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Research article

Dynamic cumulative residual Rényi entropy for Lomax distribution: Bayesian and non-Bayesian methods

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Abstract: An alternative measure of uncertainty related to residual lifetime function is the dynamic cumulative residual entropy which plays a significant role in reliability and survival analysis. This article deals with estimating dynamic cumulative residual Rényi entropy (DCRRE) for Lomax distribution using maximum likelihood and Bayesian methods of estimation. The maximum likelihood estimates and approximate confidence intervals of DCRRE are derived. Bayesian estimates and Bayesian credible intervals are derived based on gamma priors for the DCRRE under squared error, linear exponential (LINEX) and precautionary loss functions. The Metropolis-Hastings algorithm is employed to generate Markov chain Monte Carlo samples from the posterior distributions. The Bayes estimates are compared through Monte Carlo simulations. Regarding simulation results, we observe that the maximum likelihood and Bayesian estimates of the DCRRE are decreasing function on time. Further, maximum likelihood and Bayesian estimates of the DCRRE perform well as the sample size increases. Bayesian estimate of the DCRRE under LINEX loss function is more convenient than the other estimates in the most of the situations. Real data set is analyzed for clarifying purposes.

Keywords: dynamic cumulative residual Rényi entropy; Lomax distribution; Bayesian estimates; loss function
Mathematics Subject Classification: 62G07, 62C05, 62E20

1. Introduction

Entropy is a fundamental uncertainty measure of a random variable. Shannon [1] provided a quantitative measure of uncertainty as a measure of information. A flexible extension of Shannon entropy was introduced by Rényi [2]. The Rényi entropy is one parameter generalization of the Shannon entropy and can be used as a measure of randomness. Applications of Rényi entropy can be found in many fields, such as biology, genetics, electrical engineering, computer science, economics, chemistry and physics.

The Rényi entropy has been widely used in analysis of the quantum systems. It can be employed in the analysis of quantum communication protocols (Renner et al. [3]), quantum correlations (Lévay et al. [4]). Also it can be used to characterize the number of reads needed to reconstruct a DNA sequences (see, Motahari et al. [5]). Rényi entropy has numerous issues as a data analysis tools in many practical applications, especially in data that dealing with time-frequency representations. Gabarda and Cristobal [6] applied the Rényi entropy to an image fusion method and identified which pixels have a higher amount of information among the given input images. Rényi entropy can be used to calculate the risk measure of portfolio.

Entropy concepts can be related to informational market efficiency. Martina et al. [7] studied the complexity of crude oil prices using entropy measures to monitor the evolution of crude oil price movements. They found that the methods based on entropy concepts can shed light on the structure of crude oil markets as well as on its link to macroeconomic conditions and socio-political extreme events. Resconi et al. [8] provided a new suggestive reading of quantum mechanics by starting from the superposition of different Boltzmann entropies. They suggested that a quantum entropy space can be used as the fundamental arena that describing the quantum effects. Another measure of uncertainty deals with residual lifetime function is dynamic cumulative residual entropy (DCRE) which can be attractive in many fields like reliability and survival analysis. Recently, there are considerable literatures assigned to the applications, generalizations and properties of Rényi's measure of entropy.

Recently, there are many literatures studied the estimation procedure for the entropy measures. Kayal and Kumar [9] considered Bayesian estimation of entropy for exponential distribution under LINEX loss function. Seo et al. [10] produced an entropy estimate using upper record values from the generalized half-logistic distribution. Based on generalized progressive hybrid censoring scheme, Bayesian estimates of entropy for Weibull distribution were discussed by Cho et al. [11]. Based on record values, Chacko and Asha [12] studied the estimation of the entropy for generalized exponential distribution. Patra et al. [13] estimated a function of scale parameter of an exponential population under general loss function. Maximum likelihood (ML) estimate of Shannon entropy for inverse Weibull distribution under multiple censored data was discussed by Hassan and Zaky [14].

Petropoulos et al. [15] provided improved estimators of the entropy for the scale mixture of exponential distributions. Estimation of entropy for inverse Lomax distribution under multiple censored has been studied by Bantan et al. [16]. Bayesian estimate of the entropy for Lomax distribution based on upper record values has been studied by Hassan and Zaky [17].

If X is an absolutely continuous random variable with probability density function (PDF) $f(x)$, then the corresponding Rényi entropy of order β is defined as:

$$I_R(\beta)=\frac{1}{1-\beta}\log\left(\int_{-\infty}^{\infty} f^{\beta}(x)dx\right),\ \beta>0 \quad \text{and } \beta\neq 1. \tag{1}$$

Recently, measurements of uncertainty for probability distributions became more interested. Sunoj and Linu [18] defined the cumulative residual Rényi entropy (CRRE) for residual lifetime $X_t=(X-t \mid X>t)$ based on survival function instead of using PDF as follows:

$$\gamma_R(\beta)=\frac{1}{1-\beta}\log\left(\int_{0}^{\infty} \bar{F}^{\beta}(x)dx\right),\ \beta>0 \text{ and } \beta\neq 1, \tag{2}$$

where, $\bar{F}(t)=P(X>t)=1-F(t)$ is the survival (reliability) function. The main features of the CRRE are always non-negative as well as it has consistent definitions in both the continuous and discrete domains. Furthermore it can be easily computed from sample data and these computations asymptotically converge to the true values (see, Rao et al. [19]). Sunoj and Linu [18] defined the DCRRE as a dynamic form of uncertainty as follows:

$$\gamma_R(\beta)=\frac{1}{1-\beta}\log\left(\int_{t}^{\infty} \frac{\bar{F}^{\beta}(x)}{\bar{F}^{\beta}(t)}dx\right),\ \beta>0 \text{ and} \beta\neq 1. \tag{3}$$

Therefore, when $t=0$, the DCRRE tends to CRRE. In the literature, few studies had been done concerning the inferential procedures of the entropy measures incorporating DCRRE for lifetime distributions. Kamari [20] produced some properties of the DCRRE based on order statistics. Kundu et al. [21] presented the cumulative residual and past inaccuracy measures which are extensions of the corresponding cumulative entropies for truncated random variables. Renjini et al. [22] discussed the Bayesian estimates of the DCRRE for the Pareto distribution using Type II right censored data. Renjini et al. [23] discussed the Bayesian estimates of the DCRRE for Pareto distribution of the first kind under upper record values. Renjini et al. [24] obtained Bayesian estimate of the DCRRE for Pareto distribution based on complete data. Bayesian estimate of dynamic cumulative residual Shannon entropy for Pareto II distribution has been studied by Ahmadini et al. [25].

Lomax [26] proposed a very good alternative to the common lifetime distributions such as exponential, Weibull, or gamma when the experimenter presumes that the population distribution may be heavy-tailed. Lomax distribution can be applied in a variety of fields such as a business failure data model, income and wealth inequality, computer science, risk analysis and economics, actuarial science and reliability. Corbellini et al. [27] used it to model firm size and queuing problems. As an important lifetime distribution, the Lomax distribution with shape parameter α and scale parameter λ has the following PDF

$$f(x;\alpha,\lambda)=\alpha\lambda^{\alpha}(x+\lambda)^{-(\alpha+1)} \qquad , x,\alpha,\lambda>0. \tag{4}$$

The cumulative distribution function (CDF) corresponding to (4) is given by,

$$F(x;\alpha,\lambda)=1-\lambda^{\alpha}(x+\lambda)^{-\alpha} \qquad , x,\alpha,\lambda>0. \tag{5}$$

At $\lambda=1,\alpha\neq1$, Lomax distribution reduces to the beta prime distribution (also known as inverted beta distribution or beta distribution of the second kind). At $\alpha=1,\lambda\neq1$, Lomax distribution reduces to a log-logistic distribution. At $\alpha=\lambda=1$, the Lomax distribution reduces to F-distribution, F (2, 2). The reliability and hazard rate functions of Lomax distribution are given, respectively, by

$$\bar{F}(x;\alpha,\lambda)=\lambda^{\alpha}(x+\lambda)^{-\alpha},$$

and,

$$h(x;\alpha,\lambda)=\alpha(x+\lambda)^{-1}.$$

The hazard rate function of Lomax distribution takes different shapes according to values of shape parameter α. It is a constant at $\alpha=1$, and it is continuously decreasing at $\alpha>1$, which represents early failures.

Studies about the Lomax distribution have been provided by many authors. For instance, Abd-Elfattah et al. [28] obtained the Bayesian and non-Bayesian estimates of the sample size for the Lomax distribution in case of Type-I censored samples. Record values of Lomax distribution were discussed by Ahsanullah [29]. Balakrishnan and Ahsanullah [30] introduced some recurrence relations between the moments of record values from Lomax distribution. Hassan and Al-Ghamdi [31] determined the optimum test plan for simple step stress accelerated life testing. Hassan et al. [32] discussed the optimal times of changing stress level for k-level step stress accelerated life tests based on adaptive Type-II progressive hybrid censoring with product's life time following Lomax distribution. For more application about Lomax distribution, the reader can refer to [33–38].

From the above, it is evident that the Lomax distribution has been received greatest attention from theoretical and statisticians primarily due to its use in various fields. In the literature, there are no reports about the statistical inference of the DCRRE for Lomax distribution. So we need to estimate the DCRRE for Lomax distribution in view of Bayesian and non-Bayesian procedures based on complete samples. The ML estimates and approximate confidence intervals of the DCRRE are obtained. The Bayesian estimate is calculated using gamma priors under squared error (SE), LINEX and precautionary (PRE) loss functions. According to the complicated forms of the DCRRE Bayesian estimate, we employ the Markov Chain Monte Carlo (MCMC) simulation technique for numerical study.

This article can be organized as follows. Section 2 gives the DCRRE of Lomax distribution as well as obtains the ML and approximate confidence intervals of DCRRE. Section 3 presents Bayesian estimate of the DCRRE for Lomax distribution under symmetric and asymmetric loss functions. The MCMC technique is given in Section 4. A real data application is illustrated in Section 5. The paper ends with some conclusions based on the results of the numerical studies.

2. Maximum likelihood estimate of the DCRRE

This section provides an explicit expression of the DCRRE for Lomax distribution. Then, the ML estimate of the DCRRE is derived based on complete random sample.

The DCRRE of Lomax distribution is obtained by substituting (5) in (3) as follows:

$$\gamma_R(\beta) = \frac{1}{1-\beta}\log\left(\int_t^{\infty}\frac{(x+\lambda)^{-\alpha\beta}}{(t+\lambda)^{-\alpha\beta}}dx\right). \tag{6}$$

Hence, after simplification the DCRRE takes the form

$$\gamma_R(\beta) = \frac{1}{1-\beta}\log\left(\frac{t+\lambda}{\alpha\beta-1}\right),\quad \beta>0,\ \beta\neq 1 \text{ and } \alpha\beta>1. \tag{7}$$

This is the required expression of the DCRRE for Lomax distribution. To obtain the ML estimate of $\gamma_R(\beta)$, we must obtain the ML estimate of the model parameters α and λ.

Consider a simple random sample of size n drawn from PDF (4) and CDF (5), where α and λ are unknown. Then, given the sample $\underline{x}=(x_1,x_2,...,x_n)$, the likelihood function of Lomax distribution can be written as follows

$$L(\alpha,\lambda\,|\,\underline{x}) = \alpha^n\lambda^{n\alpha}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}. \tag{8}$$

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. Therefore, the logarithm of likelihood function, say $\ln\ell$, is

$$\ln\ell = n\log\alpha + n\alpha\log\lambda - (\alpha+1)\sum_{i=1}^{n}\log(x_i+\lambda).$$

The partial derivatives of the log-likelihood function with respect to α and λ, are obtained as follows:

$$\frac{\partial\ln\ell}{\partial\alpha} = \frac{n}{\alpha} + n\log\lambda - \sum_{i=1}^{n}\log(x_i+\lambda), \tag{9}$$

and

$$\frac{\partial\ln\ell}{\partial\lambda} = \frac{n\alpha}{\lambda} - (\alpha+1)\sum_{i=1}^{n}(x_i+\lambda)^{-1}. \tag{10}$$

The ML estimates of α and λ are determined by solving the equations $\partial\ln\ell/\partial\alpha=0$, and $\partial\ln\ell/\partial\lambda=0$, simultaneously. Further the resulting equations cannot be solved analytically, so, numerical technique must be applied to solve these equations, simultaneously, to obtain $\hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$. The existence and uniqueness of $\hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$ can be checked by solving, algebraically, $\partial\ln\ell/\partial\alpha=0$ and $\partial\ln\ell/\partial\lambda=0$, simultaneously and checking that, the second derivative is negative which is a maximum point.

Hence, by using the invariance property of ML estimate, then ML estimates of the DCRRE denoted by $\hat{\gamma}_{R,ML}(\beta)$ becomes

$$\hat{\gamma}_{R,ML}(\beta)=\frac{1}{1-\beta}\log\left(\frac{t+\hat{\lambda}_{ML}}{\hat{\alpha}_{ML}\beta-1}\right),\quad \beta>0,\ \beta\neq 1\,\text{and}\,\hat{\alpha}_{ML}\beta>1. \tag{11}$$

Furthermore, a confidence interval of the DCRRE is the probability that a real value of the entropy will fall between an upper and lower bounds of a probability distribution. For large sample size, the ML estimates, under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the two-sided approximate confidence limits for $\hat{\gamma}_{R,ML}(\beta)$ can be constructed, such that

$$p\left[-z_{\eta/2}\le\frac{\hat{\gamma}_{R,ML}(\beta)-\gamma_{R,ML}(\beta)}{\sigma_{(\hat{\gamma}_{R,ML}(\beta))}}\le z_{\eta/2}\right]=1-\eta, \tag{12}$$

where, $z_{\eta/2}$ is $100(1-\eta)$ the standard normal percentile, and η is the significant level. Therefore, the lower (L) and upper (U) confidence limits for $\hat{\gamma}_{R,ML}(\beta)$ can be obtained as

$$L=\hat{\gamma}_{R,ML}(\beta)-z_{\frac{\eta}{2}}\sigma_{(\hat{\gamma}_{R,ML}(\beta))}\ \text{and}\ U=\hat{\gamma}_{R,ML}(\beta)+z_{\frac{\eta}{2}}\sigma_{(\hat{\gamma}_{R,ML}(\beta))}, \tag{13}$$

where $\sigma_{(\hat{\gamma}_{R,ML}(\beta))}$ is the standard deviation. The two-sided approximate confidence limits for DCRRE will be constructed with confidence level 95 %.

3. Bayesian estimate of the DCRRE

In this section, the Bayesian estimates of $\gamma_R(\beta)$ under symmetric (SE) and asymmetric (LINEX and PRE) loss functions are obtained by assuming the priors of parameters α and λ have gamma distributions. As reported by Kayal and Kumar [9], the SE loss function is used when weights of error caused by observation and underestimation are equal. But asymmetric loss function is used when the overestimation may have more serious than underestimation or vice versa. To compute the Bayesian estimate of $\gamma_R(\beta)$, we proceed as follows. Firstly, the Bayesian estimates of α and λ are obtained and then the DCRRE estimate will be calculated by substituting the parameter estimates in (7). Additionally, the width of the Bayesian credible interval (BCI) estimates is obtained.

In Bayesian method, there are many methods that can be used in selection of prior. There are two types of the prior distributions, non-informative and informative prior (IP) distributions. The prior of parameters may be selected as independent or dependent priors. In this paper, we assume the prior of parameters α and λ, denoted by $\pi_1(\alpha)$ and $\pi_2(\lambda)$ are independent. Following Pak and Mahmoudi [39], we take $\pi_1(\alpha)$ and $\pi_2(\lambda)$ to be gamma distributions with parameters (a,b) and (c,d) as follows

$$\pi_1(\alpha)=\frac{a^{b}}{\Gamma(b)}\alpha^{b-1}e^{-\alpha a},\quad \pi_2(\lambda)=\frac{c^{d}}{\Gamma(c_2)}\lambda^{c-1}e^{-\lambda d}, \tag{14}$$

where a, b, c and d are known and non-negative. So, the joint posterior for parameters, denoted by $\pi^{*}_{1,2}(\alpha,\lambda)$, is

$$\pi_{1,2}{}^{*}(\alpha,\lambda\,|\,\underline{x})=\frac{L(\alpha,\lambda\,|\,\underline{x})\pi_1(\alpha)\pi_2(\lambda)}{\int_0^\infty\int_0^\infty L(\alpha,\lambda\,|\,\underline{x})\pi_1(\alpha)\pi_2(\lambda)d\alpha d\lambda}$$

$$=S^{-1}\,\alpha^{n+b-1}\lambda^{n\alpha+d-1}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)},$$

where,

$$S=\int_0^\infty\int_0^\infty\alpha^{n+b-1}\lambda^{n\alpha+d-1}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}d\alpha d\lambda. \tag{15}$$

So, the marginal posterior PDF of parameters α and λ is given respectively by:

$$\pi_1{}^{*}(\alpha\,|\,\underline{x})=\frac{\int_0^\infty\alpha^{n+b-1}\lambda^{n\alpha+d-1}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}d\lambda}{\int_0^\infty\int_0^\infty\alpha^{n+b-1}\lambda^{n\alpha+d-1}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}d\alpha d\lambda}, \tag{16}$$

and

$$\pi_2{}^{*}(\lambda\,|\,\underline{x})=\frac{\int_0^\infty\alpha^{n+b-1}\lambda^{n\alpha+d-1}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}d\alpha}{\int_0^\infty\int_0^\infty\alpha^{n+b-1}\lambda^{n\alpha+d-1}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}d\alpha d\lambda}. \tag{17}$$

Therefore, the Bayesian estimates of α and λ under SE loss function, say $\hat{\alpha}_{SE}$, and $\hat{\lambda}_{SE}$ can be obtained as posterior mean as follows:

$$\hat{\alpha}_{SE}=S^{-1}\int_0^\infty\int_0^\infty\alpha^{n+b}\lambda^{n\alpha+d-1}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}d\alpha d\lambda, \tag{18}$$

and,

$$\hat{\lambda}_{SE}=S^{-1}\int_0^\infty\int_0^\infty\alpha^{n+b-1}\lambda^{n\alpha+d}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}d\alpha d\lambda. \tag{19}$$

While, the Bayesian estimates of parameters α and λ under LINEX loss function, say $\hat{\alpha}_{LINEX}$, and $\hat{\lambda}_{LINEX}$ are obtained as follows:

$$\hat{\alpha}_{LINEX} = \frac{-1}{\nu}\log[\int_0^\infty e^{-\alpha\nu}\,\pi_2^*(\alpha\,|\,\underline{x})\,d\alpha],\ \nu \neq 0$$
$$= \frac{-1}{\nu}\log\left(S^{-1}\int_0^\infty\int_0^\infty \alpha^{n+b-1}\,\lambda^{n\alpha+d-1}e^{-(\nu\alpha+\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}d\alpha\,d\lambda\right), \tag{20}$$

and

$$\hat{\lambda}_{LINEX} = \frac{-1}{\nu}\log[\int_0^\infty e^{-\lambda\nu}\,\pi_2^*(\lambda\,|\,\underline{x})\,d\lambda],\ \nu \neq 0$$
$$= \frac{-1}{\nu}\log\left(S^{-1}\int_0^\infty\int_0^\infty \alpha^{n+b-1}\,\lambda^{n\alpha+d-1}e^{-(\nu\lambda+\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}d\alpha\,d\lambda\right), \tag{21}$$

where, ν is a real number. Additionally, the Bayesian estimates of parameters α and λ under PRE loss function say $\hat{\alpha}_{PRE}$ and $\hat{\lambda}_{PRE}$ are given as follows:

$$\hat{\alpha}_{PRE} = [\int_0^\infty \alpha^2\,\pi_1^*(\alpha\,|\,\underline{x})\,d\alpha]^{\frac{1}{2}}$$
$$= \left(S^{-1}\int_0^\infty\int_0^\infty \alpha^{n+b+1}\,\lambda^{n\alpha+d-1}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}\,d\alpha\,d\lambda\right)^{\frac{1}{2}}, \tag{22}$$

and

$$\hat{\lambda}_{PRE} = [\int_0^\infty \lambda^2\,\pi_2^*(\lambda\,|\,\underline{x})\,d\lambda]^{\frac{1}{2}}$$
$$= \left(S^{-1}\int_0^\infty\int_0^\infty \alpha^{n+b-1}\,\lambda^{n\alpha+d+1}e^{-(\alpha a+\lambda c)}\prod_{i=1}^{n}(x_i+\lambda)^{-(\alpha+1)}\,d\alpha\,d\lambda\right)^{\frac{1}{2}}. \tag{23}$$

Integrals (18)–(23) do not take a closed form, so the Metropolis-Hastings (M-H) and random-walk Metropolis algorithms are employed to generate the MCMC samples from posterior density functions (16) and (17), respectively. Hence, the Bayesian estimates of α and λ under SE, LINEX and PRE loss functions are obtained as the mean of the simulated samples from their posteriors. Further, once the Bayes estimates of α and λ are obtained. Therefore, the Bayesian estimate of the DCRRE, denoted by $\hat{\gamma}_{R,BE}(\beta)_{SE}$, under SE loss function is obtained, by using (7), as follows:

$$\hat{\gamma}_{R,BE}(\beta)_{SE} = \frac{1}{1-\beta}\log\left(\frac{t+\hat{\lambda}_{SE}}{\hat{\alpha}_{SE}\,\beta-1}\right). \tag{24}$$

By similar way, we obtain the Bayesian estimate of $\gamma_R(\beta)$ for Lomax distribution under LINEX, and PRE loss functions. Furthermore, the BCI is a useful summary of the posterior distribution which reflects its variation that is used to quantify the statistical uncertainty. An approximate highest posterior density interval of $\gamma_R(\beta)$ is obtained by using the same algorithm of Chen and Shao [40].

4. Monte Carlo simulation

This section evaluates the performance of the ML and Bayes estimates (BEs) based on simulation study.

The maximum likelihood estimates (MLEs) of the DCRRE for various sample sizes are computed by using Equations (9) and (10). Criteria measures include mean square error (MSE), average length (AL) and coverage probability (CP) are computed to investigate the performance of MLEs. Simulation procedures can be described as follows:

- 1000 random samples of sizes n = 30, 50, 70 and 100 are generated from Lomax distribution.
- The values of parameters are selected as $\alpha = (1.5, 2.5)$, $\lambda = (0.5, 1.5, 4)$ and $\beta = (3, 5)$.
- The true values of the DCRRE measure are selected as $\gamma_R(\beta) =$ 0.2798, 0.6264 and 0.9359 at t = 0.5, while $\gamma_R(\beta) =$ 0.077075, 0.2798 and 0.58932 at t = 1.5.
- The MLEs of α and λ are obtained from (9) and (10), then the MLE of $\gamma_R(\beta)$ is obtained by substituting $\hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$ in (11). The approximate confidence interval of $\hat{\gamma}_{R,ML}(\beta)$ using (12) are constructed with confidence level at $\eta = 0.05$.
- Compute the average MLEs of the DCRRE, MSEs, CPs, and ALs.

The M-H algorithm is one of the most famous subclasses of the MCMC method in Bayesian literature used to simulate the deviates from the posterior density and produce the good approximate results. The relative absolute biases (RABs), estimated risks (ERs) and the width of the BCI are computed to assess the behavior of the Bayesian estimates. A simulation study is done via R 3.1.2. Additionally, tables of simulation results are given. To compare the DCRRE Bayes estimates, the MCMC simulations are designed for different sample sizes under SE, LINEX and PRE loss functions. The values of parameters are selected as $\alpha = (1.5, 2.5)$, $\lambda = (0.5, 1.5, 4)$ and $\beta = (3, 5)$. The true values of the DCRRE measure are selected as $\gamma_R(\beta) =$ 0.2798, 0.6264 and 0.9359 at t = 0.5, while $\gamma_R(\beta) =$ 0.077075, 0.2798 and 0.58932 at t = 1.5. The hyper-parameters for gamma distribution are selected as a = c =1 and b = d = 4. Also, we take (v = −2, 2) for LINEX loss function. Take N = 5000 random samples of sizes n, where each sample size (n) = 10, 30, 50, 70 and 100 are generated from Lomax distribution.

The M-H algorithm is described as follows:

Step 1: Let $g(.)$ be the PDF of subject distribution;

Step 2: Initialize a starting value x_0 and determine the number of samples N;

Step 3: for i = 2 to N set $x = x_{i-1}$;

Step 4: Generate u from uniform (0,1) and generate y from $g(.)$;

Step 5: if $u \le \frac{\pi_\alpha(y)\, g(x)}{\pi_\alpha(\mathrm{x})\, g(\mathrm{y})}$ then set $x_i = y$ else set $x_i = x$;

Step 6: Set $i = i + 1$ and return to step 2 and repeat the previous steps N times.

4.1. Numerical results based on ML method

Numerical observations for DCRRE estimates using ML method at $\beta = 3$ and $\beta = 5$ are presented in Tables 1–4 and represented in Figures 1–4. Some observations related to the behavior of the DCRRE estimates are outlined as follows:

- The value of MLE, $\hat{\gamma}_{R,ML}(\beta)$, decreases as the value of λ increases for fixed value of α (see Tables 1 and 2)
- The value of MLE, $\hat{\gamma}_{R,ML}(\beta)$, decreases as α decreases for fixed value of λ (see Tables 1 and 2).
- The value of MLE, $\hat{\gamma}_{R,ML}(\beta)$, at $\alpha = 1.5, \lambda = 4$ has the smallest value compared with the corresponding estimates of the other sets of parameters (see Tables 3 and 4).
- The MSE of $\hat{\gamma}_{R,ML}(\beta)$ decreases as the sample size increases.
- For fixed value of λ, the MSE of $\hat{\gamma}_{R,ML}(\beta)$ decreases as the value of α decreases.
- For fixed value of α, the MSE of $\hat{\gamma}_{R,ML}(\beta)$ decreases as the value of λ increases.
- The MSE of the DCRRE estimates gets the smallest value at $\gamma_R(\beta) = 0.2798$, where $(\alpha = 1.5, \lambda = 1.5)$, t = 0.5 and $\beta = 3$, compared to the MSE of MLE of DCRRE measure for the corresponding other sets of parameters (see Figure 1).
- The MSE of the DCRRE estimates takes the smallest value at $\gamma_R(\beta) = 0.077075$, where $(\alpha = 1.5, \lambda = 1.5)$, t = 1.5 and $\beta = 3$, compared to MSE of MLE of DCRRE for the corresponding other sets of parameters (see Figure 2).

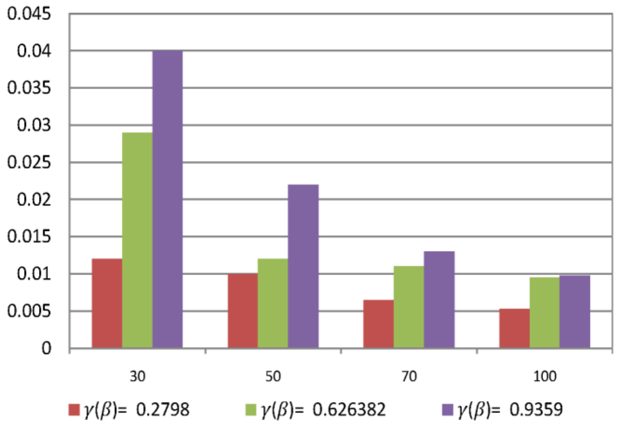

Figure 1. MSEs of DCREE estimates of Lomax under different sample size at t = 0.5 and $\beta = 3$.

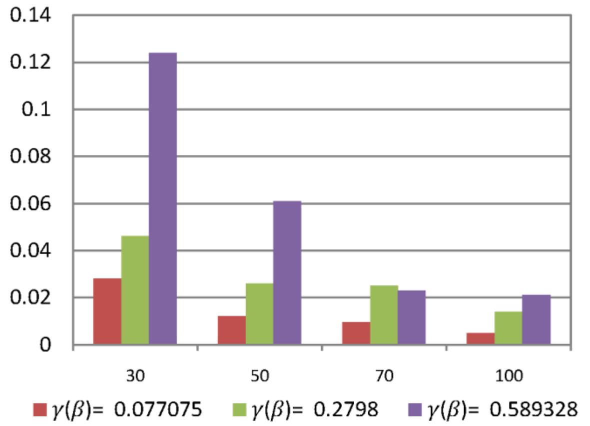

Figure 2. MSEs of DCREE estimates of Lomax under different sample size at t = 1.5 and $\beta = 3$.

- At $\beta=5$, $(\alpha=1.5, \lambda=1.5)$, where the true value of $\gamma_R(\beta)=$ 0.2946637 the MSE of $\hat{\gamma}_{R,ML}(\beta)$ has the smallest value compared to the MSE of MLEs for the corresponding other sets of parameters at t = 0.5 (see Figure 3).
- At t = 1.5, the MSE of $\hat{\gamma}_{R,ML}(\beta)$ at $(\alpha=1.5, \lambda=4.0)$, has the smallest value compared to the MSE of MLEs for the corresponding other sets of parameters (see Figure 4).

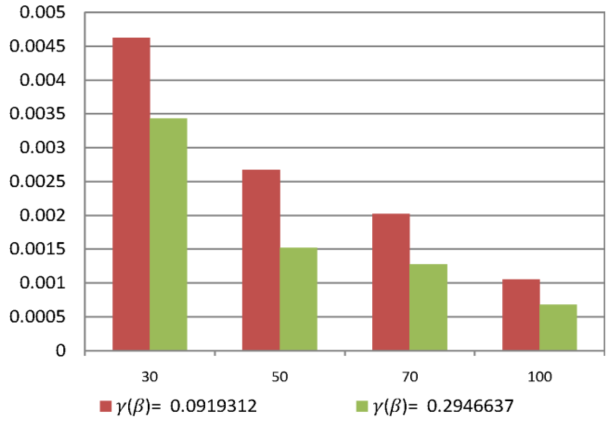

Figure 3. MSEs of DCREE estimates of Lomax under different sample size at t = 0.5 and $\beta=5$.

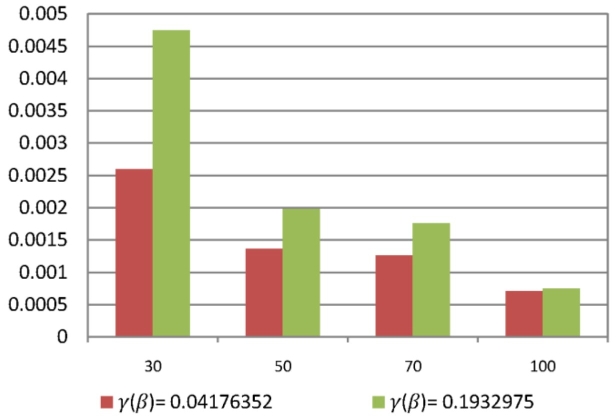

Figure 4. MSEs of DCREE estimates of Lomax under different sample size at t = 1.5 and $\beta=5$.

Generally as seen from Tables 1–4 that the CP is very close to their corresponding nominal levels in approximately most of the cases. Also, the CP increases as the sample size increases for all values of DCRRE.

- The MLE of the DCRRE decreases as the value of λ increases.
- Regarding the AL of estimates, it can be observed that, as n increases the AL of DCRRE estimates decreases.

Table 1. MSE, AL and CP of 95% CI for DCRRE estimates of Lomax distribution at t = 0.5 and $\beta=3$.

n	$(\alpha=1.5, \lambda=1.5)$ $\gamma_R(\beta)=0.2798$				$(\alpha=1.5, \lambda=0.5)$ $\gamma_R(\beta)=0.626382$				$(\alpha=2.5, \lambda=0.5)$ $\gamma_R(\beta)=0.9359$			
	Estimate	MSE	AL	CP%	Estimate	MSE	AL	CP%	Estimate	MSE	AL	CP%
30	0.269	0.012	0.422	91.0	0.655	0.029	0.657	90.0	1.029	0.04	0.697	92.0
50	0.263	0.01	0.387	94.0	0.619	0.012	0.436	92.0	0.985	0.022	0.552	94.0
70	0.278	0.006486	0.316	94.5	0.635	0.011	0.424	92.5	0.955	0.013	0.444	96.0
100	0.272	0.005312	0.284	95.5	0.632	0.0095	0.328	94.0	0.964	0.0098	0.372	97.0

Table 2. MSE, AL and CP of 95% CI for DCRRE estimates of Lomax distribution at t = 1.5 and $\beta=3$.

n	$(\alpha=1.5,\lambda=1.5)$				$(\alpha=1.5,\lambda=0.5)$				$(\alpha=2.5,\lambda=0.5)$			
	$\gamma_R(\beta)$ = 0.077075				$\gamma_R(\beta)$ = 0.2798				$\gamma_R(\beta)$ = 0.589328			
	Estimate	MSE	AL	CP%	Estimate	MSE	AL	CP%	Estimate	MSE	AL	CP%
30	0.113	0.028	0.643	92.5	0.334	0.046	0.81	94.0	0.754	0.124	1.217	92.0
50	0.087	0.012	0.412	94.0	0.276	0.026	0.628	94.5	0.648	0.061	0.943	93.0
70	0.074	0.009483	0.382	96.0	0.293	0.025	0.619	96.0	0.622	0.023	0.583	96.0
100	0.08	0.004798	0.271	96.5	0.315	0.014	0.436	97.0	0.613	0.021	0.56	96.5

Table 3. MSE, AL and CP of 95% CI for DCRRE estimates of Lomax distribution at t = 0.5 and $\beta=5$.

n	$(\alpha=1.5,\lambda=4)$				$(\alpha=1.5,\lambda=1.5)$			
	$\gamma_R(\beta)$ = 0.0919312				$\gamma_R(\beta)$ = 0.2946637			
	Estimate	MSE	AL	CP%	Estimate	MSE	AL	CP%
30	0.08684	0.004623	0.266	92.0	0.29513	0.0034	0.230	92.0
50	0.08846	0.002675	0.202	92.5	0.29149	0.0015	0.152	93.5
70	0.08373	0.002024	0.173	94.0	0.28849	0.0013	0.138	94.0
100	0.08853	0.001050	0.126	96.0	0.29245	0.0007	0.102	96.0

Table 4. MSE, MSE, AL and CP of 95% CI for DCRRE estimates of Lomax distribution at t = 1.5 and $\beta=5$.

n	$(\alpha=1.5,\lambda=4)$				$(\alpha=1.5,\lambda=1.5)$			
	$\gamma_R(\beta)$ = 0.04176352				$\gamma_R(\beta)$ = 0.1932975			
	Estimate	MSE	AL	CP%	Estimate	MSE	AL	CP%
30	0.04377	0.002594	0.200	91.0	0.213	0.0047	0.259	94.0
50	0.04585	0.001368	0.144	94.0	0.207	0.0020	0.167	95.0
70	0.03971	0.001263	0.139	97.0	0.194	0.0018	0.165	96.0
100	0.04078	0.000712	0.105	98.0	0.190	0.0007	0.106	96.5

4.2. Numerical results based on Bayesian method

Tables 5–10 and Figures 5–8, list the numerical outcomes for different estimates of the DCRRE under different loss functions at $\beta=3$. Tables 11–14 and Figures 9–12, list and describe the simulation outcomes for different estimates of the DCRRE under different loss functions at $\beta=5$. Some observations related to the behavior of the DCRRE estimates are provided as follows:

- As anticipated, the performance of all DCRRE estimates become improved as the sample size increases.
- The estimated value of the DCRRE increases as the value of α increases for fixed λ. The estimated value of the DCRRE decreases as the value of scale parameter λ increases for fixed α.
- The estimated value of the DCRRE decreases as the value of t increases for fixed values of α and λ.

- Under SE loss function, the ERs for DCRRE estimates get the smallest values at true value $\gamma_R(\beta) = 0.6264$, for most values of n (see for example Figures 5 and 6).
- The ER of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ gets the smallest values for all values of n. Also, the width of the BCI for $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ is the shortest compared to the width of the BCI in case of the SE and PRE loss functions for all n (see Table 5).

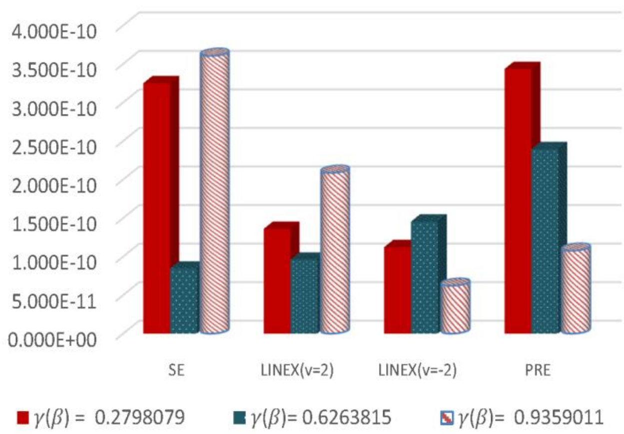

Figure 5. ERs of the DCREE estimates of Lomax under different loss functions at n =10 and t = 0.5.

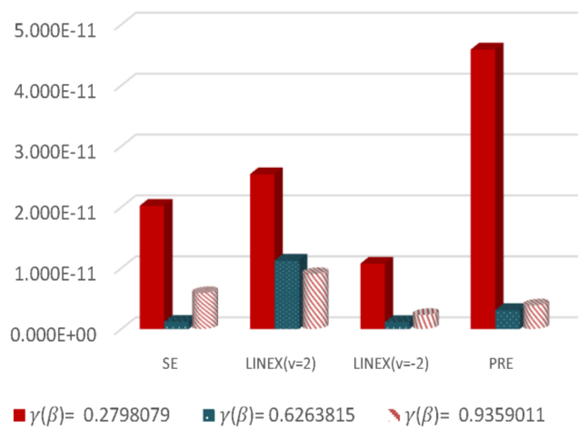

Figure 6. ERs of the DCREE estimates of Lomax under different loss functions at n = 100 and t = 0.5.

Table 5. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha,\lambda) = (1.5, 1.5)$, $t = 0.5$ and $\beta = 3$.

Sample size (n)		10	30	50	70	100
Exact value		0.2798				
SE	Estimate	0.279848	0.279552	0.2806298	0.27934	0.280126
	RAB	0.000144	0.000915	0.00293722	0.001667	0.001137
	ER	3.25E-10	8.31E-11	5.35E-11	4.35E-11	2.02E-11
	width	0.001472	0.000947	0.00087111	0.000796	0.000724
LINEX	Estimate	0.278984	0.279733	0.2796395	0.279823	0.279787
(v = 2)	RAB	0.002946	0.000269	0.00060188	5.38E-05	7.39E-05
	ER	1.36E-10	7.13E-11	5.67E-11	4.54E-11	2.54E-11
	width	0.001345	0.000973	0.00094541	0.000679	0.000662
LINEX	Estimate	0.279733	0.279778	0.2797087	0.279568	0.280088
(v = −2)	RAB	0.000268	0.000106	0.00035463	8.57E-04	0.001003
	ER	1.12E-10	2.76E-11	1.97E-11	1.15E-11	1.07E-11
	width	0.000561	0.000561	0.00054634	0.000536	0.000472
PRE	Estimate	0.279394	0.280666	0.280191	0.279949	0.280287
	RAB	0.001481	0.003065	0.00136909	0.000506	0.001712
	ER	3.43E-10	1.47E-10	9.35E-11	5.00E-11	4.59E-11
	width	0.001647	0.001361	0.00080719	0.000736	0.000583

Note: E-a: stands for 10^-a.

- The ER of $\hat{\gamma}_{R,BE}(\beta)_{SE}$ takes the smallest values for all n except n = 100. The width of the BCI for $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ is the shortest compared to the width of the BCI in case of the PRE and SE loss functions for all n (see Table 6).
- The ER of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ takes the smallest values for all values of n. The width of the BCI for $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ is the shortest compared to the width of the BCI in case of PRE and SE loss functions at n = 10 and 30 (see Table 7).
- At t = 1.5, the ERs for DCRRE estimates get the smallest values at $\gamma_R(\beta) = 0.2798$, for all n under SE and PRE loss functions (see for example Figures 7 and 8).

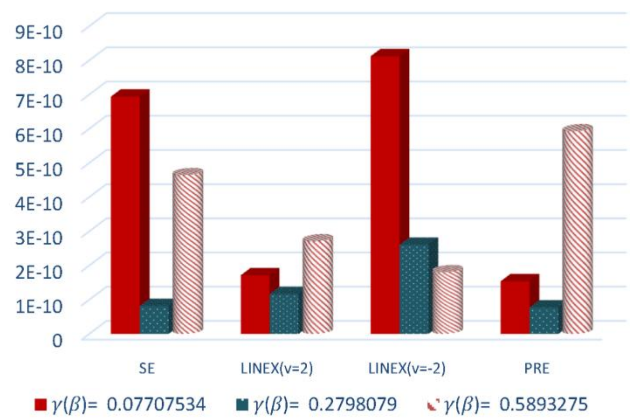

Figure 7. ERs of DCREE estimates of Lomax under different loss functions at n = 10 and t = 1.5.

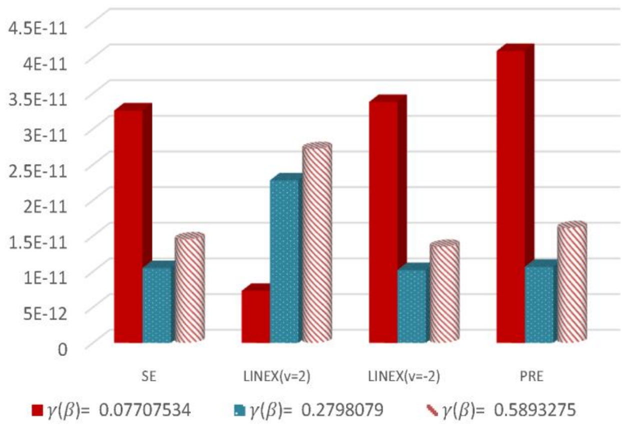

Figure 8. ERs of DCREE estimates of Lomax under different loss functions at n = 100 and t = 1.5.

- The ER of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at v = 2 takes the smallest values at n = 50, 70 and 100, while the ER of $\hat{\gamma}_{R,BE}(\beta)_{PRE}$ takes the smallest values at n = 10 and 30. The width of the BCI for $\hat{\gamma}_{R,BE}(\beta)_{SE}$ is the shortest compared to the width of the BCI in case of LINEX and PRE loss functions for all values of n (see Table 8).
- The ER of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ takes the smallest values at n = 50, 70 and 100, while the ER of $\hat{\gamma}_{R,BE}(\beta)_{PRE}$ takes the smallest values at n = 10 and 30. The width of the BCI of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ is the shortest compared to the width of the BCI in case of the SE and PRE loss functions for all values of n (see Table 9).

Table 6. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha,\lambda)=(1.5,0.5)$, t = 0.5 and $\beta=3$.

Sample size (n)		10	30	50	70	100
Exact value		0.626382				
SE	Estimate	0.625732	0.626659	0.626228	0.626467	0.625982
	RAB	0.001037	0.000444	0.00024498	0.000137	0.000638
	ER	8.44E-11	1.54E-11	4.71E-12	1.48E-12	1.20E-12
	width	0.001835	0.001319	0.00104168	0.000972	0.000502
LINEX	Estimate	0.626174	0.626484	0.6265369	0.627062	0.626144
(v = 2)	RAB	0.000331	0.000164	0.00024813	0.001086	3.78E-04
	ER	9.60E-11	6.10E-11	4.83E-11	2.26E-11	1.12E-11
	width	0.001325	0.001109	0.00099334	0.00081	0.000508
LINEX	Estimate	0.627234	0.626271	0.6259601	0.626251	0.626457
(v = −2)	RAB	0.001361	0.000176	0.00067277	2.09E-04	0.00012
	ER	1.45E-10	8.44E-11	3.55E-11	3.43E-11	1.14E-12
	width	0.001266	0.001016	0.00086437	0.000702	0.000117
PRE	Estimate	0.625288	0.625784	0.6261617	0.626237	0.626146
	RAB	0.001746	0.000954	0.00035083	0.00023	0.000376
	ER	2.39E-10	7.15E-11	9.66E-12	4.16E-12	3.11E-12
	width	0.001837	0.001343	0.00114911	0.000824	0.00082

Table 7. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha,\lambda)=(2.5,0.5)$, t = 0.5 and $\beta=3$.

Sample size (n)		10	30	50	70	100
Exact value		0.9359				
SE	Estimate	0.936015	0.934584	0.9365256	0.935385	0.936073
	RAB	0.000122	0.001408	0.00066734	0.000551	0.000183
	ER	3.60E-10	3.47E-10	7.80E-11	5.33E-11	5.90E-12
	width	0.001936	0.001746	0.00107773	0.001034	0.000659
LINEX	Estimate	0.935164	0.936662	0.9367184	0.93566	0.936114
(v = 2)	RAB	0.000787	0.000813	0.00087327	0.000257	2.27E-04
	ER	2.09E-10	1.16E-10	1.34E-11	1.16E-11	9.02E-12
	width	0.001624	0.001553	0.00131565	0.000524	0.000389
LINEX	Estimate	0.935344	0.936219	0.936625	0.936369	0.93571
(v = −2)	RAB	0.000595	0.000339	0.00077348	5.00E-04	0.000204
	ER	6.21E-11	2.01E-11	1.05E-11	1.04E-11	2.31E-12
	width	0.001493	0.001299	0.00127713	0.001012	0.000529
PRE	Estimate	0.936634	0.935286	0.9360099	0.935828	0.93604
	RAB	0.000783	0.000657	0.00011624	7.85E-05	0.000149
	ER	1.07E-10	7.56E-11	2.37E-11	1.08E-11	3.87E-12
	width	0.001758	0.001388	0.00107289	0.00069	0.000441

Table 8. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha, \lambda) = (1.5, 1.5)$, t = 1.5 and $\beta = 3$.

Sample size (n)		10	30	50	70	100
Exact value		0.077075				
SE	Estimate	0.077262	0.077398	0.077325	0.077533	0.076672
	RAB	0.002417	0.004186	0.003244	0.005942	0.005238
	ER	6.94E-10	2.08E-10	1.25E-10	4.20E-11	3.26E-11
	width	0.001643	0.001076	0.000615	0.000435	0.000163
LINEX	Estimate	0.077368	0.07687	0.077353	0.076739	0.076884
(v = 2)	RAB	0.003801	0.002665	0.003607	0.004367	2.48E-03
	ER	1.72E-10	8.44E-11	3.55E-11	2.27E-11	7.30E-12
	width	0.00185	0.001784	0.001598	0.001233	0.000677
LINEX	Estimate	0.076874	0.076719	0.076915	0.077031	0.077594
(v = −2)	RAB	0.002615	0.004622	0.002079	5.75E-04	0.006729
	ER	8.12E-10	8.54E-11	5.14E-11	3.93E-11	3.38E-11
	width	0.002806	0.002218	0.001715	0.00049	0.000182
PRE	Estimate	0.077163	0.077424	0.076188	0.077563	0.077528
	RAB	0.001138	0.00452	0.011519	0.006324	0.00587
	ER	1.54E-10	8.43E-11	5.58E-11	4.75E-11	4.09E-11
	width	0.001674	0.001589	0.001174	0.001131	0.000737

Note: E-a: stands for 10^-a.

Table 9. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha, \lambda) = (1.5, 0.5)$, t = 1.5 and $\beta = 3$.

Sample size (n)		10	30	50	70	100
Exact value		0.2798				
SE	Estimate	0.279169	0.279561	0.279304	0.280159	0.280308
	RAB	0.002282	0.000881	0.0018	0.001255	0.001789
	ER	8.15E-11	5.22E-11	5.08E-11	2.47E-11	1.05E-11
	width	0.00136	0.001081	0.001017	0.000798	0.000601
LINEX (v = 2)	Estimate	0.280571	0.279532	0.279477	0.28019	0.27874
	RAB	0.002726	0.000985	0.001182	0.001366	3.82E-03
	ER	1.16E-10	4.52E-11	2.99E-11	2.92E-11	2.28E-11
	width	0.001565	0.001479	0.001097	0.000783	0.000508
LINEX (v = −2)	Estimate	0.279694	0.280018	0.279733	0.280045	0.279167
	RAB	0.000408	0.000752	0.000268	8.46E-04	0.00229
	ER	2.61E-10	8.85E-11	1.53E-11	1.42E-11	1.02E-11
	width	0.001084	0.000861	0.000847	0.000589	0.000319
PRE	Estimate	0.279514	0.279666	0.280726	0.280197	0.279577
	RAB	0.001052	0.000509	0.003281	0.001391	0.000826
	ER	7.73E-11	4.05E-11	3.69E-11	3.03E-11	1.07E-11
	width	0.004765	0.00427	0.001586	0.001122	0.001051

- The ER of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ takes the smallest values for n = 10, 70 and 100. The width of the BCI for $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ is the shortest compared to the width of the BCI in case of the SE and PRE loss functions for most values of n (see Table 10).
- At $\beta = 5$, the ERs for DCRRE estimates get the smallest values at $\gamma_R(\beta) =$ 0.0919312, for most values of n under most selected loss functions (see for example Figures 9 and 10).

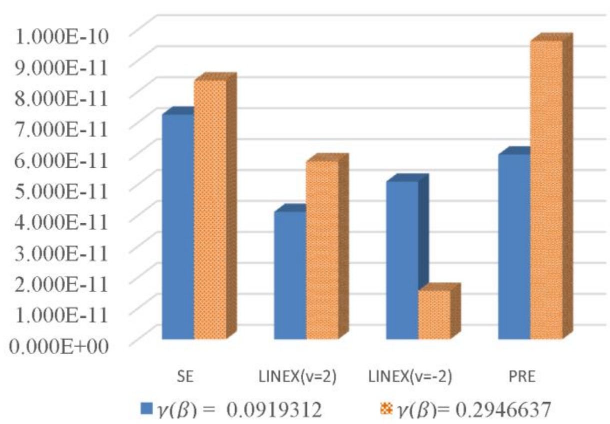

Figure 9. ERs of DCREE estimates of Lomax under different loss functions at n = 10 and t = 0.5.

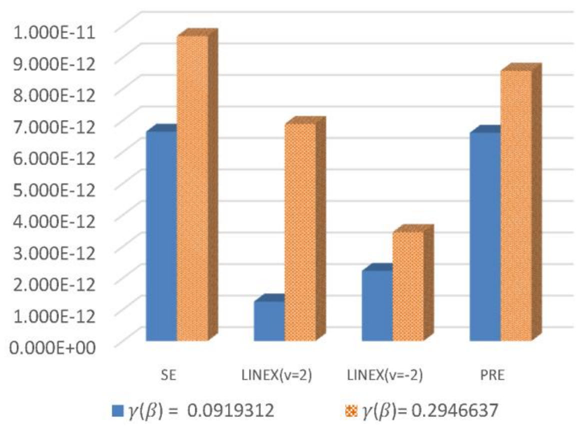

Figure 10. ERs of DCREE estimates of Lomax under different loss functions at n = 100 and t = 0.5.

- The ER of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ gets the smallest values for all values of n. While, the width of the BCI for $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = 2$ is the shortest compared to the width of the BCI in case of the SE and PRE loss functions for all n (see Table 11).
- The ER of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ gets the smallest values for all values of n. While, the width of the BCI for $\hat{\gamma}_{R,BE}(\beta)_{SE}$ is the shortest compared to the width of the BCI in case of the LINEX and PRE loss functions for all n expect at n = 10 (see Table 12).
- Under SE loss function, the ERs for DCRRE estimates get the smallest values at $\gamma_R(\beta) =$ 0.04176352, for most values of n (see for example Figures 11 and 12).
- The ER of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = 2$ gets the smallest values for n = 10, 70 and 100. While, the width of the BCI for $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = 2$ is the shortest compared to the width of the BCI in case of the SE and PRE loss functions for all n (see Table 13).
- The ER of $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = 2$ gets the smallest values for all values of n. While, the width of the BCI for $\hat{\gamma}_{R,BE}(\beta)_{LINEX}$ at $v = -2$ is the shortest compared to the width of the BCI in case of the SE and PRE loss functions for all n (see Table 14).

Table 10. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha, \lambda) = (2.5, 0.5)$, t = 1.5 and $\beta = 3$.

Sample size (n)		10	30	50	70	100
Exact value		0.589328				
SE	Estimate	0.589175	0.589863	0.589255	0.589763	0.589058
	RAB	0.000259	0.000909	0.000123	0.000739	0.000457
	ER	4.64E-10	5.74E-11	5.06E-11	3.80E-11	1.45E-11
	width	0.002794	0.001951	0.001437	0.001061	0.000778
LINEX (v = 2)	Estimate	0.589211	0.588931	0.589119	0.589373	0.588959
	RAB	0.000197	0.000673	0.000354	7.76E-05	6.26E-04
	ER	2.71E-10	9.15E-11	8.69E-11	4.18E-11	2.72E-11
	width	0.001371	0.000745	0.000648	0.000597	0.000505
LINEX (v = −2)	Estimate	0.589027	0.58976	0.588746	0.589024	0.589068
	RAB	0.00051	0.000734	0.000986	5.15E-04	0.00044
	ER	1.80E-10	9.75E-11	6.76E-11	1.84E-11	1.35E-11
	width	0.000917	0.000791	0.000773	0.000513	0.000497
PRE	Estimate	0.5895	0.589154	0.58915	0.589319	0.589044
	RAB	0.000292	0.000295	0.000301	1.37E-05	0.000481
	ER	5.93E-10	6.56E-11	6.28E-11	3.31E-11	1.61E-11
	width	0.000964	0.000738	0.000682	0.000653	0.000638

Note: E-a: stands for 10^-a.

Table 11. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha, \lambda) = (1.5, 1.5)$, t = 1.5 and $\beta = 5$.

Sample size (n)		10	30	50	70	100
Exact value		0.2946637				
SE	Estimate	0.2949246	0.2941664	0.2944852	0.2946671	0.2944438
	RAB	0.0008853	0.001688	0.0006058	1.146E-05	0.0007465
	ER	8.36E-11	6.95E-11	6.37E-11	2.28E-11	9.68E-12
	width	0.0008959	0.0007143	0.0003279	7.09E-04	6.50E-04
LINEX	Estimate	0.2944942	0.294638	0.2947687	0.2946398	0.2946735
(v = 2)	RAB	0.0005754	8.75E-05	3.56E-04	8.12E-05	3.297E-05
	ER	5.75E-11	4.33E-11	2.20E-11	1.14E-11	6.89E-12
	width	0.0005808	0.000417	0.0001934	2.38E-04	1.98E-04
LINEX	Estimate	0.2946357	0.2949186	0.2944051	0.2948607	0.2947952
(v = −2)	RAB	9.51E-05	8.65E-04	8.78E-04	0.0006685	0.0004462
	ER	1.57E-11	1.30E-11	1.24E-11	7.76E-12	3.46E-12
	width	0.0007348	0.0006429	0.0004368	3.10E-04	3.09E-04
PRE	Estimate	0.2946924	0.2944634	0.2946297	0.2949081	0.2948708
	RAB	9.71E-05	6.80E-04	1.16E-04	0.0008293	0.0007028
	ER	9.64E-11	8.02E-11	2.32E-11	1.19E-11	8.58E-12
	width	0.0007787	0.0006513	0.0004081	3.89E-04	3.34E-04

Table 12. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha,\lambda)=(1.5,1.5)$, $t=1.5$ and $\beta=5$.

Sample size (n)		10	30	50	70	100
Exact value		0.1932975				
SE	Estimate	0.1926608	0.1932249	0.1934313	0.1934544	0.1936084
	RAB	0.003293922	0.000375191	0.000692228	0.000811845	0.001608503
	ER	8.10791E-11	7.05193E-11	5.5808E-11	4.92526E-11	1.93342E-11
	width	0.001063817	0.000370056	0.000275823	0.000269164	0.000212478
LINEX	Estimate	0.193455	0.1935514	0.1934634	0.192997	0.1933625
(v = 2)	RAB	0.000814895	0.001313584	0.000858663	0.001554319	0.000336461
	ER	4.96233E-11	3.28943E-11	2.50969E-11	1.80536E-11	8.45965E-12
	width	0.000505376	0.000465587	0.000360505	0.000317531	0.000307002
LINEX	Estimate	0.1931554	0.1933025	0.1931548	0.193335	0.1933076
(v = −2)	RAB	0.000734896	2.60011E-05	0.000738031	0.000194103	5.24576E-05
	ER	4.03584E-11	5.05E-12	4.07035E-12	2.81544E-12	2.06E-12
	width	0.000434745	0.000405183	0.000352976	0.000276217	0.000245961
PRE	Estimate	0.1932864	0.193166	0.1934755	0.1935736	0.1934214
	RAB	5.72266E-05	0.000680218	0.000920906	0.001428612	0.00064088
	ER	9.45E-11	7.45763E-11	6.33743E-11	1.52514E-11	3.06928E-12
	width	0.00093006	0.000687739	0.000452429	0.00033985	0.000224581

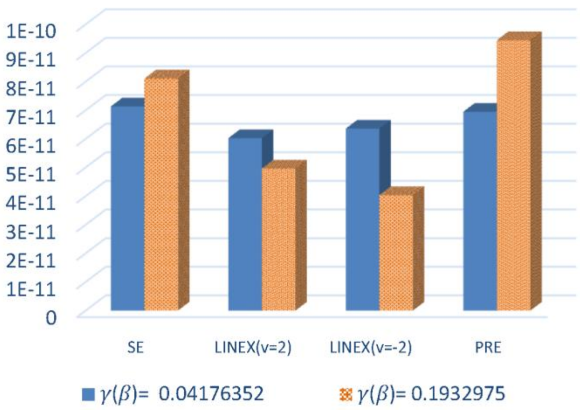

Figure 11. ERs of DCREE estimates of Lomax under different loss functions at n = 10 and t = 1.5.

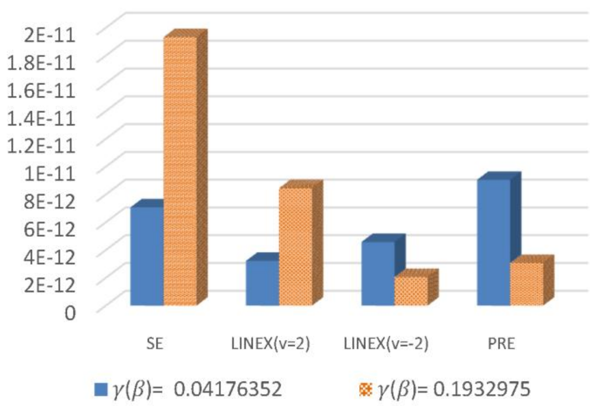

Figure 12. ERs of DCREE estimates of Lomax under different loss functions at n = 100 and t = 1.5

Table 13. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha,\lambda)=(1.5,4)$, t = 0.5 and $\beta=5$.

Sample size (n)		10	30	50	70	100
Exact value		0.0919312				
SE	Estimate	0.09201055	0.09178575	0.09181545	0.092062	0.09174895
	RAB	0.000863173	0.001582157	0.001259056	0.001422866	0.001982355
	ER	7.25936E-11	4.23112E-11	2.67945E-11	8.42203E-12	6.64231E-12
	width	0.000610588	0.000553999	0.000366307	0.000350079	0.000219156
LINEX	Estimate	0.09178774	0.09188011	0.09194927	0.09197472	0.09168103
(v = 2)	RAB	0.001560422	0.000555688	0.000196604	0.000473451	0.002721184
	ER	4.11567E-11	3.21937E-11	1.53E-11	3.78884E-12	1.25162E-12
	width	0.000334497	0.00025149	0.000246263	0.0002135	0.00018744
LINEX	Estimate	0.09188073	0.09169108	0.09190761	0.0918465	0.09182573
(v = −2)	RAB	0.000548943	0.002611855	0.000256567	0.000921266	0.00114721
	ER	5.09343E-11	1.15307E-11	1.11265E-11	4.43458E-12	2.22455E-12
	width	0.000472028	0.000455362	0.000349649	0.000302822	0.000265702
PRE	Estimate	0.09194847	0.09185285	0.09210766	0.09192755	0.09191302
	RAB	0.00018789	0.000852198	0.001919522	3.96648E-05	0.000197709
	ER	5.97E-11	2.22754E-11	1.22791E-11	8.66E-12	6.61E-12
	width	0.000566201	0.000396718	0.000261318	0.000238706	0.000236182

Note: E-a: stands for 10^-a.

Table 14. Bayes estimates, RAB, ER and width of the DCRRE for Lomax distribution for $(\alpha,\lambda)=(1.5,4)$, t = 1.5 and $\beta=5$.

Sample size (n)		10	30	50	70	100
Exact value		0.04176352				
SE	Estimate	0.04195238	0.04164627	0.04186148	0.04186715	0.04195167
	RAB	0.004522163	0.002807542	0.002345453	0.00248131	0.004505184
	ER	7.13373E-11	5.74965E-11	1.91901E-11	1.14776E-11	7.08026E-12
	width	0.00083513	0.000789892	0.00056322	0.000501208	0.000308716
LINEX	Estimate	0.04169195	0.04192397	0.04166999	0.04178557	0.04177622
(v = 2)	RAB	0.001713768	0.003841756	0.002239598	0.00052784	0.000304066
	ER	6.02454E-11	5.14854E-11	1.7497E-11	9.72E-12	3.23E-12
	width	0.000683436	0.000562861	0.000537438	0.000446217	0.000310601
LINEX	Estimate	0.04187211	0.04184263	0.04177058	0.04196117	0.04175873
(v = −2)	RAB	0.002600158	0.001894263	0.000169074	0.004732582	0.000114653
	ER	6.35843E-11	5.25171E-11	3.97E-11	2.81305E-11	4.59E-12
	width	0.000639135	0.000416648	0.000230094	0.000143375	0.00012603
PRE	Estimate	0.04162304	0.04190236	0.04174227	0.04172538	0.04199364
	RAB	0.003363745	0.003324387	0.000508727	0.000913293	0.005509934
	ER	6.94703E-11	5.8552E-11	3.03E-11	2.90967E-11	9.05905E-12
	width	0.000666527	0.000514737	0.000329824	0.000264613	0.000218476

Note: E-a: stands for 10^-a.

5. Application to real data

In this section, we provide an application to real data set to prove the importance and flexibility of the Lomax distribution compared with some other models. Also, the real data set can be used to illustrate the proposed methods in Sections 2 and 3.

The real data set was used by Jorgensen [41] which represent the active repair times (in hours) for airborne communication transceiver. The data are recorded as follows:

0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

Firstly, some preliminary data analysis is performed. The histogram, scaled total time on test (TTT) plots and the log likelihood for the real dataset are presented in Figure 13. We can observe that the shape of TTT plots is convex curve for dataset, which demonstrates decreasing failure rate.

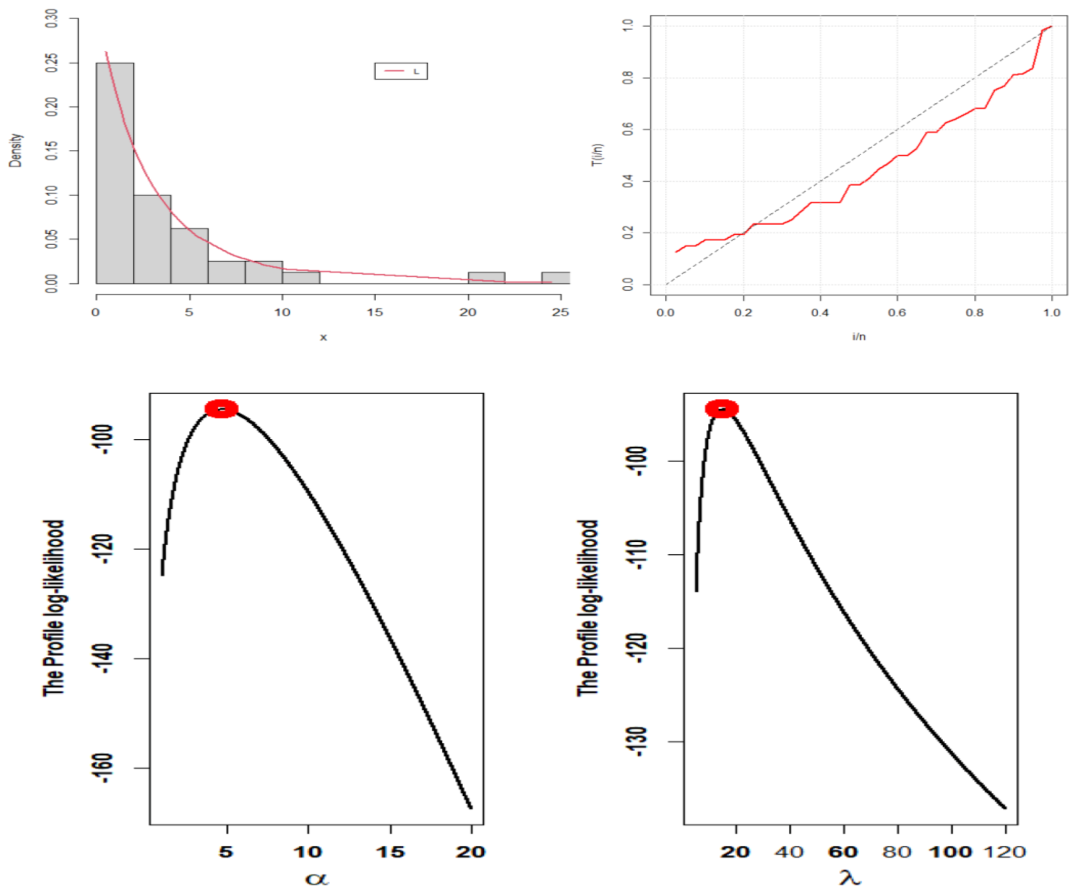

Figure 13. Estimated PDF, TTT plots and log likelihood of Lomax distribution for active repair times data for airborne communication transceiver.

Secondly, an application to a real data set is given to demonstrate the importance and flexibility of the proposed distribution compared with other one, two and three parameters fitted distributions. The suggested distributions are the alpha power Lindley (**APLi**) (Dey et al. [42]), Lindley (Li) and alpha power Weibull (**APW**) (Nassar et al. [43]) distributions. The MLE of the unknown parameters and the corresponding standard error (SE) of Lomax distribution and the other competitive models are calculated. In order to compare the different models with the Lomax model, we use measures like, Akaike information criterion (AIC), and Bayesian information criterion (BIC), consistent AIC (CAIC), Hannan-Quinn information criterion (HQIC). Also, the Kolmogorov- Smirnov (K-S) goodness of fit test and its P value (PV) for all models are calculated. Table 15 reports the MLEs and the corresponding SEs for all models. Table 16 reports numerical values of AIC, BIC, CAIC, HQIC, K-S and PVs for the different competitive distributions. From the results, we found that the Lomax distribution takes the smallest values of the considered measures and the largest value of PV among all other competitive models, and so it could be chosen as the best model.

Table 15. MLEs and their SEs (in parentheses).

Model	MLE and SE		
L (α, λ)	4.678 (3.899)	14.758 (14.657)	
APLi (α, γ)	0.015 (0.114)	0.077 (0.143)	
Li (α)	0.424 (0.0485)		
APW (α, θ, γ)	0.024 (0.0386)	1.279 (0.144)	0.058 (0.03)

Table 16. Measures of goodness-of-fit statistics of active repair times for airborne communication transceiver data.

Model	AIC	CAIC	BIC	HQIC	KS	PV
L	193.006	193.33	192.21	194.227	0.1451	0.3688
APLi	259.618	259.943	258.822	260.84	0.1499	0.3298
Li	199.583	199.688	199.185	200.193	0.2157	0.0484
APW	260.106	260.773	258.912	261.938	0.1488	0.338

Thirdly, regarding this data, the ML and Bayes estimates of the DCRRE under SE, LINEX and PRE loss functions are obtained and listed in Table 17.

As anticipated, we conclude from Table 17 that the DCRRE estimates are decreasing function on time (*t*), that is the estimated values of the DCRRE decrease as the time *t* increases.

Table 17. DCRRE estimates under ML and Bayesian methods for different loss functions at t = 0.5 and 1.5.

t	ML	SE	LINEX (v = 2)	LINEX (v = −2)	PRE
0.5	0.14384104	0.1437858	0.1439026	0.1438579	0.1437906
1.5	0.08003058	0.09838636	0.09832135	0.09835058	0.09842378

6. Conclusions

In this paper, the maximum likelihood and Bayesian methods of estimation for dynamic cumulative residual Rényi entropy of Lomax distribution are considered. The maximum likelihood estimates and approximate confidence intervals of the DCRRE are obtained. The performance of the DCRRE estimates for Lomax distribution is investigated via simulation in terms of their mean square error, average length and coverage probability. The Bayesian estimate of the DCRRE for Lomax model is obtained by considering the gamma priors under symmetric and asymmetric loss functions. The MCMC procedure is employed to compute the Bayes estimates and the Bayesian credible intervals. The performance of the DCRRE estimates for Lomax distribution is inspected through their relative absolute bias, estimated risk and the width of the credible intervals. Application to real data is provided.

Regarding the simulation results, we conclude that the mean square error of maximum likelihood and Bayesian estimates of the DCRRE decreases as the sample size increases. Also, the average length of the DCRRE estimates decreases and coverage probability increases as the sample size increases. The ER and the width of credible intervals of the DCRRE Bayes estimates decrease. For small true values of DCRRE, the width of DCRRE of BCIs under LINEX loss function is smaller than the corresponding based on SE and PRE loss functions for all selected sample size at t = 0.5 and 1.5. For large true values of DCRRE, the BCI width of the DCRRE under LINEX loss function is smaller than the corresponding based on SE and PRE loss functions for large sample size at t = 0.5 and 1.5.

Generally, the DCRRE estimates for both ML and Bayesian methods approach the true value as the sample size increases. As the time (t) increases, both ML and Bayesian estimates of the DCRRE decrease. Bayesian estimates under LINEX loss function at $v = -2$ are more suitable than other estimates under SE and PRC loss functions in most of the situations.

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Conflict of interest

The authors declare no conflict of interest.

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