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Research article

Multiple solutions for nonlocal elliptic problems driven by $p(x)$-biharmonic operator

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Abstract: In this article, we study the existence of at least three distinct weak solutions for nonlocal elliptic problems involving $p(x)$-biharmonic operator. The results are obtained by means of variational methods. We also provide an example in order to illustrate our main abstract results. We extend and improve some recent results.

Keywords: $p(x)$-biharmonic operator; nonlocal elliptic problem; three solutions; variational methods
Mathematics Subject Classification: 35J20, 35J60, 47J30

1. Introduction

In this paper we study the existence of at least three distinct weak solutions for the following problem

$$\begin{cases} T(u) = \lambda f(x, u(x)), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_λ^f}$$

where

$$T(u) = \Delta^2_{p(x)} u(x) - M\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)} dx}{p(x)}\right) \Delta_{p(x)} u(x) + \rho(x)|u(x)|^{p(x)-2} u(x),$$

$\Omega \subset \mathbb{R}^N (N \geq 2)$ is an open bounded domain with smooth boundary, $\Delta^2_{p(x)} u$ is the operator defined as $\Delta(|\Delta u|^{p(x)-2}\Delta u)$ and is called the $p(x)$-biharmonic which is a generalization of the p-biharmonic, $p(x) \in C(\overline{\Omega})$, $\rho(x) \in L^\infty(\Omega)$, $M : [0, +\infty) \to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \leq M(t) \leq m_1$ for all $t \geq 0$, $\dfrac{N}{2} < p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x) \leq p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty$, $\lambda > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function.

The Kirchhoff equation refers back to Kirchhoff [18] in 1883 in the study on the oscillations of stretched strings and plates, suggested as an extended version of the classical D'Alembert's wave equation by taking into account the effects of the changes in the length of the string during the vibrations. Kirchhoff's equation like problem (P^f_λ) model several physical and biological systems where u describes a process which depend on the average of itself. Lions in [23] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions, various problems of Kirchhoff-type have been widely investigated, we refer the reader to the papers [7, 24, 27] and the references therein.

The main interest in studying problem (P^f_λ) is given by the presence of the variable exponent $p(\cdot)$. Problems involving such kind of growth conditions benefited by a special attention in the last decade since they can model with sufficient accuracy phenomena arising in different branches of science. Two important models where operators involving variable exponents were considered come from the study of electrorheological fluids [8, 28] and elastic mechanics [34].

Fourth-order equations have various applications in areas of applied mathematics and physics such as micro-electro-mechanical systems, phase field models of multi-phase systems, thin film theory, thin plate theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells (see [4, 6, 26]). The fourth-order equation can also describe the static form change of beam or the sport of rigid body. In [22], Lazer and Mckenna have pointed out that this type of nonlinearity furnishes a model to study travelling waves in suspension bridges. Numerous authors investigated the existence and multiplicity of solutions for the problems involving $p(x)$-biharmonic operators. We refer to [10,12,16,19,21,30,31]. In the last decade, Kirchhoff type equations involving the $p(x)$-Laplacian have been investigated, for instance see [3, 9, 13–15, 17, 25].

In this paper, we are interested to discuss the existence of at least three distinct weak solutions for problem (P^f_λ). No asymptotic condition at infinity is required on the nonlinear term. In Theorem 3.1 we establish the existence of at least three distinct weak solutions for problem (P^f_λ). Theorem 3.3 is a consequence of Theorem 3.1. As a consequence of Theorem 3.3, we obtain Theorem 3.4 for the autonomous case. We present example 3.5 to illustrate Theorem 3.4.

2. Preliminaries

Let X be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r,\ r_1,\ r_2 > \inf_X \Phi,\ r_2 > r_1,\ r_3 > 0$, we define

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1, r_2) := \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3},$$

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

We shall discuss the existence of at least three distinct solutions to the problem (P^f_λ). Our main tool is based on [1, Theorem 3.3] that we now recall as follows:

Theorem 2.1. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse* *on $X^*, \Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$ *and for every $u_1, u_2 \in X$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

(c_1) $\varphi(r_1) < \beta(r_1, r_2)$;
(c_2) $\varphi(r_2) < \beta(r_1, r_2)$;
(c_3) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then for each $\lambda \in \left(\dfrac{1}{\beta(r_1, r_2)}, \dfrac{1}{\alpha(r_1, r_2, r_3)}\right)$ *the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$ and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.*

We refer the interested reader to the papers [2, 11, 20] in which Theorem 2.1 has been successfully used to ensure the existence of at least three solutions for boundary value problems.

Let Ω be a bounded domain of $\mathbb{R}^N$, denote:

$$L^{p(x)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\right\}.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by:

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\beta > 0 : \int_\Omega |\frac{u(x)}{\beta}|^{p(x)}dx \leq 1\right\}.$$

Let X be the generalized Lebesgue-Sobolev space $W^{m,p(x)}(\Omega)$ defined by putting $W^{m,p(x)}(\Omega)$ as

$$W^{m,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega)|D^\gamma u \in L^{p(x)}(\Omega),\ |\gamma| \leq m,\ m \in Z_+\right\},$$

which is equipped with the norm:

$$\|u\|_{m,p(x)} := \sum_{|\gamma|\leq m} |D^\gamma u|_{p(x)} \tag{2.1}$$

γ is the multi-index and $|\gamma|$ is the order.

The closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$ is the $W_0^{m,p(x)}(\Omega)$. It is well known [5] that, both $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces.

Proposition 2.2. *[5] Suppose* $\dfrac{1}{p(x)} + \dfrac{1}{p^0(x)} = 1$, *then $L^{p^0(x)}(\Omega)$ and $L^{p(x)}(\Omega)$ are conjugate space, and satisfy the Hölder inequality:*

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^0)^-}\right)|u|_{p(x)}|v|_{p^0(x)}, \quad u \in L^{p(x)}(\Omega),\ v \in L^{p^0(x)}(\Omega).$$

We denote $X := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$ and has the norm

$$\|u\| = \inf\left\{\sigma > 0 : \int_\Omega \left(\left|\frac{u(x)}{\sigma}\right|^{p(x)} + \left|\frac{\nabla u(x)}{\sigma}\right|^{p(x)} + \left|\frac{\Delta u(x)}{\sigma}\right|^{p(x)}\right) dx \leq 1\right\}.$$

By [32], $\|\cdot\|$, $\|\cdot\|_{2,p(\cdot)}$ and $|\Delta u|_{p(\cdot)}$ are equivalent norms of X.

A bounded operator $T : X \to \mathbb{R}$ is said to be compact if $T(B_X)$ has compact closure in $\mathbb{R}$.

Proposition 2.3. *[31] When* $p^- > \dfrac{N}{2}$, $\Omega \subset \mathbb{R}$ *is a bounded region, then* $X \mapsto C(\overline{\Omega})$ *is a compact embedding.*

According to 2.3, for each $u \in X$, there exists a constant $c > 0$ that depends on $p(\cdot), N, \Omega$:

$$||u||_\infty = \sup_{x\in\Omega} |u(x)| \le c||u||. \tag{2.2}$$

Remark 2.4. *We say that* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is an* L^1*-Carathéodory function if*

(*a*) $t \mapsto f(x, t)$ *is measurable for every* $t \in \mathbb{R}$;
(*b*) $x \mapsto f(x, t)$ *is continuous for a.e.* $x \in \Omega$;
(*c*) *for every* $\varepsilon > 0$ *there exists a function* $l_\varepsilon \in L^1(\Omega)$ *such that for a.e.* $x \in \Omega$,

$$\sup_{|t|\le\varepsilon} |f(x, t)| \le l_\varepsilon(x).$$

Corresponding to the functions f and M, we introduce the functions $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{M} : [0, +\infty) \to \mathbb{R}$, respectively, as follows

$$F(x, t) = \int_0^t f(x, \xi)d\xi \ \text{ for all } (x, t) \in \Omega \times \mathbb{R},$$

$$\widetilde{M}(t) = \int_0^t M(\xi)d\xi \ \text{ for all } t \ge 0.$$

We say that $u \in X$ is a weak solution of problem (P^f_λ) if for every $v \in X$,

$$\int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)dx + M\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)dx$$
$$+ \int_\Omega \rho(x)|u(x)|^{p(x)-2}u(x)v(x)dx - \lambda \int_\Omega f(x, u(x))v(x)dx = 0.$$

Proposition 2.5. *[5] Let* $J(u) = \int_\Omega |u|^{p(x)}dx$ *for each* $u \in L^{p(x)}(\Omega)$*, we have*

(1) $|u|_{p(x)} < 1(= 1; > 1) \Leftrightarrow J(u) < 1(= 1; > 1)$;
(2) $|u|_{p(x)} \ge 1 \Longrightarrow |u|^{p^-}_{p(x)} \le J(u) \le |u|^{p^+}_{p(x)}$;
(3) $|u|_{p(x)} \le 1 \Longrightarrow |u|^{p^+}_{p(x)} \le J(u) \le |u|^{p^-}_{p(x)}$;
(4) $|u|_{p(x)} \longrightarrow 0 \Leftrightarrow J \longrightarrow 0$.

Now for every $u \in X$, we define $I(u) := \Phi(u) - \lambda\Psi(u)$ where

$$\Phi(u) = \int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}dx + \widetilde{M}\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right) + \int_\Omega \frac{\rho(x)|u(x)|^{p(x)}}{p(x)}dx, \tag{2.3}$$

and

$$\Psi(u) = \int_\Omega F(x, u(x))dx. \tag{2.4}$$

For our convenience, set

$$\rho_0 = \min_{x\in\Omega} \rho(x), \quad M^- = \min\{1, m_0, \rho_0\} \quad \text{and} \quad M^+ = \max\{1, m_1, \rho_\infty\}.$$

Proposition 2.6. *Let $T = \Phi' : X \to X^*$ be the operator defined by*

$$T(u)(v) = \int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)dx + M\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)dx + \int_\Omega \rho(x)|u(x)|^{p(x)-2}u(x)v(x)$$

for every $u, v \in X$. Then T admits a continuous inverse on X^*.

Proof. For any $u \in X \setminus \{0\}$,

$$\lim_{\|u\|\to\infty} \frac{\langle T(u), u\rangle}{\|u\|} = \lim_{\|u\|\to\infty} \frac{\int_\Omega |\Delta u(x)|^{p(x)}dx + M\left(\int_\Omega \frac{\nabla u(x)|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u(x)|^{p(x)}dx + \int_\Omega \rho(x)|u(x)|^{p(x)}dx}{\|u\|}$$
$$\geq \lim_{\|u\|\to\infty} \frac{M^-\|u\|^{p^-}}{\|u\|} = \lim_{\|u\|\to\infty} M^-\|u\|^{p^- -1},$$

since $p^- > 1$, it follows that the map T is coercive. Since T is the Fréchet derivative of Φ, it follows that T is continuous and bounded. Using the elementary inequality [29]

$$|x-y|^\gamma \leq 2^\gamma\left(|x|^{\gamma-2}x - |y|^{\gamma-2}y\right)(x-y) \quad \text{if } \gamma \geq 2,$$

for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $N \geq 1$, we obtain for all $u, v \in X$ such that $u \neq v$,

$$\langle T(u) - T(v), u - v\rangle > 0,$$

which means that T is strictly monotone. Thus T is injective. Consequently, thanks to Minty-Browder theorem [33], the operator T is a surjection and admits an inverse mapping. Thus it is sufficient to show that T^{-1} is continuous. For this, let $(v_n)_{n=1}^\infty$ be a sequence in X^* such that $v_n \to v$ in X^*. Let u_n and u in X such that

$$T^{-1}(v_n) = u_n \quad \text{and} \quad T^{-1}(v) = u.$$

By the coercivity of T, we conclude that the sequence (u_n) is bounded in the reflexive space X. For a subsequence, we have $u_n \to \tilde{u}$ in X, which implies

$$\lim_{n\to\infty}\langle T(u_n) - T(u), u_n - \tilde{u}\rangle = \lim_{n\to\infty}\langle f_n - f, u_n - \tilde{u}\rangle = 0.$$

Therefore, by the continuity of T, we have

$$u_n \to \tilde{u} \quad \text{in } X \quad \text{and} \quad T(u_n) \to T(\tilde{u}) = T(u) \quad \text{in } X^*.$$

Moreover, since T is an injection, we conclude that $u = \tilde{u}$. □

3. Main results

Fix $x^0 \in \Omega$ and choose $s > 0$ such that $B(x^0, s) \subset \Omega$, where $B(x^0, s)$ denotes the ball with center at x^0 and radius of s. Put

$$\Theta_1 := \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_{\frac{s}{2}}^{s}\left|\frac{12(N+1)}{s^3}r - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{r}\right|^{p(x)} r^{N-1}dr,$$

$$\Theta_2 := \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\sum_{i=1}^{N}\left(\frac{12(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9(x_i - x_i^0)}{s\ell}\right)^2\right]^{\frac{p(x)}{2}} dx,$$

where $\ell = dist(x, x^0) = \sqrt{\sum_{i=1}^{N}\left(x_i - x_i^0\right)^2}$ and

$$\Theta_3 := \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left[\frac{(\frac{s}{2})^N}{N} + \int_{\frac{s}{2}}^{s}\left|\frac{4}{s^3}r^3 - \frac{12}{s^2}r^2 + \frac{9}{s}r - 1\right|^{p(x)} r^{N-1}dr\right],$$

Γ denotes the Gamma function, and

$$L := \Theta_1 + \Theta_2 + \Theta_3.$$

Theorem 3.1. *Assume that there exist positive constants θ_1, θ_2, θ_3 and $\eta \geq 1$ with $\theta_1 < \sqrt[p^-]{L}c\eta$, $\eta < \min\left\{\sqrt[p^+]{\frac{p^-M^-}{p^+c^{p^-}M^+L}}\theta_2^{\frac{p^-}{p^+}}, \theta_2\right\}$ and $\theta_2 < \theta_3$ such that*

(A_1) *$f(x,t) \geq 0$ for each $(x,t) \in \overline{\Omega} \setminus B(x^0, \frac{s}{2}) \times [-\theta_3, \theta_3]$;*
(A_2)

$$\max\left\{\frac{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}{\theta_1^{p^-}}, \frac{\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}{\theta_2^{p^-}}, \frac{\int_\Omega \sup_{|t|\leq\theta_3} F(x,t)dx}{\theta_3^{p^-} - \theta_2^{p^-}}\right\}$$
$$< \frac{p^-M^-}{p^+c^{p^-}M^+L}\frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}{\eta^{p^+}}.$$

Then for every

$$\lambda \in \left(\frac{\frac{M^+L}{p^-}\eta^{p^+}}{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx},\right.$$
$$\left.\frac{M^-}{p^+c^{p^-}}\min\left\{\frac{\theta_1^{p^-}}{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}, \frac{\theta_2^{p^-}}{\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}, \frac{\theta_3^{p^-} - \theta_2^{p^-}}{\int_\Omega \sup_{|t|\leq\theta_3} F(x,t)dx}\right\}\right),$$

problem (P_λ^f) has at least three weak solutions u_1, u_2 and u_3 such that

$$\max_{x\in\Omega}|u_1(x)| < \theta_1, \ \max_{x\in\Omega}|u_2(x)| < \theta_2 \quad \text{and} \quad \max_{x\in\Omega}|u_3(x)| < \theta_3.$$

Proof. Our goal is to apply Theorem 2.1 to the problem (P_λ^f). We consider the auxiliary problem

$$\begin{cases} T(u) = \lambda\hat{f}(x,u(x)), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \qquad (P_\lambda^{\hat{f}})$$

where $\hat{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function defined as

$$\hat{f}(x,\xi) = \begin{cases} f(x,0), & \text{if } \xi < -\theta_3, \\ f(x,\xi), & \text{if } -\theta_3 \le \xi \le \theta_3, \\ f(x,\theta_3), & \text{if } \xi > \theta_3. \end{cases}$$

If a weak solution of the problem $(P_\lambda^{\hat{f}})$ satisfies the condition $-\theta_3 \le u(x) \le \theta_3$ for every $x \in \Omega$, then, clearly it turns to be also a weak solution of (P_λ^f). Therefore, it is enough to show that our conclusion holds for (P_λ^f). We define functionals Φ and Ψ as given in (2.3) and (2.4), respectively. Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx$$

for every $v \in X$, as well as it is sequentially weakly upper semicontinuous. Recalling (2.1), we have

$$\Phi(u) \ge \frac{1}{p^+}\int_\Omega |\Delta u(x)|^{p(x)}dx + m_0\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right) + \int_\Omega \rho(x)|u(x)|^{p(x)}dx \ge \frac{M^-}{p^+}\|u\|^{p^-}$$

for all $u \in X$ with $\|u\| > 1$, which implies Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)dx + M\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)dx$$
$$+ \int_\Omega \rho(x)|u(x)|^{p(x)-2}u(x)v(x)dx$$

for every $v \in X$, while Proposition 2.6 gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested of Theorem 2.1, are verified. Define w by setting

$$w(x) := \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0,s) \\ \eta\left(\frac{4}{s^3}\ell^3 - \frac{12}{s^2}\ell^2 + \frac{9}{s}\ell - 1\right) & \text{if } x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}) \\ d & \text{if } x \in B(x^0,\frac{s}{2}). \end{cases} \tag{3.1}$$

It is easy to see that $w \in X$ and,

$$\frac{\partial w(x)}{\partial x_i} = \begin{cases} 0 & \text{if } \ x \in \overline{\Omega} \setminus B(x^0,s) \cup B(x^0,\frac{s}{2}) \\ \eta\left(\frac{12\ell(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{\ell}\right) & \text{if } \ x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}) \end{cases}$$

and

$$\frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } \ x \in \overline{\Omega} \setminus B(x^0,s) \cup B(x^0,\frac{s}{2}) \\ \eta\left(\frac{12}{s^3}\frac{(x_i - x_i^0)^2 + \ell^2}{\ell} - \frac{24}{s^2} + \frac{9}{s}\frac{\ell^2 - (x_i - x_i^0)^2}{\ell^3}\right) & \text{if } \ x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}), \end{cases}$$

and so that

$$\sum_{i=1}^{N}\frac{\partial^2 w(x)}{\partial x_i^2}=\begin{cases}0 & \text{if } \ x\in\overline{\Omega}\setminus B(x^0,s)\cup B(x^0,\frac{s}{2})\\ \eta\left(\frac{12l(N+1)}{s^3}-\frac{24N}{s^2}+\frac{9}{s}\frac{N-1}{\ell}\right) & \text{if } \ x\in B(x^0,s)\setminus B(x^0,\frac{s}{2}).\end{cases}$$

It is easy to see that $w\in X$ and, in particular, since

$$\int_\Omega |\Delta w(x)|^p dx \le \eta^{p^+}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_{\frac{s}{2}}^{s}\left|\frac{12(N+1)}{s^3}r-\frac{24N}{s^2}+\frac{9(N-1)}{s}\frac{1}{r}\right|^{p(x)}r^{N-1}dr,$$

$$\int_\Omega |\nabla w(x)|^p dx = \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})}\left[\sum_{i=1}^{N}\eta^2\left(\frac{12l(x_i-x_i^0)}{s^3}-\frac{24(x_i-x_i^0)}{s^2}+\frac{9}{s}\frac{(x_i-x_i^0)}{l}\right)^2\right]^{\frac{p(x)}{2}}dx$$

$$\le \eta^{p^+}$$

$$\times\int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})}\left[\sum_{i=1}^{N}\left(\frac{12l(x_i-x_i^0)}{s^3}-\frac{24(x_i-x_i^0)}{s^2}+\frac{9}{s}\frac{(x_i-x_i^0)}{l}\right)^2\right]^{\frac{p(x)}{2}}dx$$

and

$$\int_\Omega |w(x)|^p dx \le \eta^{p^+}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left(\frac{(\frac{s}{2})^N}{N}+\int_{\frac{s}{2}}^{s}\left|\frac{4}{s^3}r^3-\frac{12}{s^2}r^2+\frac{9}{s}r-1\right|^{p(x)}r^{N-1}dr\right).$$

In particular, one has

$$\begin{aligned}\frac{M^-L}{p^+}\eta^{p^-} &\le\frac{1}{p^+}\left(\Theta_1\eta^{p^-}+m_0\Theta_2\eta^{p^-}+\rho_0\Theta_3\eta^{p^-}\right)\le\Phi(w)\\ &\le\frac{1}{p^-}\left(\Theta_1\eta^{p^+}+m_1\Theta_2\eta^{p^+}+\rho_\infty\Theta_3\eta^{p^+}\right)\le\frac{M^+L}{p^-}\eta^{p^+}.\end{aligned}$$

On the other hand, bearing (A_1) in mind, from the definition of Ψ, we infer

$$\Psi(w)=\int_\Omega F(x,w(x))dx\ge\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx.$$

Choose $r_1=\frac{M^-}{p^+}\left(\frac{\theta_1}{c}\right)^{p^-}$, $r_2=\frac{M^-}{p^+}\left(\frac{\theta_2}{c}\right)^{p^-}$ and $r_3=\frac{M^-}{p^+}\left(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{c^{p^-}}\right)$. From the conditions

$$\theta_1<\sqrt[p^-]{L}c\eta,\quad \sqrt[p^-]{\frac{p^+M^+L}{p^-M^-}}c\eta^{\frac{p^+}{p^-}}<\theta_2$$

and $\theta_2<\theta_3$, we achieve $r_1<\Phi(w)<r_2$ and $r_3>0$. For all $u\in X$ with $\Phi(u)<r_1$, taking (2.1) and (2.2) into account, one has

$$\|u\|\le\max\left\{(p^+r_1)^{\frac{1}{p^+}},(p^+r_1)^{\frac{1}{p^-}}\right\}.$$

So, thanks to the embedding $X\hookrightarrow C^0(\overline{\Omega})$, one has $\|u\|_\infty<\theta_1$. From the definition of r_1, it follows that

$$\Phi^{-1}(-\infty,r_1)=\{u\in X;\Phi(u)<r_1\}\subseteq\{u\in X;|u|_\infty\le\theta_1\}.$$

Hence, one has

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)}\int_\Omega F(x,u(x))dx \le \int_\Omega \sup_{|t|\le\theta_1} F(x,t)dx.$$

As above, we can obtain that

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)}\int_\Omega F(x,u(x))dx \le \int_\Omega \sup_{|t|\le\theta_2} F(x,t)dx$$

and

$$\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\int_\Omega F(x,u(x))dx \le \int_\Omega \sup_{|t|\le\theta_3} F(x,t)dx.$$

Therefore, since $0\in\Phi^{-1}(-\infty,r_1)$ and $\Phi(0)=\Psi(0)=0$, one has

$$\begin{aligned}\varphi(r_1) &= \inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{(\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u))-\Psi(u)}{r_1-\Phi(u)}\\ &\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1}\\ &= \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_1)}\int_\Omega F(x,u(x))dx}{r_1} \le \frac{\int_\Omega \sup\limits_{|t|\le\theta_1} F(x,t)dx}{\frac{M^-}{p^+}\left(\frac{\theta_1}{c}\right)^{p^-}},\end{aligned}$$

$$\varphi(r_2)\le\frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2}=\frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2)}\int_\Omega F(x,u(x))dx}{r_2}\le\frac{\int_\Omega \sup\limits_{|t|\le\theta_2} F(x,t)dx}{\frac{M^-}{p^+}\left(\frac{\theta_2}{c}\right)^{p^-}},$$

and

$$\gamma(r_2,r_3)\le\frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3}=\frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\int_\Omega F(x,u(x))dx}{r_3}\le\frac{\int_\Omega \sup\limits_{|t|\le\theta_3} F(x,t)dx}{\frac{M^-}{p^+}\left(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{c^{p^-}}\right)}.$$

On the other hand, for each $u\in\Phi^{-1}(-\infty,r_1)$ one has

$$\beta(r_1,r_2)\ge\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx-\int_\Omega\sup\limits_{|t|\le\theta_1}F(x,t)dx}{\Phi(w)-\Phi(u)}\ge\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx-\int_\Omega\sup\limits_{|t|\le\theta_1}F(x,t)dx}{\frac{M^+L}{p^-}\eta^{p^+}}.$$

Due to (A_2) we get

$$\alpha(r_1,r_2,r_3)<\beta(r_1,r_2).$$

Therefore, (b_1) and (b_2) of Theorem 2.1 are verified. Finally, we verify that $\Phi - \lambda\Psi$ satisfies the assumption 2 of Theorem 2.1. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions of the problem (P_λ^f). Since we assumed f is nonnegative, for fixed $\lambda > 0$, we have $\lambda f(k, su_1 + (1-s)u_2) \geq 0$ for all $s \in [0,1]$, and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$ for every $s \in [0,1]$. Hence, Theorem 2.1 implies that for every

$$\lambda \in \left(\frac{\frac{M^+ L}{p^-}\eta^{p^+}}{\int_{B(x^0, \frac{s}{2})} F(x,\eta)dx - \int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}, \right.$$

$$\left. \frac{M^-}{p^+ c^{p^-}} \min\left\{ \frac{\theta_1^{p^-}}{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}, \frac{\theta_2^{p^-}}{\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}, \frac{\theta_3^{p^-} - \theta_2^{p^-}}{\int_\Omega \sup_{|t|\leq\theta_3} F(x,t)dx} \right\} \right),$$

the functional $\Phi - \lambda\Psi$ has three critical points $u_i,\ i = 1,2,3$, in X such that $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$ and $\Phi(u_3) < r_2 + r_3$, that is,

$$\max_{x\in\Omega} |u_1(x)| < \theta_1, \ \max_{x\in\Omega} |u_2(x)| < \theta_2 \quad \text{and} \quad \max_{x\in\Omega} |u_3(x)| < \theta_3.$$

Then, taking into account the fact that the solutions of the problem (P_λ^f) are exactly critical points of the functional $\Phi - \lambda\Psi$ we have the desired conclusion. □

Remark 3.2. *If f is non-negative, then the weak solution ensured in Theorem 3.1 is non-negative. Indeed, let u_0 be the weak solution of the problem (P_λ^f) ensured in Theorem 3.1, then u_0 is nonnegative. Arguing by a contradiction, assume that the set $\mathcal{A} = \{x \in \Omega : u_0(x) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(x) = \min\{0, u_0(x)\}$ for all $x \in \Omega$. Clearly, $\bar{v} \in X$ and one has*

$$\int_\Omega |\Delta u_0(x)|^{p(x)-2}\Delta u_0(x)\Delta\bar{v}(x)dx + M\left(\int_\Omega \frac{|\nabla u_0(x)|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u_0(x)|^{p(x)-2}\nabla u_0(x)\nabla\bar{v}(x)dx$$
$$+ \int_\Omega \rho(x)|u_0(x)|^{p(x)-2}u_0(x)\bar{v}(x)dx - \lambda\int_\Omega f(x,u_0(x))\bar{v}(x)dx = 0$$

for every $\bar{v} \in X$. Thus we have

$$0 \leq M^-\|u\|_{(\mathcal{A})} \leq \int_{\mathcal{A}} |\Delta u_0(x)|^{p(x)} + M\left(\int_{\mathcal{A}} \frac{|\nabla u_0(x)|^{p(x)}}{p(x)}dx\right)\int_{\mathcal{A}} |\nabla u_0(x)|^{p(x)}dx$$
$$+ \int_{\mathcal{A}} \rho(x)|u_0(x)|^{p(x)}dx = \lambda\int_{\mathcal{A}} f(x,u_0(x))u_0(x)dx \leq 0,$$

i.e.,

$$\|u_0\|_{(\mathcal{A})} \leq 0$$

which contradicts with this fact that u_0 is a non-trivial weak solution. Hence, the set $\mathcal{A}$ is empty, and u_0 is positive.

Theorem 3.3. *Assume that there exist positive constants* θ_1, θ_4 *and* $\eta \geq 1$ *with* $\theta_1 < \min\left\{\eta^{\frac{p^+}{p^-}}, \sqrt[p^-]{Lc\eta}\right\}$ *and* $\eta < \min\left\{\sqrt[p^+]{\frac{p^- M^-}{2c^{p^-} p^+ M^+ L}}\theta_4^{\frac{p^-}{p^+}}, \theta_4\right\}$ *such that*

(A_3) $f(x,t) \geq 0$ *for each* $(x,t) \in \overline{\Omega} \setminus B(x^0, \frac{s}{2}) \times [-\theta_4, \theta_4]$;

(A_4)

$$\max\left\{\frac{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}{\theta_1^{p^-}}, \frac{2\int_\Omega \sup_{|t|\leq\theta_4} F(x,t)dx}{\theta_4^{p^-}}\right\} < \frac{p^- M^-}{p^+ c^{p^-} M^+ L + p^- M^-} \frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx}{\eta^{p^+}}.$$

Then for every

$$\lambda \in \Lambda' := \left(\frac{\left(p^+ c^{p^-} M^+ L + p^- M^-\right)\eta^{p^+}}{p^- p^+ c^{p^-} \int_{B(x^0,\frac{s}{2})} F(x,\eta)dx}, \frac{M^-}{p^+ c^{p^-}} \min\left\{\frac{\theta_1^{p^-}}{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}, \frac{\theta_4^{p^-}}{2\int_\Omega \sup_{|t|\leq\theta_4} F(x,t)dx}\right\}\right),$$

problem (P_λ^f) *has at least three weak solutions* u_1, u_2 *and* u_3 *such that*

$$\max_{x\in\Omega} |u_1(x)| < \theta_1, \ \max_{x\in\Omega} |u_2(x)| < \frac{1}{\sqrt[p^-]{2}}\theta_4 \quad \textit{and} \quad \max_{x\in\Omega} |u_3(x)| < \theta_4.$$

Proof. Choose $\theta_2 = \frac{1}{\sqrt[p^-]{2}}\theta_4$ and $\theta_3 = \theta_4$. So, from (A_4) one has

$$\begin{aligned}\frac{\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}{\theta_2^{p^-}} &= \frac{2\int_\Omega \sup_{|t|\leq\frac{1}{\sqrt[p^-]{2}}\theta_4} F(x,t)dx}{\theta_4^{p^-}} \leq \frac{2\int_\Omega \sup_{|t|\leq\theta_4} F(x,t)dx}{\theta_4^{p^-}} \\ &< \frac{p^- M^-}{p^+ c^{p^-} M^+ L + p^- M^-} \frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx}{\eta^{p^+}},\end{aligned} \tag{3.2}$$

and

$$\frac{\int_\Omega \sup_{|t|\leq\theta_3} F(x,t)dx}{\theta_3^{p^-} - \theta_2^{p^-}} = \frac{2\int_\Omega \sup_{|t|\leq\theta_4} F(x,t)dx}{\theta_4^{p^-}} < \frac{p^- M^-}{p^+ c^{p^-} M^+ L + p^- M^-} \frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx}{\eta^{p^+}}. \tag{3.3}$$

Moreover, since $\theta_1 < \eta^{\frac{p^+}{p^-}}$, from (A_4) we have

$$\frac{p^- M^-}{p^+ c^{p^-} M^+ L} \frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}{\eta^{p^+}} > \frac{p^- M^-}{p^+ c^{p^-} M^+ L} \frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx}{\eta^{p^+}}$$

$$
\begin{aligned}
&-\frac{p^-M^-}{p^+c^{p^-}M^+L}\frac{\displaystyle\int_\Omega \sup_{|t|\le\theta_1} F(x,t)dx}{\theta_1^{p^-}}\\
>&\frac{p^-M^-}{p^+c^{p^-}M^+L}\left(\frac{\displaystyle\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^{p^+}}-\frac{p^-M^-}{p^+c^{p^-}M^+L+p^-M^-}\frac{\displaystyle\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^{p^+}}\right)\\
=&\frac{p^-M^-}{p^+c^{p^-}M^+L+p^-M^-}\frac{\displaystyle\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^{p^+}}.
\end{aligned}
$$

Hence, from (A_4), (3.2) and (3.3), it is easy to observe that the assumption (A_2) of Theorem 3.1 is satisfied, and it follows the conclusion. □

The following result is a consequence of Theorem 3.3.

Theorem 3.4. *Let $f : \Omega\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\xi f(x,\xi) > 0$ for all $(x,\xi)\in\Omega\times\mathbb{R}\backslash\{0\}$. Assume that*

(A_5) $\lim_{\xi\to 0}\dfrac{f(x,\xi)}{|\xi|^{p^--1}} = \lim_{|\xi|\to+\infty}\dfrac{f(x,\xi)}{|\xi|^{p^--1}} = 0.$

Then for every $\lambda > \overline{\lambda}$ where

$$
\overline{\lambda} = \frac{p^+c^{p^-}M^+L+p^-M^-}{p^-p^+c^{p^-}}\max\left\{\inf_{\eta\ge 1}\frac{\eta^{p^+}}{\displaystyle\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx};\ \inf_{\eta\le -1}\frac{(-\eta)^{p^+}}{\displaystyle\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}\right\},
$$

problem (P_λ^f) possesses at least four distinct non-trivial solutions.

Proof. Set

$$
f_1(x,\xi)=\begin{cases} f(x,\xi), & \text{if } (x,\xi)\in\Omega\times[0,+\infty),\\ 0, & \text{otherwise,}\end{cases}
$$

and

$$
f_2(x,\xi)=\begin{cases} -f(x,-\xi), & \text{if } (x,\xi)\in\Omega\times[0,+\infty),\\ 0, & \text{otherwise,}\end{cases}
$$

and define $F_1(x,\xi) := \displaystyle\int_0^\xi f_1(x,t)dt$ for every $(x,\xi)\in\Omega\times\mathbb{R}$. Fix $\lambda > \lambda^*$, and let $\eta\ge 1$ such that $\lambda > \dfrac{\left(p^+c^{p^-}M^+L+p^-M^-\right)\eta^{p^+}}{p^-p^+c^{p^-}\displaystyle\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}$. From

$$
\lim_{\xi\to 0}\frac{f_1(x,\xi)}{|\xi|^{p^--1}} = \lim_{|\xi|\to+\infty}\frac{f_1(x,\xi)}{|\xi|^{p^--1}} = 0,
$$

there is $\theta_1 > 0$ such that

$$\theta_1 < \min\left\{\eta^{\frac{p^+}{p^-}},\ \sqrt[p^-]{Lc\eta}\right\} \text{ and } \frac{\int_\Omega F_1(x,\theta_1)dx}{\theta_1^{p^-}} < \frac{M^-}{\lambda p^+ c^{p^-}},$$

and $\theta_4 > 0$ such that

$$\eta < \min\left\{\sqrt[p^+]{\frac{p^- M^-}{2p^+ c^{p^-} M^+ L}}\theta_4^{\frac{p^-}{p^+}}, \theta_4\right\}$$

and

$$\frac{\int_\Omega F_1(x,\theta_4)dx}{\theta_4^{p^-}} < \frac{M^-}{2\lambda p^+ c^{p^-}}.$$

Then, (A_4) in Theorem 3.3 is satisfied,

$$\lambda \in \left(\frac{\left(p^+ c^{p^-} M^+ L + p^- M^-\right)\eta^{p^+}}{p^- p^+ c^{p^-} \int_{B(x^0,\frac{s}{2})} F_1(x,\eta)dx}, \frac{M^-}{p^+ c^{p^-}} \min\left\{\frac{\theta_1^{p^-}}{\int_\Omega \sup_{|t|\le\theta_1} F_1(x,t)dx}, \frac{\theta_4^{p^-}}{2\int_\Omega \sup_{|t|\le\theta_4} F_1(x,t)dx}\right\}\right).$$

Hence, the problem $(P_\lambda^{f_1})$ admits two positive solutions u_1, u_2, which are positive solutions of the problem (P_λ^{f}). Next, arguing in the same way, from

$$\lim_{\xi\to 0}\frac{f_2(x,\xi)}{|\xi|^{p^- -1}} = \lim_{|\xi|\to+\infty}\frac{f_2(x,\xi)}{|\xi|^{p^- -1}} = 0,$$

we ensure the existence of two positive solutions u_3, u_4 for the problem $(P_\lambda^{f_2})$. Clearly, $-u_3$, $-u_4$ are negative solutions of the problem (P_λ^{f}) and the conclusion is achieved. □

Example 3.5. *Let* $\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 9\right\}$. *Consider the problem*

$$\begin{cases} \Delta^2_{p(x,y)}u(x) - M\left(\int_\Omega \frac{|\nabla u(x)|^{p(x,y)}dx}{p(x)}\right)\Delta_{p(x,y)}u(x) + |u(x)|^{p(x,y)-2}u(x) = \lambda f(x,y,u), & (x,y)\in\Omega, \\ u = \Delta u = 0, \quad (x,y)\in\partial\Omega, \end{cases}$$

where $M(t) = \frac{3}{2} + \frac{\sin(t)}{2}$ *for each* $t \in [0,\infty)$, $p(x,y) = x^2 + y^2 + 4$ *for all* $(x,y)\in\Omega$ *and*

$$f(x,y,t) = \begin{cases} 5(x^2+y^2)t^4, & \text{if } t \le 1, (x,y)\in\Omega, \\ (x^2+y^2)\frac{5}{\sqrt{t}}, & \text{if } t > 1, (x,y)\in\Omega. \end{cases}$$

By the expression of f, *we have*

$$F(x,y,t) = \begin{cases} (x^2+y^2)t^5, & \text{if } t \le 1, (x,y)\in\Omega, \\ (x^2+y^2)(10\sqrt{t}-9), & \text{if } t > 1, (x,y)\in\Omega. \end{cases}$$

Direct calculations give $M^- = 1$, $M^+ = 2$, $p^- = 4$ *and* $p^+ = 13$. *It is clear that*

$$\lim_{\xi\to 0}\frac{f(x,\xi)}{|\xi|^3} = \lim_{|\xi|\to+\infty}\frac{f(x,\xi)}{|\xi|^3} = 0.$$

Hence, by applying Theorem 3.4, there is $\lambda^* > 0$ *such that for each* $\lambda > \lambda^*$, *the problem possesses at least four distinct non-trivial solutions.*

As a special case, we present a simple consequence of Theorem 3.3 when f dose not depend upon x. To be precise, consider the following problem

$$\begin{cases} T(u) = \lambda f(u(x)), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega \end{cases} \tag{3.4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continues function.

Put

$$F(t) = \int_0^t f(\xi)d\xi \text{ for all } t \in \mathbb{R}.$$

Theorem 3.6. *Assume that there exist positive constants* θ_1, θ_2, θ_3 *and* $\eta \geq 1$ *with* $\theta_1 < \sqrt[p^-]{L}c\eta$, $\eta < \min\left\{ \sqrt[p^+]{\frac{p^-M^-}{p^+c^{p^-}M^+L}}\theta_2^{\frac{p^-}{p^+}}, \theta_2 \right\}$ *and* $\theta_2 < \theta_3$ *such that*

(A_7) $f(t) \geq 0$ *for each* $t \in [-\theta_3, \theta_3]$;

(A_8)

$$\max\left\{ \frac{F(\theta_1)}{\theta_1^{p^-}}, \frac{F(\theta_2)}{\theta_2^{p^-}}, \frac{F(\theta_3)}{\theta_3^{p^-} - \theta_2^{p^-}} \right\} < \frac{p^-M^-}{p^+c^{p^-}\text{meas}(\Omega)M^+L}\frac{\text{meas}(B(x^0, \frac{s}{2}))F(\eta) - \text{meas}(\Omega)F(\theta_1)}{\eta^{p^+}}.$$

Then for every

$$\lambda \in \left(\frac{\frac{M^+L}{p^-}\eta^{p^+}}{\text{meas}(B(x^0, \frac{s}{2}))F(\eta) - \text{meas}(\Omega)F(\theta_1)}, \frac{M^-}{p^+c^{p^-}\text{meas}(\Omega)}\min\left\{ \frac{\theta_1^{p^-}}{F(\theta_1)}, \frac{\theta_2^{p^-}}{F(\theta_2)}, \frac{\theta_3^{p^-} - \theta_2^{p^-}}{F(\theta_3)} \right\} \right),$$

problem (P_λ^f) *has at least three weak solutions* u_1, u_2 *and* u_3 *such that*

$$\max_{x\in\Omega}|u_1(x)| < \theta_1, \ \max_{x\in\Omega}|u_2(x)| < \theta_2 \quad \textit{and} \quad \max_{x\in\Omega}|u_3(x)| < \theta_3.$$

Theorem 3.7. *Assume that there exist positive constants* θ_1, θ_4 *and* $\eta \geq 1$ *with* $\theta_1 < \min\left\{ \eta^{\frac{p^+}{p^-}}, \sqrt[p^-]{L}c\eta \right\}$ *and*

$$\eta < \min\left\{ \sqrt[p^+]{\frac{p^-M^-}{2p^+c^{p^-}M^+L}}\theta_4^{\frac{p^-}{p^+}}, \theta_4 \right\}$$

such that

(A_9) $f(t) \geq 0$ *for each* $t \in [-\theta_4, \theta_4]$;

(A_{10})

$$\max\left\{\frac{F(\theta_1)}{\theta_1^{p^-}}, \frac{2F(\theta_4)}{\theta_4^{p^-}}\right\} < \frac{\text{meas}(B(x^0, \frac{s}{2}))p^- M^-}{\text{meas}(\Omega)\,(p^+ c^{p^-} M^+ L + p^- M^-)}\frac{F(\eta)}{\eta^{p^+}}.$$

Then for every

$$\lambda \in \Lambda' := \left(\frac{\left(p^+ c^{p^-} M^+ L + p^- M^-\right)\eta^{p^+}}{p^- p^+ c^{p^-}\text{meas}(B(x^0, \frac{s}{2}))F(\eta)}, \frac{M^-}{p^+ c^{p^-}\text{meas}(\Omega)}\min\left\{\frac{\theta_1^{p^-}}{F(\theta_1)}, \frac{\theta_4^{p^-}}{2F(\theta_4)}\right\}\right),$$

problem (P_λ^f) *has at least three weak solutions* u_1, u_2 *and* u_3 *such that*

$$\max_{x\in\Omega}|u_1(x)| < \theta_1,\ \max_{x\in\Omega}|u_2(x)| < \frac{1}{\sqrt[p^-]{2}}\theta_4 \quad \text{and} \quad \max_{x\in\Omega}|u_3(x)| < \theta_4.$$

Conflict of interest

All authors confirm that there are no competing interests between them.

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