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Research article

Dynamics of a nonlinear SIQRS computer virus spreading model with two delays

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Abstract: In this paper, a Susceptible-Infected-Quarantined-Susceptible (SIQRS) computer virus propagation model with nonlinear infection rate and two-delay is formulated. The local stability of virus-free equilibrium without delay is examined. Furthermore, we also expound and prove that time-delay plays a crucial role in sufficient conditions for the local stability of the virus-existence equilibrium and the occurrence of Hopf bifurcation at the critical value. Especially, direction and stability of the Hopf bifurcation are demonstrated. Finally, some numerical simulations are presented in order to verify the theoretical results.

Keywords: delays; nonlinear infection rate; quarantine strategy; bifurcation; stability; SIQRS computer virus propagation model
Mathematics Subject Classification: 34C23

1. Introduction

Since the first personal computer came out in 1980, computers gradually appeared in our daily life. In 1995, the emergence of the Internet further promoted the computers into all fields of production and living. By June 2020, China's Internet users had reached 940 million, an increase of 36.25 million compared with March 2020, and the Internet penetration rate reached 67.0%, an increase of 2.5 percentage points compared with March 2020 [1]. Computer network is a sharp double-edged sword, bringing conveniences as well as disasters. In October 2019, a total of 44.23 million new viruses was found in the National Computer Virus Emergency Response Center and 218.04 million computers were infected, which was 1.78% higher than that in September, and the main transmission channels were "phishing", "webpage pegging" and loopholes [2]. The propagation of computer virus has become more rapid and harmful, posing serious changes. In the early October 2019, Demant, the world's largest hearing aid manufacturer, was invaded by blackmail virus, resulting in a direct economic loss of more than 95 million dollars. Fractional derivative equations are often used to study the dynamic behavior of systems, which can help us understand the evolution law of the system [3–6]. Consequently, it is of great practical significance to analyze the propagation of computer virus to protect computers against viruses by use fractional derivative equations.

Some mathematical models, which characterized the spread of computer viruses over the internet, were proposed to help us study the problem quantitatively. There are many similarities between computer virus and biological virus, such as infectivity, destructiveness, variability and so on. Based on these similarities, J. O. Kephart and S. R. White applied the mathematical models of epidemics to the computer virus propagation model creatively [7]. On this foundation, many computer virus models have been established [8–10]. Singh et al. [11] considered a fractional epidemiological SIR model with an arbitrary order derivative having nonsingular kernel, and discussed the existence of the solution. Considering that the recovered nodes may become susceptible again once some new viruses appear or the known computer viruses mutate, Chen et al. [12] presented a new SIRS model. But they all assumed the infection rate in models is bilinear. But in fact, this situation is not the case. In most realistic situations, the bilinear infection rate is always impossible to achieve due to the increase of the susceptible computers and infectious computers. In view of the nonlinear infection rate, both of the inhibition effect owing to the uncertain behavior of susceptible computers and the crowding effect of infectious computers are considered at the same time.

Considering that the network topology in the proliferation of virus may lead to nonlinear infection rate, MadhuSudanan et al. [13] formulated a computer viruses model with nonlinear infection rate and incubation period delay:

$$\begin{cases} \frac{dS(t)}{dt} = (1-p)b - \frac{\beta S(t-\tau)I(t-\tau)}{1+\sigma S(t-\tau)} - dS(t) + \delta R(t), \\ \frac{dI(t)}{dt} = \frac{\beta S(t-\tau)I(t-\tau)}{1+\sigma S(t-\tau)} - (d+\alpha+\gamma)I(t), \\ \frac{dR(t)}{dt} = pb + \gamma I(t) - (d+\delta)R(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$, $R(t)$ represent the number of susceptible computers, infected computers and recovered computers at time t, respectively. The meanings of all the parameters in system (1.1) can be referred to [13].

Quarantine strategy generally refers to the control of individuals with abnormal performance, so as to prevent others from being infected by viruses. Quarantine strategy is an important measure for the treatment of infectious diseases. It can not only conduct centralized management and treatment for infected individuals, but also effectively control the source of infection and greatly reduce the number of contacts. Later, inspired by the biological infectious disease model, many scholars applied the quarantine strategy to the research of computer virus model, and put forward a series of models accordingly [14, 15]. Hence, quarantine strategy should be introduced into the computer virus model. The effect of Anti-virus can protect recovered computers from the known viruses, however, as time goes on, Anti-virus may lose function as a result of the emergence of new viruses and the variation of known viruses, and the update speed of anti-virus software is always slower than that of new virus. So it needs a short time before entering susceptible state, called the temporary immune time delay. Considering the effect of quarantine strategy and the existence of temporary immune time delay, we investigate a new SIQRS computer virus model with two delays:

$$\begin{cases} \frac{dS(t)}{dt} &= (1-p)b - \frac{\beta S(t-\tau_1)I(t-\tau_1)}{1+\sigma S(t-\tau_1)} - dS(t) + \delta R(t-\tau_2), \\ \frac{dI(t)}{dt} &= \frac{\beta S(t-\tau_1)I(t-\tau_1)}{1+\sigma S(t-\tau_1)} - (d+\alpha_1+\gamma+\varepsilon)I(t), \\ \frac{dQ(t)}{dt} &= \varepsilon I(t) - (\eta+d+\alpha_2)Q(t), \\ \frac{dR(t)}{dt} &= pb + \gamma I(t) + \eta Q(t) - dR(t) - \delta R(t-\tau_2), \end{cases} \tag{1.2}$$

where $Q(t)$ is the number of quarantine computers at time t; α_1 is the death rate of infected computers due to virus; α_2 is the death rate of quarantine computers due to virus; ε is the quarantine rate of infected computers; η is the recovered rate of the quarantine computers; τ_1 is the incubation period delay; τ_2 is the temporary immune time delay before the recovered computers come into the susceptible status.

2. Local asymptotic stability of the virus-free equilibrium

When the system (1.2) reaches the virus-free equilibrium, there is no virus, namely $I_0^* = 0$. Let us equate system (1.2) to be zero, we can obtain:

$$\begin{cases} \delta R_0^* + (1-p)b - \frac{\beta S_0^* I_0^*}{1+\sigma S_0^*} - dS_0^* &= 0, \\ \frac{\beta S_0^* I_0^*}{1+\sigma S_0^*} - (d+\alpha_1+\gamma+\varepsilon)I_0^* &= 0, \\ \varepsilon I_0^* - (\eta+d+\alpha_2)Q_0^* &= 0, \\ \gamma I_0^* + \eta Q_0^* + pb - (d+\delta)R_0^* &= 0, \end{cases} \tag{2.1}$$

Then, then system (1.2) has a virus-free equilibrium $E_0^*(S_0^*, I_0^*, Q_0^*, R_0^*)$. Here,

$$\begin{aligned} S_0^* &= \frac{\delta b + bd(1-p)}{d(d+\delta)}, \\ I_0^* &= 0, \\ Q_0^* &= 0, \\ R_0^* &= \frac{pb}{d+\delta}. \end{aligned}$$

The basic regeneration number is the critical threshold to determine whether there is a virus in system (1.2). According to the way in [16], it is easy to obtain the basic regeneration number of system (1.2). Let $X = (I, S, Q, R)^T$, then system (1.2) can be equivalent to $\frac{dX(t)}{dt} = F - V$, where

$$F = \begin{pmatrix} \frac{\beta SI}{1+\sigma S} \\ 0 \\ 0 \\ 0 \end{pmatrix}, V = \begin{pmatrix} (d+\alpha_1+\eta+\gamma)I \\ -(1-p)b + \frac{\beta SI}{1+\sigma S} + dS + \delta R \\ (\eta+d+\alpha_2)Q - \varepsilon I \\ dR + \delta R - \eta Q - \gamma I - pb \end{pmatrix}.$$

The infected compartment is I, giving $m = 1$, then the Jacobian matrixes of F and V at $E_0^*(S_0^*, I_0^*, Q_0^*, R_0^*)$ are

$$F' = \left(\frac{\beta S_0^*}{1+\sigma S_0^*}\right), V' = \left(d+\alpha_1+\eta+\gamma\right).$$

Then

$$R_0 = \frac{\beta S_0^*}{(1+\sigma S_0^*)(d+\alpha_1+\gamma+\varepsilon)}. \tag{2.2}$$

If $R_0 < 1$, then system (1.2) has a virus-free equilibrium $E_0^*(S_0^*, I_0^*, Q_0^*, R_0^*)$. The Jacobian matrix of system (1.2) at $E_0^*(S_0^*, I_0^*, Q_0^*, R_0^*)$ is

$$J(E_0^*) = \begin{pmatrix} -d & -\frac{\beta S_0^*}{1+\sigma S_0^*} & 0 & \delta \\ 0 & \frac{\beta S_0^*}{1+\sigma S_0^*} - (d+\alpha_1+\gamma+\varepsilon) & 0 & 0 \\ 0 & \varepsilon & -(\eta+d+\alpha_2) & 0 \\ 0 & \gamma & \eta & -(d+\delta) \end{pmatrix},$$

The corresponding characteristic equation becomes

$$(\lambda+d)(\lambda - \frac{\beta S_0^*}{1+\sigma S_0^*} + d + \alpha_1 + \gamma + \varepsilon)(\lambda + d + \eta + \alpha_2)(\lambda + d + \delta) = 0. \tag{2.3}$$

Then the eigenvalues of Eq.(2.3) are

$$\begin{aligned} \lambda_1 &= -d < 0, \\ \lambda_2 &= \frac{\beta S_0^*}{1+\sigma S_0^*} - (d+\alpha_1+\gamma+\varepsilon) < 0, \\ \lambda_3 &= -(d+\eta+\alpha_2) < 0, \\ \lambda_4 &= -(d+\delta) < 0, \end{aligned}$$

So, when all the eigenvalues are less than zero, the virus-free equilibrium of system (1.2) is locally stable according to Routh-Hurwitz criteria.

3. Local asymptotic stability of the virus-existence equilibrium and the occurrence of Hopf bifurcation

If $R_0 = \frac{\beta S_0^*}{(1+\sigma S_0^*)(d+\alpha_1+\gamma+\varepsilon)} > 1$, then system (1.2) has a unique virus-existence equilibrium $E^*(S^*, I^*, Q^*, R^*)$. Here,

$$\begin{aligned} S^* &= \frac{d+\alpha_1+\gamma+\varepsilon}{\beta - \sigma(d+\alpha_1+\gamma+\varepsilon)}, \\ I^* &= \frac{(d+\delta)dS^* - (d+\delta)(1-p)b - \delta pb}{k_1 + \delta\gamma - k_2}, \\ Q^* &= \frac{\varepsilon}{\eta+d+\alpha_2} I^*, \\ R^* &= \frac{pb + \gamma I^* + \eta Q^*}{d+\delta}, \end{aligned}$$

where

$$k_1 = \frac{\delta\eta\varepsilon}{\eta+d+\alpha_2},$$

$$k_2 = \frac{(\delta+d)\beta S^*}{1+\sigma S^*}.$$

The linearized part of system (1.2) is

$$\begin{cases} \frac{dS(t)}{dt} = l_{11}S(t) + m_{11}S(t-\tau_1) + m_{12}I(t-\tau_1) + n_{14}R(t-\tau_2), \\ \frac{dI(t)}{dt} = m_{21}S(t-\tau_1) + l_{22}I(t) + m_{22}I(t-\tau_1), \\ \frac{dQ(t)}{dt} = l_{32}I(t) + l_{33}Q(t), \\ \frac{dR(t)}{dt} = l_{42}I(t) + l_{43}Q(t) + l_{44}R(t) + n_{44}R(t-\tau_2), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} l_{11} &= -d, m_{11} = -\frac{\beta I^*}{(1+\sigma S^*)^2}, m_{12} = -\frac{\beta S^*}{1+\sigma S^*}, n_{14} = \delta, \\ m_{21} &= \frac{\beta I^*}{(1+\sigma S^*)^2}, m_{22} = \frac{\beta S^*}{1+\sigma S^*}, l_{22} = -(d+\alpha_1+\gamma+\varepsilon), \\ l_{32} &= \varepsilon, l_{33} = -(\eta+d+\alpha_2), l_{42} = \gamma, l_{43} = \eta, l_{44} = -d, n_{44} = -\delta, \end{aligned}$$

From the system (3.1), we can obtain that

$$\begin{aligned} X_0(\lambda) + X_1(\lambda)e^{-\lambda\tau_1} &+ X_2(\lambda)e^{-\lambda\tau_2} + X_3(\lambda)e^{-\lambda(\tau_1+\tau_2)} \\ &+ X_4(\lambda)e^{-2\lambda\tau_2} + X_5(\lambda)e^{-\lambda(2\tau_1+\tau_2)} = 0, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} X_0(\lambda) &= \lambda^4 + \lambda^3(-l_{11} - l_{22} - l_{33} - l_{44}) + \lambda^2(l_{11}l_{22} + l_{11}l_{33} + l_{11}l_{44} + l_{22}l_{33} + l_{22}l_{44} + l_{33}l_{44}) \\ &\quad +\lambda(-l_{11}l_{22}l_{33} - l_{11}l_{22}l_{44} - l_{11}l_{33}l_{44} - l_{22}l_{33}l_{44}) + l_{11}l_{22}l_{33}l_{44}, \\ X_1(\lambda) &= \lambda^3(-m_{11} - m_{22}) + \lambda^2(l_{11}m_{22} + l_{22}m_{11} + l_{33}m_{11} + l_{33}m_{22} + l_{44}m_{11} + l_{44}m_{22}) \\ &\quad +\lambda(-l_{33}l_{44}m_{22} - l_{11}l_{44}m_{22} - l_{11}l_{33}m_{22} - l_{33}l_{44}m_{11} - l_{22}l_{44}m_{11} - l_{22}l_{33}m_{11}) \\ &\quad +l_{22}l_{33}l_{44}m_{22} + l_{22}l_{33}l_{44}m_{11}, \\ X_2(\lambda) &= -n_{44}\lambda^3 + \lambda^2(l_{11}n_{44} + l_{22}n_{44} + l_{33}n_{44}) + \lambda(-l_{11}l_{22}n_{44} - l_{11}l_{33}n_{44} - l_{22}l_{33}n_{44}) \\ &\quad +l_{11}l_{22}l_{33}n_{44}, \\ X_3(\lambda) &= \lambda^2(m_{11}n_{44} + m_{22}n_{44}) + \lambda(-l_{11}m_{22}n_{44} - l_{22}m_{22}n_{44} - l_{33}m_{11}n_{44} - l_{33}m_{22}n_{44} \\ &\quad +l_{42}m_{21}n_{44}) + (l_{11}l_{33}m_{22}n_{44} + l_{22}l_{33}m_{11}n_{44} + l_{32}l_{43}m_{21}n_{14} - l_{33}l_{42}m_{21}n_{14}), \\ X_4(\lambda) &= \lambda^2(m_{11}m_{22} + m_{12}m_{21}) + \lambda(-l_{33}m_{11}m_{22} - l_{44}m_{11}m_{22} - l_{33}m_{12}m_{21} - l_{44}m_{12}m_{21}) \\ &\quad +l_{33}l_{44}m_{11}m_{22} + l_{33}l_{44}m_{12}m_{21}, \\ X_5(\lambda) &= \lambda(-m_{11}m_{22}n_{44} - m_{12}m_{21}n_{44}) + l_{33}m_{11}m_{22}n_{44} + l_{33}m_{12}m_{21}n_{44}. \end{aligned}$$

Case 1. $\tau_1 = \tau_2 = 0$, Eq (3.2) becomes

$$\lambda^4 + X_0^3\lambda^3 + X_0^2\lambda^2 + X_0^1\lambda + X_0^0 = 0, \tag{3.3}$$

where $X_i^j (i = 0, 1, 2, 3, 4, 5; j = 0, 1, 2, 3, 4)$ represents the coefficient of λ^j in $X_i(\lambda)$.

Lemma 1 [13]. According to Routh-Hurwitz criteria, when $R_0 > 1$, the virus-existence equilibrium $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable.

Case 2. $\tau_1 > 0$, $\tau_2 = 0$. Then, Eq (3.2) becomes

$$[X_0(\lambda) + X_2(\lambda)] + [X_1(\lambda) + X_3(\lambda)]e^{-\lambda\tau_1} + [X_4(\lambda) + X_5(\lambda)]e^{-2\lambda\tau_1} = 0. \tag{3.4}$$

Taking $\lambda = i\omega_1$ into Eq (3.4) and separating the real and imaginary parts, we obtain

$$\begin{cases} A_{11}\cos\tau_1\omega_1 + A_{21}\sin\tau_1\omega_1 + B_{11} = -A_{31}\sin 2\tau_1\omega_1 + A_{41}\cos 2\tau_1\omega_1, \\ A_{21}\cos\tau_1\omega_1 - A_{11}\sin\tau_1\omega_1 + B_{21} = -A_{31}\cos 2\tau_1\omega_1 - A_{41}\sin 2\tau_1\omega_1, \end{cases} \tag{3.5}$$

with

$$\begin{aligned} A_{11} &= X_1^0 - X_1^2\omega_1^2 + X_3^0 - X_3^2\omega_1^2, \\ A_{21} &= X_1^1\omega_1 - X_1^3\omega_1^3 + X_3^1\omega_1, \\ A_{31} &= X_4^1\omega_1 + X_5^1\omega_1, \\ A_{41} &= X_4^2\omega_1^2 - X_4^0 - X_5^0, \\ B_{11} &= \omega_1^4 - X_0^2\omega_1^2 + X_0^0 - X_2^2\omega_1^2 + X_2^0, \\ B_{21} &= X_0^1\omega_1^1 - X_0^3\omega_1^3 - X_2^3\omega_1^3 - X_2^1\omega_1, \end{aligned}$$

Because $\cos^2\tau_1\omega_1 + \sin^2\tau_1\omega_1 = 1$, $\sin\tau_1\omega_1 = \pm\sqrt{1-\cos^2\tau_1\omega_1}$.

(1) If $\sin\tau_1\omega_1 = \sqrt{1-\cos^2\tau_1\omega_1}$, after calculation, we have

$$\begin{aligned} A_{11}^2 + A_{21}^2 + B_{11}^2 + B_{21}^2 - A_{31}^2 - A_{41}^2 + 2(A_{11}B_{11} + A_{21}B_{21})\cos\tau_1\omega_1 \\ +2(B_{11}A_{21} - A_{11}B_{21})\sqrt{1-\cos^2\tau_1\omega_1} = 0. \end{aligned} \tag{3.6}$$

Let $f_1(\omega_1) = \cos\tau_1\omega_1$, and we suppose that (G_1): $f_1(\omega_1) = \cos\tau_1\omega_1$ has at least a positive root ω_{11}, which makes Eq (3.6) true. Thus,

$$\tau_{11}^{(i)} = \frac{1}{\omega_{11}} \times [\arccos(f_1(\omega_{11})) + 2i\pi], i = 0, 1, 2, \cdots. \tag{3.7}$$

(2) If $\sin\tau_1\omega_1 = -\sqrt{1-\cos^2\tau_1\omega_1}$, after calculation, we have

$$\begin{aligned} A_{11}^2 + A_{21}^2 + B_{11}^2 + B_{21}^2 - A_{31}^2 - A_{41}^2 + 2(A_{11}B_{11} + A_{21}B_{21})\cos\tau_1\omega_1 \\ +2(A_{11}B_{21} - A_{21}B_{11})\sqrt{1-\cos^2\tau_1\omega_1} = 0. \end{aligned} \tag{3.8}$$

Let $g_1(\omega_1) = \cos\tau_1\omega_1$, and we suppose that (G_2): $g_1(\omega_1) = \cos\tau_1\omega_1$ has at least a positive root ω_{12}, which makes Eq (3.8) true. Thus,

$$\tau_{12}^{(i)} = \frac{1}{\omega_{12}} \times [\arccos(g_1(\omega_{12})) + 2i\pi], i = 0, 1, 2, \cdots. \tag{3.9}$$

Define

$$\tau_{10} = \min\{\tau_{11}^{(i)}, \tau_{12}^{(i)}\}, i = 0, 1, 2, \cdots, \tag{3.10}$$

where $\tau_{11}^{(i)}$ and $\tau_{12}^{(i)}$ are defined by Eq (3.7) and Eq (3.9), respectively.

Multiplying $e^{\lambda\tau_1}$ on both sides of Eq (3.4), and then after deriving from τ to λ, we can get

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = -\frac{[X_0'(\lambda)+X_2'(\lambda)]e^{\lambda\tau_1}+[X_1'(\lambda)+X_3'(\lambda)]+[X_4'(\lambda)+X_5'(\lambda)]e^{-\lambda\tau_1}}{-\lambda[X_0(\lambda)+X_2(\lambda)]e^{-\lambda\tau_1}+\lambda[X_4(\lambda)+X_5(\lambda)]e^{\lambda\tau_1}} - \frac{\tau_1}{\lambda}. \tag{3.11}$$

According to the Hopf bifurcation theorem [17], if the surmise (G_3): $\mathrm{Re}[d\lambda/d\tau_1]^{-1}_{\tau_1=\tau_{10}} \neq 0$ is true, the virus-existence equilibrium $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable. So, we have Theorem 1.

Theorem 1. For system (1.2), when $R_0 > 1$ and the conditions (G_1)-(G_3) hold, then $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable when $\tau_1 \in [0, \tau_{10})$; there is a Hopf bifurcation at $E^*(S^*, I^*, Q^*, R^*)$ when $\tau_1 = \tau_{10}$.

Case 3. $\tau_1 = 0$, $\tau_2 > 0$. Then Eq (3.2) becomes

$$[X_0(\lambda)+X_1(\lambda)+X_4(\lambda)]+[X_2(\lambda)+X_3(\lambda)+X_5(\lambda)]e^{-\lambda\tau_2} = 0, \tag{3.12}$$

Substituting $\lambda = i\omega_2$ into Eq (3.12), we obtain

$$\begin{cases} C_{11}\cos\tau_2\omega_2 + C_{21}\sin\tau_2\omega_2 = D_{11}, \\ C_{21}\cos\tau_2\omega_2 - C_{11}\sin\tau_2\omega_2 = D_{21}, \end{cases} \tag{3.13}$$

with

$$\begin{aligned} C_{11} &= -X_2^2\omega_2^2 + X_2^0 + X_3^0 + X_5^0 - X_3^2\omega_2^2, \\ C_{21} &= -X_2^3\omega_2^3 + X_2^1\omega_2 + X_3^1\omega_2 + X_5^1\omega_2, \\ D_{11} &= X_0^2\omega_2^2 + X_1^2\omega_2^2 + X_4^2\omega_2^2 - \omega_2^4 - X_0^0 - X_1^0 - X_4^0, \\ D_{21} &= X_0^3\omega_2^3 + X_1^3\omega_2^3 - X_0^1\omega_2 - X_1^1\omega_2 - X_4^1\omega_2, \end{aligned}$$

Squaring both sides of two equations in Eq (3.13), and adding them up, we obtain

$$C_{11}^2 + C_{21}^2 = D_{11}^2 + D_{21}^2. \tag{3.14}$$

We suppose that (G_4): Eq (3.14) has at least one positive real root ω_{20}. Then, from Eq (3.13), we derive

$$\tau_2^{(i)} = \frac{1}{\omega_{20}} \times \left[\arccos\frac{C_{11}D_{11}+C_{21}D_{21}}{C_{11}^2+C_{21}^2} + 2i\pi\right], \tag{3.15}$$

where $i = 0, 1, 2, \cdots$.

Define

$$\tau_{20} = \min\{\tau_2^{(i)}, i = 0, 1, 2, \cdots\}, \tag{3.16}$$

and $\tau_2^{(i)}$ is defined by Eq (3.15).

Taking the derivative of λ with respect to τ, we obtain

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = -\frac{X_0'+X_1'+X_4'}{\lambda[X_0+X_1+X_4]} + \frac{X_2'+X_3'+X_5'}{\lambda[X_2+X_3+X_5]} - \frac{\tau_2}{\lambda}, \tag{3.17}$$

Thus, it is easy to obtain the expression of $\mathrm{Re}[d\lambda/d\tau_2]^{-1}_{\tau_2=\tau_{20}}$. According to the Hopf bifurcation theorem [17], if the hypothesis (G_5): $\mathrm{Re}[d\lambda/d\tau_2]^{-1}_{\tau_2=\tau_{20}} \neq 0$ is true, the virus-existence equilibrium $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable. In conclusion, Theorem 2 can be obtained.

Theorem 2. For system (1.2), when $R_0 > 1$ and the conditions (G_4)-(G_5) hold, then $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable when $\tau_2 \in [0, \tau_{20})$; there is a Hopf bifurcation at $E^*(S^*, I^*, Q^*, R^*)$ when $\tau_2 = \tau_{20}$.

Case 4. $\tau_1 = \tau_2 = \tau_*$. Then Eq (3.2) becomes

$$X_0(\lambda) + [X_1(\lambda) + X_2(\lambda)]e^{-\lambda\tau_*} + [X_3(\lambda) + X_4(\lambda)]e^{-2\lambda\tau_*} + X_5(\lambda)e^{-3\lambda\tau_*} = 0, \tag{3.18}$$

Multiplying $e^{\lambda\tau_*}$ on both sides of Eq (3.18), then we obtain

$$X_0(\lambda)e^{\lambda\tau_*} + [X_1(\lambda) + X_2(\lambda)] + [X_3(\lambda) + X_4(\lambda)]e^{-\lambda\tau_*} + X_5(\lambda)e^{-2\lambda\tau_*} = 0, \tag{3.19}$$

Substituting $\lambda = i\omega_3$ into Eq (3.19), we obtain

$$\begin{cases} A_{12}\cos\tau_*\omega_3 + A_{22}\sin\tau_*\omega_3 = -A_{32}\sin 2\tau_*\omega_3 + A_{42}\cos 2\tau * \omega_3, \\ A'_{22}\cos\tau_*\omega_3 - A_{12}\sin\tau * \omega_3 = -A_{32}\cos 2\tau * \omega_3 - A_{42}\sin 2\tau_*\omega_3, \end{cases} \tag{3.20}$$

with

$$\begin{aligned} A_{12} &= X_0^3\omega_3^3 - X_0^1\omega_3 - X_3^1\omega_3 - X_4^1\omega_3, \\ A_{22} &= \omega_3^4 - X_0^2\omega_3^2 - X_3^2\omega_3^2 + X_4^2\omega_3^2 + X_0^0 - X_3^0 - X_4^0, \\ A'_{22} &= \omega_3^4 - X_0^2\omega_3^2 - X_3^2\omega_3^2 - X_4^2\omega_3^2 + X_0^0 + X_3^0 + X_4^0, \\ A_{32} &= X_5^1\omega_3, \\ A_{42} &= -X_5^0, \\ B_{12} &= -X_1^2\omega_3^2 + X_1^0 - X_2^2\omega_3^2 + X_2^0, \\ B_{22} &= -X_1^3\omega_3^3 - X_2^3\omega_3^3 + X_1^1\omega_3 + X_2^1\omega_3, \end{aligned}$$

Squaring both sides of two equations in Eq (3.20), and adding them up, we obtain

$$\begin{aligned} &(A_{12}\cos\tau_*\omega_3 + A_{22}\sin\tau_*\omega_3 + B_{12})^2 + (A'_{22}\cos\tau_*\omega_3 - A_{12}\sin\tau_*\omega_3 + B_{22})^2 \\ &= A_{32}^2 + A_{42}^2. (3.21) \end{aligned}$$

Because $\cos^2\tau_*\omega_3 + \sin^2\tau_*\omega_3 = 1$, $\sin\tau_*\omega_3 = \pm\sqrt{1-\cos^2\tau_*\omega_3}$.

(1) If $\sin\tau_*\omega_3 = \sqrt{1-\cos^2\tau_*\omega_3}$, after calculation, we have

$$\begin{aligned} &A_{12}^2 + A_{22}^2 + B_{12}^2 + B_{22}^2 - A_{32}^2 - A_{42}^2 + 2(A_{12}A_{22} + A_{12}B_{12} + A_{22}B_{12})\cos\tau_*\omega_3 \\ &+2(A'_{22}B_{22} - A'_{22}A_{12} - A_{12}B_{22})\sqrt{1-\cos^2\tau_*\omega_3} = 0. \end{aligned} \tag{3.22}$$

Let $f_2(\omega_3) = \cos\tau_*\omega_3$, and we suppose that ($G_6$): $f_2(\omega_3) = \cos\tau_*\omega_3$ has at least a positive root ω_{31}, which makes Eq (3.22) true. Thus,

$$\tau_{*1}^{(i)} = \frac{1}{\omega_{31}} \times [\arccos(f_2(\omega_{31})) + 2i\pi], i = 0, 1, 2, \cdots. \tag{3.23}$$

(2) If $\sin\tau_*\omega_3 = -\sqrt{1-\cos^2\tau_*\omega_3}$, after calculation, we have

$$\begin{aligned}A_{12}^2 + A_{22}^2 + B_{12}^2 + B_{22}^2 - A_{32}^2 - A_{42}^2 + 2(A_{12}A_{22} + A_{12}B_{12} + A_{22}B_{12})\cos\tau_*\omega_3 \\ -2(A'_{22}B_{22} - A'_{22}A_{12} - A_{12}B_{22})\sqrt{1-\cos^2\tau_*\omega_3} = 0.\end{aligned} \tag{3.24}$$

Let $g_2(\omega_3) = \cos\tau_*\omega_3$, and we suppose that ($G_7$): $g_2(\omega_3) = \cos\tau_*\omega_3$ has at least a positive root ω_{32}, which makes Eq (3.24) true. Thus,

$$\tau_{*2}^{(i)} = \frac{1}{\omega_{32}} \times [\arccos(g_2(\omega_{32})) + 2i\pi], i = 0, 1, 2, \cdots . \tag{3.25}$$

Define

$$\tau_{*0} = \min\{\tau_{*1}^{(i)}, \tau_{*2}^{(i)}\}, i = 0, 1, 2, \cdots , \tag{3.26}$$

where $\tau_{*1}^{(i)}$ and $\tau_{*2}^{(i)}$ are defined by Eq (3.23) and Eq (3.25), respectively.

Then after deriving from τ to λ, we can get

$$\left[\frac{d\lambda}{d\tau_*}\right]^{-1} = -\frac{X'_0(\lambda)e^{\lambda\tau_*} + [X'_1(\lambda) + X'_2(\lambda)] + [X'_3(\lambda) + X'_4(\lambda)]e^{-\lambda\tau_*} + X'_5(\lambda)e^{-2\lambda\tau_*}}{-\lambda X_0(\lambda)e^{\lambda\tau_*} + \lambda[X_3(\lambda) + X_4(\lambda)]e^{-\lambda\tau_*} + 2\lambda X_5(\lambda)e^{-2\lambda\tau_*}} - \frac{\tau_*}{\lambda}. \tag{3.27}$$

Based on the Hopf bifurcation theorem [17], if the surmise (G_8): $\mathrm{Re}[d\lambda/d\tau_*]^{-1}_{\tau_*=\tau_{*0}} \neq 0$ is true, the virus-existence equilibrium $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable. Therefore, Theorem 3 can be obtained.

Theorem 3. For system (1.2), when $R_0 > 1$ and the conditions (G_6)-(G_8) hold, then $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable when $\tau_* \in [0, \tau_{*0})$; there is a Hopf bifurcation at $E^*(S^*, I^*, Q^*, R^*)$ when $\tau_* = \tau_{*0}$.

Case 5. $\tau_1 > 0$, $\tau_2 \in (0, \tau_{20})$. For convenience, let ω_4 be equal to ω_1. Then, this case is similar as in Case 2.

$$[X_0(\lambda) + X_1(\lambda) + X_4(\lambda)] + [X_1(\lambda) + X_3(\lambda)]e^{-\lambda\tau_1} + [X_4(\lambda) + X_5(\lambda)]e^{-2\lambda\tau_1} = 0. \tag{3.28}$$

$$\begin{cases} A_{13}\cos\tau_{1*}\omega_4 + A_{23}\sin\tau_{1*}\omega_4 + B_{13} = -A_{33}\sin 2\tau_{1*}\omega_4 + A_{43}\cos 2\tau_{1*}\omega_4, \\ A_{23}\cos\tau_{1*}\omega_4 - A_{13}\sin\tau_{1*}\omega_4 + B_{23} = -A_{33}\cos 2\tau_{1*}\omega_4 - A_{43}\sin 2\tau_{1*}\omega_4, \end{cases} \tag{3.29}$$

with

$$\begin{aligned} A_{13} &= X_1^0 - X_1^2\omega_4^2 + X_3^0 - X_3^2\omega_4^2, \\ A_{23} &= X_1^1\omega_4 - X_1^3\omega_4^3 + X_3^1\omega_4, \\ A_{33} &= X_4^1\omega_4 + X_5^1\omega_4, \\ A_{43} &= X_4^2\omega_4^2 - X_4^0 - X_5^0, \\ B_{13} &= \omega_4^4 - X_0^2\omega_4^2 + X_0^0 - X_2^2\omega_4^2 + X_2^0, \\ B_{23} &= X_0^1\omega_4^1 - X_0^3\omega_4^3 - X_2^3\omega_4^3 - X_2^1\omega_4, \end{aligned}$$

Because $\cos^2\tau_{1*}\omega_4 + \sin^2\tau_{1*}\omega_4 = 1$, $\sin\tau_{1*}\omega_4 = \pm\sqrt{1-\cos^2\tau_{1*}\omega_4}$.

(1) If $\sin\tau_{1*}\omega_4 = \sqrt{1-\cos^2\tau_{1*}\omega_4}$, after calculation, we have

$$A_{13}^2 + A_{23}^2 + B_{13}^2 + B_{23}^2 - A_{33}^2 - A_{43}^2 + 2(A_{13}B_{13} + A_{23}B_{23})\cos\tau_{1*}\omega_4$$

$$+2(B_{13}A_{23} - A_{13}B_{23})\sqrt{1-\cos^2\tau_{1*}\omega_4} = 0. \tag{3.30}$$

Let $f_3(\omega_4) = \cos\tau_{1*}\omega_4$, and we suppose that (G_9): $f_3(\omega_4) = \cos\tau_{1*}\omega_4$ has at least a positive root ω_{41}, which makes Eq (3.30) true. Thus,

$$\tau_{1*1}^{(i)} = \frac{1}{\omega_{41}} \times [\arccos(f_3(\omega_{41})) + 2i\pi], i = 0, 1, 2, \cdots. \tag{3.31}$$

(2) If $\sin\tau_{1*}\omega_4 = -\sqrt{1-\cos^2\tau_{1*}\omega_4}$, after calculation, we have

$$\begin{aligned} &A_{13}^2 + A_{23}^2 + B_{13}^2 + B_{23}^2 - A_{33}^2 - A_{43}^2 + 2(A_{13}B_{13} + A_{23}B_{23})\cos\tau_{1*}\omega_4 \\ &+2(A_{13}B_{23} - A_{23}B_{13})\sqrt{1-\cos^2\tau_{1*}\omega_4} = 0. \end{aligned} \tag{3.32}$$

Let $g_3(\omega_4) = \cos\tau_{1*}\omega_4$, and we suppose that (G_{10}): $g_3(\omega_4) = \cos\tau_{1*}\omega_4$ has at least a positive root ω_{42}, which makes Eq (3.32) true. Thus,

$$\tau_{1*2}^{(i)} = \frac{1}{\omega_{42}} \times [\arccos(g_3(\omega_{42})) + 2i\pi], i = 0, 1, 2, \cdots. \tag{3.33}$$

Define

$$\tau_{1*0} = \min\{\tau_{1*1}^{(i)}, \tau_{1*2}^{(i)}\}, i = 0, 1, 2, \cdots, \tag{3.34}$$

where $\tau_{1*1}^{(i)}$ and $\tau_{1*2}^{(i)}$ are defined by Eq (3.31) and Eq (3.33), respectively. Multiplying $e^{\lambda\tau_{1*}}$ on both sides of Eq (3.28), and then after deriving from τ to λ, we can get

$$\left[\frac{d\lambda}{d\tau_{1*}}\right]^{-1} = -\frac{[X_0'(\lambda) + X_2'(\lambda)]e^{\lambda\tau_{1*}} + [X_1'(\lambda) + X_3'(\lambda)] + [X_4'(\lambda) + X_5'(\lambda)]e^{-\lambda\tau_{1*}}}{-\lambda[X_0(\lambda) + X_2(\lambda)]e^{-\lambda\tau_{1*}} + \lambda[X_4(\lambda) + X_5(\lambda)]e^{\lambda\tau_{1*}}} - \frac{\tau_{1*}}{\lambda}. \tag{3.35}$$

According to the Hopf bifurcation theorem [17], if the surmise (G_{11}): $\mathrm{Re}[d\lambda/d\tau_{1*}]^{-1}_{\tau_{1*}=\tau_{1*0}} \neq 0$ is true, the virus-existence equilibrium $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable. Thus, we have Theorem 4.

Theorem 4. For system (1.2), when $R_0 > 1$ and the conditions (G_9)-(G_{11}) hold, then $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable when $\tau_{1*} \in [0, \tau_{1*0})$; there is a Hopf bifurcation at $E^*(S^*, I^*, Q^*, R^*)$ when $\tau_{1*} = \tau_{1*0}$.

Case 6. $\tau_1 \in [0, \tau_{10})$, $\tau_2 > 0$. Assume that $\lambda = i\omega_5$ is the root of Eq (3.2). For convenience, let ω_5 be equal to ω_2. Then, this case is similar as in Case 3. Then we can get

$$[X_0(\lambda) + X_1(\lambda) + X_4(\lambda)] + [X_2(\lambda) + X_3(\lambda) + X_5(\lambda)]e^{-\lambda\tau_{2*}} = 0, \tag{3.36}$$

Substituting $\lambda = i\omega_5$ into Eq.(3.36), we obtain

$$\begin{cases} C_{12}\cos\tau_{2*}\omega_5 + C_{22}\sin\tau_{2*}\omega_5 = D_{12}, \\ C_{22}\cos\tau_{2*}\omega_5 - C_{12}\sin\tau_{2*}\omega_5 = D_{22}, \end{cases} \tag{3.37}$$

with

$$C_{12} = -X_2^2\omega_5^2 + X_2^0 + X_3^0 + X_5^0 - X_3^2\omega_5^2,$$

$$\begin{aligned} C_{22} &= -X_2^3\omega_5^3 + X_2^1\omega_5 + X_3^1\omega_5 + X_5^1\omega_5, \\ D_{12} &= X_0^2\omega_5^2 + X_1^2\omega_5^2 + X_4^2\omega_5^2 - \omega_5^4 - X_0^0 - X_1^0 - X_4^0, \\ D_{22} &= X_0^3\omega_5^3 + X_1^3\omega_5^3 - X_0^1\omega_5 - X_1^1\omega_5 - X_4^1\omega_5, \end{aligned}$$

Squaring both sides of two equations in Eq.(3.37), and adding them up, we obtain

$$C_{12}^2 + C_{22}^2 = D_{12}^2 + D_{22}^2. \tag{3.38}$$

We suppose that (G_{12}): Eq (3.38) has at least one positive real root ω_{50}. Then, from Eq (3.36), we derive

$$\tau_{2*}^{(i)} = \frac{1}{\omega_{50}} \times \Big[\arccos \frac{C_{13}D_{13} + C_{13}D_{13}}{C_{13}^2 + C_{23}^2} + 2i\pi \Big], \tag{3.39}$$

where $i = 0, 1, 2, \cdots$.

Define

$$\tau_{2*0} = \min\{\tau_{2*}^{(i)}, 1 = 0, 1, 2, \cdots\}, \tag{3.40}$$

and $\tau_{2*}^{(i)}$ is defined by Eq (3.39).

Taking the derivative of λ with respect to τ, we obtain

$$\left[\frac{d\lambda}{d\tau_{2*}}\right]^{-1} = -\frac{X_0' + X_1' + X_4'}{\lambda[X_0 + X_1 + X_4]} + \frac{X_2' + X_3' + X_5'}{\lambda[X_2 + X_3 + X_5]} - \frac{\tau_{2*}}{\lambda}, \tag{3.41}$$

Thus, it is easy to obtain the expression of $\mathrm{Re}[d\lambda/d\tau_{2*}]^{-1}_{\tau_{2*}=\tau_{2*0}}$. Based on the Hopf bifurcation theorem [17], when the hypothesis (G_{13}): $\mathrm{Re}[d\lambda/d\tau_{2*}]^{-1}_{\tau_{2*}=\tau_{2*0}} \neq 0$ is true. In conclusion, Theorem 5 can be gotten.

Theorem 5. For system (1.2), when $R_0 > 1$ and the conditions (G_{12})-(G_{13}) hold, then $E^*(S^*, I^*, Q^*, R^*)$ is locally asymptotically stable when $\tau_{2*} \in [0, \tau_{2*0})$; there is a Hopf bifurcation at $E^*(S^*, I^*, Q^*, R^*)$ when $\tau_{2*} = \tau_{2*0}$.

4. Direction and stability of Hopf bifurcation

Center manifold theory is one of the important theories for studying Hopf bifurcation. Considering this idea, in this section, we use the method in [18,19] to study direction and stability of Hopf bifurcation of system (1.2). We assume that $\tau_2^* < \tau_1^*$, where $\tau_2^* \in (0, \tau_{20})$. Let $\tau_1 = \tau_1^* + \varpi(\varpi \in \mathrm{R})$, $\chi_1 = S(\tau_1 t), \chi_2 = I(\tau_1 t), \chi_3 = Q(\tau_1 t), \chi_4 = R(\tau_1 t)$. System (1.2) becomes

$$\dot{\chi}(t) = L_\varpi(\chi_t) + F(\varpi, \chi_t), \tag{4.1}$$

where $\chi(t) = (\chi_1, \chi_2, \chi_3, \chi_4)^T \in C = C([-1, 0], \mathrm{R}^4)$ and L_ϖ: $C \to \mathrm{R}^4$ and F: $\mathrm{R} \times C \to \mathrm{R}^4$ are defined as

$$L_\varpi\varphi = (\tau_1^* + \varpi)\Big(L'\varphi(0) + M'\varphi\Big(-\frac{\tau_2^*}{\tau_1^*}\Big) + N'\varphi(-1)\Big), \tag{4.2}$$

and

$$F(\varpi, \varphi) = (\tau_1^* + \varpi)[F_1, F_2, 0, 0]^T, \tag{4.3}$$

with

$$L' = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ 0 & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & l_{42} & l_{43} & l_{44} \end{pmatrix}, M' = \begin{pmatrix} 0 & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N' = \begin{pmatrix} 0 & 0 & 0 & n_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_{44} \end{pmatrix},$$

and

$$\begin{aligned} F_1 &= h_{11}\varphi_1(0)\varphi_2(0) + h_{12}\varphi_1^2(0) + h_{13}\varphi_1^2(0)\varphi_2(0) + h_{14}\varphi_1^3(0) + \cdots, \\ F_2 &= h_{21}\varphi_1(0)\varphi_2(0) + h_{22}\varphi_1^2(0) + h_{23}\varphi_1^2(0)\varphi_2(0) + h_{24}\varphi_1^3(0) + \cdots, \end{aligned}$$

$$\begin{aligned} h_{11} &= -\frac{\beta}{(1+\sigma S^*)^2}, h_{12} = -\frac{2\sigma\beta I^*}{(1+\sigma S^*)^3}, h_{13} = \frac{2\sigma\beta}{(1+\sigma S^*)^3}, h_{14} = \frac{6\sigma^2\beta I^*}{(1+\sigma S^*)^4}, \\ h_{21} &= \frac{\beta}{(1+\sigma S^*)^2}, h_{22} = \frac{2\sigma\beta I^*}{(1+\sigma S^*)^3}, h_{23} = -\frac{2\sigma\beta I^*}{(1+\sigma S^*)^3}, h_{24} = -\frac{6\sigma^2\beta I^*}{(1+\sigma S^*)^4}. \end{aligned}$$

According to Riesz representation theorem, there exists $\eta(\vartheta, \varpi)$ and $\vartheta \in [-1, 0)$ such that

$$L_{\varpi}\varphi = \int_{-1}^{0} d\eta(\vartheta, \varpi)\varphi(\vartheta). \tag{4.4}$$

In fact, we choose

$$\eta(\vartheta, \varpi) = \begin{cases} (\tau_1^* + \varpi)(L' + M' + N'), & \vartheta = 0, \\ (\tau_1^* + \varpi)(M' + N'), & \vartheta \in [-\frac{\tau_2^*}{\tau_1^*}, 0), \\ (\tau_1^* + \varpi)(N'), & \vartheta \in (-1, -\frac{\tau_2^*}{\tau_1^*}), \\ 0, & \vartheta = -1. \end{cases} \tag{4.5}$$

with $\phi(\vartheta)$ is the Dirac delta function.

For $\varphi \in C([-1, 0], R^4)$, define

$$A(\varpi)\varphi = \begin{cases} \frac{d\varphi(\vartheta)}{d\vartheta}, & -1 \le \vartheta < 0, \\ \int_{-1}^{0} d\eta(\vartheta, \varpi)\varphi(\vartheta), & \vartheta = 0, \end{cases}$$

and

$$R(\varpi)\varphi = \begin{cases} 0, & -1 \le \vartheta < 0, \\ F(\varpi, \varphi), & \vartheta = 0. \end{cases}$$

Then system (1.2) is equivalent to

$$\dot{\chi}(t) = A(\varpi)\chi_t + R(\varpi)\chi_t. \tag{4.6}$$

For $\psi \in C^1([0, 1], (R^4)^*)$, define

$$A^*(\psi) = \begin{cases} -\frac{d\psi(s)}{ds}, & 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(s, 0)\psi(-s), & s = 0, \end{cases}$$

and the bilinear inner form for $A(0)$ and A^*

$$\langle\psi(s),\varphi(\vartheta)\rangle=\bar{\psi}(0)\varphi(0)-\int_{\vartheta=-1}^{0}\int_{\zeta=0}^{\vartheta}\bar{\psi}(\zeta-\vartheta)d\eta(\vartheta)\varphi(\zeta)d\zeta, \tag{4.7}$$

where $\eta(\vartheta)=\eta(\vartheta,0)$.

Let $u(\vartheta)=(1,u_2,u_3,u_4)^Te^{i\tau_1^*\omega_1^*\vartheta}$ and $u^*(s)=D(1,u_2^*,u_3^*,u_4^*)^Te^{i\tau_1^*\omega_1^*s}$. Based on definitions of $A(0)$ and $A^*(0)$, one can obtain

$$\begin{aligned}
u_2 &= \frac{m_{21}e^{-i\tau_1^*\omega_1^*}}{i\omega_1^*-l_{22}-m_{22}e^{-i\tau_1^*\omega_1^*}},\\
u_3 &= \frac{l_{32}u_2}{i\omega_1^*-l_{33}},\\
u_4 &= \frac{l_{42}u_2+l_{43}u_3}{i\omega_1^*-l_{44}-n_{44}e^{-i\tau_2^*\omega_1^*}},\\
u_2^* &= -\frac{i\omega_1^*+l_{11}}{m_{21}e^{i\tau_1^*\omega_1^*}},\\
u_3^* &= -\frac{l_{43}u_2^*}{i\tau_1^*\omega_1^*+l_{33}},\\
u_4^* &= -\frac{n_{14}e^{i\tau_2^*\omega_1^*}}{l_{44}+n_{14}e^{i\tau_2^*\omega_1^*}+i\omega_1^*}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\bar{D} = {} & [1+u_2\bar{u}_2^*+u_3\bar{u}_3^*+u_4\bar{u}_4^*+\tau_1^*e^{-i\tau_1^*\omega_1^*}u_2(m_{12}+m_{22}\bar{u}_2^*)\\
&+\tau_1^*e^{-i\tau_1^*\omega_1^*}m_{21}\bar{u}_2^*+\tau_2^*e^{-i\tau_2^*\omega_1^*}u_4(n_{14}+n_{44}\bar{u}_4^*)]^{-1}.
\end{aligned}$$

Next, g_{20}, g_{11}, g_{02} and g_{21} can be obtained with aid of the method in [20]:

$$\begin{aligned}
g_{20} &= 2\tau_1^*\bar{D}[h_{11}u_2+h_{12}+\bar{u}_2^*(h_{21}u_2+h_{22})],\\
g_{11} &= \tau_1^*\bar{D}[h_{11}(u_2+\bar{u}_2)+2h_{12}+\bar{u}_2^*h_{21}(u_2+\bar{u}_2)+2h_{22}\bar{u}_2^*)],\\
g_{20} &= 2\tau_1^*\bar{D}[h_{11}\bar{u}_2+h_{12}+\bar{u}_2^*h_{21}\bar{u}_2+h_{22}\bar{u}_2^*)],\\
g_{21} &= 2\tau_1^*\bar{D}\Big[h_{11}\Big(u_2W_{11}^{(1)}(0)+\frac{1}{2}W_{20}^{(1)}(0)\bar{u}_2+W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0)\Big)\\
&\quad+h_{12}(W_{11}^{(2)}(0)+W_{20}^{(1)}(0))+h_{13}(2u_2+\bar{u}_2)+3h_{14}\\
&\quad+\bar{u}_2^*h_{21}\Big(W_{11}^{(1)}(0)u_2+\frac{1}{2}W_{20}^{(1)}(0)\bar{u}_2+W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0)\Big)\\
&\quad+h_{22}\bar{u}_2^*(W_{11}^{(2)}(0)+W_{20}^{(1)}(0))+h_{22}(2u_2+\bar{u}_2)+3h_{24}\Big],
\end{aligned}$$

with

$$W_{20}(\vartheta) = \frac{ig_{20}u(0)}{\tau_1^*\omega_1^*}e^{i\tau_1^*\omega_1^*\vartheta}+\frac{i\bar{g}_{02}\bar{u}(0)}{3\tau_1^*\omega_1^*}e^{-i\tau_1^*\omega_1^*\vartheta}+P_1e^{2i\tau_1^*\omega_1^*\vartheta},$$

$$W_{11}(\theta) = -\frac{ig_{11}u(0)}{\tau_1^*\omega_1^*}e^{i\tau_1^*\omega_1^*\vartheta} + \frac{i\bar{g}_{11}\bar{u}(0)}{\tau_1^*\omega_1^*}e^{-i\tau_1^*\omega_1^*\vartheta} + P_2.$$

P_1 and P_2 can be computed by

$$P_1 = 2\begin{pmatrix} l_{11}^* & -m_{12}e^{-i\tau_1^*\omega_1^*} & 0 & -n_{14}e^{-i\tau_2^*\omega_1^*} \\ -m_{21}e^{-i\tau_1^*\omega_1^*} & l_{22}^* & 0 & 0 \\ 0 & -l_{32} & l_{33}^* & 0 \\ 0 & -l_{42} & -l_{43} & l_{44}^* \end{pmatrix}^{-1} \times \begin{pmatrix} P_1^{(1)} \\ P_1^{(2)} \\ 0 \\ 0 \end{pmatrix},$$

$$P_2 = -\begin{pmatrix} l_{11}+m_{11} & m_{12} & 0 & n_{14} \\ m_{21} & l_{22}+m_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & l_{42} & l_{43} & l_{44}+n_{44} \end{pmatrix}^{-1} \times \begin{pmatrix} P_2^{(1)} \\ P_2^{(2)} \\ 0 \\ 0 \end{pmatrix}.$$

where

$$\begin{aligned} l_{11}^* &= 2i\omega_1^* - l_{11} - m_{11}e^{-i\tau_1^*\omega_1^*}, \\ l_{22}^* &= 2i\omega_1^* - l_{22} - m_{22}e^{-i\tau_1^*\omega_1^*}, \\ l_{33}^* &= 2i\omega_1^* - l_{33}, \\ l_{44}^* &= 2i\omega_1^* - l_{44} - n_{44}e^{-i\tau_2^*\omega_1^*}, \end{aligned}$$

and

$$\begin{aligned} P_1^{(1)} &= h_{11}u_2 + h_{12}, \\ P_1^{(2)} &= h_{21}u_2 + h_{22}, \\ P_2^{(1)} &= h_{11}(u_2+\bar{u}_2) + 2h_{12}, \\ P_2^{(2)} &= h_{21}(u_2+\bar{u}_2) + 2h_{22}. \end{aligned}$$

Then, we can obtain

$$\begin{aligned} &C_1(0) = \tfrac{i}{2\tau_1^*\omega_1^*}\Big(g_{11}g_{20} - 2|g_{11}|^2 - \tfrac{|g_{02}|^2}{3}\Big) + \tfrac{g_{21}}{2} \\ &\mu_2 = -\tfrac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_1^*)\}}, \\ &\beta_2 = 2\mathrm{Re}\{C_1(0)\}, \\ &T_2 = -\tfrac{\mathrm{Im}\{C_1(0)\}+\mu_2\mathrm{Im}\{\lambda'(\tau_1^*)\}}{\tau_1^*\omega_1^*}, \end{aligned} \tag{4.8}$$

Thus, we have Theorem 6 about the Hopf bifurcation at τ_1^*.

Theorem 6. For system (1.2), the following results hold. If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical); If $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable); If $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solutions increase (decrease).

5. Numerical simulations

In order to identify the correctness of above results, some parameters are used to numerical simulations. The values of all parameters are shown in Table 1.

Table 1. Estimation for values of the parameters.

Parameter	Value	Source
b	1	[13]
p	0.9	[13]
β	0.65	[13]
σ	0.4	[13]
d	0.1	assumed
δ	0.7	[13]
ε	0.14	assumed
γ	0.3	[13]
η	0.1	assumed
α_1	0.1	assumed
α_2	0.18	assumed

Then, system (1.2) takes the form

$$\begin{cases} \frac{dS(t)}{dt} &= 0.1 - \frac{0.65S(t-\tau_1)I(t-\tau_1)}{1+0.4S(t-\tau_1)} - 0.1S(t) + 0.7R(t-\tau_2), \\ \frac{dI(t)}{dt} &= \frac{0.65S(t-\tau_1)I(t-\tau_1)}{1+0.4S(t-\tau_1)} - 0.64I(t), \\ \frac{dQ(t)}{dt} &= 0.14I(t) - 0.48Q(t), \\ \frac{dR(t)}{dt} &= 0.9 + 0.1Q(t) + 0.3I(t) - 0.7R(t-\tau_2) - 0.1R(t), \end{cases} \tag{5.1}$$

from which we can obtain $R_0 = 1.98 > 1$ and $E^*(1.6244, 2.0598, 0.7739, 2.0011)$.

To verify Theorem 1, we use Matlab software and obtain $\omega_{10} = 0.0786$ and $\tau_{10} = 9.3985$. Figure 1 shows that system (5.1) is locally asymptotically stable when $\tau_1 \in [0, \tau_{10})$, $\tau_2 = 0$ and a Hopf bifurcation arises when $\tau_1 = \tau_{10}$. After that, from Figure 2, system (5.1) becomes unstable when $\tau_1 > \tau_{10}$.

In the same way, we apply Matlab software to verify Theorem 2. Then, we obtain $\omega_{20} = 0.3307$ and $\tau_{20} = 2.2352$. From Figure 3, system (5.1) is locally asymptotically stable when $\tau_1 = 0$, $\tau_2 \in [0, \tau_{20})$, and a Hopf bifurcation arises when $\tau_2 = \tau_{20}$. Otherwise, system (5.1) becomes unstable when $\tau_2 > \tau_{20}$ in Figure 4.

A short calculation revealed that $\omega_{30} = 0.3833$ and $\tau_{*0} = 1.9282$. Afterwards, Figure 5 shows that system (5.1) is locally asymptotically stable when $\tau_1 = \tau_2 \in [0, \tau_{*0})$, and it can be seen a Hopf bifurcation when $\tau_1 = \tau_2 = \tau_{*0}$. Figure 6 shows that system (5.1) becomes unstable when $\tau_1 = \tau_2 > \tau_{*0}$.

It is easy to obtain $\omega_{40} = 0.0710$ and $\tau_{1*0} = 10.4056$. When $\tau_1 \in [0, \tau_{1*0})$, $\tau_2 \in [0, \tau_{20})$, system (1.2) is locally asymptotically stable, and system (5.1) undergoes a Hopf bifurcation when $\tau_1 = \tau_{1*0}$. Once $\tau_1 > \tau_{1*0}$, system (5.1) becomes unstable. The corresponding simulations are shown in Figure 7 and Figure 8.

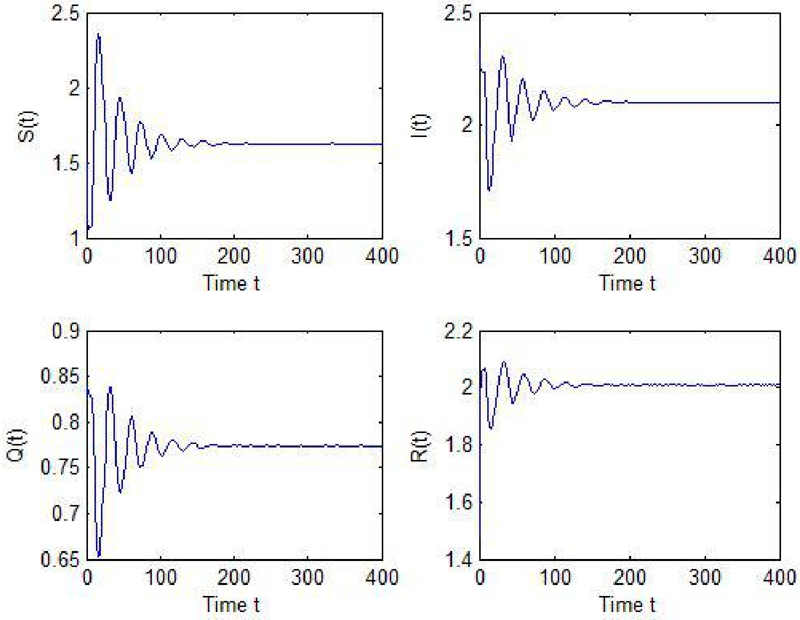

Figure 1. Evolutions of S, I, Q, R for $\tau_1 = 7.2010 < \tau_{10}$ of system (5.1) versus time t.

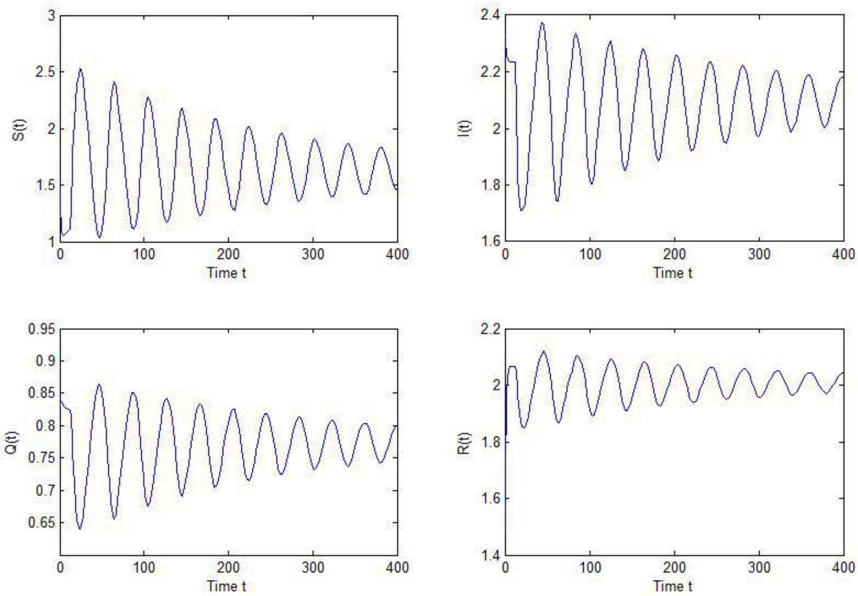

Figure 2. Evolutions of S, I, Q, R for $\tau_1 = 11.9806 > \tau_{10}$ of system (5.1) versus time t.

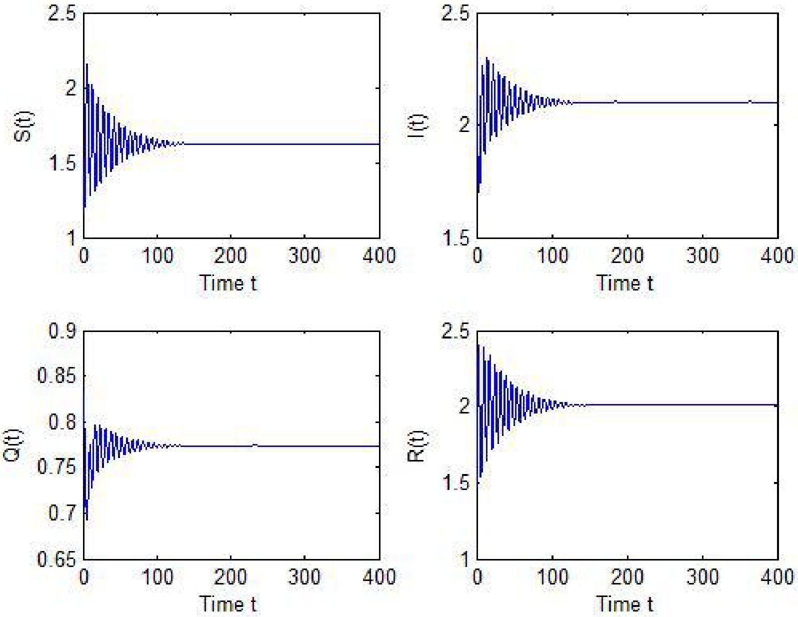

Figure 3. Evolutions of S, I, Q, R for $\tau_2 = 2.0803 < \tau_{20}$ of system (5.1) versus time t.

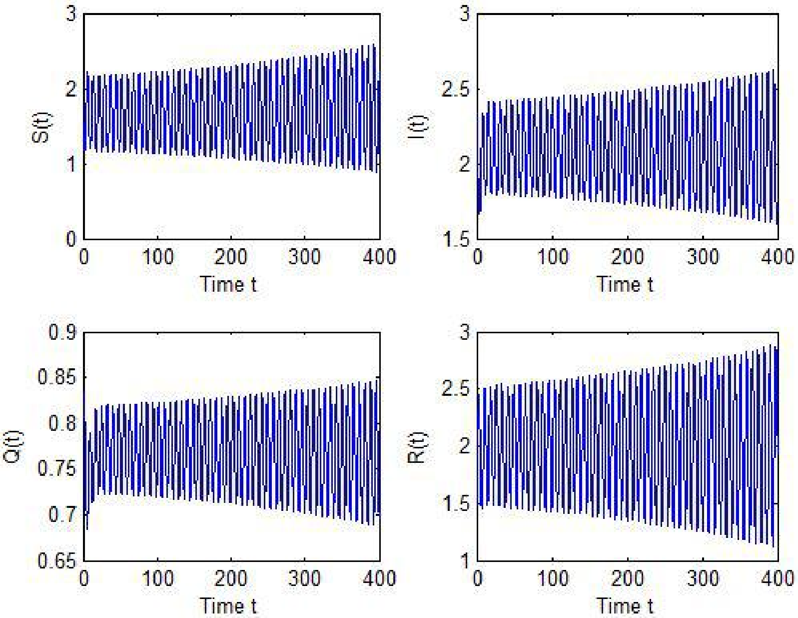

Figure 4. Evolutions of S, I, Q, R for $\tau_2 = 2.4012 > \tau_{20}$ of system (5.1) versus time t.

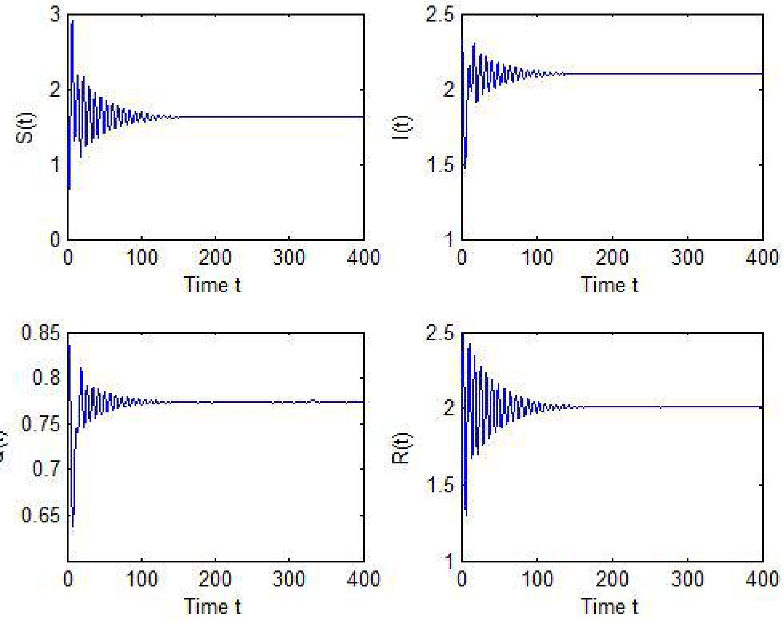

Figure 5. Evolutions of S, I, Q, R for $\tau_1 = \tau_2 = 1.8457 < \tau_{*0}$ of system (5.1) versus time t.

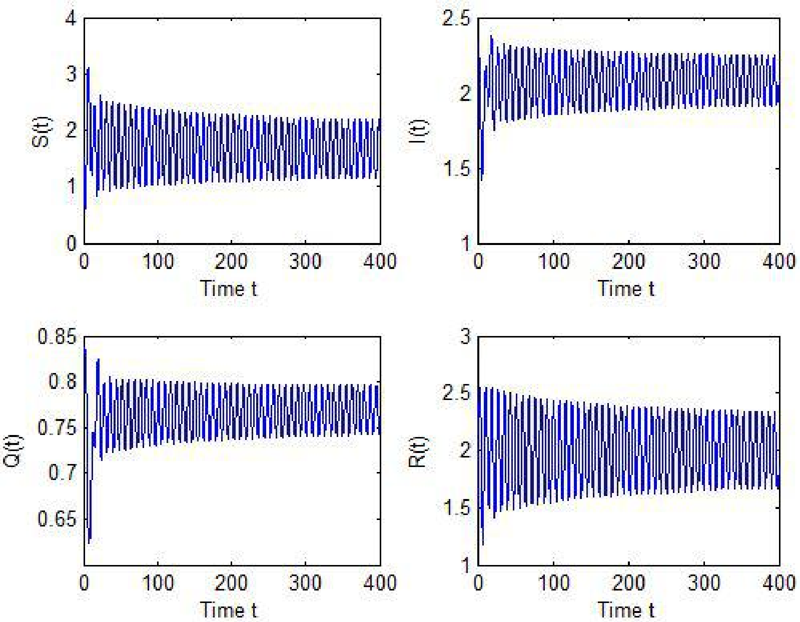

Figure 6. Evolutions of S, I, Q, R for $\tau_1 = \tau_2 = 1.9796 > \tau_{*0}$ of system (5.1) versus time t.

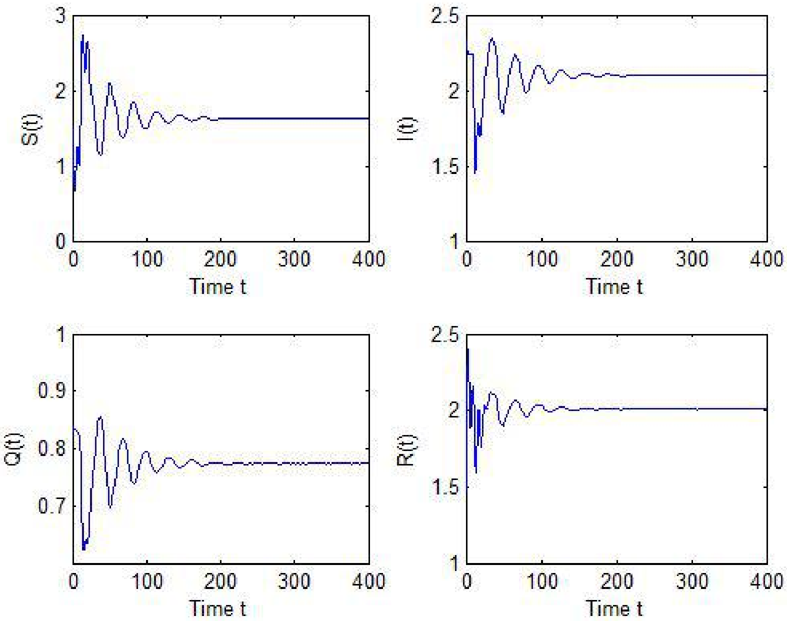

Figure 7. Evolutions of S, I, Q, R for $\tau_1 = 8.8464 < \tau_{1*0}$, $\tau_2 = 1.5764 \in [0, \tau_{20})$ of system (5.1) versus time t.

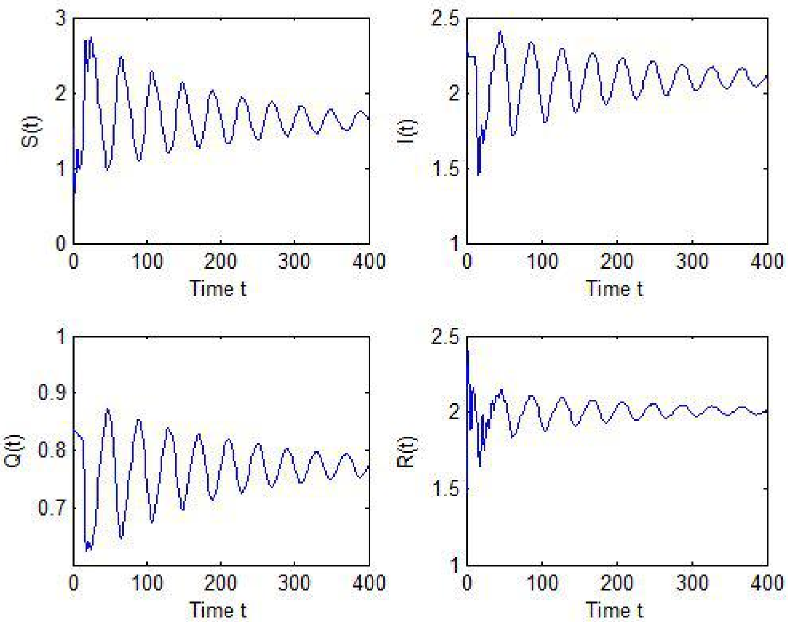

Figure 8. Evolutions of S, I, Q, R for $\tau_1 = 12.7054 > \tau_{1*0}$, $\tau_2 = 1.5764 \in [0, \tau_{20})$ of system (5.1) versus time t.

Through simple calculation, $\omega_{50} = 0.3971$ and $\tau_{2*0} = 1.8612$ can be got. As Figure 9 shows, system (5.1) is locally asymptotically stable when $\tau_1 \in [0, \tau_{10})$, $\tau_2 \in [0, \tau_{2*0})$ and a Hopf bifurcation arises when $\tau_2 = \tau_{2*0}$. And we can see that system (5.1) becomes unstable when $\tau_2 > \tau_{2*0}$ in Figure 10.

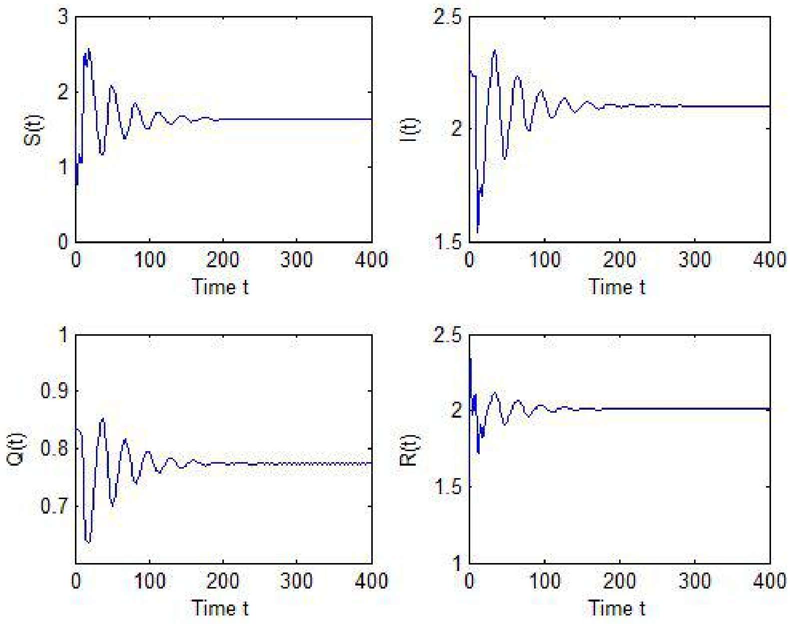

Figure 9. Evolutions of S, I, Q, R for $\tau_1 = 8.7421 \in [0, \tau_{10})$, $\tau_2 = 1.3413 < \tau_{2*0}$ of system (5.1) versus time t.

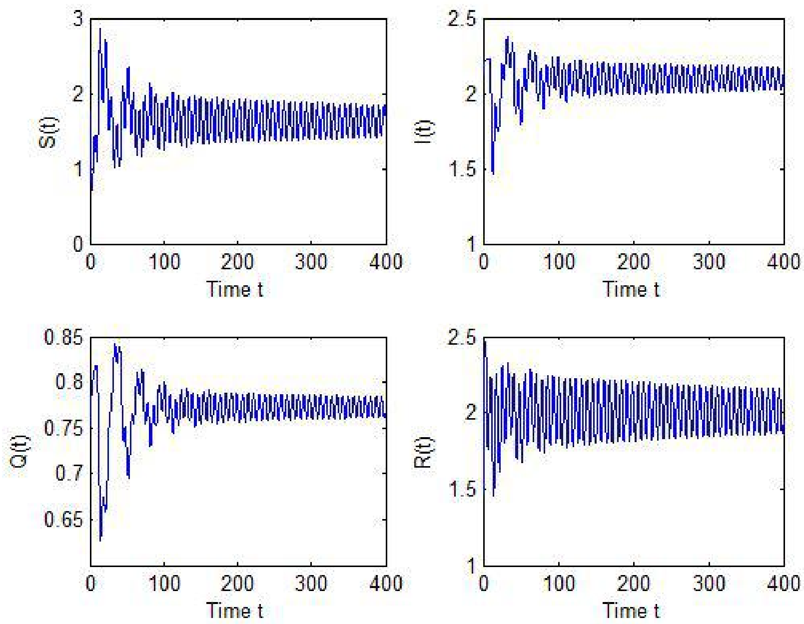

Figure 10. Evolutions of S, I, Q, R for $\tau_1 = 8.7421 \in [0, \tau_{10})$, $\tau_2 = 1.9735 > \tau_{2*0}$ of system (5.1) versus time t.

6. Conclusions

In this paper, we propose a novel Susceptible-Infected-Quarantined-Recovered (SIQRS) computer virus propagation model with quarantine strategy based on the model formulated in [13]. We consider not only incubation period delay, but also temporary immunization time delay when we observe the dynamics of the SIQRS model. The local stability of the virus-free equilibrium and the virus-existent equilibrium also has been discussed. Furthermore, we analyze the local stability and Hopf bifurcation under another five cases with different delays. If τ is less than the key value, system (1.2) is local stable; otherwise, there is a Hopf bifurcation. Then, we determine the direction of Hopf bifurcation and the stability of bifurcating periodic solutions by using the normal form and center manifold theorem. Ultimately, some numerical simulations are used to prove the validity of the theoretical results.

Compared with the model in [13], our novel model consider quarantine strategy , which is used to the prevention and cure of computer virus, so our model is closer to the actual situation. Furthermore, incubation period is one of the significant characteristics of computer virus, and it is very important to take the latency delay into account. Nowadays, antivirus software, which enable computers to gain temporary immunity, plays a very important role in the defense of computer virus. Temporary immunity delay is a common phenomena in real life. In our SIQRS model, the above cases are taken into account at the same time, and our model has more reference value over the existing ones. Global asymptotic stability is as important as local asymptotic stability, and it will be studied in the future.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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