Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(4): 3874–3888.
DOI:10.3934/math.2021230
Received: 25 September 2020
Accepted: 24 January 2021
Published: 01 February 2021

Research article

Uniqueness on linear difference polynomials of meromorphic functions

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Abstract: Suppose that $f(z)$ is a meromorphic function with hyper order $\sigma_2(f) < 1$. Let $L(z, f) = b_1(z)f(z + c_1) + b_2(z)f(z + c_2) + \cdots + b_n(z)f(z + c_n)$ be a linear difference polynomial, where $b_1(z), b_2(z), \cdots, b_n(z)$ are nonzero small functions relative to $f(z)$, and $c_1, c_2, \cdots, c_n$ are distinct complex numbers. We investigate the uniqueness results about $f(z)$ and $L(z, f)$ sharing small functions. These results promote the existing results on differential cases and difference cases of Brück conjecture. Some sufficient conditions to show that $f(z)$ and $L(z, f)$ cannot share some small functions are also presented.

Keywords: Nevanlinna theory; meromorphic functions; uniqueness; linear difference polynomial; deficiency
Mathematics Subject Classification: 30D35, 39A10

1. Introduction

In this paper, a meromorphic function $f(z)$ will always mean meromorphic in the complex plane. We assume that the reader is familiar with the fundamental results and standard notations of Nevanlinna's value distribution theory, such as the proximity function $m(r, f)$, the counting function $N(r, f)$, the characteristic function $T(r, f)$ and the first main theorem, for details, see e.g., Hayman [14], Yang and Yi [25]. For the meromorphic function $f(z)$, we use $\mathcal{S}(f)$ to denote the family of all meromorphic functions $\alpha(z)$ that satisfy $T(r, \alpha) = S(r, f)$, where $S(r, f) = o(T(r, f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. For convenience, we also include all constant functions in $\mathcal{S}(f)$. Functions in the set $\mathcal{S}(f)$ are called small functions with respect to $f(z)$. In addition, we denote the set of all entire functions in $\mathcal{S}(f)$ as $\mathcal{S}_e(f)$.

Let $f(z)$ and $g(z)$ be two meromorphic functions, and let $a(z)$ be a small function with respect to

$f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM, if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicities. If $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros ignoring multiplicities, it is said that $f(z)$ and $g(z)$ share $a(z)$ IM.

For two meromorphic functions $f(z)$ and $g(z)$, the famous five-value and four-value theorems due to Nevanlinna [21] say: If $f(z)$ and $g(z)$ share five distinct values in $\widehat{\mathbb{C}}$ IM, then $f(z)\equiv g(z)$; if $f(z)$ and $g(z)$ share four distinct values in $\widehat{\mathbb{C}}$ CM, then $f(z)\equiv g(z)$ or $f(z)$ is a Möbius transformation of $g(z)$. Gundersen [11], Mues [20] and Wang [24] generalized "4CM" to "2CM+2IM" independently. But the problem of whether "1CM+3IM=4CM" or not is still open.

There are many papers about meromorphic functions sharing some values with their derivatives, see e.g., [3, 5, 12, 18, 22, 26]. For example, Brück [3] raised the following conjecture.

Conjecture. *Let $f(z)$ be a nonconstant entire function such that $\sigma_2(f)<\infty$ and $\sigma_2(f)$ is not a positive integer. If $f(z)$ and $f'(z)$ share one finite value a CM, then*

$$\frac{f'(z)-a}{f(z)-a}=\tau$$

for some constant $\tau\neq 0$.

The conjecture has been verified in the special cases when $a=0$ [3], or when $f(z)$ is of finite order [12], or when $\sigma_2(f)<1/2$ [5].

Let $f(z)$ be a meromorphic function in the complex plane. The order of growth of $f(z)$ is denoted by $\sigma(f)$, the hyper-order of $f(z)$ is denoted by $\sigma_2(f)$, and the exponents of convergence of the zeros and the poles of $f(z)$ are denoted by $\lambda(f)$ and $\lambda(1/f)$ respectively, see e.g., [14]. If the meromorphic function $f(z)$ satisfies

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}=\lim_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

we say that $f(z)$ is of regular growth, see e.g. [8]. The deficiency of $a(z)\in\mathcal{S}(f)$ is defined by

$$\delta(a,f)=1-\limsup_{r\to\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)}.$$

If $\delta(a,f)>0$, then $a(z)$ is called a small deficient function of $f(z)$, see e.g., [14].

2. Uniqueness of linear difference polynomials concerning deficient values

With the development of complex differences and difference equations, a number of articles focused on uniqueness of meromorphic functions sharing values with their shifts or difference operators, see e.g., [4, 6, 7, 15–17, 19, 23]. Here, for a nonzero constant c, the difference operators $\Delta_c^n f$ are defined (see [2]) by

$$\Delta_c f(z)=f(z+c)-f(z),\ \Delta_c^{n+1}f(z)=\Delta_c^n f(z+c)-\Delta_c^n f(z),\ n=1,2,\cdots.$$

Now, we recall the following result, which can be seen as the difference analogue of Brück conjecture.

Theorem 1 ([15]). *Let $f(z)$ be a meromorphic function with $\sigma(f)<2$, and let $c\in\mathbb{C}$. If $f(z)$ and $f(z+c)$ share $a\in\mathbb{C}$ and ∞ CM, then*

$$\frac{f(z+c)-a}{f(z)-a}=\tau$$

for some constant τ.

In [15], Heittokangas et al. give the example $f(z)=e^{z^2}+1$ to show that $\sigma(f)<2$ can not be relaxed. Theorem 1 dealt with the uniqueness of a meromorphic function sharing values with its shift. It is well known that $\Delta_c f(z)=f(z+c)-f(z)$ is regarded as the difference counterpart of $f'(z)$. Cui and Chen [7] dealt with the uniqueness of a meromorphic function sharing values with its difference operator, and proved the following result.

Theorem 2 ([7]). *Let $f(z)$ be a nonconstant meromorphic function of finite order, and c be a nonzero finite complex constant. Let a, b be two distinct finite complex constants and n be a positive integer. If $\Delta_c^n f(z)$ and $f(z)$ share a, b, ∞ CM, then $\Delta_c^n f(z)\equiv f(z)$.*

Regarding Theorems 1 and 2, we pose the following questions.

Question 1. Since $\sigma(f)<2$ can not be relaxed in Theorem 1, can we replace it with other conditions?

Question 2. What can be said if $\Delta_c^n f(z)$ and $f(z)$ share two values in Theorem 2?

Question 3. What happens if $f(z+\eta)$ and $\Delta_c^n f(z)$ in Theorems 1 and 2 are generalized to linear difference polynomials?

In this paper, we consider the case that $f(z)$ has a small deficient function and we generalized $f(z+\eta)$ and $\Delta_c^n f(z)$ to linear difference polynomial $L(z,f)$ of the form

$$L(z,f)=b_1(z)f(z+c_1)+b_2(z)f(z+c_2)+\cdots+b_n(z)f(z+c_n), \tag{2.1}$$

where $b_1(z), b_2(z), \cdots, b_n(z)\in \mathcal{S}(f)/\{0\}$, and $c_1, c_2, \cdots, c_n$ are distinct complex numbers. We discuss the case $f(z)$ and $L(z,f)$ share some small functions and get the following result.

Theorem 3. *Let $f(z)$ be a transcendental meromorphic function with $\sigma_2(f)<1$, let $a_1(z), a_2(z)\in\mathcal{S}(f)$ be such that $a_1(z)\not\equiv a_2(z)$ and $\sigma(a_j)<1 (j=1,2)$, and let $L(z,f)$ be a linear difference polynomial of the form (2.1) with $b_i(z)\in\mathcal{S}(f)/\{0\}$, $\sigma(b_i)<1 (i=1,\cdots,n)$ and $a_1(z)\not\equiv L(z,a_2(z))$. If $\delta(a_2,f)>0$, and $f(z)$ and $L(z,f)$ share $a_1(z)$ and ∞ CM, then*

$$\frac{L(z,f)-a_1(z)}{f(z)-a_1(z)}=\tau$$

for some constant τ. In particular, if the deficient function $a_2(z)\equiv 0$, then $L(z,f)\equiv f(z)$.

For the special cases $\Delta_c^n f(z)$ and $f(z+c)$, we obtain the following corollary, which extends Theorems 1 and 2 to some extent.

Corollary 1. *Let $f(z)$ be a transcendental meromorphic function with $\sigma_2(f)<1$, let a, b be two distinct finite complex constants, and let c be a nonzero finite complex constant.*

(i) If $\delta(b,f)>0$, and $\Delta_c^n f(z)$ and $f(z)$ share a,∞ CM, then

$$\frac{\Delta_c^n f(z)-a}{f(z)-a}=\tau$$

for some constant τ. In particular, if the deficient value $b=0$, then $\Delta_c^n f(z)\equiv f(z)$.

(ii) If $\delta(b,f)>0$, and $f(z+c)$ and $f(z)$ share a,∞ CM, then $f(z+c)\equiv f(z)$.

We give an example to show that Theorem 3 may not hold, if $a_1(z) \equiv a_2(z)$.

Example 1. *Let* $f(z) = e^{z^2} + e^z$*,* $L(z, f) = f(z + 2\pi i)$ *and* $a_1(z) \equiv a_2(z) \equiv e^z$*. We see that* $L(z, f)$ *and* $f(z)$ *share* e^z, ∞ *CM,* $\delta(e^z, f) = 1 > 0$*. Obviously,*

$$\frac{L(z, f) - e^z}{f(z) - e^z} = e^{4\pi iz - 4\pi^2}$$

is not a constant.

Example 2 below shows that "$L(z, f) \equiv f(z)$" in Theorem 3 and "$\Delta_c^n f(z) \equiv f(z)$" in Corollary 1 (i) may not hold, if the deficient function of $f(z)$ is not identically zero.

Example 2. *Let* $f(z) = e^{\pi iz} + 6$ *and* $L(z, f) = \Delta_1 f(z) = f(z+1) - f(z) = -2e^{\pi iz}$*. We see that* $\Delta_1 f(z)$ *and* $f(z)$ *share* $4, \infty$ *CM and* $\delta(6, f) = 1 > 0$*. Obviously,*

$$\frac{\Delta_1 f(z) - 4}{f(z) - 4} = -2$$

and $\Delta_1 f(z) \not\equiv f(z)$*.*

To study the relation between two entire functions with deficient values while their derivatives share some value is an interesting topic in the uniqueness theory. Yi and Yang get the following result on this topic.

Theorem 4 ([25, Theorem 9.16])**.** *Let* f *and* g *be non-constant entire functions. If* $\delta(0, f) + \delta(0, g) > 1$*, and* f' *and* g' *share 1 CM, then* $f \equiv g$ *or* $f'g' \equiv 1$*.*

Similarly, we study the relation between $f(z)$ and $L(z, f)$ from this point of view and get the following result. Since $L(z, f)$ is the linear difference polynomial of $f(z)$, the result is more specific.

Theorem 5. *Let* $f(z)$ *be a transcendental entire function with* $\sigma_2(f) < 1$*, let* $a_1(z), a_2(z) \in \mathcal{S}_e(f)$ *be such that* $a_1(z) \not\equiv a_2(z)$ *and* $\sigma(a_j) < 1 (j = 1, 2)$*, and let* $L(z, f)$ *be a linear difference polynomial of the form (2.1) with* $b_i(z) \in \mathcal{S}_e(f)/\{0\}$*,* $\sigma(b_i) < 1 (i = 1, \cdots, n)$ *and* $a_1(z) \not\equiv L(z, a_2(z))$*. If* $\delta(a_2, f) + \delta(a_2, L(z, f)) > 1$*, and* $f(z)$ *and* $L(z, f)$ *share* $a_1(z)$ *CM, then* $L(z, f) \equiv f(z)$*.*

By Corollary 1, using a similar proof as in proof of Theorem 5, we get the following corollary, which extends Theorem 4 to some extent.

Corollary 2. *Let* $f(z)$ *be a transcendental entire function with* $\sigma_2(f) < 1$*, let* a, b *be two distinct finite complex constants, and let* c *be a nonzero finite complex constant.*

(i) If $\delta(b, f) + \delta(b, \Delta_c^n f(z)) > 1$*, and* $f(z)$ *and* $\Delta_c^n f(z)$ *share* a *CM, then* $\Delta_c^n f(z) \equiv f(z)$*.*

(ii) If $\delta(b, f) + \delta(b, f(z + c)) > 1$*, and* $f(z)$ *and* $f(z + c)$ *share* a *CM, then* $f(z + c) \equiv f(z)$*.*

In order to prove our theorems, we need the following lemmas. The first of these lemmas is a version of the difference analogue of the logarithmic derivative lemma.

Lemma 1 ([13])**.** *Let* $f(z)$ *be a nonconstant meromorphic function and* $c \in \mathbb{C}$*. If* $\sigma_2(f) < 1$ *and* $\varepsilon > 0$*, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{1-\sigma_2(f)-\varepsilon}}\right)$$

for all r *outside of a set of finite logarithmic measure.*

By [1, Lemma 1], [9, p. 66] and [13, Lemma 8.3], we immediately deduce the following lemma.

Lemma 2. *Let $f(z)$ be a non-constant meromorphic function of $\sigma_2(f) < 1$, and let $c \neq 0$ be an arbitrary complex number. Then*

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f),$$
$$N(r, f(z+c)) = N(r, f(z)) + S(r, f).$$

Lemma 3 ([18])**.** *Suppose that h is a non-constant meromorphic function satisfying*

$$\overline{N}(r, h) + \overline{N}(r, 1/h) = S(r, h).$$

Let $f = a_0h^p + a_1h^{p-1} + \cdots + a_p$, and $g = b_0h^q + b_1h^{q-1} + \cdots + b_q$ be polynomials in h with coefficients $a_0, a_1, \cdots, a_p, b_0, b_1, \cdots, b_q$ being small functions of h and $a_0b_0a_p \not\equiv 0$. If $q \leq p$, then $m(r, g/f) = S(r, h)$.

Lemma 4 ([27])**.** *Let $f_1(z), f_2(z)$ and $f_3(z)$ be meromorphic functions that satisfy*

$$\sum_{j=1}^{3} f_j(z) \equiv 1.$$

If $f_1(z) \not\equiv$ constant, and

$$\sum_{j=1}^{3} N_2\left(r, \frac{1}{f_j(z)}\right) + \sum_{j=1}^{3} \overline{N}\left(r, f_j(z)\right) < (\lambda + o(1))T(r), r \in I,$$

where $0 \leq \lambda < 1, T(r) = \max_{1\leq j\leq 3}\{T(r, f_j(z))\}$, and I has infinite linear measure, then either $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

Proof of Theorem 3. Since $f(z)$ and $L(z, f)$ share $a_1(z)$ and ∞ CM, we have

$$\frac{L(z, f) - a_1(z)}{f(z) - a_1(z)} = e^{h(z)}, \tag{2.2}$$

where $h(z)$ is an entire function. Since $L(z, f)$ is a linear difference polynomial of $f(z)$ with small meromorphic coefficients, by (2.1), (2.2) and Lemma 2, we have

$$T(r, e^{h(z)}) = O(T(r, f)),$$

and so

$$S(r, e^h) = S(r, f).$$

Now we prove that $h(z)$ is a constant. Suppose that, on the contrary, $h(z)$ is not a constant. Since $\sigma(a_j) < 1 (j = 1, 2)$, $\sigma(b_i) < 1 (i = 1, \cdots, n)$ and $e^{h(z)}$ is of regular growth with $\sigma(e^h) \geq 1$, we have

$$\begin{cases} T(r, a_j(z)) = S(r, e^h) \ (j = 1, 2), \\ T(r, b_i(z)) = S(r, e^h) \ (i = 1, \cdots, n), \\ T(r, L(z, a_2)) = S(r, e^h). \end{cases} \tag{2.3}$$

Since $L(z,f)$ is linear, we get from (2.2) that

$$L(z, f - a_2) - e^{h(z)}(f(z) - a_2(z)) = a_1(z) - L(z, a_2) - (a_1(z) - a_2(z))e^{h(z)}.$$

Since $a_1(z) \not\equiv L(z, a_2)$ and $a_1(z) \not\equiv a_2(z)$, we have $a_1(z) - L(z, a_2) - (a_1(z) - a_2(z))e^{h(z)} \not\equiv 0$. Dividing the above equality by $(a_1(z) - L(z, a_2) - (a_1(z) - a_2(z))e^{h(z)})(f(z) - a_2(z))$, we obtain

$$\frac{1}{a_1(z) - L(z, a_2) - (a_1(z) - a_2(z))e^{h(z)}} \left(\frac{L(z, f - a_2)}{f(z) - a_2(z)} - e^{h(z)} \right) = \frac{1}{f(z) - a_2(z)}. \tag{2.4}$$

We deduce from Lemma 3 and (2.3) that

$$m\left(r, \frac{1}{a_1(z) - L(z, a_2) - (a_1(z) - a_2(z))e^{h(z)}}\right) = S(r, e^h),$$

$$m\left(r, \frac{e^{h(z)}}{a_1(z) - L(z, a_2) - (a_1(z) - a_2(z))e^{h(z)}}\right) = S(r, e^h).$$

Furthermore, by Lemma 1, we get

$$m\left(r, \frac{L(z, f - a_2)}{f(z) - a_2(z)}\right) = S(r, f).$$

So by (2.4) we obtain

$$m\left(r, \frac{1}{f(z) - a_2}\right) = S(r, e^h) + S(r, f) = S(r, f),$$

which gives $\delta(a_2, f) = 0$, contradicting $\delta(a_2, f) > 0$. Hence we proved that $h(z)$ is a constant. Set $e^{h(z)} = \tau$. We have

$$\frac{L(z, f) - a_1(z)}{f(z) - a_1(z)} = \tau. \tag{2.5}$$

Next we consider the case $a_2(z) \equiv 0$. Since $a_1(z) \not\equiv a_2(z)$, we have $a_1(z) \not\equiv 0$. By (2.5), we have

$$L(z, f) - \tau f(z) = (1 - \tau)a_1(z).$$

If $\tau \neq 1$, then dividing the above equality by $(1 - \tau)a_1(z)f(z)$, we obtain

$$\frac{1}{(1 - \tau)a_1(z)} \frac{L(z, f)}{f(z)} - \frac{\tau}{(1 - \tau)a_1(z)} = \frac{1}{f(z)}.$$

So by Lemma 1, we get

$$m\left(r, \frac{1}{f(z)}\right) = S(r, f),$$

which gives $\delta(0, f) = 0$, contradicting $\delta(0, f) > 0$. Hence $\tau = 1$ and $L(z, f) \equiv f(z)$. □

Proof of Corollary 1. (i) If $a \neq \Delta_c^n b = 0$, we see from Theorem 3 that Corollary 1 (i) holds. Next we consider the case $a = 0$, $b \neq 0$. Since $f(z)$ and $\Delta_c^n f(z)$ share a and ∞ CM, we have

$$\frac{\Delta_c^n f(z)}{f(z)} = e^{h(z)}, \tag{2.6}$$

where $h(z)$ is an entire function. Lemma 1 gives

$$T(r, e^{h(z)}) = m(r, e^{h(z)}) = S(r, f).$$

Suppose that $h(z)$ is not a constant. Since $\Delta_c^n f(z) = \Delta_c^n (f(z) - b)$ and $b \neq 0$, we get from (2.6) that

$$\frac{1}{be^{h(z)}} \frac{\Delta_c^n (f(z) - b)}{f(z) - b} - \frac{1}{b} = \frac{1}{f(z) - b}.$$

By Lemma 1, we have

$$m\left(r, \frac{1}{f(z) - b}\right) = S(r, f),$$

and so $\delta(b, f) = 0$, contradicting $\delta(b, f) > 0$. Hence $h(z)$ is a constant and Corollary 1 (i) holds.

(ii) By Theorem 3, we have

$$f(z + c) - a = \tau(f(z) - a), \tag{2.7}$$

where τ is a constant. If $\tau \neq 1$, then we get from (2.7) that

$$\frac{1}{(\tau - 1)(a - b)} \frac{f(z + c) - b}{f(z) - b} - \frac{\tau}{(\tau - 1)(a - b)} = -\frac{1}{f(z) - b}.$$

We also have

$$m\left(r, \frac{1}{f(z) - b}\right) = S(r, f),$$

and so $\delta(b, f) = 0$, contradicting $\delta(b, f) > 0$. Hence $\tau = 1$ and $f(z + c) \equiv f(z)$. □

Proof of Theorem 5. Since $\delta(a_2, f) + \delta(a_2, L(z, f)) > 1$, we have $\delta(a_2, f) > 0$. So by Theorem 3, we have

$$\frac{L(z, f) - a_1(z)}{f(z) - a_1(z)} = \tau, \tag{2.8}$$

where τ is a constant. By (2.8), we have

$$T(r, L(z, f)) = T(r, f) + S(r, f). \tag{2.9}$$

If $a_2(z) \equiv 0$, then Theorem 3 gives $L(z, f) \equiv f(z)$. So we consider the case $a_2(z) \not\equiv 0$. Suppose that $\tau \neq 1$. Setting $\delta(a_2) = \delta(a_2, f) + \delta(a_2, L(z, f))$, for any given ε with

$$0 < \varepsilon < \min\left\{\frac{\delta(a_2, f)}{2}, \frac{\delta(a_2, L(z, f))}{2}, \frac{\delta(a_2) - 1}{2}\right\},$$

there is a constant r_0 such that for all $r > r_0$, we have

$$(\delta(a_2, f) - \varepsilon)T(r, f) \le m\left(r, \frac{1}{f(z) - a_2(z)}\right),$$

$$(\delta(a_2, L(z, f)) - \varepsilon)T(r, L(z, f)) \le m\left(r, \frac{1}{L(z, f) - a_2(z)}\right).$$

So by (2.9) and Nevanlinna's first fundamental theorem, we get

$$N\left(r, \frac{1}{f(z) - a_2(z)}\right) \le (1 - \delta(a_2, f) + \varepsilon)T(r, f) + S(r, f), \tag{2.10}$$

$$\begin{aligned} N\left(r, \frac{1}{L(z, f) - a_2(z)}\right) &\le (1 - \delta(a_2, L(z, f)) + \varepsilon)T(r, L(z, f)) + S(r, f) \\ &= (1 - \delta(a_2, L(z, f)) + \varepsilon)T(r, f) + S(r, f). \end{aligned} \tag{2.11}$$

Since $\tau \neq 1$ and $a_2(z) \not\equiv 0$, we obtain from (2.8) that

$$\frac{L(z, f) - a_2(z)}{(\tau - 1)a_2(z)} - \frac{\tau(f(z) - a_2(z))}{(\tau - 1)a_2(z)} + \frac{a_1(z)}{a_2(z)} = 1. \tag{2.12}$$

We write (2.12) as

$$F_1(z) + F_2(z) + F_3(z) \equiv 1,$$

where

$$F_1(z) = \frac{L(z, f) - a_2(z)}{(\tau - 1)a_2(z)}, \quad F_2(z) = -\frac{\tau(f(z) - a_2(z))}{(\tau - 1)a_2(z)}, \quad F_3(z) = \frac{a_1(z)}{a_2(z)}.$$

Set $T(r) = \max\limits_{1 \le j \le 3}\{T(r, F_j(z)\}$. Then

$$T(r) = T(r, f) + S(r, f).$$

Since $f(z)$ is entire, by (2.10) and (2.11) we get

$$\sum_{j=1}^{3} N\left(r, \frac{1}{F_j(z)}\right) + \sum_{j=1}^{3} N(r, F_j(z)) \le (2 - \delta(a_2) + 2\varepsilon)T(r, f) + S(r, f).$$

Since $F_1(z)$ is not a constant and $2 - \delta(a_2) + 2\varepsilon < 1$, we deduce from Lemma 4 that $F_2(z) \equiv 1$ or $F_3(z) \equiv 1$, which is impossible. So we proved that $\tau = 1$ and $L(z, f) \equiv f(z)$. □

3. The case that an entire function cannot share values with its difference polynomials

Whether two meromorphic functions can share some values under certain conditions is an important topic in the uniqueness theory. The following result shows that $f(z)$ and $\Delta^n f(z)$ can not have any finite CM sharing value if $\sigma(f) < 1$.

Theorem 6 ([28]). *Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 1$. Then $f(z)$ and $\Delta^n f(z)$ cannot share a finite value a CM.*

Next we obtain some sufficient conditions to show that $f(z)$ and $L(z, f)$ cannot share some small functions CM.

Theorem 7. *Let $f(z)$ be a transcendental entire function with $\sigma_2(f) < 1$, let $L(z, f)$ be a linear difference polynomial of the form (2.1) with $b_i(z) \in \mathcal{S}_e(f)/\{0\} (i = 1, 2, \cdots, n)$, and let $a(z) \in \mathcal{S}_e(f)$ be such that $a(z) \not\equiv L(z, a)$ and $L(z, f) \not\equiv L(z, a)$. If $\delta(a, f) = 1$, then $f(z)$ and $L(z, f)$ cannot share either $a(z)$ or $L(z, a)$ CM.*

The following example satisfies Theorem 7.

Example 3. *Let $f(z) = e^z + 1$, $L(z, f) = f(z+1) - f(z)$ and $a(z) = 1$. We see that $L(z, a) = 0$, $\delta(a, f) = 1$, $a(z) \not\equiv L(z, a)$ and $L(z, f) \not\equiv L(z, a)$. Obviously, $f(z)$ and $L(z, f)$ cannot share either $a(z)$ or $L(z, a)$ CM.*

Examples 4 and 5 below show, respectively, the conditions "$a(z) \not\equiv L(z, a)$" and "$L(z, f) \not\equiv L(z, a)$" in Theorem 7 cannot be omitted.

Example 4. *Let $f(z) = e^z + 1$, $L(z, f) = 2f(z + 2\pi i) - f(z + \pi i) = 3e^z + 1$ and $a(z) = 1$. We see that $a(z) \equiv L(z, a)$, and $f(z)$ and $L(z, f)$ share $a(z)$ CM.*

Example 5. *Let $f(z) = e^z + 1$, $L(z, f) = f(z + 2\pi i) + f(z + \pi i) = 2$ and $a(z) = 1$. We see that $L(z, f) \equiv L(z, a)$, and $f(z)$ and $L(z, f)$ share $a(z)$ CM.*

Since the condition "$a(z) \not\equiv L(z, a)$" in Theorem 7 cannot be omitted, we naturally ask: Can it be replaced by other conditions? We discuss this problem and get the following result.

Theorem 8. *Let $f(z)$ be a finite order transcendental entire function, let $a(z) \not\equiv 0$ be an entire function with $\sigma(a) < \sigma(f)$, $\lambda(f - a) < \sigma(f)$ if $\sigma(f) < 2$ and $\lambda(f - a) < \sigma(f) - 1$ if $\sigma(f) \geq 2$, and let $L(z, f)$ be a linear difference polynomial of the form (2.1) with nonzero constant coefficients $b_1, b_2, \cdots, b_n$ such that $b_1 + b_2 + \cdots + b_n \neq 1$. If $n \geq 2$ and $L(z, f) \not\equiv L(z, a)$, then $f(z)$ and $L(z, f)$ cannot share either $a(z)$ or $L(z, a)$ CM.*

By Theorem 8, we easily get the following corollary.

Corollary 3. *Let $f(z)$ be a finite order transcendental entire function, let $a(z) \not\equiv 0$ be an entire function with $\sigma(a) < \sigma(f)$, $\lambda(f - a) < \sigma(f)$ if $\sigma(f) < 2$ and $\lambda(f - a) < \sigma(f) - 1$ if $\sigma(f) \geq 2$. If $\Delta_c^n f(z) \not\equiv \Delta_c^n a(z)$, then $f(z)$ and $\Delta_c^n f(z)$ cannot share either $a(z)$ or $\Delta_c^n a(z)$ CM.*

Examples 6 and 7 below show respectively that "$n \geq 2$" and "$b_1 + b_2 + \cdots + b_n \neq 1$" in Theorem 8 cannot be omitted.

Example 6. *Let $f(z) = e^{z^2} + e^z$, $L(z, f) = e^{-1} f(z + 1)$ and $a(z) = e^z$. We see that $L(z, f) \not\equiv L(z, a)$ and $n = 1$. Obviously, $L(z, f)$ and $f(z)$ share $a(z)$ and $L(z, a)$ CM.*

Example 7. *Let $f(z) = e^z + 1$, $L(z, f) = 2f(z + \pi i) - f(z)$ and $a(z) = 1$. We see that $L(z, f) \not\equiv L(z, a)$ and $b_1 + b_2 + \cdots + b_n = 1$. Obviously, $L(z, f)$ and $f(z)$ share $a(z)$ and $L(z, a)$ CM.*

In order to prove the theorems, we need the following lemmas.

Lemma 5 ([2]). *Let $g(z)$ be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{g(z+c)}{g(z)} \to 1 \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \le h$.

Lemma 6 ([10, pp.69–70] or [25, p.82]). *Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$ are meromorphic functions and that $g_1(z), g_2(z), \cdots, g_n(z)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(2) $g_j(z) - g_k(z)$ are not constants for $1 \le j < k \le n$;

(3) for $1 \le j \le n, 1 \le h < k \le n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty,\ r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \cdots, n)$.

Proof of Theorem 7. Letting $f(z) - a(z) = g(z)$, since $\delta(a, f) = 1$ and $f(z)$ is entire, we obtain

$$N\left(r, \frac{1}{g(z)}\right) + N(r, g(z)) = S(r, g). \tag{3.1}$$

First we prove that $f(z)$ and $L(z, f)$ cannot share $a(z)$ CM. Suppose that, on the contrary, $f(z)$ and $L(z, f)$ share $a(z)$ CM, we have

$$L(z, f) - a(z) = (f(z) - a(z))e^{h(z)},$$

where $h(z)$ is an entire function, and so

$$L(z, g) + L(z, a) - a(z) = g(z)e^{h(z)}. \tag{3.2}$$

We see from (2.1) that

$$L(z, g) = b_1(z)g(z + c_1) + b_2(z)g(z + c_2) + \cdots + b_n(z)g(z + c_n) = A(z)g(z),$$

where

$$A(z) = b_1(z)\frac{g(z + c_1)}{g(z)} + b_2(z)\frac{g(z + c_2)}{g(z)} + \cdots + b_n(z)\frac{g(z + c_n)}{g(z)}.$$

Since $L(z, f) \not\equiv L(z, a)$, we have $L(z, g) \not\equiv 0$ and so $A(z) \not\equiv 0$. Since $b_i(z) \in \mathcal{S}_e(f)/\{0\}(i = 1, 2, \cdots, n)$, we deduce from (3.1), Lemma 1 and Lemma 2 that

$$\begin{aligned} &T(r, A(z)) \\ &= N(r, A(z)) + m(r, A(z)) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n} N\left(r, \frac{g(z+c_i)}{g(z)}\right) + \sum_{i=1}^{n} m\left(r, \frac{g(z+c_i)}{g(z)}\right) + \sum_{i=1}^{n} T(r, b_i(z)) + S(r,g) \\
&\le \sum_{i=1}^{n} \left(N(r, g(z+c_i)) + N\left(r, \frac{1}{g(z)}\right)\right) + S(r,g) \\
&= S(r,g).
\end{aligned} \tag{3.3}
$$

(3.2) can be written as

$$
A(z)g(z) + L(z,a) - a(z) = g(z)e^{h(z)}. \tag{3.4}
$$

Since $L(z,a) - a(z) \not\equiv 0$, by (3.1), (3.3) and Nevanlinna's second fundamental theorem, we have

$$
\begin{aligned}
&T(r, g(z)) + S(r,g) \\
&= T(r, A(z)g(z)) \\
&\le N(r, A(z)g(z)) + N\left(r, \frac{1}{A(z)g(z)}\right) + N\left(r, \frac{1}{A(z)g(z) + L(z,a) - a(z)}\right) + S(r,g) \\
&= N\left(r, \frac{1}{A(z)g(z) + L(z,a) - a(z)}\right) + S(r,g).
\end{aligned} \tag{3.5}
$$

By (3.1), (3.5) and comparing the counting functions of zeros of both sides of (3.4), we get a contradiction. So $f(z)$ and $L(z,f)$ cannot share $a(z)$ CM.

Second we prove that $f(z)$ and $L(z,f)$ cannot share $L(z,a)$ CM. Suppose that, on the contrary, $f(z)$ and $L(z,f)$ share $L(z,a)$ CM, we have

$$
L(z,f) - L(z,a) = (f(z) - L(z,a))e^{h(z)},
$$

where $h(z)$ is an entire function. So

$$
L(z,g) = (g(z) + a(z) - L(z,a))e^{h(z)}. \tag{3.6}
$$

Similarly, we can prove that

$$
N\left(r, \frac{1}{L(z,g)}\right) = S(r,g),
$$

$$
N\left(r, \frac{1}{g(z) + a(z) - L(z,a)}\right) = T(r,g) + S(r,g).
$$

Comparing the counting functions of zeros of both sides of (3.6), we also get a contradiction. So $f(z)$ and $L(z,f)$ cannot share $L(z,a)$ CM. □

Proof of Theorem 8. Since $f(z)$ is entire, $\sigma(a) < \sigma(f)$ and $\lambda(f-a) < \sigma(f)$, by Hadamard's factorization theorem, we get

$$
f(z) - a(z) = H(z)e^{g(z)}, \tag{3.7}
$$

where $g(z)$ is a polynomial, $H(z)$ is an entire function satisfying $\lambda(H) = \sigma(H) < \sigma(f) = \sigma(e^g) = \deg g(z)$. So $f(z)$ is of regular growth and we obtain

$$T(r, a) = S(r, f), \quad \delta(a, f) = 1.$$

If $L(z, a) \not\equiv a(z)$, we see from Theorem 7 that $f(z)$ and $L(z, f)$ cannot share either $a(z)$ or $L(z, a)$ CM. So in the following, we discuss the case $L(z, a) \equiv a(z)$, i.e.,

$$b_1 a(z + c_1) + b_2 a(z + c_2) + \cdots + b_n a(z + c_n) \equiv a(z).$$

Since $a(z) \not\equiv 0$, we affirm that $\sigma(a) \geq 1$. Otherwise, by Lemma 5, there exists an ε-set E_1 such that

$$\frac{a(z + c_i)}{a(z)} = 1 + o_i(1) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E_1,\ i = 1, 2, \cdots, n,$$

where for $i = 1, 2, \cdots, n$, $o_i(1) \to 0$ as $z \to \infty$ in $\mathbb{C} \setminus E_1$. So we have

$$(b_1 + b_2 + \cdots + b_n) + b_1 o_1(1) + b_2 o_2(1) + \cdots + b_n o_n(1) \equiv 1 \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E_1.$$

Since $b_1 o_1(1) + b_2 o_2(1) + \cdots + b_n o_n(1) \to 0$ as $z \to \infty$ in $\mathbb{C} \setminus E_1$, we get $b_1 + b_2 + \cdots + b_n = 1$, contradicting the hypothesis. So $\sigma(a) \geq 1$ and

$$m := \deg g(z) = \sigma(f) \geq 2. \tag{3.8}$$

Suppose that $f(z)$ and $L(z, f)$ share $a(z)$ CM, we have

$$L(z, f) - a(z) = (f(z) - a(z))e^{h(z)},$$

where $h(z)$ is an entire function. By $L(z, a) \equiv a(z)$ and (3.7), we have

$$L(z, He^g) = H(z)e^{h(z)+g(z)}.$$

Considering (2.1), we have

$$b_1 \frac{H(z + c_1)}{H(z)} e^{g(z+c_1)-g(z)} + b_2 \frac{H(z + c_2)}{H(z)} e^{g(z+c_2)-g(z)} + \cdots + b_n \frac{H(z + c_n)}{H(z)} e^{g(z+c_n)-g(z)} = e^{h(z)}. \tag{3.9}$$

Setting

$$g(z) = d_m z^m + d_{m-1} z^{m-1} + \cdots (m \geq 2, d_m \neq 0),$$

we have for every $j = 1, 2, \cdots, n$,

$$g(z + c_j) - g(z) = c_j m d_m z^{m-1} + O(z^{m-2}). \tag{3.10}$$

By (3.7), (3.8) and the hypothesis of $\lambda(f - a)$, we have

$$\lambda(f - a) = \lambda(H) = \sigma(H) < m - 1.$$

So by Lemma 2, we have

$$\sigma\left(\frac{H(z+c_j)}{H(z)}\right) < m-1, \quad j = 1, 2, \cdots, n.$$

Set

$$s_j(z) = b_j \frac{H(z+c_j)}{H(z)} e^{O(z^{m-2})}, \quad j = 1, 2, \cdots, n. \tag{3.11}$$

Then for every $j = 1, 2, \cdots, n$, $\sigma(s_j) < m-1$. While $e^{\eta z^{m-1}}$ is of regular growth with order $m-1$ provided $\eta \neq 0$. So for every $j = 1, 2, \cdots, n$,

$$T(r, s_j) = S(r, e^{\eta z^{m-1}})$$

provided $\eta \neq 0$. By (3.9)–(3.11), we see that

$$s_1(z)e^{c_1 m d_m z^{m-1}} + s_2(z)e^{c_2 m d_m z^{m-1}} + \cdots + s_n(z)e^{c_n m d_m z^{m-1}} = e^{h(z)}. \tag{3.12}$$

By (3.12), we easily see that $\deg h(z) \leq m-1$. Now we divide our discussion into two cases.

Case 1. $c_1 c_2 \cdots c_n \neq 0$.

Subcase 1.1. $\deg h(z) < m-1$. Then $T(r, e^h) = S(r, e^{\eta z^{m-1}})$ provided $\eta \neq 0$. Since $c_1, c_2, \cdots, c_n$ are distinct complex numbers, by (3.12) and Lemma 6, we have $e^{h(z)} \equiv 0$. This is impossible.

Subcase 1.2. $\deg h(z) = m-1$. By setting

$$h(z) = l_{m-1}z^{m-1} + \cdots, (l_{m-1} \neq 0),$$

we have

$$e^{h(z)} = u(z)e^{l_{m-1}z^{m-1}},$$

where $u(z) \not\equiv 0$ and $T(r, u) = S(r, e^{\eta z^{m-1}})$ provided $\eta \neq 0$. If for all $j = 1, 2, \cdots, n$, $l_{m-1} \neq c_j m d_m$, then by (3.12) and Lemma 6, we have $u(z) \equiv 0$, a contradiction. Otherwise, without loss of generality, we set $l_{m-1} = c_1 m d_m$. (3.12) can be written as

$$(s_1(z) - u(z))e^{c_1 m d_m z^{m-1}} + s_2(z)e^{c_2 m d_m z^{m-1}} + \cdots + s_n(z)e^{c_n m d_m z^{m-1}} = 0. \tag{3.13}$$

Since $n \geq 2$, we deduce from (3.13) and Lemma 6 that $s_2(z) \equiv \cdots \equiv s_n(z) \equiv 0$. So by (3.11), we have $b_2 = \cdots = b_n = 0$, contradicting $b_j \neq 0, j = 1, 2, \cdots, n$.

Case 2. One of $c_1, c_2, \cdots, c_n$ is a zero. Without loss of generality, we set $c_n = 0$. So (3.12) can be written as

$$s_1(z)e^{c_1 m d_m z^{m-1}} + s_2(z)e^{c_2 m d_m z^{m-1}} + \cdots + s_n(z) = e^{h(z)}.$$

Using a similar proof as in Case 1, we can also deduce a contradiction. □

4. Conclusions

Using the theory of meromorphic functions and the Nevanlinna theory, this paper study the uniqueness results about $f(z)$ and a linear difference polynomial $L(z, f)$ and promote the existing results on differential cases and difference cases of Br*ü*ck conjecture. Meanwhile, some sufficient conditions to show that $f(z)$ and $L(z, f)$ cannot share some small functions are also presented.

Acknowledgments

We are very grateful to the anonymous referees for their careful review and valuable suggestions. This research was funded by the National Natural Science Foundation of China (11801093, 11871260), the Natural Science Foundation of Guangdong Province (2018A030313508, 2020A1515010459), Guangdong Young Innovative Talents Project (2018KQNCX117) and Characteristic Innovation Project of Guangdong Province(2019KTSCX119).

Conflict of interest

The authors declare no conflict of interest.

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