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Research article

Some geometric properties of certain meromorphically multivalent functions associated with the first-order differential subordination

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Abstract: A new subclass $\mathcal{G}_n(A, B, \lambda)$ of meromorphically multivalent functions defined by the first-order differential subordination is introduced. Some geometric properties of this new subclass are investigated. The sharp upper bound on $|z| = r < 1$ for the functional $\mathrm{Re}\{(1-\lambda)z^p f(z) - \frac{\lambda}{p} z^{p+1} f'(z)\}$ over the class $\mathcal{G}_n(A, B, 0)$ is obtained.

Keywords: analytic function; meromorphically multivalent function; differential subordination; coefficient estimate; sharp bound
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1. Introduction

Throughout our present discussion, we assume that

$$n,\ p \in \mathbb{N},\ -1 \le B < 1,\ B < A \ \text{ and } \lambda < 0. \tag{1.1}$$

Let $\Sigma_n(p)$ be the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^{k-p} \tag{1.2}$$

which are analytic in the punctured open unit disk $\mathbb{U}^* = \{z : 0 < |z| < 1\}$. The class $\Sigma_n(p)$ is closed under the Hadamard product

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^{k-p} = (f_1 * f_2)(z),$$

where

$$f_j(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,j} z^{k-p} \in \Sigma_n(p) \quad (j = 1, 2).$$

For functions $f(z)$ and $g(z)$ analytic in $\mathbb{U} = \{z : |z| < 1\}$, we say that $f(z)$ is subordinate to $g(z)$ and write $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

If the function $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In this paper we introduce and investigate the following subclass of $\Sigma_n(p)$.

Definition. A function $f(z) \in \Sigma_n(p)$ is said to be in the class $\mathcal{G}_n(A, B, \lambda)$ if it satisfies the first-order differential subordination:

$$(1-\lambda) z^p f(z) - \frac{\lambda}{p} z^{p+1} f'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{1.3}$$

Recently, several authors (see, e.g., [1–8, 10–16] and the references cited therein) introduced and studied various subclasses of meromorphically multivalent functions. Certain properties such as distortion bounds, inclusion relations and coefficient estimates are given. In this note we obtain inclusion relation, coefficient estimate and sharp bounds on $\mathrm{Re}\,(z^p f(z))$ for functions $f(z)$ belonging to the class $\mathcal{G}_n(A, B, \lambda)$. Furthermore, we investigate a new problem, that is, to find

$$\max_{|z|=r<1} \mathrm{Re} \left\{ (1-\lambda) z^p f(z) - \frac{\lambda}{p} z^{p+1} f'(z) \right\},$$

where $f(z)$ varies in the class

$$\mathcal{G}_n(A, B, 0) = \left\{ f(z) \in \Sigma_n(p) : z^p f(z) \prec \frac{1+Az}{1+Bz} \right\}. \tag{1.4}$$

We need the following lemma in order to derive the main results for the class $\mathcal{G}_n(A, B, \lambda)$.

Lemma [9]. Let $g(z)$ be analytic in $\mathbb{U}$ and $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z),$$

where $\mathrm{Re}\mu \geq 0$ and $\mu \neq 0$, then $g(z) \prec h(z)$.

2. Geometric properties of functions in class $\mathcal{G}_n(A, B, \lambda)$

Theorem 1. *Let* $0 < \alpha_1 < \alpha_2$. *Then* $Q_n(A, B, \alpha_2) \subset Q_n(A, B, \alpha_1)$.

Proof. Suppose that

$$g(z) = z^{1-p} f'(z) \tag{2.1}$$

for $f(z) \in \mathcal{Q}_n(A, B, \alpha_2)$. Then the function $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = p$. By using (1.3) and (2.1), we have

$$\begin{aligned}(1-\alpha_2) z^{1-p} f'(z) + \frac{\alpha_2}{p-1} z^{2-p} f''(z) &= g(z) + \frac{\alpha_2}{p-1} z g'(z) \\ &\prec p \frac{1+Az}{1+Bz}.\end{aligned} \tag{2.2}$$

An application of the above Lemma yields

$$g(z) \prec p \frac{1+Az}{1+Bz}. \tag{2.3}$$

By noting that $0 < \frac{\alpha_1}{\alpha_2} < 1$ and that the function $\frac{1+Az}{1+Bz}$ is convex univalent in $\mathbb{U}$, it follows from (2.1)–(2.3) that

$$\begin{aligned}&(1-\alpha_1) z^{1-p} f'(z) + \frac{\alpha_1}{p-1} z^{2-p} f''(z) \\ &= \frac{\alpha_1}{\alpha_2}\left((1-\alpha_2) z^{1-p} f'(z) + \frac{\alpha_2}{p-1} z^{2-p} f''(z)\right) + \left(1 - \frac{\alpha_1}{\alpha_2}\right) g(z) \\ &\prec p \frac{1+Az}{1+Bz}.\end{aligned}$$

This shows that $f(z) \in \mathcal{Q}_n(A, B, \alpha_1)$. The proof of Theorem 1 is completed. □

Theorem 2. *Let $f(z) \in \mathcal{Q}_n(A, B, \alpha)$. Then, for $|z| = r < 1$,*

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) \geq p\left(1 - (p-1)(A-B) \sum_{m=1}^{\infty} \frac{B^{m-1} r^{nm}}{\alpha nm + p - 1}\right), \tag{2.4}$$

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > p\left(1 - (p-1)(A-B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{\alpha nm + p - 1}\right), \tag{2.5}$$

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) \leq p\left(1 + (p-1)(A-B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1} r^{nm}}{\alpha nm + p - 1}\right) \tag{2.6}$$

and

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) < p\left(1 + (p-1)(A-B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1}}{\alpha nm + p - 1}\right) \quad (B \neq -1). \tag{2.7}$$

All the bounds are sharp for the function $f_n(z)$ given by

$$f_n(z) = z^p + p(p-1)(A-B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1} z^{nm+p}}{(nm+p)(\alpha nm + p - 1)} \quad (z \in \mathbb{U}). \tag{2.8}$$

Proof. It is known that for $|\xi| \le \sigma$ $(\sigma < 1)$ that

$$\left|\frac{1+A\xi}{1+B\xi} - \frac{1-AB\sigma^2}{1-B^2\sigma^2}\right| \le \frac{(A-B)\sigma}{1-B^2\sigma^2} \tag{2.9}$$

and

$$\frac{1-A\sigma}{1-B\sigma} \le \operatorname{Re}\left(\frac{1+A\xi}{1+B\xi}\right) \le \frac{1+A\sigma}{1+B\sigma}. \tag{2.10}$$

Let $f(z) \in \mathcal{Q}_n(A, B, \alpha)$. Then we can write

$$(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z) = p\frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{U}), \tag{2.11}$$

where $w(z) = w_n z^n + w_{n+1}z^{n+1} + \cdots$ is analytic and $|w(z)| < 1$ for $z \in \mathbb{U}$. By the Schwarz lemma, we know that $|w(z)| \le |z|^n$ $(z \in \mathbb{U})$. It follows from (2.11) that

$$\frac{(1-\alpha)(p-1)}{\alpha}z^{\frac{(1-\alpha)(p-1)}{\alpha}-1}f'(z) + z^{\frac{(1-\alpha)(p-1)}{\alpha}}f''(z) = \frac{p(p-1)}{\alpha}z^{\frac{p-1}{\alpha}-1}\left(\frac{1+Aw(z)}{1+Bw(z)}\right),$$

which implies that

$$\left(z^{\frac{(1-\alpha)(p-1)}{\alpha}}f'(z)\right)' = \frac{p(p-1)}{\alpha}z^{\frac{p-1}{\alpha}-1}\left(\frac{1+Aw(z)}{1+Bw(z)}\right).$$

After integration we arrive at

$$\begin{aligned} f'(z) &= \frac{p(p-1)}{\alpha}z^{-\frac{(1-\alpha)(p-1)}{\alpha}}\int_0^z \xi^{\frac{p-1}{\alpha}-1}\left(\frac{1+Aw(\xi)}{1+Bw(\xi)}\right)d\xi \\ &= \frac{p(p-1)}{\alpha}z^{p-1}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1+Aw(tz)}{1+Bw(tz)}\right)dt. \end{aligned} \tag{2.12}$$

Since

$$|w(tz)| \le t^n r^n \quad (|z| = r < 1;\ 0 \le t \le 1),$$

we get from (2.12) and left-hand inequality in (2.10) that, for $|z| = r < 1$,

$$\begin{aligned} \operatorname{Re}\left(\frac{f'(z)}{z^{p-1}}\right) &\ge \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1-At^n r^n}{1-Bt^n r^n}\right)dt \\ &= p - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}r^{nm}}{\alpha nm + p - 1}, \end{aligned} \tag{2.13}$$

and, for $z \in \mathbb{U}$,

$$\begin{aligned} \operatorname{Re}\left(\frac{f'(z)}{z^{p-1}}\right) &> \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1-At^n}{1-Bt^n}\right)dt \\ &= p - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{\alpha nm + p - 1}. \end{aligned}$$

Similarly, by using (2.12) and the right-hand inequality in (2.10), we have (2.6) and (2.7) (with $B \neq -1$).

Furthermore, for the function $f_n(z)$ given by (2.8), we find that $f_n(z) \in \mathcal{A}_n(p)$,

$$f_n'(z) = pz^{p-1} + p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}z^{nm+p-1}}{\alpha nm + p - 1} \tag{2.14}$$

and

$$(1-\alpha)z^{1-p}f_n'(z) + \frac{\alpha}{p-1}z^{2-p}f_n''(z) = p + p(A-B)\sum_{m=1}^{\infty}(-B)^{m-1}z^{nm} = p\frac{1+Az^n}{1+Bz^n}.$$

Hence $f_n(z) \in \mathcal{Q}_n(A, B, \alpha)$ and, from (2.14), we conclude that the inequalities (2.4) to (2.7) are sharp. The proof of Theorem 2 is completed. □

Corollary. *Let $f(z) \in \mathcal{Q}_n(A, B, \alpha)$. If*

$$(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{\alpha nm + p - 1} \leq 1, \tag{2.15}$$

then $f(z)$ is p-valent close-to-convex in $\mathbb{U}$.

Proof. Let $f(z) \in \mathcal{Q}_n(A, B, \alpha)$ and (2.15) be satisfied. Then, by using (2.5) in Theorem 2, we see that

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > 0 \quad (z \in \mathbb{U}).$$

This shows that $f(z)$ is p-valent close-to-convex in $\mathbb{U}$. The proof of the corollary is completed. □

Theorem 3. *Let $f(z) \in \mathcal{Q}_n(A, B, \alpha)$. Then, for $|z| = r < 1$,*

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) \geq 1 - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}r^{nm}}{(nm+p)(\alpha nm + p - 1)}, \tag{2.16}$$

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) \leq 1 + p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}r^{nm}}{(nm+p)(\alpha nm + p - 1)} \tag{2.17}$$

and

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) > 1 - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{(nm+p)(\alpha nm + p - 1)}. \tag{2.18}$$

All of the above bounds are sharp.

Proof. It is obvious that

$$\begin{aligned} f(z) &= \int_0^z f'(\xi)d\xi = z\int_0^1 f'(tz)dt \\ &= z^p\int_0^1 t^{p-1}\frac{f'(tz)}{(tz)^{p-1}}dt \quad (z \in \mathbb{U}). \end{aligned} \tag{2.19}$$

Making use of (2.4) in Theorem 2, it follows from (2.19) that

$$\begin{aligned}\operatorname{Re}\left(\frac{f(z)}{z^p}\right) &= \int_0^1 t^{p-1}\operatorname{Re}\left(\frac{f'(tz)}{(tz)^{p-1}}\right)dt\\ &\geq \int_0^1 t^{p-1}\left(p - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}(rt)^{nm}}{\alpha nm + p - 1}\right)dt\\ &= 1 - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}r^{nm}}{(nm+p)(\alpha nm + p - 1)},\end{aligned}$$

which gives (2.16).

Similarly, we deduce from (2.6) in Theorem 2 and (2.19) that (2.17) holds true.

Also, with the help of (2.13), we find that

$$\begin{aligned}\operatorname{Re}\left(\frac{f'(tz)}{(tz)^{p-1}}\right) &\geq \frac{p(p-1)}{\alpha}\int_0^1 u^{\frac{p-1}{\alpha}-1}\left(\frac{1-A(utr)^n}{1-B(utr)^n}\right)du\\ &> p - p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}t^{nm}}{\alpha nm + p - 1} \quad (|z| = r < 1;\ 0 < t \leq 1).\end{aligned}$$

From this and (2.19), we obtain (2.18).

Furthermore, it is easy to see that the inequalities (2.16)–(2.18) are sharp for the function $f_n(z)$ given by (2.8). Now the proof of Theorem 3 is completed. □

Theorem 4. *Let $f(z) \in \mathcal{Q}_n(A, B, \alpha)$ and $AB \leq 1$. Then, for $|z| = r < 1$,*

$$|f(z)| \leq r^p + p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}r^{nm+p}}{(nm+p)(\alpha nm + p - 1)} \tag{2.20}$$

and

$$|f(z)| < 1 + p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}}{(nm+p)(\alpha nm + p - 1)}. \tag{2.21}$$

The above bounds are sharp.

Proof. Since $AB \leq 1$, it follows from (2.9) that

$$\left|\frac{1+A\xi}{1+B\xi}\right| \leq \left|\frac{1-AB\sigma^2}{1-B^2\sigma^2}\right| + \frac{(A-B)\sigma}{1-B^2\sigma^2} = \frac{1+A\sigma}{1+B\sigma} \quad (|\xi| \leq \sigma < 1). \tag{2.22}$$

By virtue of (2.12) and (2.22), we have, for $|z| = r < 1$,

$$\begin{aligned}\left|\frac{f'(uz)}{(uz)^{p-1}}\right| &\leq \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left|\frac{1+Aw(utz)}{1+Bw(utz)}\right|dt\\ &\leq \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1+A(utr)^n}{1+B(utr)^n}\right)dt \qquad (2.23)\\ &< \frac{p(p-1)}{\alpha}\int_0^1 t^{\frac{p-1}{\alpha}-1}\left(\frac{1+Au^nt^n}{1+Bu^nt^n}\right)dt. \qquad (2.24)\end{aligned}$$

By noting that

$$|f(z)| \leq r^p \int_0^1 u^{p-1} \left| \frac{f'(uz)}{(uz)^{p-1}} \right| du,$$

we deduce from (2.23) and (2.24) that the desired inequalities hold true.

The bounds in (2.20) and (2.21) are sharp with the extremal function $f_n(z)$ given by (2.8). The proof of Theorem 4 is completed. □

Theorem 5. *Let* $f(z) \in \mathcal{Q}_1(A, B, \alpha)$ *and*

$$g(z) \in \mathcal{Q}_1(A_0, B_0, \alpha_0) \quad (-1 \leq B_0 < 1;\ B_0 < A_0;\ \alpha_0 > 0).$$

If

$$p(p-1)(A_0 - B_0) \sum_{m=1}^{\infty} \frac{B_0^{m-1}}{(m+p)(\alpha_0 m + p - 1)} \leq \frac{1}{2}, \tag{2.25}$$

then $(f * g)(z) \in \mathcal{Q}_1(A, B, \alpha)$, *where the symbol* $*$ *denotes the familiar Hadamard product of two analytic functions in* $\mathbb{U}$.

Proof. Since $g(z) \in \mathcal{Q}_1(A_0, B_0, \alpha_0)$, we find from the inequality (2.18) in Theorem 3 and (2.25) that

$$\mathrm{Re}\left(\frac{g(z)}{z^p}\right) > 1 - p(p-1)(A_0 - B_0) \sum_{m=1}^{\infty} \frac{B_0^{m-1}}{(m+p)(\alpha_0 m + p - 1)} \geq \frac{1}{2} \quad (z \in \mathbb{U}).$$

Thus the function $\frac{g(z)}{z^p}$ has the following Herglotz representation:

$$\frac{g(z)}{z^p} = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in \mathbb{U}), \tag{2.26}$$

where $\mu(x)$ is a probability measure on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$.

For $f(z) \in \mathcal{Q}_1(A, B, \alpha)$, we have

$$z^{1-p}(f * g)'(z) = (z^{1-p} f'(z)) * (z^{-p} g(z))$$

and

$$z^{2-p}(f * g)''(z) = (z^{2-p} f''(z)) * (z^{-p} g(z)).$$

Thus

$$\begin{aligned}
&(1-\alpha) z^{1-p}(f * g)'(z) + \frac{\alpha}{p-1} z^{2-p}(f * g)''(z) \\
&= (1-\alpha)\Big((z^{1-p} f'(z)) * (z^{-p} g(z))\Big) + \frac{\alpha}{p-1}\Big((z^{2-p} f''(z)) * (z^{-p} g(z))\Big) \\
&= h(z) * \frac{g(z)}{z^p},
\end{aligned} \tag{2.27}$$

where

$$h(z) = (1-\alpha) z^{1-p} f'(z) + \frac{\alpha}{p-1} z^{2-p} f''(z) \prec p \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \tag{2.28}$$

In view of the fact that the function $\frac{1+Az}{1+Bz}$ is convex univalent in $\mathbb{U}$, it follows from (2.26) to (2.28) that

$$(1-\alpha)z^{1-p}(f*g)'(z)+\frac{\alpha}{p-1}z^{2-p}(f*g)''(z)=\int_{|x|=1}h(xz)d\mu(x)\prec p\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}).$$

This shows that $(f*g)(z)\in\mathcal{Q}_1(A,B,\alpha)$. The proof of Theorem 5 is completed. □

Theorem 6. *Let*

$$f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k}\in\mathcal{Q}_n(A,B,\alpha). \tag{2.29}$$

Then

$$|a_{p+k}|\leq\frac{p(p-1)(A-B)}{(p+k)(\alpha k+p-1)}\quad(k\geq n). \tag{2.30}$$

The result is sharp for each $k\geq n$.

Proof. It is known that, if

$$\varphi(z)=\sum_{j=1}^{\infty}b_jz^j\prec\psi(z)\quad(z\in\mathbb{U}),$$

where $\varphi(z)$ is analytic in $\mathbb{U}$ and $\psi(z)=z+\cdots$ is analytic and convex univalent in $\mathbb{U}$, then $|b_j|\leq 1$ $(j\in\mathbb{N})$.

By using (2.29), we have

$$\begin{aligned}\frac{(1-\alpha)z^{1-p}f'(z)+\frac{\alpha}{p-1}z^{2-p}f''(z)-p}{p(A-B)}&=\frac{1}{p(p-1)(A-B)}\sum_{k=n}^{\infty}(p+k)(\alpha k+p-1)a_{p+k}z^k\\&\prec\frac{z}{1+Bz}\quad(z\in\mathbb{U}).\end{aligned} \tag{2.31}$$

In view of the fact that the function $\frac{z}{1+Bz}$ is analytic and convex univalent in $\mathbb{U}$, it follows from (2.31) that

$$\frac{(p+k)(\alpha k+p-1)}{p(p-1)(A-B)}|a_{p+k}|\leq 1\quad(k\geq n),$$

which gives (2.30).

Next we consider the function $f_k(z)$ given by

$$f_k(z)=z^p+p(p-1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}z^{km+p}}{(km+p)(\alpha km+p-1)}\quad(z\in\mathbb{U};\ k\geq n).$$

Since

$$(1-\alpha)z^{1-p}f_k'(z)+\frac{\alpha}{p-1}z^{2-p}f_k''(z)=p\frac{1+Az^k}{1+Bz^k}\prec p\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U})$$

and

$$f_k(z)=z^p+\frac{p(p-1)(A-B)}{(p+k)(\alpha k+p-1)}z^{p+k}+\cdots$$

for each $k\geq n$, the proof of Theorem 6 is completed. □

Theorem 7. *Let $f(z) \in \mathcal{Q}_n(A, B, 0)$. Then, for $|z| = r < 1$,*

(i) if $M_n(A, B, \alpha, r) \geq 0$, we have

$$\begin{aligned}&\operatorname{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\}\\ &\geq \frac{p[p-1-((p-1)(A+B)+\alpha n(A-B))r^n + (p-1)ABr^{2n}]}{(p-1)(1-Br^n)^2};\end{aligned} \tag{2.32}$$

(ii) if $M_n(A, B, \alpha, r) \leq 0$, we have

$$\operatorname{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} \geq \frac{p(4\alpha^2 K_A K_B - L_n^2)}{4\alpha(p-1)(A-B)r^{n-1}(1-r^2)K_B}, \tag{2.33}$$

where

$$\begin{cases} K_A = 1 - A^2 r^{2n} - nAr^{n-1}(1-r^2), \\ K_B = 1 - B^2 r^{2n} - nBr^{n-1}(1-r^2), \\ L_n = 2\alpha(1-ABr^{2n}) - \alpha n(A+B)r^{n-1}(1-r^2) - (p-1)(A-B)r^{n-1}(1-r^2), \\ M_n(A, B, \alpha, r) = 2\alpha K_B(1-Ar^n) - L_n(1-Br^n). \end{cases} \tag{2.34}$$

The above results are sharp.

Proof. Equality in (2.32) occurs for $z = 0$. Thus we assume that $0 < |z| = r < 1$.

For $f(z) \in \mathcal{Q}_n(A, B, 0)$, we can write

$$\frac{f'(z)}{pz^{p-1}} = \frac{1 + Az^n\varphi(z)}{1 + Bz^n\varphi(z)} \quad (z \in \mathbb{U}), \tag{2.35}$$

where $\varphi(z)$ is analytic and $|\varphi(z)| \leq 1$ in $\mathbb{U}$. It follows from (2.35) that

$$\begin{aligned}&(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\\ &= \frac{f'(z)}{z^{p-1}} + \frac{\alpha p(A-B)(nz^n\varphi(z) + z^{n+1}\varphi'(z))}{(p-1)(1+Bz^n\varphi(z))^2}\\ &= \frac{f'(z)}{z^{p-1}} + \frac{\alpha np}{(p-1)(A-B)}\left(\frac{f'(z)}{pz^{p-1}} - 1\right)\left(A - B\frac{f'(z)}{pz^{p-1}}\right) + \frac{\alpha p(A-B)z^{n+1}\varphi'(z)}{(p-1)(1+Bz^n\varphi(z))^2}.\end{aligned} \tag{2.36}$$

By using the Carathéodory inequality:

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - r^2},$$

we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{z^{n+1}\varphi'(z)}{(1+Bz^n\varphi(z))^2}\right\} &\geq -\frac{r^{n+1}(1-|\varphi(z)|^2)}{(1-r^2)|1+Bz^n\varphi(z)|^2}\\ &= -\frac{r^{2n}|A - \frac{Bf'(z)}{pz^{p-1}}|^2 - |\frac{f'(z)}{pz^{p-1}} - 1|^2}{(A-B)^2 r^{n-1}(1-r^2)}.\end{aligned} \tag{2.37}$$

Put $\frac{f'(z)}{pz^{p-1}} = u + iv \quad (u, v \in \mathbb{R})$. Then (2.36) and (2.37), together, yield

$$\begin{aligned}
\operatorname{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} &\geq p\left(1 + \frac{\alpha n(A+B)}{(p-1)(A-B)}\right)u - \frac{\alpha npA}{(p-1)(A-B)} \\
&- \frac{\alpha npB}{(p-1)(A-B)}(u^2 - v^2) - \frac{\alpha p[r^{2n}((A-Bu)^2 + (Bv)^2) - ((u-1)^2 + v^2)]}{(p-1)(A-B)r^{n-1}(1-r^2)} \\
&= p\left(1 + \frac{\alpha n(A+B)}{(p-1)(A-B)}\right)u - \frac{\alpha np}{(p-1)(A-B)}(A + Bu^2) - \frac{\alpha p(r^{2n}(A-Bu)^2 - (u-1)^2)}{(p-1)(A-B)r^{n-1}(1-r^2)} \\
&+ \frac{\alpha p}{(p-1)(A-B)}\left(nB + \frac{1 - B^2 r^{2n}}{r^{n-1}(1-r^2)}\right)v^2. \qquad (2.38)
\end{aligned}$$

We note that

$$\begin{aligned}
\frac{1 - B^2 r^{2n}}{r^{n-1}(1-r^2)} \geq \frac{1 - r^{2n}}{r^{n-1}(1-r^2)} &= \frac{1}{r^{n-1}}\left(1 + r^2 + r^4 + \cdots + r^{2(n-2)} + r^{2(n-1)}\right) \\
&= \frac{1}{2r^{n-1}}\left[(1 + r^{2(n-1)}) + (r^2 + r^{2(n-2)}) + \cdots + (r^{2(n-1)} + 1)\right] \\
&\geq n \geq -nB. \qquad (2.39)
\end{aligned}$$

Combining (2.38) and (2.39), we have

$$\begin{aligned}
\operatorname{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} &\geq p\left(1 + \frac{\alpha n(A+B)}{(p-1)(A-B)}\right)u - \frac{\alpha np}{(p-1)(A-B)}(A + Bu^2) \\
&+ \frac{\alpha p((u-1)^2 - r^{2n}(A-Bu)^2)}{(p-1)(A-B)r^{n-1}(1-r^2)} \\
&=: \psi_n(u). \qquad (2.40)
\end{aligned}$$

Also, (2.10) and (2.35) imply that

$$\frac{1 - Ar^n}{1 - Br^n} \leq u = \operatorname{Re}\left(\frac{f'(z)}{pz^{p-1}}\right) \leq \frac{1 + Ar^n}{1 + Br^n}.$$

We now calculate the minimum value of $\psi_n(u)$ on the segment $\left[\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right]$. Obviously, we get

$$\psi_n'(u) = p\left(1 + \frac{\alpha n(A+B)}{(p-1)(A-B)}\right) - \frac{2\alpha npB}{(p-1)(A-B)}u + \frac{2\alpha p((1 - B^2 r^{2n})u - (1 - ABr^{2n}))}{(p-1)(A-B)r^{n-1}(1-r^2)},$$

$$\psi_n''(u) = \frac{2\alpha p}{(p-1)(A-B)}\left(\frac{1 - B^2 r^{2n}}{r^{n-1}(1-r^2)} - nB\right) \geq \frac{2\alpha np(1-B)}{(p-1)(A-B)} > 0 \quad \text{(see (2.36))} \qquad (2.41)$$

and $\psi_n'(u) = 0$ if and only if

$$\begin{aligned}
u = u_n &= \frac{2\alpha(1 - ABr^{2n}) - \alpha n(A+B)r^{n-1}(1-r^2) - (p-1)(A-B)r^{n-1}(1-r^2)}{2\alpha(1 - B^2 r^{2n} - nBr^{n-1}(1-r^2))} \\
&= \frac{L_n}{2\alpha K_B} \quad \text{(see (2.31))}. \qquad (2.42)
\end{aligned}$$

Since

$$
\begin{aligned}
&2\alpha K_B(1+Ar^n)-L_n(1+Br^n)\\
&\quad=2\alpha\left[(1+Ar^n)(1-B^2r^{2n})-(1+Br^n)(1-ABr^{2n})\right]\\
&\qquad+\alpha nr^{n-1}(1-r^2)\left[(A+B)(1+Br^n)-2B(1+Ar^n)\right]+(p-1)(A-B)r^{n-1}(1-r^2)(1+Br^n)\\
&\quad=2\alpha(A-B)r^n(1+Br^n)+\alpha n(A-B)r^{n-1}(1-r^2)(1-Br^n)+(p-1)(A-B)r^{n-1}(1-r^2)(1+Br^n)\\
&\quad>0,
\end{aligned}
$$

we see that

$$u_n<\frac{1+Ar^n}{1+Br^n}. \tag{2.43}$$

But u_n is not always greater than $\frac{1-Ar^n}{1-Br^n}$. The following two cases arise.

(i) $u_n\le\frac{1-Ar^n}{1-Br^n}$, that is, $M_n(A,B,\alpha,r)\ge 0$ (see (2.34)). In view of $\psi_n'(u_n)=0$ and (2.41), the function $\psi_n(u)$ is increasing on the segment $\left[\frac{1-Ar^n}{1-Br^n},\frac{1+Ar^n}{1+Br^n}\right]$. Therefore, we deduce from (2.40) that, if $M_n(A,B,\alpha,r)\ge 0$, then

$$
\begin{aligned}
&\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z)+\frac{\alpha}{p-1}z^{2-p}f''(z)\right\}\ge\psi_n\left(\frac{1-Ar^n}{1-Br^n}\right)\\
&\quad=p\left(1+\frac{\alpha n(A+B)}{(p-1)(A-B)}\right)\left(\frac{1-Ar^n}{1-Br^n}\right)-\frac{\alpha np}{(p-1)(A-B)}\left(A+B\left(\frac{1-Ar^n}{1-Br^n}\right)^2\right)\\
&\quad=p\frac{1-Ar^n}{1-Br^n}-\frac{\alpha np}{(p-1)(A-B)}\left(1-\frac{1-Ar^n}{1-Br^n}\right)\left(A-B\frac{1-Ar^n}{1-Br^n}\right)\\
&\quad=\frac{p[p-1-((p-1)(A+B)+\alpha n(A-B))r^n+(p-1)ABr^{2n}]}{(p-1)(1-Br^n)^2}.
\end{aligned}
$$

This proves (2.32).

Next we consider the function $f(z)$ given by

$$f(z)=p\int_0^z t^{p-1}\frac{1-At^n}{1-Bt^n}dt\in\mathcal{Q}_n(A,B,0).$$

It is easy to find that

$$(1-\alpha)r^{1-p}f'(r)+\frac{\alpha}{p-1}r^{2-p}f''(r)=\frac{p[p-1-((p-1)(A+B)+\alpha n(A-B))r^n+(p-1)ABr^{2n}]}{(p-1)(1-Br^n)^2},$$

which shows that the inequality (2.32) is sharp.

(ii) $u_n\ge\frac{1-Ar^n}{1-Br^n}$, that is, $M_n(A,B,\alpha,r)\le 0$. In this case, we easily see that

$$\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z)+\frac{\alpha}{p-1}z^{2-p}f''(z)\right\}\ge\psi_n(u_n). \tag{2.44}$$

In view of (2.34), $\psi_n(u)$ in (2.40) can be written as follows:

$$\psi_n(u) = \frac{p(\alpha K_B u^2 - L_n u + \alpha K_A)}{(p-1)(A-B)r^{n-1}(1-r^2)}. \tag{2.45}$$

Therefore, if $M_n(A, B, \alpha, r) \leq 0$, then it follows from (2.42), (2.44) and (2.45) that

$$\begin{aligned}\mathrm{Re}\left\{(1-\alpha)z^{1-p}f'(z) + \frac{\alpha}{p-1}z^{2-p}f''(z)\right\} &\geq \frac{p(\alpha K_B u_n^2 - L_n u_n + \alpha K_A)}{(p-1)(A-B)r^{n-1}(1-r^2)} \\ &= \frac{p(4\alpha^2 K_A K_B - L_n^2)}{4\alpha(p-1)(A-B)r^{n-1}(1-r^2)K_B}.\end{aligned}$$

To show that the inequality (2.33) is sharp, we take

$$f(z) = p\int_0^z t^{p-1}\frac{1+At^n\varphi(t)}{1+Bt^n\varphi(t)}dt \quad \text{and} \quad \varphi(z) = -\frac{z-c_n}{1-c_n z} \quad (z \in \mathbb{U}),$$

where $c_n \in \mathbb{R}$ is determined by

$$\frac{f'(r)}{pr^{p-1}} = \frac{1+Ar^n\varphi(r)}{1+Br^n\varphi(r)} = u_n \in \left[\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right).$$

Clearly, $-1 \leq \varphi(r) < 1$, $-1 \leq c_n < 1$, $|\varphi(z)| \leq 1$ $(z \in \mathbb{U})$, and so $f(z) \in Q_n(A, B, 0)$. Since

$$\varphi'(r) = -\frac{1-c_n^2}{(1-c_n r)^2} = -\frac{1-|\varphi(r)|^2}{1-r^2},$$

from the above argument we obtain that

$$(1-\alpha)r^{1-p}f'(r) + \frac{\alpha}{p-1}r^{2-p}f''(r) = \psi_n(u_n).$$

The proof of Theorem 7 is completed. □

3. Conclusions

In this paper, we have introduced and investigated some geometric properties of the class $\mathcal{G}_n(A, B, \lambda)$ which is defined by using the principle of first-order differential subordination. For this function class, we have derived the sharp upper bound on $|z| = r < 1$ for the following functional:

$$\mathrm{Re}\left\{(1-\lambda)z^p f(z) - \frac{\lambda}{p}z^{p+1}f'(z)\right\}$$

over the class $\mathcal{G}_n(A, B, 0)$. We have also obtained other properties of the function class $\mathcal{G}_n(A, B, \lambda)$.

Motivated by a recently-published survey-cum-expository review article by Srivastava [15], the interested reader's attention is drawn toward the possibility of investigating the basic (or q-) extensions of the results which are presented in this paper. However, as already pointed out by Srivastava, their further extensions using the so-called (p, q)-calculus will be rather trivial and inconsequential variations of the suggested extensions which are based upon the classical q-calculus, the additional paremeter p being redundant or superfluous (see, for details, [15, p. 340]).

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Conflicts of interest

The authors agree with the contents of the manuscript, and there are no conflicts of interest among the authors.

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