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Research article

On a combination of fractional differential and integral operators associated with a class of normalized functions

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Abstract: Recently, the combined fractional operator (CFO) is introduced and discussed in Baleanu et al. [1] in real domain. In this paper, we extend CFO to the complex domain and study its geometric properties in some normalized analytic functions including the starlike and convex functions. Moreover, we employ the complex CFO to modify a class of Briot-Bouquet differential equations in a complex region. As a consequence, the upper solution is illustrated by using the concept of subordination inequality.

Keywords: fractional calculus; fractional differential operator; univalent function; analytic function; subordination and superordination; open unit disk; Briot-Bouquet differential equation
Mathematics Subject Classification: 30C45, 30C55

1. Introduction

The topic of fractional calculus operators attracts the attention of many investigators to present different types of solutions such as numerical, approximation and analytic solutions for fractional differential equations (see [2–4]). The classical fractional calculus' operators, like the Riemann-Liouville, appeared in pure and theoretical studies while some new or modified operators can be useful for applications. The authors of [1] used the conventional differential operators in combination with the operators from Anderson and Ulness [5]. Baleanu et al. established a relation between their operator and the Mittag-Leffler function when they resolved some classes of differential equations. The purpose for which they produced this mixed operator was generating a common operator that permits demonstrating real information from a collection of procedures and systems. Recently, Ibrahim and Jahangiri [6] presented a new fractional differential operator in a complex region. It is an extension of Anderson-Ulness operator into a complex domain as well as of the differential operator of Salagean [7]. Recently, the Srivastava-Owa fractional differential and integral operators are employed in different applications, such as in image processing for denoising [8] and enhance images [9]. Also, the conformable operator is utilized to define a new model of of economic order quantity [10].

In this work, we introduce a generalized fractional operator in a complex domain and study its geometric properties. We employ this operator to define different classes of univalent functions (one-one analytic normalized functions in the complex domain). In addition, we present the operator in some structures of the subordination and superordination inequalities. Finally, as an application, we extend a category of Briot-Bouquet differential equations in a complex region and determine its upper bound solution by utilizing the generalized fractional operator.

2. Materials and method

In this section, we present the methodology to define the complex CFO.

2.1. Complex fractional calculus

Let $\mathcal{S}$ be the class of analytic functions in the open unit disk $\cup := \{\xi \in \mathbb{C} : |\xi| < 1\}$ and $\mathcal{S}[\phi, n]$ be the subclass of $\mathcal{S}$ having the function

$$\phi(\xi) = \phi + \phi_n \xi^n + \phi_{n+1}\xi^{n+1} + \dots,$$

where ϕ, ϕ_n, ϕ_{n+1}, $n = 1, 2, \dots$, are the coefficients constants of the analytic function $\phi(\xi)$. Let $\wedge$ be the subclass of $\mathcal{S}$ indicating the function $\phi(\xi) = \xi + \phi_2\xi^2 + \dots$ (see [11]). Here, we give the definitions of the Riemann-Liouville fractional operators (integral and derivative) in the complex plane $\mathbb{C}$ as the following:

Definition 2.1. The integral of arbitrary order $\wp$, where $\Re\{\wp\} > 0$ for a function $h(\xi)$, is

$$I_\xi^\wp h(\xi) = \left(\frac{1}{\Gamma(\wp)}\right)\left(\int_0^\xi h(\zeta)(\xi-\zeta)^{\wp-1}d\zeta\right); \quad \Re\{\wp\} > 0. \tag{2.1}$$

Here, h is in a simply-connected region of ($\mathbb{C}$) having $(0, 0)$ and the multiplicity of $(\xi-\zeta)^{\wp-1}$ is vanished by indicating $\log(\xi - \zeta)$ when $\Re(\xi - \zeta) > 0$.

Definition 2.2. The Srivastava and Owa fractional derivative of order $0 \le \wp < 1$ (see [12]);

$$D_\xi^\wp h(\xi) = \left(\frac{1}{\Gamma(1-\wp)}\right)\frac{d}{d\xi}\left(\int_0^\xi \frac{h(\zeta)}{(\xi-\zeta)^\wp}d\zeta\right); \tag{2.2}$$

such that h is analytic in a simply-connected region of $\mathbb{C}$ involving $(0, 0)$ and the multiplicity of $(\xi-\zeta)^{-\wp}$ is isolated by using $\log(\xi - \zeta)$ when $\Re(\xi - \zeta) > 0$.

In our discussion, we deal with the class of analytic functions in the open unit disk $\mathcal{H} = \mathcal{H}(\cup)$. For m a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, m] = \{f \in \mathcal{H}, f(\xi) = a + a_m\xi^m + \dots\}.$$

Definition 2.3. Let $k = 1, 2, ...$ Then for $k-1 < \wp < k$ and analytic function $\phi(\xi)$; the Caputo derivative of order $\wp$ is given by the following construction

$$ {}^{C}D_{\xi}^{\wp}h(\xi) = \begin{cases} \frac{1}{\Gamma(k-\wp)} \int_0^{\xi} h^{(k)}(\zeta)(\xi-\zeta)^{k-\wp-1}d\zeta, & \text{if } k-1 < \wp < k; \\ \\ \frac{d^n}{d\xi^n}h(\xi), \text{ if } \wp = k. \end{cases} \tag{2.3} $$

Where h and its k-derivatives are analytic in simply-connected region in $\mathbb{C}$ having the origin and the multiplicity of $(\xi-\zeta)^{k-\wp-1}$ is deleted by utilizing $\log(\xi-\zeta)$ when $\Re(\xi-\zeta) > 0$.

Remark 2.4. The properties of $D_{\xi}^{\wp}h(\xi)$ and ${}^{C}D_{\xi}^{\wp}h(\xi)$ are as follows:

- $$ D_{\xi}^{\wp}(\xi)^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\wp)}(\xi)^{\alpha-\wp}, \quad I_{\xi}^{\wp}(\xi)^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\wp)}(\xi)^{\alpha+\wp}. $$

  Moreover, when $h^{(k)}(0) = 0$ for all $k = 1, 2, ...,$ then

  $$ D_{\xi}^{\wp}h(\xi) = {}^{C}D_{\xi}^{\wp}h\,(\xi). $$

- $$ D_{\xi}^{\wp}h(\xi) = \frac{d^k}{d\xi^k}I_{\xi}^{k-\wp}h\,(\xi). $$

- $$ {}^{C}D_{\xi}^{\wp}h\,(\xi) = I_{\xi}^{k-\wp}\frac{d^k}{d\xi^k}h\,(\xi). $$

2.2. *Complex differential operator*

Recently, Ibrahim and Jahangiri [6] introduced a definition of a conformable differential operator in a complex domain, specifically for function $\phi \in \wedge$ as follows:

Definition 2.5. Let β be non-negative number and let $[[\beta]]$ be the integer part of β. For $\phi \in \wedge$, the complex conformable derivative $\mathcal{D}^{\beta}\,\phi$ of order β in defined by

$$ \begin{aligned} \mathcal{D}^{\beta}\,\phi(\xi) &= \mathcal{D}^{\beta-[[\beta]]}(\mathcal{D}^{[[\beta]]}\,\phi(\xi)) \\ &= \frac{\kappa_1(\beta-[[\beta]],\xi)}{\kappa_1(\beta-[[\beta]],\xi)+\kappa_0(\beta-[[\beta]],\xi)}\,(\mathcal{D}^{[[\beta]]}\phi(\xi)) \\ &+ \frac{\kappa_0(\beta-[[\beta]],\xi)}{\kappa_1(\beta-[[\beta]],\xi)+\kappa_0(\beta-[[\beta]],\xi)}\,(\xi(\mathcal{D}^{[[\beta]]}\phi(\xi))'), \end{aligned} \tag{2.4} $$

where for $\wp = \beta - [[\beta]] \in [0, 1)$,

$$ \begin{aligned} &\mathcal{D}^{0}\phi(\xi) = \phi(\xi) \\ &\mathcal{D}^{\wp}\phi(\xi) = \frac{\kappa_1(\wp,\xi)}{\kappa_1(\wp,\xi)+\kappa_0(\wp,\xi)}\,\phi(\xi) + \frac{\kappa_0(\wp,\xi)}{\kappa_1(\wp,\xi)+\kappa_0(\wp,\xi)}\,(\xi\,\phi'(\xi)) \\ &\mathcal{D}^{1}\phi(\xi) = \xi\,\phi(\xi)',\ ..., \\ &\mathcal{D}^{[[\wp]]}\phi(\xi) = \mathcal{D}\left(\mathcal{D}^{[[\wp]]-1}\phi(\xi)\right), \end{aligned} \tag{2.5} $$

where the functions $\kappa_1, \kappa_0 : [0,1] \times \cup \to \cup$ are analytic in $\cup$ and thus $\kappa_1(\wp,\xi) \neq -\kappa_0(\wp,\xi)$,

$$\lim_{\wp\to 0}\kappa_1(\wp,\xi) = 1, \quad \lim_{\wp\to 1}\kappa_1(\wp,\xi) = 0, \quad \kappa_1(\wp,\xi) \neq 0,\ \forall\xi \in \cup,\ \wp \in (0,1),$$

and

$$\lim_{\wp\to 0}\kappa_0(\wp,\xi) = 0, \quad \lim_{\wp\to 1}\kappa_0(\wp,\xi) = 1, \quad \kappa_0(\wp,\xi) \neq 0,\ \forall\xi \in \cup\wp \in (0,1).$$

Example 2.6. Let $\phi \in \wedge$ taking the expansion formula $\phi(\xi) = \xi + \sum_{n=2}^{\infty}\phi_n\,\xi^n$ then

$$\begin{aligned}
&\mathcal{D}^{\wp}\phi(\xi)\\
&= \frac{\kappa_1(\wp,\xi)}{\kappa_1(\wp,\xi)+\kappa_0(\wp,\xi)}\,\phi(\xi) + \frac{\kappa_0(\wp,\xi)}{\kappa_1(\wp,\xi)+\kappa_0(\wp,\xi)}\,(\xi\,\phi'(\xi))\\
&= \frac{\kappa_1(\wp,\xi)}{\kappa_1(\wp,\xi)+\kappa_0(\wp,\xi)}\left(\xi + \sum_{n=2}^{\infty}\phi_n\,\xi^n\right) + \frac{\kappa_0(\wp,\xi)}{\kappa_1(\wp,\xi)+\kappa_0(\wp,\xi)}\left(\xi + \sum_{n=2}^{\infty}n\phi_n\,\xi^n\right)\\
&= \xi + \sum_{n=2}^{\infty}\left(\frac{\kappa_1(\wp,\xi)+n\,\kappa_0(\wp,\xi)}{\kappa_1(\wp,\xi)+\kappa_0(\wp,\xi)}\right)\phi_n\,\xi^n.
\end{aligned} \tag{2.6}$$

Thus, $\mathcal{D}^{\wp}\phi(\xi) \in \wedge$, whenever, $\phi \in \wedge$. We consider the following functions with fractional indices $\kappa_0(\wp,\xi) = \wp\xi^{1-\wp}$ and $\kappa_1(\wp,\xi) = (1-\wp)\xi^{\wp}$ in the sequel.

2.3. *Structure of the combined fractional operator*

By using the next property of ${}^{C}D_{\xi}^{\wp}$, we define the combined operator as follows:

$${}^{C}D_{\xi}^{\wp}\phi(\xi) = I_{\xi}^{1-\wp}\phi'(\xi) = \frac{1}{\Gamma(1-\wp)}\int_0^{\xi}\phi'(\zeta)(\xi-\zeta)^{-\wp}d\zeta.$$

Now, by replacing the term $\phi'(\xi)$ by the complex conformable differential operator in (2.5): $\mathcal{D}^{\wp}\phi(\xi)$, we receive the following hybrid operator

$$\begin{aligned}
{}^{H}D_{\xi}^{\wp}\phi(\xi) = \frac{1}{\Gamma(1-\wp)}\int_0^{\xi}&\left(\frac{\kappa_1(\wp,\zeta)}{\kappa_1(\wp,\zeta)+\kappa_0(\wp,\zeta)}\,\phi(\zeta) + \frac{\kappa_0(\wp,\zeta)}{\kappa_1(\wp,\zeta)+\kappa_0(\wp,\zeta)}\,(\zeta\,\phi'(\zeta))\right)\\
&\times(\xi-\zeta)^{-\wp}d\zeta.
\end{aligned} \tag{2.7}$$

A special case of (2.7) can be recognized, when κ_1 and κ_0 are constants depending on the fractional value $\wp$. We have the following construction of a linear hybrid operator:

$$
\begin{aligned}
{}^{L}D_{\xi}^{\wp}\phi(\xi) &= \frac{1}{\Gamma(1-\wp)}\int_0^{\xi}\left(\frac{\kappa_1(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\,\phi(\zeta)+\frac{\kappa_0(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\,(\zeta\,\phi'(\zeta))\right)\\
&\qquad\qquad\times(\xi-\zeta)^{-\wp}d\zeta\\
&=\left(\frac{\kappa_1(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\right)I_{\xi}^{1-\wp}\phi(\xi)\\
&+\left(\frac{\kappa_0(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\right)\frac{1}{\Gamma(1-\wp)}\int_0^{\xi}(\zeta\,\phi'(\zeta))(\xi-\zeta)^{-\wp}d\zeta\\
&=\left(\frac{\kappa_1(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\right)I_{\xi}^{1-\wp}\phi(\xi)+\left(\frac{\kappa_0(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\right)\frac{1}{\Gamma(1-\wp)}\\
&\qquad\left(\int_0^{\xi}[(\zeta\phi(\zeta))'-\phi(\zeta)](\xi-\zeta)^{-\wp}d\zeta\right)\\
&=\left(\frac{\kappa_1(\wp)-\kappa_0(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\right)I_{\xi}^{1-\wp}\phi(\xi)+\left(\frac{\kappa_0(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\right){}^{C}D_{\xi}^{\wp}\,(\xi\phi(\xi))\,.
\end{aligned}
\tag{2.8}
$$

It is clear that ${}^{L}D_{\xi}^{\wp}\phi(\xi)$ indicates a linear combination of the Srivastava and Owa integral operator for the analytic normalized function ϕ and the complex Caputo differential operator for $\xi\phi$. For example, when $\kappa_1 = 1-\wp$ and $\kappa_0 = \wp$, we get

$$
{}^{L}D_{\xi}^{\wp}\phi(\xi) = (1-2\wp)\,I_{\xi}^{1-\wp}\phi(\xi)+\wp\,{}^{C}D_{\xi}^{\wp}\,(\xi\phi(\xi))\,.
$$

Remark 2.7.
- The operators ${}^{H}D_{\xi}^{\wp}$ and ${}^{L}D_{\xi}^{\wp}$ are non-local singular operators where they involve the kernel term $(\xi-\zeta)^{-\wp}$. Therefore, every analytic univalent function ϕ admits integrable singularity when $\xi = \zeta$ of the integral, since $\wp \in (0,1)$.
- The endpoint limits when $\wp \in [0,1]$ satisfies

$$
\lim_{\wp\to 0}{}^{H}D_{\xi}^{\wp}\phi(\xi) = \lim_{\wp\to 0}{}^{L}D_{\xi}^{\wp}\phi(\xi) = \int_0^{\xi}\phi(\zeta)d\zeta;
$$

$$
\lim_{\wp\to 1}{}^{H}D_{\xi}^{\wp}\phi(\xi) = \lim_{\wp\to 1}{}^{L}D_{\xi}^{\wp}\phi(\xi) = (\xi\phi(\xi))' - I_{\xi}^{1-\wp}\phi(\xi).
$$

Theorem 2.8. *Let $\phi \in \wedge$. Then the operator $\Omega^{\wp}\phi \in \wedge$, where*

$$
\Omega^{\wp}\phi(\xi) := \Gamma(3-\wp)\left(\frac{{}^{L}D_{\xi}^{\wp}\phi(\xi)}{\xi^{1-\wp}}\right).
$$

Proof. For $\phi \in \wedge$ and by using Remark 2.4 and Eq (2.8), we have

$$
\begin{aligned}
&\Gamma(3-\wp)\left(\frac{{}^{L}D_{\xi}^{\wp}\phi(\xi)}{\xi^{1-\wp}}\right)\\
&=\Gamma(3-\wp)\left(\frac{I_{\xi}^{1-\wp}\left(\frac{\kappa_1(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\,\phi(\zeta)+\frac{\kappa_0(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\,(\zeta\,\phi'(\zeta))\right)}{\xi^{1-\wp}}\right)\\
&=\Gamma(3-\wp)\left(\frac{I_{\xi}^{1-\wp}\left(\xi+\sum_{n=2}^{\infty}K_n(\wp)\phi_n\,\xi^n\right)}{\xi^{1-\wp}}\right)\\
&=\Gamma(3-\wp)\left(\frac{\frac{\Gamma(2)}{\Gamma(3-\wp)}\xi^{1+1-\wp}+\sum_{n=2}^{\infty}K_n(\wp)\phi_n\,\frac{\Gamma(n+1)}{\Gamma(n+2-\wp)}\xi^{n+1-\wp}}{\xi^{1-\wp}}\right)\\
&=\xi+\sum_{n=2}^{\infty}K_n(\wp)\phi_n\left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\xi^n,
\end{aligned}
\tag{2.9}
$$

where $K_n(\wp) := \left(\frac{\kappa_1(\wp)+n\,\kappa_0(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}\right)$. Hence, $\Omega^{\wp} \in \wedge$. □

Remark 2.9.
- Note that when $\wp \to 1$, we have

$$\Omega\phi(\xi) = \left({}^{L}D_{\xi}\phi(\xi)\right).$$

Consequently, we get the following expansion

$$\Omega\phi(\xi) = \xi + \sum_{n=2}^{\infty} n\phi_n\,\xi^n,$$

which is reduced to the well-known Salagean operator. Therefore, one can generalize the Salagean classes of analytic functions [13–15] by utilizing the fractional operator $\Omega^{\wp}$.

- The fractional operator $\Omega^{\wp}$ acts on the convex function $\phi(\xi) = \xi/(1-\xi)$ as follows:

$$\Omega^{\wp}\phi(\xi) = \xi + \sum_{n=2}^{\infty} K_n(\wp)\left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\xi^n.$$

If $K_n(\wp) \le \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)$, $\forall n \ge 2$, then $\Omega^{\wp}\phi(\xi)$ is also convex in $\cup$.

Next part of this paper indicates some classes of analytic univalent functions involving the operator $\Omega^{\wp}$.

3. Results

In this section, we aim to investigate the geometric properties of the operator $\Omega^{\wp}$. For this purpose, we need the following definitions from the geometric function theory:

A function $\phi \in \wedge$ is starlike through the origin if the linear slice linking the origin to all added point of ϕ deceits completely in $\phi(\xi : |\xi| < 1)$. A univalent function (injective function of a complex variable) ϕ is convex in $\cup$ if the linear slice linking every two points of $\phi(\xi : |\xi| < 1)$ sets wholly in $\phi(\xi : |\xi| < 1)$. We denote these classes by $\mathcal{S}^*$ and C for starlike and convex correspondingly. Next, we assume that the class $\mathcal{P}$ includes all mappings ψ analytic in $\cup$ with a positive real part in $\cup$ realizing $\psi(0) = 1$, $\psi'(0) > 0$. Consistently, $\phi \in \mathcal{S}^* \Leftrightarrow \xi\phi'(\xi)/\phi(\xi) \in \mathcal{P}$ and $\phi \in C \Leftrightarrow 1 + \xi\phi''(\xi)/\phi'(\xi) \in \mathcal{P}$. Consistently, $\Re(\xi\phi'(\xi)/\phi(\xi)) > 0$ for the starlikeness and $\Re(1 + \xi\phi''(\xi)/\phi'(\xi)) > 0$ for the convexity.

Let f and g be analytic functions in $\cup$. The function f is said to be subordinate to g, written $f \prec g$ or $f(\xi) \prec g(\xi)$, if there exists a function ω analytic in $\cup$, with $\omega(0) = 0$ and $|\omega(\xi)| < 1$, and such that $f(\xi) = g(\omega(\xi))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\cup) \subset g(\cup)$ (see [16]).

Linking the definitions of starlike and convex function in the subordination concept, Ma and Minda [17] formulated the next sub-classes

$$\frac{\xi\phi'(\xi)}{\phi(\xi)} \prec \curlyvee(\xi), \quad \curlyvee \in \mathcal{P}$$

and

$$1 + \frac{\xi\phi''(\xi)}{\phi'(\xi)} \prec \curlywedge(\xi), \quad \curlywedge \in \mathcal{P}.$$

We request the following result [18]:

Lemma 3.1. *Suppose that $h, \hbar \in \cup$, then the subordination $h \prec \hbar$ yields*

$$\int_0^{2\pi} |h(\zeta)|^p d\theta \leq \int_0^{2\pi} |\hbar(\zeta)|^p d\theta, \tag{3.1}$$

where $\zeta = re^{i\theta}, r \in (0, 1)$ and $p \in \mathbb{R}^+$.

Definition 3.2. Let $\wp \in [0, 1]$. A function $\phi \in \wedge$ is in the class $\mathfrak{S}^*_\wp(\sigma)$ if and only if

$$\frac{\xi(\Omega^\wp\phi(\xi))'}{\Omega^\wp\phi(\xi)} \prec \omega(\xi), \quad \xi \in \cup,$$

where ω is univalent function with a positive real part in $\cup$ achieving

$$\omega(0) = 1, \quad \Re(\omega'(\xi)) > 0.$$

Theorem 3.3. *Let $\phi \in \wedge$ and $\wp \in [0, 1]$. If $\phi \in C$ (the class of convex functions in the open unit disk) then*

$$|\Omega^\wp\phi(\xi)| \leq r(rF(1, 2; 3 - \wp; r))', \tag{3.2}$$

where F is Gauss hypergeometric function. The equality occurs for the Koebe function of the first type $K(\xi) = \xi/(1 - \xi)$, $\xi \in \cup$.

Proof. Let $\phi \in C$ then the coefficients satisfy $|\phi_n| < 1$ for all n. Moreover, we have the following limit

$$\lim_{\wp \to 1} K_n(\wp) = \lim_{\wp \to 1} \left(\frac{\kappa_1(\wp) + n\,\kappa_0(\wp)}{\kappa_1(\wp) + \kappa_0(\wp)} \right) = n.$$

A calculation implies

$$\begin{aligned}
|\Omega^{\wp}\phi(\xi)| &= \Big|\sum_{n=1}^{\infty} K_n(\wp)\phi_n \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\xi^n\Big| \\
&\leq \Gamma(3-\wp)\sum_{n=1}^{\infty} n\left(\frac{\Gamma(n+1)}{\Gamma(n+2-\wp)}\right) r^n \\
&= \Gamma(3-\wp) r \sum_{n=0}^{\infty}(n+1)\left(\frac{\Gamma(n+2)}{\Gamma(n+3-\wp)}\right) r^n \\
&= \Gamma(3-\wp) r \sum_{n=0}^{\infty}\left(\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(n+3-\wp)}\right)\frac{(n+1)r^n}{n!} \\
&= \frac{\Gamma(3-\wp)r}{\Gamma(3-\wp)}\sum_{n=0}^{\infty}\left(\frac{(1)_n(2)_n}{(3-\wp)_n}\right)\frac{(n+1)r^n}{n!} \\
&= r\sum_{n=0}^{\infty}\left(\frac{(1)_n(2)_n}{(3-\wp)_n}\right)\frac{(n+1)r^n}{n!} \\
&= r(rF(1,2;3-\wp;r))',
\end{aligned}$$

where $(b)_n = \frac{\Gamma(b+n)}{\Gamma(b)}$ is the Pochhammer symbol. Lastly, by assuming the Koebe function $K(\xi)$, with $\phi_n = 1$ in the above conclusion, we have the sharp result. □

Theorem 3.4. *Let $\phi \in \wedge$ and $\wp \in [0,1]$. If ϕ is univalent in $\cup$ then*

$$|\Omega^{\wp}\phi(\xi)| \leq r(rF(2,2;3-\wp;r))', \tag{3.3}$$

where F is Gauss hypergeometric function. The equality occurs for the Koebe function of the second type $K(\xi) = \xi/(1-\xi)^2$, $\xi \in \cup$.

Proof. Let ϕ univalent in $\cup$ then by De Branges' Theorem, the coefficients satisfy $|\phi_n| < n$ for all n.

Moreover, we have the following limit $\lim_{\wp\to 1} K_n(\wp) = n$. A calculation implies

$$\begin{aligned}
|\Omega^{\wp}\phi(\xi)| &= \Big|\sum_{n=1}^{\infty} K_n(\wp)\phi_n \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\xi^n\Big| \\
&\leq \Gamma(3-\wp)\sum_{n=1}^{\infty} n^2 \left(\frac{\Gamma(n+1)}{\Gamma(n+2-\wp)}\right) r^n \\
&= \Gamma(3-\wp) r \sum_{n=0}^{\infty}(n+1)^2 \left(\frac{\Gamma(n+2)}{\Gamma(n+3-\wp)}\right) r^n \\
&= \Gamma(3-\wp) r \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+2)(n+1)\Gamma(n+1)}{\Gamma(n+3-\wp)}\right)\frac{(n+1)r^n}{n!} \\
&= \Gamma(3-\wp) r \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+2)\Gamma(n+2)}{\Gamma(n+3-\wp)}\right)\frac{(n+1)r^n}{n!} \\
&= \frac{\Gamma(3-\wp) r}{\Gamma(3-\wp)} \sum_{n=0}^{\infty} \left(\frac{(2)_n(2)_n}{(3-\wp)_n}\right)\frac{(n+1)r^n}{n!} \\
&= r \sum_{n=0}^{\infty} \left(\frac{(2)_n(2)_n}{(3-\wp)_n}\right)\frac{(n+1)r^n}{n!} \\
&= r(rF(2,2;3-\wp;r))'.
\end{aligned}$$

By replacing the Koebe function $K(\xi)$, with $\phi_n = n$ in the above conclusion, we can receive the sharp outcome. □

Theorem 3.5. *Let $\phi \in \wedge$ and $\Theta(\xi)$ is univalent convex in $\cup$ achieving the subordination inequality*

$$\frac{\xi(\Omega^{\wp}\phi(\xi))'}{\Omega^{\wp}\phi(\xi)} \prec \Theta(\xi). \tag{3.4}$$

Then

$$\Omega^{\wp}\phi(\xi) \prec \xi \exp\Big(\int_0^{\xi} \frac{\Theta(\vartheta(\iota))-1}{\iota} d\iota\Big), \tag{3.5}$$

where $\vartheta(\xi)$ is analytic in $\cup$, with $\vartheta(0) = 0$, $|\vartheta(\xi)| < 1$ and it is the upper limit in the above integral. Furthermore, for $|\xi| = \iota$, $\Omega^{\wp}\phi(\xi)$ fulfills the formula

$$\exp\Big(\int_0^1 \frac{\Theta(\vartheta(-\iota))-1}{\iota} d\iota\Big) \leq \Big|\frac{\Omega^{\wp}\phi(\xi)}{\xi}\Big| \leq \exp\Big(\int_0^1 \frac{\Theta(\vartheta(\iota))-1}{\iota} d\iota\Big).$$

Proof. In view of the definition of the subordination, there occurs a Schwarz map with $\vartheta(0) = 0$ and $|\vartheta(\xi)| < 1$ satisfying

$$\frac{\xi(\Omega^{\wp}\phi(\xi))'}{\Omega^{\wp}\phi(\xi)} = \Theta(\vartheta(\xi)), \quad \xi \in \cup.$$

This implies that

$$\frac{(\Omega^{\wp}\phi(\xi))'}{\Omega^{\wp}\phi(\xi)} - \frac{1}{\xi} = \frac{\Theta(\vartheta(\xi))-1}{\xi}.$$

Integrate both sides, we have

$$\log \Omega^{\wp}\phi(\xi) - \log \xi = \int_0^{\xi} \frac{\Theta(\vartheta(\iota)) - 1}{\iota} d\iota.$$

A computation yields

$$\log \left(\frac{\Omega^{\wp}\phi(\xi)}{\xi} \right) = \int_0^{\xi} \frac{\Theta(\vartheta(\iota)) - 1}{\iota} d\iota. \tag{3.6}$$

Then, for some Schwarz function, we get the inequality

$$\Omega^{\wp}\phi(\xi) \prec \xi \exp\Big(\int_0^{\xi} \frac{\Theta(\vartheta(\iota)) - 1}{\iota} d\iota \Big).$$

Moreover, Θ maps the disk $0 < |\xi| < \iota < 1$ onto a territory which is symmetric convex w.r.t x-axis, which means

$$\Theta(-\iota|\xi|) \leq \Re(\Theta(\vartheta(\iota\xi))) \leq \Theta(\iota|\xi|), \quad \iota \in (0, 1),$$

therefore, we have the relations

$$\Theta(-\iota) \leq \Theta(-\iota\,|\xi|), \quad \Theta(\iota\,|\xi|) \leq \Theta(\iota).$$

By using the above relations, we conclude that

$$\int_0^1 \frac{\Theta(\Psi(-\iota|\xi|)) - 1}{\iota} d\iota \leq \Re\Big(\int_0^1 \frac{\Theta(\vartheta(\iota)) - 1}{\iota} d\iota \Big) \leq \int_0^1 \frac{\Theta(\vartheta(\iota|\xi|)) - 1}{\iota} d\iota,$$

which implies that

$$\int_0^1 \frac{\Theta(\vartheta(-\iota|\xi|)) - 1}{\iota} d\iota \leq \log \Big| \frac{\Omega^{\wp}\phi(\xi)}{\xi} \Big| \leq \int_0^1 \frac{\Theta(\vartheta(\iota|\xi|)) - 1}{\iota} d\iota,$$

and

$$\exp\left(\int_0^1 \frac{\Theta(\vartheta(-\iota|\xi|)) - 1}{\iota} d\iota \right) \leq \Big| \frac{\Omega^{\wp}\phi(\xi)}{\xi} \Big| \leq \exp\left(\int_0^1 \frac{\Theta(\vartheta(\iota|\xi|)) - 1}{\iota} d\iota \right).$$

We indicate that

$$\exp\Big(\int_0^1 \frac{\Theta(\vartheta(-\iota)) - 1}{\iota} d\iota \Big) \leq \Big| \frac{\Omega^{\wp}\phi(\xi)}{\xi} \Big| \leq \exp\Big(\int_0^1 \frac{\Theta(\vartheta(\iota)) - 1}{\iota} d\iota \Big).$$

□

Theorem 3.6. *Suppose that $\phi \in \wedge$ with non-negative connections and $\wp \in [0, 1]$. If Θ is univalent convex in $\cup$, then there is a solution satisfying*

$$\Omega^{\wp}\phi(\xi) \prec \xi \exp\Big(\int_0^{\xi} \frac{\Theta(\vartheta(\iota)) - 1}{\iota} d\iota \Big), \tag{3.7}$$

where $\vartheta(\xi)$ is analytic in $\cup$, with $\vartheta(0) = 0$ and $|\vartheta(\xi)| < 1$.

Proof. We check the following formula for the real parts:

$$\begin{aligned}
&\Re\left(\frac{\xi(\Omega^{\wp}\phi(\xi))'}{\Omega^{\wp}\phi(\xi)}\right) > 0\\
&\Leftrightarrow \Re\left(\frac{\xi + \sum_{n=2}^{\infty} nK_n(\wp)\phi_n \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\xi^n}{\xi + \sum_{n=2}^{\infty} K_n(\wp)\phi_n \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\xi^n}\right) > 0\\
&\Leftrightarrow \Re\left(\frac{1 + \sum_{n=2}^{\infty} nK_n(\wp)\phi_n \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\xi^{n-1}}{1 + \sum_{n=2}^{\infty} K_n(\wp)\phi_n \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\xi^{n-1}}\right) > 0\\
&\Leftrightarrow \Re\left(\frac{1 + \sum_{n=2}^{\infty} nK_n(\wp) \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\phi_n}{1 + \sum_{n=2}^{\infty} K_n(\wp) \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\phi_n}\right) > 0, \quad \xi \to 1\\
&\Leftrightarrow \left(1 + \sum_{n=2}^{\infty} nK_n(\wp) \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\phi_n\right) > 0.
\end{aligned}$$

Moreover, we indicate that $(\Omega^{\wp}\phi(0) = 0$, which yields

$$\frac{\xi(\Omega^{\wp}\phi(\xi))'}{\Omega^{\wp}\phi(\xi)} \in \mathcal{P}.$$

Hence, according to Theorem 3.4, we have (3.7). □

Example 3.7. In this example, we illustrate some special forms of Θ when $\mu \in [0,1]$.

(1) $\Theta(\xi) = \mu + (1-\mu)\sqrt{1+\xi}$,
(2) $\Theta(\xi) = \mu + (1-\mu)e^{\xi}$,
(3) $\Theta(\xi) = \mu + (1-\mu)(1+\sin(\xi))$,
(4) $\Theta(\xi) = \mu + (1-\mu)e^{e^{\xi}-1}$.

Then in view of Theorem 3.5, the subordination $\frac{\xi(\Omega^{\wp}\phi(\xi))'}{\Omega^{\wp}\phi(\xi)} \prec \Theta(\xi)$ implies

$$\Omega^{\wp}\phi(\xi) \prec \xi \exp\Big(\int_0^{\xi} \frac{\Theta(\vartheta(\iota)) - 1}{\iota} d\iota\Big).$$

For $\Theta(\xi) = \mu + (1-\mu)\sqrt{1+\xi}, \mu = 0$, we have

$$\Omega^{\wp}\phi(\xi) \prec \frac{e^{(2\sqrt{\xi+1})}(1-\sqrt{\xi+1})}{(\sqrt{\xi+1}+1)}.$$

Moreover, for $\Theta(\xi) = \mu + (1-\mu)(1+\sin(\xi)), \mu = 0$, we obtain

$$\Omega^{\wp}\phi(\xi) \prec \xi + \xi^2 + \xi^3/2 + \xi^4/9 - \xi^5/72 + O(\xi^6).$$

Remark 3.8. If $\wp = \mu = 0$ in Example 3.7, we have the sub-classes in [13–15] respectively.

Theorem 3.9. *Let $\phi \in \wedge$ be convex univalent in $\cup$ and $\wp \in [0,1]$. Then*

$$\int_0^{2\pi} |\Omega^{\wp}\phi(\xi)|^p d\theta \leq \int_0^{2\pi} \left|\xi\left(\frac{1+\xi}{1-\xi}\right)^{\delta}\right|^p d\theta, \quad p>0 \tag{3.8}$$

and

$$\int_0^{2\pi} \left|(\Omega^{\wp}\phi(\xi))'\right|^p d\theta \leq \int_0^{2\pi} \left|\left(\frac{1+\xi}{1-\xi}\right)^{\delta}\right|^p d\theta, \quad p>0, \tag{3.9}$$

where $\left(\frac{1+\xi}{1-\xi}\right)^{\delta}$ is the Janowski function of order $\delta \geq 1$.

Proof. Let

$$\sigma(\xi,\delta) = \xi\Big(\frac{1+\xi}{1-\xi}\Big)^{\delta}, \quad \zeta \in \cup, \delta \geq 1. \tag{3.10}$$

Then, a computation implies that

$$\begin{aligned}
\sigma(\zeta,1) &= \xi + 2\xi^2 + 2\xi^3 + 2\xi^4 + 2\xi^5 + 2\xi^6 + O(\xi^7)\\
\sigma(\xi,2) &= \xi + 4\xi^2 + 8\xi^3 + 12\xi^4 + 16\xi^5 + 20\xi^6 + ...\\
\sigma(\xi,3) &= \xi + 6\xi^2 + 18\xi^3 + 38\xi^4 + ...\\
\sigma(\xi,4) &= \xi + 8\xi^2 + 16\xi^3 + 24\xi^4 + ...\\
&\vdots
\end{aligned}$$

Since $\phi(\xi)$ is convex then its coefficients satisfy the inequality $|\phi_n| \leq 1$ for all n and $\lim_{\wp\to 0} K_n(\wp) = 1$, and

$$\lim_{\wp\to 0} \frac{\Gamma(2+\wp)\Gamma(n+1)}{\Gamma(n+1+\wp)} = 1.$$

Moreover, the coefficients of $\Omega^{\wp}$ achieve the inequality

$$\left|K_n(\wp)\phi_n\left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)}\right)\right| \leq 1. \tag{3.11}$$

Which means that $\Omega^{\wp}\phi(\xi)$ is majorized by the function $\sigma(\xi,\delta)$ for all $\delta \geq 1$. By the properties of majority [19], we obtain

$$\Omega^{\wp}\phi(\xi) \prec \sigma(\xi,\delta), \quad \xi \in \cup. \tag{3.12}$$

Thus, according to Lemma 3.1, we conclude that

$$\int_0^{2\pi} |\Omega^{\wp}\phi(\xi)|^p d\theta \leq \int_0^{2\pi} \left|\xi\left(\frac{1+\xi}{1-\xi}\right)^{\delta}\right|^p d\theta, \quad p>0.$$

Similarly for the $(\Omega^{\wp})'$ we have

$$\int_0^{2\pi} \left|(\Omega^{\wp}\phi(\xi))'\right|^p d\theta \leq \int_0^{2\pi} \left|\left(\frac{1+\xi}{1-\xi}\right)^{\delta}\right|^p d\theta, \quad p>0.$$

□

Remark 3.10. The condition on ϕ which is convex univalent in $\cup$, can be replaced by the following condition

$$|\phi_n| \leq \left|\left(\frac{\Gamma(n+2-\wp)}{K_n(\wp)\Gamma(3-\wp)\Gamma(n+1)}\right)\right|;$$

or ϕ is univalent then $|\phi| \leq n$ and $\delta \geq 2$.

3.1. Corresponding integral formula

In this part, we focus on the integral operator $\Upsilon^{\wp}\phi(\xi), \phi \in \wedge$, corresponding to the operator $\Omega^{\wp}\phi(\xi)$. The formula of $\Upsilon^{\wp}\phi(\xi)$ is given by the expansion

$$\Upsilon^{\wp}\phi(\xi) = \sum_{n=1}^{\infty} \phi_n \left(\frac{\Gamma(n+2-\wp)}{K_n(\wp)\Gamma(3-\wp)\Gamma(n+1)} \right) \xi^n, \quad \phi_1 = 1. \tag{3.13}$$

Definition 3.11. The Hadamard product of two functions ρ_1 and $\rho_2 \in \wedge$ is defined by the formula:

$$\begin{aligned} \rho_1(\xi) * \rho_2(\xi) &= \left(\xi + \sum_{n=2}^{\infty} a_n \xi^n \right) * \left(\xi + \sum_{n=2}^{\infty} b_n \xi^n \right) \\ &= \left(\xi + \sum_{n=2}^{\infty} a_n b_n \xi^n \right), \xi \in \cup. \end{aligned} \tag{3.14}$$

Definition 3.12. The Mittag-Leffler function $\Xi_{a,b}$ is an entire function defined by the series

$$\Xi_{a,b}(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(an+b)}, \quad a > 0,$$

where $\Gamma(\theta)$ is the gamma function.

We have the following propositions.

Proposition 3.13. *Let $\phi \in \wedge$ and $\wp \in [0, 1]$. Then*

$$(\Omega^{\wp} * \Upsilon^{\wp}) * \phi(\xi) = (\Upsilon^{\wp} * \Omega^{\wp}) * \phi(\xi) = \phi(\xi), \quad \xi \in \cup.$$

Proof. By applying the Hadamard product definition, we have

$$\begin{aligned} &(\Omega^{\wp} * \Upsilon^{\wp}) * \phi(\xi) \\ &= \left(\sum_{n=1}^{\infty} K_n(\wp) \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)} \right) \xi^n \right) * \left(\sum_{n=1}^{\infty} \left(\frac{\Gamma(n+2-\wp)}{K_n(\wp)\Gamma(3-\wp)\Gamma(n+1)} \right) \xi^n \right) * \phi(\xi) \\ &= \left(\sum_{n=1}^{\infty} \left(\frac{\Gamma(n+2-\wp)}{K_n(\wp)\Gamma(3-\wp)\Gamma(n+1)} \right) \xi^n \right) * \left(\sum_{n=1}^{\infty} K_n(\wp) \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)} \right) \xi^n \right) * \phi(\xi) \\ &= (\Upsilon^{\wp} * \Omega^{\wp}) * \phi(\xi) \\ &= \left(\sum_{n=1}^{\infty} \xi^n \right) * \phi(\xi) \\ &= \phi(\xi). \end{aligned}$$

□

Proposition 3.14. *Let $\phi \in \wedge$ and $\wp \in [0, 1]$. Then*

$$\Omega^{\wp}\phi(\xi) = \psi_1(\xi) * \Xi_{1,(1+\wp)} * \phi(\xi)$$

and

$$\Upsilon^{\wp}\phi(\xi) = \psi_2(\xi) * \Xi_{1,1} * \phi(\xi),$$

where $\Xi_{a,b}$ *indicates the Mittag-Leffler function and*

$$\psi_1(\xi) = \sum_{n=0}^{\infty} K_n(\wp)\Gamma(3-\wp)\Gamma(n+1)\xi^n$$

and

$$\psi_2(\xi) = \sum_{n=1}^{\infty} \left(\frac{\Gamma(n+2-\wp)}{K_n(\wp)\Gamma(3-\wp)} \right) \xi^n.$$

Proof. Let $\phi \in \wedge$ such that $\phi_0 = 0$ and $\phi_1 = 1$. Thus, we have

$$\begin{aligned}
&\Omega^{\wp}\phi(\xi) \\
&= \sum_{n=0}^{\infty} K_n(\wp)\phi_n \left(\frac{\Gamma(3-\wp)\Gamma(n+1)}{\Gamma(n+2-\wp)} \right) \xi^n \\
&= \left(\sum_{n=0}^{\infty} K_n(\wp)\Gamma(3-\wp)\Gamma(n+1)\xi^n \right) * \left(\sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(n+2-\wp)} \right) * \left(\sum_{n=0}^{\infty} \phi_n \xi^n \right) \\
&= \psi_1(\xi) * \Xi_{1,(1+\wp)} * \phi(\xi).
\end{aligned}$$

Now, for the integral $\Upsilon^{\wp}$, we have

$$\begin{aligned}
&\Upsilon^{\wp}\phi(\xi) \\
&= \sum_{n=1}^{\infty} \phi_n \left(\frac{\Gamma(n+2-\wp)}{K_n(\wp)\Gamma(3-\wp)\Gamma(n+1)} \right) \xi^n \\
&= \left(\sum_{n=1}^{\infty} \left(\frac{\Gamma(n+2-\wp)}{K_n(\wp)\Gamma(3-\wp)} \right) \xi^n \right) * \left(\sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(n+1)} \right) * \left(\sum_{n=0}^{\infty} \phi_n \xi^n \right) \\
&= \psi_2(\xi) * \Xi_{1,1} * \phi(\xi).
\end{aligned}$$

□

Theorem 3.15. *Let* $\phi \in \wedge$ *and* $\wp \in [0, 1]$. *If* ϕ *is univalent then*

$$|\Upsilon^{\wp}\phi(\xi)| \le rF(1, (3-\wp); 2; r), \quad r < 1. \tag{3.15}$$

Proof. Likely of Theorem 3.3 and for a univalent function ϕ, we have

$$\begin{aligned}
|\Upsilon^{\wp}\phi(\xi)| &= \Big| \sum_{n=1}^{\infty} \phi_n \left(\frac{\Gamma(n+2-\wp)}{K_n(\wp)\Gamma(3-\wp)\Gamma(n+1)} \right) \xi^n \Big| \\
&\le \frac{1}{\Gamma(3-\wp)} \sum_{n=1}^{\infty} \left(\frac{n\Gamma(n+2-\wp)}{n\Gamma(n+1)} \right) r^n \\
&= \frac{r}{\Gamma(3-\wp)} \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1)\Gamma(n+3-\wp)}{\Gamma(n+2)} \right) \frac{r^n}{n!} \\
&= r \sum_{n=0}^{\infty} \left(\frac{(1)_n(3-\wp)_n}{(2)_n} \right) \frac{r^n}{n!} \\
&= rF(1, (3-\wp); 2; r), \quad r < 1.
\end{aligned}$$

□

Note that, when $\phi \in \wedge$ is convex, then we obtain the same result in Theorem 3.15.

4. Conclusions

We considered two fractional operators (differential and integral) in the open unit disk. We showed that these operators are preserving the normalized class ($h(0) = h'(0) - 1 = 0$). We proved that the fractional operators are bounded by the Gauss hypergeometric function and they are represented by a convoluted formula with the Mittag-Leffler functions. We indicated that the differential operator ${}^{L}D_{\xi}^{\wp}\phi(\xi)$ is a linear combination of the Srivastava-Owa differential operator for the analytic normalized function ϕ and the complex Caputo differential operator for $\xi\phi$. For future work, one can suggest the mixed conformable operators in other classes of analytic functions like the multi-valent and harmonic classes.

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Conflict of interest

The authors declare no conflict of interest.

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