. Positive solutions

In many realistic problems, the positive solution is more significant. In this section we will study this question.

Theorem 9. *Suppose that* (H_4) *holds. Then there exists* $\lambda^* > 0$ *such that for any* $0 < \lambda < \lambda^*$ *problem (P) has at least one positive solution.*

Proof. Let $D = [0,1], D_0 = [t_1,t_2] \subset (0,1) \subset D, \eta = \min_{t_1\leq t\leq t_2} \frac{3}{4}t^3(1-t)^2 > 0$. By ($H_4$) there exist $b_1 > 0$ and $R_1 > 0$ such that

$$f(t,u) \geq -b_1, \forall t \in [0,1], u \geq 0,$$

$$f(t,u) \geq \eta^{-1}Nb_1, \forall t \in [t_1,t_2], u \geq R_1, \tag{21}$$

where $N > \frac{1-(t_2-t_1)}{t_2-t_1}$ is a natural number. Let

$$f_2(t,u) = \begin{cases} f(t,u), & u \geq 0, \\ f(t,-u), & u < 0. \end{cases}$$

Then

$$f_2(t,u) \geq -b_1, \forall t \in [0,1], u \in \mathbb{R}. \tag{22}$$

Let

$$(A_2u)(t) = \int_0^1 G(t,s) f_2(s,u(s))ds.$$

Obviously, $A_2 : E \to E$ is a completely continuous operator.

From Remark 6 and the proof of Theorem 5 there exists $R_0 > 0$, for any $R > R_0$,

$$\deg(I - \lambda A_2, T_R, \theta) = 0, \forall \lambda > 0. \tag{23}$$

Take $0 < r < R_0$. Let $m = \max_{0\leq t\leq 1, |u|\leq r} |f_2(t,u)|$, $M = \max_{0\leq s,t\leq 1} G(t,s)$, $\overline{\lambda} = r(mM)^{-1}$. For any $0 < \lambda < \overline{\lambda}, u \in \partial T_r$, we have

$$\|\lambda A_2 u\| = \max_{0\leq t\leq 1} |\int_0^1 \lambda G(t,s) f_2(s,u(s))ds| < \overline{\lambda} Mm = r = \|u\|.$$

Thus

$$\deg(I - \lambda A_2, T_r, \theta) = 1, \ \forall\ 0 < \lambda < \overline{\lambda}. \tag{24}$$

From (23) and (24) we have that for any $0 < \lambda < \overline{\lambda}$, there exists $u_\lambda \in C[0,1]$ with $\|u_\lambda\| > r$ such that $u_\lambda = \lambda A_2 u_\lambda$. Now we show

$$\lim_{\lambda\to 0^+,\ u_\lambda=\lambda A_2 u_\lambda,\ \|u_\lambda\|>r} \|u_\lambda\| = +\infty. \tag{25}$$

In fact, if (25) doesn't hold, then there exist $\lambda_n > 0, u_{\lambda_n} \in C[0,1]$ such that $\lambda_n \to 0, r < \|u_{\lambda_n}\| < c$ ($c > 0$ is a constant), and

$$u_{\lambda_n} = \lambda_n A_2 u_{\lambda_n}. \tag{26}$$

Since A_2 is completely continuous, then $\{u_{\lambda_n}\}$ has a subsequence (assume without loss of generality that it is $\{u_{\lambda_n}\}$) converging to $u_* \in C[0,1]$. Let $n \to +\infty$ in (26), we have $u_* = \theta$, which is a contradiction of $\|u_{\lambda_n}\| > r > 0$. Then (25) holds.

Next we show that there exists $R = R(\overline{\lambda}) > 0$ such that if $0 < \lambda_0 \leq \overline{\lambda}, \|u_0\| \geq R$ and $u_0 = \lambda_0 A u_0$, then $u_0(t) \geq 0$. Take

$$R = R(\overline{\lambda}) = \max\{2\eta^{-1}R_1,\ 2\eta^{-1}\overline{\lambda} b_1 M,\ 2\overline{\lambda} b_1 M\}. \tag{27}$$

Assume that $0 < \lambda_0 \leq \overline{\lambda}, \|u_0\| \geq R$ and $u_0 = \lambda_0 A u_0$. Take any $\bar{t} \in [t_1, t_2]$, by (G_3) we have that for any $\tau \in [0,1]$,

$$\begin{aligned} u_0(\bar{t}) &= \lambda_0 \int_0^1 G(\bar{t},s)[f_2(s,u_0(s)) + b_1]ds - \lambda_0 \int_0^1 b_1 G(\bar{t},s)ds \\ &\geq \lambda_0 \eta \int_0^1 G(\tau,s) f_2(s,u_0(s))ds - \overline{\lambda} b_1 M \\ &= \eta u_0(\tau) - \overline{\lambda} b_1 M. \end{aligned} \tag{28}$$

On account of the continuity of u_0, there exists $t^* \in [0,1]$ such that $u_0(t^*) = \|u_0\|$. Take $\tau = t^*$ in (28), by (27) we have

$$\begin{aligned} u_0(\bar{t}) &\geq \eta\|u_0\| - \overline{\lambda} b_1 M \geq \eta R - \overline{\lambda} b_1 M \\ &= \tfrac{\eta}{2}R + \tfrac{\eta}{2}R - \overline{\lambda} b_1 M \geq \tfrac{\eta}{2}R \geq R_1. \end{aligned}$$

Thus $u_0(\bar{t}) \geq R_1$, for any $\bar{t} \in [t_1, t_2]$. By (21) we have

$$f_2(s, u_0(s)) \geq \eta^{-1} N b_1, \forall s \in D_0 = [t_1, t_2]. \tag{29}$$

It follows from (G_1) and (G_3) that for any $s \in [t_1, t_2]$ and $t, \tau \in [0,1]$

$$G(t,s) = G(s,t) \geq \eta G(\tau, t) = \eta G(t, \tau). \tag{30}$$

Take $D_i \subset D$ $(i = 1, 2, \cdots, N)$ such that $\text{mes}D_i = \text{mes}D_0$, $\bigcup\limits_{i=1}^{N} D_i \supset D \backslash D_0$. By (29) and (30) we have that for any $t \in D, s \in D_0, \tau \in D_i$ $(i = 1, 2, \cdots, N)$,

$$\frac{1}{N} G(t,s) f_2(s, u_0(s)) \geq b_1 G(t, \tau). \tag{31}$$

Notice that $\text{mes}D_i = \text{mes}D_0 (i = 1, 2, \cdots, N)$, then

$$\frac{1}{N}\int_{D_0} G(t,s) f_2(s, u_0(s)) ds \geq \int_{D_i} b_1 G(t,\tau) d\tau, \ i = 1, 2, \cdots, N.$$

Thus

$$\begin{aligned} \int_{D_0} G(t,s) f_2(s, u_0(s)) ds &\geq \sum_{i=1}^{N} \int_{D_i} b_1 G(t,\tau) d\tau \\ &\geq \int_{D\backslash D_0} b_1 G(t,\tau) d\tau \\ &= \int_{D\backslash D_0} b_1 G(t,s) ds. \end{aligned} \tag{32}$$

By (32) and (21) we have that for any $t \in [0,1]$,

$$u_0(t) = \lambda_0 \int_{D_0} G(t,s) f_2(s, u_0(s)) ds + \lambda_0 \int_{D\backslash D_0} G(t,s) f_2(s, u_0(s)) ds \geq 0.$$

For R in (27), by (25), there exists $\lambda^* < \overline{\lambda}$ such that if $0 < \lambda \leq \lambda^*, \|u_\lambda\| \geq r$ and $u_\lambda = \lambda A_2 u_\lambda$, then $\|u_\lambda\| \geq R$, thus $u_\lambda(t) \geq 0$. By the definition of A_2 and f_2 we have

$$u_\lambda(t) = \lambda \int_0^1 G(t,s) f_2(s, u_\lambda(s)) ds = \lambda \int_0^1 G(t,s) f(s, u_\lambda(s)) ds = \lambda A u_\lambda(t).$$

So $u_\lambda(t)$ is a positive solution of problem (P). $\square$

Remark 10. In Theorem 9 we obtain the existence of positive solutions for the semipositone boundary value problem (P) without that assuming (1) holds.

Remark 11. Since we only study the existence of positive solutions for the boundary value problem (P), which is irrelevant to the value of $f(t,u)$ when $u \le 0$, we only suppose that $f(t,u)$ is bounded below when $u \ge 0$. The nonlinear term $f(t,u)$ may be unbounded from below when $u \le 0$.

Example 12. Consider the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda\left[(\sqrt{t}+1)u^3 - \sqrt[3]{u}\right], \ 0 \le t \le 1, \\ u(0) = u'(0) = u(1) = u''(1) = 0. \end{cases} \tag{P_1}$$

In this example, $f(t,u) = (\sqrt{t}+1)u^3 - \sqrt[3]{u}$, then

$$\lim_{u\to+\infty} \frac{f(t,u)}{u} = \lim_{u\to+\infty}\left((\sqrt{t}+1)u^2 - \frac{1}{\sqrt[3]{u^2}}\right) = +\infty,$$

which means that (H_4) holds. By Theorem 9 there exists $\lambda^* > 0$ such that for any $0 < \lambda < \lambda^*$ BVP (P_1) has at least one positive solution.

Remark 13. In Example 12, the nonlinear term f doesn't satisfy (1) and (H_3), but the existence of positive solutions of BVP (P_1) is obtained by using our result.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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