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Research article

More inequalities on numerical radii of sectorial matrices

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Abstract: In this article, we refine some numerical radius inequalities of sectorial matrices recently obtained by Bedrani , Kittaneh and Sababheh. Among other results, it is shown that if $A_i \in \mathbb{M}_n(\mathbb{C})$ with $W(A_i) \subseteq S_\alpha$, $i = 1, 2 \cdots, n$, and $a_1, \cdots, a_n$ are positive real numbers with $\sum_{j=1}^n a_j = 1$, then

$$\omega^t\left(\sum_{i=1}^n a_i A_i\right) \le \cos^{2t}(\alpha)\omega\left(\sum_{i=1}^n a_i A_i^t\right),$$

where $t \in [-1, 0]$. An improvement of the Heinz-type inequality for the numerical radii of sectorial matrices is also given. Moreover, we present some numerical radius inequalities of sectorial matrices involving positive linear maps.

Keywords: sectorial matrices; numerical radius; function; operator mean; positive linear maps
Mathematics Subject Classification: 15A45, 15A60

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. For $A \in \mathbb{M}_n(\mathbb{C})$, the conjugate transpose of A is denoted by A^*, and the matrices $\Re A = \frac{1}{2}(A + A^*)$ and $\Im A = \frac{1}{2i}(A - A^*)$ are called the real part and imaginary part of A, respectively ([6, p. 6] and [12, p. 7]), Moreover, A is called accretive if $\Re A > 0$. For two Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, we write $A \ge B$ (or $B \le A$) if $A - B$ is positive semidifinite. A linear map $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is called positive if it maps positive definite matrices to positive definite matrices and is said to be unital if it maps identity matrices to identity matrices.

The numerical range of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \|x\| = 1\},$$

while the operator norm of A is defined by

$$\|A\| = \max\{|\langle Ax, y\rangle| : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1\}.$$

Let $|||\cdot|||$ denote any unitarily invariant norm on $A \in \mathbb{M}_n(\mathbb{C})$, which satisfies $|||UAV||| = |||A|||$ for any unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. The numerical radius of A is defined by $\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}$. Note that numerical radius is weakly unitarily invariant instead of unitarily invariant, that is, for $A \in \mathbb{M}_n(\mathbb{C})$, $\omega(U^*AU) = \omega(A)$ for every unitary $U \in \mathbb{M}_n(\mathbb{C})$. It is well-known that

$$\omega(A) \le \|A\| \tag{1.1}$$

for $A \in \mathbb{M}_n(\mathbb{C})$.

For $\alpha \in [0, \frac{\pi}{2})$, S_α denotes the sectorial region in the complex plane as follows:

$$S_\alpha = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \le (\Re z)\tan\alpha\}.$$

If $W(A) \subseteq S_0$, then A is positive definite, and if $W(A), W(B) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then $W(A+B) \subseteq S_\alpha$, A is nonsingular and $\Re(A)$ is positive definite. Moreover, $W(A) \subseteq S_\alpha$ implies $W(X^*AX) \subseteq S_\alpha$ for any nonzero $n \times m$ matrix X, thus $W(A^{-1}) \subseteq S_\alpha$. Recently, Tan and Chen [21] also proved that for any positive linear map Φ, $W(A) \subseteq S_\alpha$ implies that $W(\Phi(A)) \subseteq S_\alpha$. Recent developments on sectorial matrices can be found in [10, 13–18, 21, 23].

For two positive definite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$ and $0 \le \lambda \le 1$, the weighted geometric mean is defined by $A\sharp_\lambda B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}$, and the weighted harmonic mean is defined by $A!_\lambda B = ((1-\lambda)A^{-1} + \lambda B^{-1})^{-1}$, while the weighted arithmetic mean is defined by $A\nabla_\lambda B = (1-\lambda)A + \lambda B$. In particular, when $\lambda = \frac{1}{2}$, we denote the geometric mean, harmonic mean and arithmetic mean by $A\sharp B$, $A!B$ and $A\nabla B$, respectively. When $\lambda \notin [0, 1]$, we still define $A\sharp_\lambda B$ as above, which is then not needed to be a matrix mean.

For two accretive matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, Drury [9] defined the geometric mean of A and B as follows

$$A\sharp B = \left(\frac{2}{\pi}\int_0^\infty (tA + t^{-1}B)^{-1}\,\frac{dt}{t}\right)^{-1}. \tag{1.2}$$

This new geometric mean defined by (1.2) possesses some similar properties compared to the geometric mean of positive matrices. For instance, $A\sharp B = B\sharp A$, $(A\sharp B)^{-1} = A^{-1}\sharp B^{-1}$. Moreover, if $A, B \in \mathbb{M}_n(\mathbb{C})$ with $W(A), W(B) \subset S_\alpha$, then $W(A\sharp B) \subset S_\alpha$.

Later, Raissouli, Moslehian and Furuichi [20] defined the following weighted geometric mean of two accretive matrices $A, B \in \mathbb{M}_n(\mathbb{C})$,

$$A\sharp_\lambda B = \frac{\sin\lambda\pi}{\pi}\int_0^\infty t^{\lambda-1}(A^{-1} + tB^{-1})^{-1}\,\frac{dt}{t}, \tag{1.3}$$

where $\lambda \in [0, 1]$. If $\lambda = \frac{1}{2}$, then the formula (1.3) coincides with the formula (1.2).

Very recently, Bedrani, Kittaneh and Sababheh [2] defined a more general operator mean for two accretive matrices $A, B \in \mathbb{M}_n(\mathbb{C})$,

$$A\sigma_f B = \int_0^1 ((1-s)A^{-1} + sB^{-1})^{-1}\,dv_f(s), \tag{1.4}$$

where f: $(0,\infty) \to (0,\infty)$ is an operator monotone function with $f(1) = 1$ and v_f is the probability measure characterizing σ_f. For more information about operator mean, more generally, operator monotone functions that preserve the ordering of real parts of operators, we refer the readers to the recent work of Gaál and Pálfia [11]. Particularly, if $A, B \in \mathbb{M}_n(\mathbb{C})$ with $W(A), W(B) \subset S_\alpha$, then $W(A\sigma_f B) \subset S_\alpha$.

Moreover, they also characterize the operator monotone function for an accretive matrix: let $A \in \mathbb{M}_n(\mathbb{C})$ be accretive and $f : (0,\infty) \to (0,\infty)$ be an operator monotone function with $f(1) = 1$,

$$f(A) = \int_0^1 ((1-s)I + sA^{-1})^{-1}\, dv_f(s), \tag{1.5}$$

where v_f is the probability measure satisfying $f(x) = \displaystyle\int_0^1 ((1-s) + sx^{-1})^{-1}\, dv_f(s)$.

Recently, Mao et al. [19] defined the Heinz mean for two sector matrices $A, B \in \mathbb{M}_n(\mathbb{C})$ with $W(A), W(B) \subseteq S_\alpha$ as

$$H_t(A,B) = \frac{A\sharp_t B + A\sharp_{1-t} B}{2}, \qquad t \in [0,1].$$

Ando [1] proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive definite, then for any positive linear map Φ,

$$\Phi(A\sigma_f B) \leq \Phi(A)\sigma_f\Phi(B). \tag{1.6}$$

Ando's formula (1.6) is known as a matrix Hölder inequality.

To reduce the brackets, we denote $(\Phi(A))^t$ by $\Phi^t(A)$ throughout this paper. The famous Choi's inequality [5, p. 41] says: if Φ is a positive unital linear map and $A > 0$, then

$$\Phi^t(A) \leq \Phi(A^t), \qquad t \in [-1,0]. \tag{1.7}$$

$$\Phi^t(A) \geq \Phi(A^t), \qquad t \in [0,1]. \tag{1.8}$$

For the sake of convenience, we shall need the following notation.

$$\mathfrak{m} = \{f(x),\ \text{where } f : (0,\infty) \to (0,\infty) \text{ is an operator monotone function with } f(1) = 1\}.$$

In a recent paper [3], Bedrani, Kittaneh and Sababheh studied the numerical radius inequalities of sectorial matrices. They [3] obtained relation between $\omega^{-t}(A)$ and $\omega(A^{-t})$ as follows.

Theorem 1.1. Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subseteq S_\alpha$. Then for $t \in [0,1]$

$$\cos(t\alpha)\cos^{2t}(\alpha)\omega^{-t}(A) \leq \omega(A^{-t}).$$

They also [3] gave the Heinz-type inequality for the numerical radii of sectorial matrices below.

Theorem 1.2. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$. Then for $t \in [0,1]$

$$\cos^4(\alpha)\omega(A\sharp B) \leq \omega(H_t(A,B)) \leq \frac{\sec^4(\alpha)}{2}\omega(A+B).$$

In this paper, we intend to improve upon the bounds of Theorem 1.1 and 1.2. Furthermore, we shall present some numerical radius inequalities of sectorial matrices involving positive linear maps.

2. Main results

We begin this section with some lemmas which will be necessary for proving our main results.

Lemma 2.1. (see [2]) Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subseteq S_\alpha$. If $f \in \mathfrak{m}$, then

$$f(\Re A) \le \Re(f(A)) \le \sec^2(\alpha) f(\Re A).$$

In Lemma 2.1, letting $f(x) = x^t, t \in [0,1]$, we have

$$\cos^2(\alpha)\Re A^t \le \Re^t A \le \Re A^t. \tag{2.1}$$

The following lemma gives a better bound of (2.1).

Lemma 2.2. (see [8]) Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subseteq S_\alpha$ and $t \in [0,1]$. Then

$$\cos^{2t}(\alpha)\Re A^t \le \Re^t A \le \Re A^t$$

The famous Löwner-Heinz inequality says that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are such that $A \ge B \ge 0$ and $t \in [0,1]$, then $A^t \ge B^t$. Inspired by Lemma 2.2, a sectorial matrix version is as follows: If $A, B \in \mathbb{M}_n(\mathbb{C})$ with $W(A), W(B) \subseteq S_\alpha$ such that $\Re A \ge \Re B \ge 0$ and $t \in [0,1]$, then $\Re A^t \ge \cos^{2t}(\alpha)\Re B^t$. This is because $\Re A^t \ge \Re^t A \ge \Re^t B \ge \cos^{2t}(\alpha)\Re B^t$.

Next we present a reverse of Lemma 2.2.

Lemma 2.3. (see [8]) Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subseteq S_\alpha$ and $t \in [-1,0]$. Then

$$\Re A^t \le \Re^t A \le \cos^{2t}(\alpha)\Re A^t.$$

Lately, Bedrani, Kittaneh and Sababheh [2] obtained the following inequality for general operator mean of sectorial matrices.

Lemma 2.4. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$. Then

$$\Re A \sigma_f \Re B \le \Re(A \sigma_f B) \le \sec^2(\alpha)(\Re A \sigma_f \Re B).$$

Lemma 2.5. (see [3]) Let $A \in \mathbb{M}_n$ be such that $W(A) \subset S_\alpha$. Then

$$\cos(\alpha)\omega(A) \le \omega(\Re A) \le \omega(A).$$

Lemma 2.6. (see [3]) Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subseteq S_\alpha$. If $f \in \mathfrak{m}$, then

$$|||A\sigma_f B||| \le |||A|||\sigma_f|||B|||,$$

for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}_n(\mathbb{C})$.

Lemma 2.7. (see [6, p.74], [24]) Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subseteq S_\alpha$. Then for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}_n(\mathbb{C})$,

$$\cos(\alpha)|||A||| \le |||\Re A||| \le |||A|||.$$

Lemma 2.8. (see [7]) Let $A_1, A_2, \cdots, A_k \in \mathbb{M}_n(\mathbb{C})$ be positive and $a_1, \cdots, a_k$ be positive real numbers with $\sum_{j=1}^{k} a_j = 1$. Then for every unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}_n(\mathbb{C})$,

$$\left|\left|\left| f\left(\sum_{i=1}^{k} a_i A_i\right)\right|\right|\right| \leq \left|\left|\left| \sum_{i=1}^{k} a_i f(A_i)\right|\right|\right|$$

for every nonnegative convex function f on $[0, \infty)$.

Now we are ready to give our first main result.

Theorem 2.9. Let $A_i \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A_i) \subseteq S_\alpha$, $i = 1, 2\cdots, k$, and $a_1, \cdots, a_k$ be positive real numbers with $\sum_{j=1}^{k} a_j = 1$. Then for $t \in [-1, 0]$,

$$\omega^t\left(\sum_{i=1}^{k} a_i A_i\right) \leq \cos^{2t}(\alpha)\omega\left(\sum_{i=1}^{k} a_i A_i^t\right).$$

Proof. Compute

$$\begin{aligned}
\omega^t\left(\sum_{i=1}^{k} a_i A_i\right) &\leq \omega^t\left(\Re \sum_{i=1}^{k} a_i A_i\right) \quad \text{(by Lemma 2.5)}\\
&= \left\|\Re \sum_{i=1}^{k} a_i A_i\right\|^t\\
&= \left\|\sum_{i=1}^{k} a_i \Re A_i\right\|^t\\
&= \left\|(\sum_{i=1}^{k} a_i \Re A_i)^t\right\|\\
&\leq \left\|\sum_{i=1}^{k} a_i \Re^t A_i\right\| \quad \text{(by Lemma 2.8)}\\
&\leq \cos^{2t}(\alpha)\left\|\sum_{i=1}^{k} a_i \Re A_i^t\right\| \quad \text{(by Lemma 2.3)}\\
&= \cos^{2t}(\alpha)\left\|\Re(\sum_{i=1}^{k} a_i A_i^t)\right\|\\
&= \cos^{2t}(\alpha)\omega(\Re(\sum_{i=1}^{k} a_i A_i^t))\\
&\leq \cos^{2t}(\alpha)\omega(\sum_{i=1}^{k} a_i A_i^t), \quad \text{(by Lemma 2.5)}
\end{aligned}$$

which completes the proof.

Corollary 2.10. Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subseteq S_\alpha$. Then for $t \in [-1, 0]$,

$$\omega^t(A) \leq \cos^{2t}(\alpha)\omega(A^t).$$

Proof. The result directly derived from Theorem 2.9 by substituting $k = 1$.

We remark that Corollary 2.10 is a refinement of Theorem 1.1.

Corollary 2.11. Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subseteq S_\alpha$. Then

$$\omega^{-1}(A) \leq \sec^2(\alpha)\omega(A^{-1}).$$

Proof. The result is directly derived from Corollary 2.10 by substituting $t = -1$.

Thanks to Corollary 2.11, considerable refinements of Theorem 3.6 and 3.12 in [4] are given below.

Corollary 2.12. Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subseteq S_\alpha$ and $B > 0$. Then for $t \in (1, 2)$,

$$\cos^5(\alpha)\omega^{-1}(B^{-2})\omega^{1-t}(A)\omega^{t-2}(B) \leq \omega(A\sharp_t B).$$

Proof. The result directly derived from Theorem 3.6 in [4] and Corollary 2.11.

Corollary 2.13. Let $B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(B) \subseteq S_\alpha$ and $A > 0$. Then for $t \in (-1, 0)$,

$$\cos^5(\alpha)\omega^{-1}(A^{-2})\omega^{-(t+1)}(A)\omega^{t}(B) \leq \omega(A\sharp_t B).$$

Proof. The result directly derived from Theorem 3.12 in [4] and Corollary 2.11.

Next we give a complement of Theorem 2.9.

Theorem 2.14. Let $A_i \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A_i) \subseteq S_\alpha$, $i = 1, 2 \cdots, k$, and $a_1, \cdots, a_k$ be positive real numbers with $\sum_{j=1}^k a_j = 1$. Then for $t \in [-1, 0]$,

$$\omega\left(\left(\sum_{i=1}^k a_i A_i\right)^t\right) \leq \sec(t\alpha)\cos^{2t}(\alpha)\omega\left(\sum_{i=1}^k a_i A_i^t\right).$$

Proof. We have

$$\begin{aligned}
\omega\left(\left(\sum_{i=1}^k a_i A_i\right)^t\right) &\leq \left\|(\sum_{i=1}^k a_i A_i)^t\right\| \quad \text{(by (1.1))} \\
&\leq \sec(t\alpha)\left\|\Re(\sum_{i=1}^k a_i A_i)^t\right\| \quad \text{(by Lemma 2.7)} \\
&\leq \sec(t\alpha)\left\|(\sum_{i=1}^k a_i \Re A_i)^t\right\| \quad \text{(by Lemma 2.7)} \\
&\leq \sec(t\alpha)\left\|\sum_{i=1}^k a_i \Re^t A_i\right\| \quad \text{(by convexity)} \\
&\leq \sec(t\alpha)\cos^{2t}(\alpha)\left\|\sum_{i=1}^k a_i \Re A_i^t\right\| \quad \text{(by Lemma 2.3)} \\
&= \sec(t\alpha)\cos^{2t}(\alpha)\left\|\Re(\sum_{i=1}^k a_i A_i^t)\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \sec(t\alpha)\cos^{2t}(\alpha)\omega\left(\Re(\sum_{i=1}^{k} a_i A_i^t)\right)\\
&\leq \sec(t\alpha)\cos^{2t}(\alpha)\omega\left(\sum_{i=1}^{k} a_i A_i^t\right), \quad \text{(by Lemma 2.5)}
\end{aligned}
$$

completing the proof.

Lemma 2.15. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $t \in [-1, 0]$. If $f \in \mathfrak{m}$, then for any positive unital linear map Φ,

$$\Re(\Phi^t(A)\sigma_f \Phi^t(B)) \leq \cos^{2t}(\alpha)\Re(\Phi(A^t\sigma_f B^t)).$$

Proof. We have the following chain of inequalities

$$
\begin{aligned}
\Re(\Phi^t(A)\sigma_f\Phi^t(B)) &= \Re\left(\int_0^1 ((1-s)\Phi^{-t}(A) + s\Phi^{-t}(B))^{-1}\, d\nu_f(s)\right)\\
&= \int_0^1 \Re((1-s)\Phi^{-t}(A) + s\Phi^{-t}(B))^{-1}\, d\nu_f(s)\\
&\leq \int_0^1 ((1-s)\Re\Phi^{-t}(A) + s\Re\Phi^{-t}(B))^{-1}\, d\nu_f(s) \quad \text{(by Lemma 2.3)}\\
&\leq \int_0^1 ((1-s)(\Re\Phi(A))^{-t} + s(\Re\Phi(B))^{-t})^{-1}\, d\nu_f(s)\, \text{(by Lemma 2.2)}\\
&\leq \int_0^1 ((1-s)\Phi\Re^{-t}(A) + s\Phi\Re^{-t}(B))^{-1}\, d\nu_f(s) \quad \text{(by (1.8))}\\
&= \int_0^1 (\Phi((1-s)\Re^{-t}(A) + t\Re^{-t}(B)))^{-1}\, d\nu_f(s)\\
&\leq \int_0^1 \Phi(((1-s)\Re^{-t}(A) + t\Re^{-t}(B))^{-1})\, d\nu_f(s) \quad \text{(by (1.7))}\\
&\leq \cos^{2t}(\alpha)\Phi\left(\int_0^1 ((1-s)\Re^{-1}(A^t) + t\Re^{-1}(B^t))^{-1}\, d\nu_f(s)\right) \text{(by Lemma 2.3)}\\
&= \cos^{2t}(\alpha)\Phi(\Re A^t\sigma_f \Re B^t)\\
&\leq \cos^{2t}(\alpha)\Phi(\Re(A^t\sigma_f B^t))\\
&= \cos^{2t}(\alpha)\Re(\Phi(A^t\sigma_f B^t)),
\end{aligned}
$$

which completes the proof.

Theorem 2.16. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $t \in [-1, 0]$. If $f \in \mathfrak{m}$, then for any positive unital linear map Φ,

$$\omega(\Phi^t(A)\sigma_f\Phi^t(B)) \leq \sec(t\alpha)\cos^{2t}(\alpha)\omega(\Phi(A^t\sigma_f B^t)).$$

Proof. Compute

$$\omega(\Phi^t(A)\sigma_f\Phi^t(B)) \leq \|\Phi^t(A)\sigma_f\Phi^t(B)\| \quad \text{(by (1.1))}$$

$$
\begin{aligned}
&\le \sec(t\alpha)\|\Re(\Phi^t(A)\sigma_f\Phi^t(B))\| \quad \text{(by Lemma 2.7)}\\
&\le \sec(t\alpha)\cos^{2t}(\alpha)\|\Re(\Phi(A^t\sigma_f B^t))\| \quad \text{(by Lemma 2.15)}\\
&= \sec(t\alpha)\cos^{2t}(\alpha)\omega(\Re(\Phi(A^t\sigma_f B^t)))\\
&\le \sec(t\alpha)\cos^{2t}(\alpha)\omega(\Phi(A^t\sigma_f B^t)). \quad \text{(by Lemma 2.5)}
\end{aligned}
$$

This completes the proof.

The following result presents a reverse of Theorem 2.16.

Theorem 2.17. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $t \in [0, 1]$. If $f \in \mathfrak{m}$, then for any positive unital linear map Φ,

$$\cos^{2t}(\alpha)\cos^3(t\alpha)\omega(\Phi(A^t\sigma_f B^t)) \le \omega(\Phi^t(A)\sigma_f\Phi^t(B)),$$

$$\cos^{2t}(\alpha)\cos^3(t\alpha)\omega(\Phi(A^t\sigma_f B^t)) \le \omega(\Phi^t(A))\sigma_f\omega(\Phi^t(B)).$$

Proof. We estimate

$$
\begin{aligned}
\cos^{2t}(\alpha)\cos^3(t\alpha)\omega(\Phi(A^t\sigma_f B^t)) &\le \cos^{2t}(\alpha)\cos^2(t\alpha)\omega(\Re\Phi(A^t\sigma_f B^t)) \quad \text{(by Lemma 2.5)}\\
&= \cos^{2t}(\alpha)\cos^2(t\alpha)\|\Re\Phi(A^t\sigma_f B^t)\|\\
&= \cos^{2t}(\alpha)\cos^2(t\alpha)\|\Phi\Re(A^t\sigma_f B^t)\|\\
&\le \cos^{2t}(\alpha)\|\Phi(\Re A^t\sigma_f \Re B^t)\| \quad \text{(by Lemma 2.4)}\\
&\le \cos^{2t}(\alpha)\|\Phi(\Re A^t)\sigma_f\Phi(\Re B^t)\| \quad \text{(by (1.6))}\\
&\le \|\Phi(\Re^t A)\sigma_f\Phi(\Re^t B)\| \quad \text{(by Lemma 2.2)}\\
&\le \|\Phi^t(\Re A)\sigma_f\Phi^t(\Re B)\| \quad \text{(by (1.8))}\\
&= \|\Re^t(\Phi(A))\sigma_f\Re^t(\Phi(B))\|\\
&\le \|\Re(\Phi^t(A))\sigma_f\Re(\Phi^t(B))\| \quad \text{(by Lemma 2.2)}\\
&\le \|\Re(\Phi^t(A)\sigma_f\Phi^t(B))\| \quad \text{(by Lemma 2.4)}\\
&= \omega(\Re(\Phi^t(A)\sigma_f\Phi^t(B)))\\
&\le \omega(\Phi^t(A)\sigma_f\Phi^t(B)), \quad \text{(by Lemma 2.5)}
\end{aligned}
$$

which proves the first inequality. To prove the second inequality, compute

$$
\begin{aligned}
\cos^{2t}(\alpha)\cos^3(t\alpha)\omega(\Phi(A^t\sigma_f B^t)) &\le \|\Re(\Phi^t(A))\sigma_f\Re(\Phi^t(B))\|\\
&\le \|\Re(\Phi^t(A))\|\sigma_f\|\Re(\Phi^t(B))\|\\
&= \omega(\Re(\Phi^t(A)))\sigma_f\omega(\Re(\Phi^t(B)))\\
&\le \omega(\Phi^t(A))\sigma_f\omega(\Phi^t(B)),
\end{aligned}
$$

where the first inequality is obtained by the preceding proof, the second one is by Lemma 2.6 and the last one is due to Lemma 2.5. This completes the proof.

Let $t = 1$ in Theorem 2.17, one can obtain

$$\cos^5(\alpha)\omega(\Phi(A\sigma_f B)) \le \omega(\Phi(A)\sigma_f\Phi(B)), \tag{2.2}$$

$$\cos^5(\alpha)\omega(\Phi(A\sigma_f B)) \le \omega(\Phi(A))\sigma_f\omega(\Phi(B)). \tag{2.3}$$

Next we are attempting to refine inequalities (2.2) and (2.3).

Theorem 2.18. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$. If $f \in \mathfrak{m}$, then for any positive unital linear map Φ,

$$\omega(\Phi(A\sigma_f B)) \leq \sec^3(\alpha)\omega(\Phi(A)\sigma_f\Phi(B)), \tag{2.4}$$

$$\omega(\Phi(A\sigma_f B)) \leq \sec^3(\alpha)\omega(\Phi(A))\sigma_f\omega(\Phi(B)). \tag{2.5}$$

Proof. To prove inequality (2.4), compute

$$\begin{aligned}
\omega(\Phi(A\sigma_f B)) &\leq \|\Phi(A\sigma_f B)\| \quad \text{(by (1.1))} \\
&\leq \sec(\alpha)\|\Re\Phi(A\sigma_f B)\| \quad \text{(by Lemma 2.7)} \\
&= \sec(\alpha)\|\Phi\Re(A\sigma_f B)\| \\
&\leq \sec^3(\alpha)\|\Phi(\Re(A)\sigma_f\Re(B))\| \quad \text{(by Lemma 2.4)} \\
&\leq \sec^3(\alpha)\|\Phi(\Re(A))\sigma_f\Phi(\Re(B))\| \quad \text{(by (1.6))} \\
&= \sec^3(\alpha)\|\Re(\Phi(A))\sigma_f\Re(\Phi(B))\| \\
&\leq \sec^3(\alpha)\|\Re(\Phi(A)\sigma_f\Phi(B))\| \quad \text{(by Lemma 2.4)} \\
&= \sec^3(\alpha)\omega(\Re(\Phi(A)\sigma_f\Phi(B))) \\
&\leq \sec^3(\alpha)\omega(\Phi(A)\sigma_f\Phi(B)). \quad \text{(by Lemma 2.5)}
\end{aligned}$$

Next we prove inequality (2.5).

$$\begin{aligned}
\omega(\Phi(A\sigma_f B)) &\leq \sec^3(\alpha)\|\Re(\Phi(A))\sigma_f\Re(\Phi(B))\| \\
&\leq \sec^3(\alpha)\|\Re(\Phi(A))\|\sigma_f\|\Re(\Phi(B))\| \\
&= \sec^3(\alpha)\omega(\Re(\Phi(A)))\sigma_f\omega(\Re(\Phi(B))) \\
&\leq \sec^3(\alpha)\omega(\Phi(A))\sigma_f\omega(\Phi(B)),
\end{aligned}$$

where the first inequality is obtained by the preceding proof, the second one is by Lemma 2.6 and the last one is due to Lemma 2.5. This completes the proof.

We remark that (2.4) coincides with Theorem 3.7 in [3] when setting $\Phi(X) = X$ for every $X \in \mathbb{M}_n(\mathbb{C})$.

Theorem 2.19. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$. Then for $t \in (0, 1)$,

$$\cos^3(\alpha)\omega(A\sharp B) \leq \omega(H_t(A, B)) \leq \frac{\sec^3(\alpha)}{2}\omega(A + B).$$

Proof. To prove the first inequality, We have

$$\begin{aligned}
\omega(A\sharp B) &\leq \|A\sharp B\| \quad \text{(by (1.1))} \\
&\leq \sec(\alpha)\|\Re(A\sharp B)\| \quad \text{(by Lemma 2.7)} \\
&\leq \sec^3(\alpha)\|\Re(H_t(A, B))\| \quad \text{(by Theorem 2.9 in [19])} \\
&= \sec^3(\alpha)\omega(\Re(H_t(A, B))) \\
&\leq \sec^3(\alpha)\omega(H_t(A, B)). \quad \text{(by Lemma 2.5)}
\end{aligned}$$

Next we show the second inequality

$$\begin{aligned}\omega(H_t(A,B)) &\leq \|H_t(A,B)\| \quad \text{(by (1.1))}\\ &\leq \sec(\alpha)\|\Re(H_t(A,B))\| \quad \text{(by Lemma 2.7)}\\ &\leq \frac{\sec^3(\alpha)}{2}\|\Re(A+B)\| \quad \text{(by Theorem 2.9 in [19])}\\ &= \frac{\sec^3(\alpha)}{2}\omega(\Re(A+B))\\ &\leq \frac{\sec^3(\alpha)}{2}\omega(A+B), \quad \text{(by Lemma 2.5)}\end{aligned}$$

which completes the proof.

We remark that Theorem 2.19 is an improvement of Theorem 1.2.

Consider a partitioned matirx $A \in \mathbb{M}_n(\mathbb{C})$ in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. If A_{11} is invertible, we denote the Schur complement of A_{11} in A by $S(A) = A_{22} - A_{21}A_{11}^{-1}A_{12}$. Whenever we mention $S(B)$, we assume $B \in \mathbb{M}_n(\mathbb{C})$ has the partition mentioned above and the relevant inverse exists.

In [25], the author gave the mean inequalities for the Schur complement of sectorial matrices. Next we try to derive the numerical radius inequalities of the Schur complement of sectorial matrices.

Theorem 2.20. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $t \in (0,1)$. Then

$$\omega(S(A)\nabla_t S(B)) \leq \sec^3(\alpha)\omega(S(A\nabla_t B)), \tag{2.6}$$

$$\omega(S(A)\sharp_t S(B)) \leq \sec^5(\alpha)\omega(S(A\nabla_t B)), \tag{2.7}$$

$$\omega(S(A)!_t S(B)) \leq \sec^5(\alpha)\omega(S(A\nabla_t B)). \tag{2.8}$$

Proof. First we prove inequality (2.6)

$$\begin{aligned}\omega(S(A)\nabla_t S(B)) &\leq \|S(A)\nabla_t S(B)\| \quad \text{(by (1.1))}\\ &\leq \sec(\alpha)\|\Re(S(A)\nabla_t S(B))\| \quad \text{(by Lemma 2.7)}\\ &\leq \sec^3(\alpha)\|\Re(S(A\nabla_t B))\| \quad \text{(by Theorem 1.2 in [25])}\\ &= \sec^3(\alpha)\omega(\Re(S(A\nabla_t B)))\\ &\leq \sec^3(\alpha)\omega(S(A\nabla_t B)). \quad \text{(by Lemma 2.5)}\end{aligned}$$

To show inequality (2.7), we have

$$\begin{aligned}\omega(S(A)\sharp_t S(B)) &\leq \|S(A)\sharp_t S(B)\| \quad \text{(by (1.1))}\\ &\leq \sec(\alpha)\|\Re(S(A)\sharp_t S(B))\| \quad \text{(by Lemma 2.7)}\end{aligned}$$

$$
\begin{aligned}
&\leq \sec^5(\alpha)\|\Re(S(A\sharp_t B))\| \quad \text{(by Theorem 1.2 in [25])}\\
&= \sec^5(\alpha)\omega(\Re(S(A\sharp_t B)))\\
&\leq \sec^5(\alpha)\omega(S(A\sharp_t B)). \quad \text{(by Lemma 2.5)}
\end{aligned}
$$

Now we prove inequality (2.8)

$$
\begin{aligned}
\omega(S(A)!_t S(B)) &\leq \|S(A)!_t S(B)\| \quad \text{(by (1.1))}\\
&\leq \sec(\alpha)\|\Re(S(A)!_t S(B))\| \quad \text{(by Lemma 2.7)}\\
&\leq \sec^5(\alpha)\|\Re(S(A!_t B))\| \quad \text{(by Theorem 1.2 in [25])}\\
&= \sec^5(\alpha)\omega(\Re(S(A!_t B)))\\
&\leq \sec^5(\alpha)\omega(S(A!_t B)), \quad \text{(by Lemma 2.5)}
\end{aligned}
$$

completing the proof.

The celebrated Bellman type operator inequality states that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite and $f : (0, \infty) \to (0, \infty)$ is operator convex, then for any positive unital linear map Φ,

$$
f(\Phi(A\nabla_t B)) \geq \Phi(f(A)\nabla_t f(B)). \tag{2.9}
$$

We shall generalize the settings of the Bellman type operator inequality to sectorial matrices as follows.

Lemma 2.21. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $t \in [0, 1]$. If $f \in \mathfrak{m}$, then for any positive unital linear map Φ,

$$
\Re f(\Phi(A\nabla_t B)) \geq \cos^2(\alpha)\Re\Phi(f(A)\nabla_t f(B)).
$$

Proof. We estimate

$$
\begin{aligned}
\Re f(\Phi(A\nabla_t B)) &\geq f(\Re\Phi(A\nabla_t B)) \quad \text{(by Lemma 2.1)}\\
&= f(\Phi\Re(A\nabla_t B))\\
&\geq \Phi(f(\Re A)\nabla_t f(\Re B)) \quad \text{(by (2.9))}\\
&\geq \cos^2(\alpha)\Phi(\Re f(A)\nabla_t \Re f(B)) \quad \text{(by Lemma 2.1)}\\
&= \cos^2(\alpha)\Phi(\Re(f(A)\nabla_t f(B)))\\
&= \cos^2(\alpha)\Re\Phi(f(A)\nabla_t f(B)),
\end{aligned}
$$

completing the proof.

We remark that in Lemma 2.21 putting $t = 0$, we get Theorem 6.3 in [2].

Theorem 2.22. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $t \in [0, 1]$. If $f \in \mathfrak{m}$, then for any positive unital linear map Φ,

$$
\omega(f(\Phi(A\nabla_t B))) \geq \cos^3(\alpha)\omega(\Phi(f(A)\nabla_t f(B))).
$$

Proof. We estimate

$$
\omega(f(\Phi(A\nabla_t B))) \geq \omega(\Re f(\Phi(A\nabla_t B))) \quad \text{(by Lemma 2.5)}
$$

$$
\begin{aligned}
&= \|\Re f(\Phi(A\nabla_t B))\| \\
&\geq \cos^2(\alpha)\|\Re\Phi(f(A)\nabla_t f(B))\| \quad \text{(by Lemma 2.21)} \\
&= \cos^2(\alpha)\omega(\Re\Phi(f(A)\nabla_t f(B))) \quad \text{(by (6.8) in [2])} \\
&\geq \cos^3(\alpha)\omega(\Phi(f(A)\nabla_t f(B))). \quad \text{(by Lemma 2.5)}
\end{aligned}
$$

This completes the proof.

The following corollary is a complement of Proposition 3.3 in [3].

Corollary 2.23. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $t \in [0, 1]$. If $f \in \mathfrak{m}$, then

$$\omega(f(A\nabla_t B)) \geq \cos^3(\alpha)\omega(f(A)\nabla_t f(B)).$$

Proof. Let $\Phi(X) = X$ for every $X \in \mathbb{M}_n(\mathbb{C})$ in Theorem 2.22, we get the desired result.

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Conflict of interest

The author declares that he has no conflict of interest.

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