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Research article

A note on the bounds of Roman domination numbers

Zepeng Li*

School of Information Science and Engineering, Lanzhou University, Lanzhou 730000, China

* **Correspondence:** Email: lizp@lzu.edu.cn.

Abstract: Let G be a graph and $f : V(G) \to \{0, 1, 2\}$ be a mapping. f is said to be a Roman dominating function of G if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight $w(f)$ of a Roman dominating function f is the value $w(f) = \sum_{u\in V(G)} f(u)$, and the minimum weight of a Roman dominating function is the Roman domination number $\gamma_R(G)$. f is said to be a Roman {2}-dominating function of G if $\sum_{v\in N(u)} f(v) \geq 2$ for every vertex u with $f(u) = 0$, where $N(u)$ is the set of neighbors of u in G. The weight of a Roman {2}-dominating function f is the sum $\sum_{v\in V} f(v)$ and the minimum weight of a Roman {2}-dominating function is the Roman {2}-domination number $\gamma_{\{R2\}}(G)$. Chellali et al. (2016) proved that $\gamma_R(G) \geq \frac{\Delta+1}{\Delta}\gamma(G)$ for every nontrivial connected graph G with maximum degree Δ. In this paper, we generalize this result on nontrivial connected graph G with maximum degree Δ and minimum degree δ. We prove that $\gamma_R(G) \geq \frac{\Delta+2\delta}{\Delta+\delta}\gamma(G)$, which also implies that $\frac{3}{2}\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ for any nontrivial regular graph. Moreover, we prove that $\gamma_R(G) \leq 2\gamma_{\{R2\}}(G) - 1$ for every graph G and there exists a graph I_k such that $\gamma_{\{R2\}}(I_k) = k$ and $\gamma_R(I_k) = 2k - 1$ for any integer $k \geq 2$.

Keywords: domination number; Roman domination; Roman {2}-domination
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1. Introduction

In this paper, we shall only consider graphs without multiple edges or loops. Let $G = (V(G), E(G))$ be a graph, $v \in V(G)$, the *neighborhood* of v in G is denoted by $N(v)$. That is to say $N(v) = \{u|uv \in E(G), u \in V(G)\}$. The *degree* of a vertex v is denoted by $d(v)$, i.e. $d(v) = |N(v)|$. A graph is *trivial* if it has a single vertex. The maximum degree and the minimum degree of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Denote by K_n the complete graph on n vertices.

A subset D of the vertex set of a graph G is a *dominating set* if every vertex not in D has at least one neighbor in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D of G with $|D| = \gamma(G)$ is called a γ-set of G.

Roman domination of graphs is an interesting variety of domination, which was proposed by Cockayne et al. [6]. A *Roman dominating function* (RDF) of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight $w(f)$ of a Roman dominating function f is the value $w(f) = \sum_{u\in V(G)} f(u)$. The minimum weight of an RDF on a graph G is called the *Roman domination number* $\gamma_R(G)$ of G. An RDF f of G with $w(f) = \gamma_R(G)$ is called a γ_R-function of G. The problems on domination and Roman domination of graphs have been investigated widely, for example, see list of references [8–10, 13] and [3, 7, 12], respectively.

In 2016, Chellali et al. [5] introduced a variant of Roman dominating functions, called Roman {2}-dominating functions. A *Roman* {2}*-dominating function* (*R*{2}*DF*) of G is a function $f : V \to \{0, 1, 2\}$ such that $\sum_{u\in N(v)} f(u) \geq 2$ for every vertex $v \in V$ with $f(v) = 0$. The weight of a Roman {2}-dominating function f is the sum $\sum_{v\in V} f(v)$. The *Roman* {2}*-domination number* $\gamma_{\{R2\}}(G)$ is the minimum weight of an R{2}DF of G. Note that if f is an R{2}DF of G and v is a vertex with $f(v) = 0$, then either there is a vertex $u \in N(v)$ with $f(u) = 2$, or at least two vertices $x, y \in N(v)$ with $f(x) = f(y) = 1$. Hence, an RDF of G is also an R{2}DF of G, which is also mentioned by Chellali et al [5]. Moreover, they showed that the decision problem for Roman {2}-domination is **NP**-complete, even for bipartite graphs.

In fact, a Roman {2}-dominating function is essentially the same as a *weak* {2}*-dominating function*, which was introduced by Brešar et al. [1] and studied in literatures [2, 11, 14, 15].

For a mapping $f : V(G) \to \{0, 1, 2\}$, let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f such that $V_i = \{x : f(x) = i\}$ for $i = 0, 1, 2$. Note that there exists a 1-1 correspondence between the function f and the partition (V_0, V_1, V_2) of $V(G)$, so we will write $f = (V_0, V_1, V_2)$.

Chellali et al. [4] obtained the following lower bound of Roman domination number.

Lemma 1. *(Chellali et al. [4]) Let G be a nontrivial connected graph with maximum degree Δ. Then $\gamma_R(G) \geq \frac{\Delta+1}{\Delta}\gamma(G)$.*

In this paper, we generalize this result on nontrivial connected graph G with maximum degree Δ and minimum degree δ. We prove that $\gamma_R(G) \geq \frac{\Delta+2\delta}{\Delta+\delta}\gamma(G)$. As a corollary, we obtain that $\frac{3}{2}\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ for any nontrivial regular graph G. Moreover, we prove that $\gamma_R(G) \leq 2\gamma_{\{R2\}}(G) - 1$ for every graph G and there exists a graph I_k such that $\gamma_{\{R2\}}(I_k) = k$ and $\gamma_R(I_k) = 2k - 1$ for any integer $k \geq 2$.

2. A lower bound of Roman domination number

Lemma 2. *(Cockayne et al. [6]) Let $f = (V_0, V_1, V_2)$ be a γ_R-function of an isolate-free graph G with $|V_1|$ as small as possible. Then*

(i) No edge of G joins V_1 and V_2;

(ii) V_1 is independent, namely no edge of G joins two vertices in V_1;

(iii) Each vertex of V_0 is adjacent to at most one vertex of V_1.

Theorem 3. *Let G be a nontrivial connected graph with maximum degree $\Delta(G) = \Delta$ and minimum degree $\delta(G) = \delta$. Then*

$$\gamma_R(G) \geq \frac{\Delta + 2\delta}{\Delta + \delta}\gamma(G). \tag{2.1}$$

Moreover, if the equality holds, then

$$\gamma(G) = \frac{n(\Delta+\delta)}{\Delta\delta+\Delta+\delta} \text{ and } \gamma_R(G) = \frac{n(\Delta+2\delta)}{\Delta\delta+\Delta+\delta}.$$

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R-function of G with V_1 as small as possible. By Lemma 2, we know that $N(v) \subseteq V_0$ for any $v \in V_1$ and $N(v_1) \cap N(v_2) = \emptyset$ for any $v_1, v_2 \in V_1$. So we have

$$|V_1| \leq \frac{|V_0|}{\delta} \tag{2.2}$$

Since G is nontrivial, it follows that $V_2 \neq \emptyset$. Note that every vertex in V_2 is adjacent to at most Δ vertices in V_0; we have

$$|V_0| \leq \Delta|V_2| \tag{2.3}$$

By Formulae (2.2) and (2.3), we have

$$|V_1| \leq \frac{\Delta}{\delta}|V_2| \tag{2.4}$$

By the definition of an RDF, every vertex in V_0 has at least one neighbor in V_2. So $V_1 \cup V_2$ is a dominating set of G. Together with Formula (2.4), we can obtain that

$$\gamma(G) \leq |V_1| + |V_2| \leq \frac{\Delta}{\delta}|V_2| + |V_2| = \frac{\Delta+\delta}{\delta}|V_2|.$$

Note that f is a γ_R-function of G; we have

$$\gamma_R(G) = |V_1| + 2|V_2| = (|V_1| + |V_2|) + |V_2| \geq \gamma(G) + \frac{\delta}{\Delta+\delta}\gamma(G) = \frac{\Delta+2\delta}{\Delta+\delta}\gamma(G).$$

Moreover, if the equality in Formula (2.1) holds, then by previous argument we obtain that $|V_1| = \frac{|V_0|}{\delta}$, $|V_0| = \Delta|V_2|$, and $V_1 \cup V_2$ is a γ-set of G. Then we have

$$n = |V_0| + |V_1| + |V_2| = |V_0| + \frac{|V_0|}{\delta} + \frac{|V_0|}{\Delta} = \frac{\Delta\delta+\Delta+\delta}{\Delta\delta}|V_0|.$$

Hence, we have

$$|V_0| = \frac{n\Delta\delta}{\Delta\delta+\Delta+\delta}, \quad |V_1| = \frac{n\Delta}{\Delta\delta+\Delta+\delta}, \text{ and } |V_2| = \frac{n\delta}{\Delta\delta+\Delta+\delta}.$$

So

$$\gamma_R(G) = |V_1| + 2|V_2| = \frac{n(\Delta+2\delta)}{\Delta\delta+\Delta+\delta} \text{ and } \gamma(G) = |V_1| + |V_2| = \frac{n(\Delta+\delta)}{\Delta\delta+\Delta+\delta}$$

since $V_1 \cup V_2$ is a γ-set of G. This completes the proof. □

Now we show that the lower bound in Theorem 3 can be attained by constructing an infinite family of graphs. For any integers $k \geq 2$, $\delta \geq 2$ and $\Delta = k\delta$, we construct a graph H_k from $K_{1,\Delta}$ by adding k news vertices such that each new vertex is adjacent to δ vertices of $K_{1,\Delta}$ with degree 1 and no two new vertices has common neighbors. Then add some edges between the neighbors of each new vertex u such that $\delta(H_k) = \delta$ and the induced subgraph of $N(u)$ in H_k is not complete. The resulting graph H_k is a connected graph with maximum degree $\Delta(G) = \Delta$ and maximum degree $\delta(G) = \delta$. It can be checked that $\gamma(H_k) = k + 1$ and $\gamma_R(H_k) = k + 2 = \frac{\Delta+2\delta}{\Delta+\delta}\gamma(G)$.

For example, if $k = 2$, $\delta = 3$ and $\Delta = k\delta = 6$, then the graph H_2 constructed by the above method is shown in Figure 1, where u_1 and u_2 are new vertices.

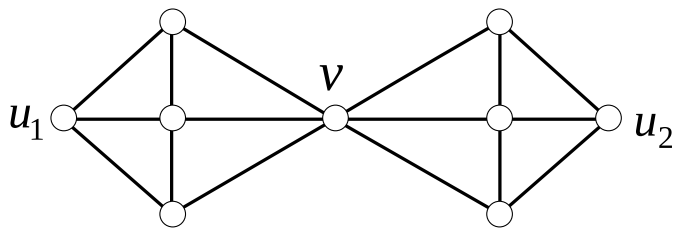

Figure 1. An example to illustrate the construction of H_k.

Furthermore, by Theorem 3, we can obtain a lower bound of the Roman domination number on regular graphs.

Corollary 4. *Let G be an r-regular graph, where* $r \geq 1$. *Then*

$$\gamma_R(G) \geq \frac{3}{2}\gamma(G) \tag{2.5}$$

Moreover, if the equality holds, then

$$\gamma(G) = \frac{2n}{r+2} \quad \text{and} \quad \gamma_R(G) = \frac{3n}{r+2}.$$

Proof. Since G is r-regular, we have $\Delta(G) = \delta(G) = r$. By Theorem 3 we can obtain that this corollary is true. □

For any integer $n \geq 2$, denote by G_{2n} the $(2n-2)$-regular graph with $2n$ vertices, namely G_{2n} is the graph obtained from K_{2n} by deleting a perfect matching. It can be checked that $\gamma(G_{2n}) = 2$ and $\gamma_R(G_{2n}) = 3 = \frac{3}{2}\gamma(G)$ for any $n \geq 2$. Hence, the bound in Corollary 4 is attained.

Note that $\gamma_R(G) \leq 2\gamma(G)$ for any graph G; we can conclude the following result.

Corollary 5. *Let G be an r-regular graph, where* $r \geq 1$. *Then*

$$\frac{3}{2}\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G).$$

3. Relationship between Roman domination and Roman {2}-domination numbers

Chellali et al. [5] obtain the following bounds for the Roman {2}-domination number of a graph G.

Lemma 6. *(Chellali et al. [5]) For every graph G,* $\gamma(G) \leq \gamma_{\{R2\}}(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Lemma 7. *(Chellali et al. [5]) If G is a connected graph of order n and maximum degree $\Delta(G) = \Delta$, then*

$$\gamma_{\{R2\}}(G) \geq \frac{2n}{\Delta + 2}.$$

Theorem 8. *For every graph G, $\gamma_R(G) \leq 2\gamma_{\{R2\}}(G) - 1$. Moreover, for any integer $k \geq 2$, there exists a graph I_k such that $\gamma_{\{R2\}}(I_k) = k$ and $\gamma_R(I_k) = 2k - 1$.*

Proof. Let $f = (V_0, V_1, V_2)$ be an $\gamma_{\{R2\}}$-function of G. Then $\gamma_{\{R2\}}(G) = |V_1| + 2|V_2|$ and $\gamma_R(G) \leq 2|V_1| + 2|V_2|$ since $V_1 \cup V_2$ is a dominating set of G. If $|V_2| \geq 1$, then $\gamma_R(G) \leq 2|V_1| + 2|V_2| = 2\gamma_{\{R2\}}(G) - 2|V_2| \geq 2\gamma_{\{R2\}}(G) - 2$. If $|V_2| = 0$, then every vertex in V_0 is adjacent to at least two vertices in V_1. So for any vertex $u \in V_1$, $f' = (V_0, \{u\}, V_1 \setminus \{u\})$ is an RDF of G. Then we have $\gamma_R(G) \leq 1 + 2|V_1 \setminus \{u\}| = 2|V_1| - 1 = 2\gamma_{\{R2\}}(G) - 1$.

For any integer $k \geq 2$, let I_k be the graph obtained from K_k by replacing every edge of K_k with two paths of length 2. Then $\Delta(I_k) = 2(k - 1)$ and $\delta(I_k) = 2$. We first prove that $\gamma_{\{R2\}}(I_k) = k$. Since $V(I_k) = |V(K_k)| + 2|E(K_k)| = k + 2 \cdot \frac{k(k-1)}{2} = k^2$, by Lemma 7 we can obtain $\gamma_{\{R2\}}(I_k) \geq \frac{2|V(I_k)|}{\Delta(I_k)+2} = \frac{2k^2}{2(k-1)+2} = k$. On the other hand, let $f(x) = 1$ for each $x \in V(I_k)$ with $d(x) = 2(k - 1)$ and $f(y) = 0$ for each $y \in V(I_k)$ with $d(y) = 2$. It can be seen that f is an R{2}DF of I_k and $w(f) = k$. Hence, $\gamma_{\{R2\}}(I_k) = k$.

We now prove that $\gamma_R(I_k) = 2k - 1$. Let $g = \{V_1', V_2', V_3'\}$ be a γ_R-function of I_k such that $|V_1'|$ is minimum. For each 4-cycle $C = v_1v_2v_3v_4v_1$ of I_k with $d(v_1) = d(v_3) = 2(k-1)$ and $d(v_2) = d(v_4) = 2$, we have $w_g(C) = g(v_1)+g(v_2)+g(v_3)+g(v_4) \geq 2$. If $w_g(C) = 2$, then by Lemma 2(iii) we have $g(v_i) \in \{0, 2\}$ for any $i \in \{1, 2, 3, 4\}$. Hence, one of v_1 and v_3 has value 2 and $g(v_2) = g(v_4) = 0$. If $w_g(C) = 3$, then by Lemma 2(i) we have $\{g(v_1), g(v_3)\} = \{1, 2\}$ or $\{g(v_2), g(v_4)\} = \{1, 2\}$. When $\{g(v_2), g(v_4)\} = \{1, 2\}$, let $\{g'(v_1), g'(v_2)\} = \{1, 2\}$, $g'(v_2) = g'(v_4) = 0$ and $g'(x) = g(x)$ for any $x \in V(I_k) \setminus \{v_1, v_2, v_3, v_4\}$. Then g' is also a γ_R-function of I_k. If $w_g(C) = 4$, then exchange the values on C such that v_1, v_3 have value 2 and v_2, v_4 have value 0. So we obtain that I_k has a γ_R-function h such that $h(y) = 0$ for any $y \in V(I_k)$ with degree 2. Note that any two vertices of I_k with degree $2(k - 1)$ belongs to a 4-cycle considered above; we can obtain that there is exactly one vertex z of I_k with degree $2(k - 1)$ such that $h(z) = 1$. Hence, $\gamma_R(I_k) = w(h) = 2k - 1$. □

Note that the graph I_k constructed in Theorem 8 satisfies that $\gamma(I_k) = k = \gamma_{\{R2\}}(I_k)$. By Theorem 8, it suffices to prove that $\gamma(I_k) = k$. Let $A = \{v : v \in V(I_k), d(v) = 2(k - 1)\}$ and $B = V(I_k) \setminus A$. We will prove that I_k has a γ-set containing no vertex of B. Let D be a γ-set of I_k. If D contains a vertex $u \in B$. Since the degree of u is 2, let u_1 and u_2 be two neighbors of u in I_k. Then $d(u_1) = d(u_2) = 2(k - 1)$ and, by the construction of I_k, u_1 and u_2 have two common neighbors u, u' with degree 2. Hence, at least one of u', u_1, and u_2 belongs to D. Let $D' = (D \setminus \{u, u'\}) \cup \{u_1, u_2\}$. Then D' is also a γ-set of I_k. Hence, we can obtain a γ-set of I_k containing no vertex of B by performing the above operation for each vertex $v \in D \cap B$. So A is a γ-set of I_k and $\gamma(I_k) = |A| = k$.

By Lemma 6 and Theorem 8, we can obtain the following corollary.

Corollary 9. *For every graph G, $\gamma_{\{R2\}}(G) \leq \gamma_R(G) \leq 2\gamma_{\{R2\}}(G) - 1$.*

Theorem 10. *For every graph G, $\gamma_R(G) \leq \gamma(G) + \gamma_{\{R2\}}(G) - 1$.*

Proof. By Lemma 6 we can obtain that $\gamma_R(G) \leq 2\gamma(G) \leq \gamma(G) + \gamma_{\{R2\}}(G)$. If the equality holds, then $\gamma_R(G) = 2\gamma(G)$ and $\gamma(G) = \gamma_{\{R2\}}(G)$. So $\gamma_R(G) = 2\gamma_{\{R2\}}(G)$, which contradicts Theorem 8. Hence, we have $\gamma_R(G) \leq \gamma(G) + \gamma_{\{R2\}}(G) - 1$. □

4. Conclusions

In this paper, we prove that $\gamma_R(G) \geq \frac{\Delta+2\delta}{\Delta+\delta}\gamma(G)$ for any nontrivial connected graph G with maximum degree Δ and minimum degree δ, which improves a result obtained by Chellali et al. [4]. As a corollary, we obtain that $\frac{3}{2}\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ for any nontrivial regular graph G. Moreover, we prove that $\gamma_R(G) \leq 2\gamma_{\{R2\}}(G) - 1$ for every graph G and the bound is achieved. Although the bounds in Theorem 3 and Theorem 8 are achieved, characterizing the graphs that satisfy the equalities remain a challenge for further work.

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Conflict of interest

The author declares that they have no conflict of interest.

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