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Research article

Asymptotic behavior of ground states for a fractional Choquard equation with critical growth

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Abstract: In this paper, we are concerned with the following fractional Choquard equation with critical growth:

$$(-\Delta)^s u + \lambda V(x)u = (|x|^{-\mu} * F(u))f(u) + |u|^{2_s^*-2}u \text{ in } \mathbb{R}^N,$$

where $s \in (0,1)$, $N > 2s$, $\mu \in (0,N)$, $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent, V is a steep well potential, $F(t) = \int_0^t f(s)ds$. Under some assumptions on f, the existence and asymptotic behavior of the positive ground states are established. In particular, if $f(u) = |u|^{p-2}u$, we obtain the range of p when the equation has the positive ground states for three cases $2s < N < 4s$ or $N = 4s$ or $N > 4s$.

Keywords: fractional Choquard equation; critical growth; ground states; asymptotic behavior
Mathematics Subject Classification: 35J50, 35Q40, 58E05

1. Introduction and the main results

The fractional Laplacian operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = C_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where the symbol P. V. stands for the Cauchy principal value and $C_{N,s}$ is a dimensional constant depending on N and s, precisely given by

$$C_{N,s} = \left[\frac{1-\cos\zeta_1}{|\zeta|^{N+2s}} d\zeta \right]^{-1}.$$

The nonlocal operators can be seen as the infinitesimal generators of Lévy stable diffusion processes [1]. Moreover, they allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media. The more physical background can be found in [9, 10, 16] and the references therein.

There are many papers considered the existence, multiplicity and qualitative properties of solutions for the fractional equations in the last decades, we refer to [2, 7, 8, 11] for the subcritical case and to [19, 24, 25, 28] for critical case, respectively.

It is worth mentioning that some authors have been investigated the following Schrödinger equation

$$(-\Delta)^s u + \lambda V(x)u = g(u) \ \text{ in } \ \mathbb{R}^N, \tag{1.1}$$

where V satisfies the following assumptions:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x) \geq 0$, $\Omega := int(V^{-1}(0))$ is non-empty with smooth boundary.

(V_2) There exists $M > 0$ such that $\left|\{x \in \mathbb{R}^N | V(x) \leq M\}\right| < \infty$, where $|\cdot|$ denotes the Lebesgue measure.

Note that the function V satisfying (V_1) and (V_2) is called the deepening potential well, which was first proposed by Bartsch and Wang in [5]. When $s = 1$ and $g(u) = |u|^{p-2}u$ with $2 < p < 2^*$, Bartsch and Wang [6] showed that, for λ large, (1.1) has a positive least energy solution, they also proved that a certain concentration behaviour of the solutions occur as $\lambda \to \infty$. In [13], Clapp and Ding actually generalized the results of [6] into the critical case. For more results to the Schrödinger equation with deepening potential well, we also cite [3, 4, 21, 25–27, 31] with no attempt to provide the full list of references.

Especially, if $s \in (0, 1)$ and $g(u) = (|x|^{-\mu} * F(u))f(u)$, then (1.1) goes back to the following fractional Choquard equation

$$(-\Delta)^s u + \lambda V(x)u = (|x|^{-\mu} * F(u))f(u) \ \text{ in } \ \mathbb{R}^N. \tag{1.2}$$

There are many works involving the existence, multiplicity and qualitative properties for solutions of (1.2) in the recent periods, we can refer to [12, 14, 18, 24, 30] as well as to the references therein. Very recently, under the assumption of $(V_1) - (V_2)$, Guo and Hu in [20] have proved the existence of the least energy solution to (1.2) with subcritical growth, which localizes near the bottom of potential well $int(V^{-1}(0))$ as λ large enough. It is a natural question that whether one can establish the similar results if nonlinearity is at critical growth, which inspired our present article. In this paper, we are concerned with the existence and asymptotic behavior of ground states for the following fractional Choquard equation with critical growth

$$(-\Delta)^s u + \lambda V(x)u = (|x|^{-\mu} * F(u))f(u) + |u|^{2_s^*-2}u \ \text{ in } \ \mathbb{R}^N, \tag{Q_λ}$$

where $s \in (0, 1)$, $N > 2s$, $\mu \in (0, N)$, where $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent, $F(t) = \int_0^t f(s)ds$, f satisfies the following assumptions:

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$, and there exist $c_1 > 0$ and $\frac{2N-\mu}{N} \leq p_1 \leq p_2 < \frac{2N-\mu}{N-2s}$ with $p_1 > \frac{2N-\mu}{2N-4s}$ such that $|f(t)| \leq c_1(|t|^{p_1-1} + |t|^{p_2-1})$ for all $t > 0$.

(f_2) There exist $q > 1$ and $c_2 > 0$ such that $f(t) \geq c_2|t|^{q-1}$ for all $t > 0$.

(f_3) $\frac{f(t)}{t}$ is nondecreasing in $(0, +\infty)$.

Remark 1.1. *From $(f_1)-(f_2)$, we have $p_1 \le q \le p_2$. We point out that Ambrosetti-Rabinowitz condition is not necessary in present paper.*

Remark 1.2. *Taking $f(t) = |t|^{p-2}t$, where $p \in [\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2s})$ with $p > \frac{2N-\mu}{2N-4s}$, then f satisfies $(f_1) - (f_3)$. We also remark that besides the usual power function, there are many other functions that satisfy our assumptions. For example, we may choose suitable μ, s, p and q such that $2 \le q \le p < \frac{2N-\mu}{N-2s}$. By a direct calculation, the assumption $(f_1) - (f_3)$ hold if we choose*

$$g(t) = |t|^{q-1} + |t| \ln(1 + |t|^{p-2}).$$

To statement our main results of this paper, let us introduce the following fractional Choquard equation:

$$\begin{cases}(-\Delta)^s u = (|x|^{-\mu} * F(u))f(u) + |u|^{2_s^*-2}u \text{ in } \Omega,\\ u \neq 0 \text{ in } \Omega,\\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,\end{cases} \tag{Q_0}$$

where $s \in (0, 1)$, $N > 2s$, $\mu \in (0, N)$, which acts as a limit role for (Q_λ) as $\lambda \to \infty$. Our main results of this paper are stated as follows:

Theorem 1.1. *Assume that $(V_1) - (V_2)$ and $(f_1) - (f_3)$ hold. Then, equation (Q_λ) has at least a positive ground state for λ large enough.*

Theorem 1.2. *Under the assumptions of Theorem 1.1, suppose that u_{λ_n} is one of the positive ground states of equation (Q_{λ_n}) with $\lambda_n \to \infty$. Then, up to a subsequence, $u_{\lambda_n} \to u$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$. Moreover, u is a positive ground state of equation (Q_0).*

In particular, by taking $f(u) = |u|^{p-2}u$ in (Q_λ) and (Q_0), we obtain the following fractional Choquard equations:

$$(-\Delta)^s u + \lambda V(x)u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u + |u|^{2_s^*-2}u \text{ in } \mathbb{R}^N \tag{P_λ}$$

and

$$\begin{cases}(-\Delta)^s u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u + |u|^{2_s^*-2}u \text{ in } \Omega,\\ u \neq 0 \text{ in } \Omega,\\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,\end{cases} \tag{P_0}$$

where $s \in (0, 1)$, $N > 2s$, $\mu \in (0, N)$.

As a direct result of Theorem 1.1 and Theorem 1.2, we have

Theorem 1.3. *Assume that $\mu \in (0, N)$ and $(V_1) - (V_2)$ hold. Then, equation (P_λ) has at least a positive ground state for λ large enough if one of the following cases occurs:*

(a) $2s < N < 4s$, $p \in (\frac{2N-\mu}{2N-4s}, \frac{2N-\mu}{N-2s})$.

(b) $N = 4s$, $p \in (\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2s})$.

(c) $N > 4s$, $p \in [\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2s})$.

Furthermore, suppose that u_{λ_n} is one of the positive ground states of equation (P_{λ_n}) with $\lambda_n \to \infty$. Then, up to a subsequence, $u_{\lambda_n} \to u$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$. Moreover, u is a positive ground state of equation (P_0).

Remark 1.3. *By Hardy-Littlewood-Sobolev inequality (see [22]), the energy functional corresponding to equation* (P_λ) *belongs to* C^1 *if* $p \in [\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2s}]$. *However, we need to put further restriction on p to overcome the difficulties caused by the estimates of convolution term. It seems that the condition* $p > \frac{2N-\mu}{2N-4s}$ *is essential for the proof of Lemma 2.8 below. Under the assumptions* $(V_1) - (V_2)$, *whether or not the existence and asymptotic behavior of ground states of equation* (P_λ) *can be established is an interesting question for the case* $N = 4s$ *with* $p = \frac{2N-\mu}{N}$ *and the case* $2s < N < 4s$ *with* $p \in (\frac{2N-\mu}{N}, \frac{2N-\mu}{2N-4s})$.

Compared with the nonlocal nonlinearity, the term $(|x|^{-\mu} * F(u))f(u)$ depends not only the pointwise value of $f(u)$, but also on $|x|^{-\mu} * F(u)$, which leads to some estimates about nonlocal term are likely to be confronted with some difficulties. In order to overcome them, some new variational techniques will be employed in our paper. Another difficulty of the problem (Q_λ) stems from that we can not verify that the energy functional corresponding to equation (Q_λ) satisfies the $(PS)_c$ condition under the any level set due to the fact that $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is noncompact. On the contrary, we can only check that the functional satisfies the $(PS)_c$ condition under a certain level set. Consequently, we have to make some more precise estimations involving critical term and nonlocal term.

The paper is organized as follows. In Section 2, we will introduce the variational frame and prove several Lemmas. In Section 3, we focus on the proofs of the main results.

Notation. Throughout this paper, $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively. $|\cdot|_r$ denotes the norm in $L^r(\Omega)$ for $1 \le r \le \infty$. $B_\rho(x)$ denotes the ball of radius ρ centered at x. C denote various positive constants whose value may change from line to line but are not essential to the analysis of the proof.

2. Variational frame and some Lemmas

Before proving our main results, it is necessary to introduce some useful definitions and notations. Firstly, fractional Sobolev spaces are the convenient setting for our problem, so we will give some stretches of the fractional order Sobolev spaces. We recall that, for any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$ is defined as follows:

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) | \int_{\mathbb{R}^N} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi < \infty\},$$

whose norm is defined as

$$||u||^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the inner

$$[u, v] := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy$$

and the norm

$$[u] := \Big(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \Big)^{\frac{1}{2}}.$$

The embedding $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous and for any $s \in (0,1)$, there exists a best constant $S_s > 0$ such that

$$S_s := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)} \frac{[u]^2}{|u|_{2_s^*}^2}$$

The fractional laplacian, $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$, is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N.$$

Also, by the Plancherel formular in Fourier analysis, we have

$$[u]^2_{H^s(\mathbb{R}^N)} = \frac{2}{C(s)}|(-\Delta)^{\frac{s}{2}}u|_2^2.$$

As a consequence, the norms on $H^s(\mathbb{R}^N)$ defined below

$$u \longmapsto \Big(\int_{\mathbb{R}^N} |u|^2 dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\frac{1}{2}},$$
$$u \longmapsto \Big(\int_{\mathbb{R}^N} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi\Big)^{\frac{1}{2}},$$
$$u \longmapsto \Big(\int_{\mathbb{R}^N} |u|^2 dx + |(-\Delta)^{\frac{s}{2}}u|_2^2\Big)^{\frac{1}{2}}$$

are equivalent. For more details on fractional Sobolev spaces, we refer the reader to [15] and the references therein. In this paper, the definition of fractional Sobolev space $H^s(\mathbb{R}^N)$ is chosen by

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N)\big|[u] < +\infty\}$$

equipped with the inner

$$\langle u, v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^N} uvdx$$

whose associated norm we denote by $\|\cdot\|$.

Now, for fixed $\lambda > 0$, we define the following fractional Sobolev space

$$E_\lambda = \{u \in H^s(\mathbb{R}^N)| \int_{\mathbb{R}^N} \lambda V(x)u^2 dx < +\infty\}$$

equipped with the inner product

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^N} \lambda V(x)uvdx$$

whose associated norm we denote by $\|\cdot\|_\lambda$. Define

$$E_0 = \{u \in H^s(\mathbb{R}^N)|u(x) = 0 \text{ in } \Omega\}.$$

Obviously, E_0 is a closed subspace of $H^s(\mathbb{R}^N)$, and hence is a Hilbert space.

Lemma 2.1. *[25] Let* $0 < s < 1$*, then there exists a constant* $C = C(s) > 0$*, such that*

$$|u|_{2_s^*}^2 \leq C[u]^2$$

for any $u \in H^s(\mathbb{R}^N)$*. Moreover, the embedding* $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ *is continuous for any* $r \in [2, 2_s^*]$ *and is locally compact whenever* $r \in [1, 2_s^*)$*.*

Because we are concerned with the nonlocal problems, we would like to recall the well-known Hardy-Littlewood-Sobolev inequality.

Lemma 2.2. *[22] Suppose* $\mu \in (0, N)$*, and* $s, r > 1$ *with* $\frac{1}{s} + \frac{1}{r} = 1 + \frac{\mu}{N}$*. Let* $g \in L^s(\mathbb{R}^N)$*,* $h \in L^r(\mathbb{R}^N)$*, there exists a sharp constant* $C(s, \mu, r, N)$*, independent of g and h, such that*

$$\int_{\mathbb{R}^N} (|x|^{-\mu} * g)h dx \leq C(s, \mu, r, N)|g|_s |h|_r.$$

Since we are looking for ground states of (Q_λ) when λ is large enough, without loss of generality, we assume $\lambda \geq 1$ in the rest of the paper. We have the following embedding result.

Lemma 2.3. *Assume that* $V(x)$ *satisfies* (V_2)*. Then the embedding* $E_\lambda \hookrightarrow H^s(\mathbb{R}^N)$ *is continuous for any* $\lambda \geq 1$*. Moreover, there exists* τ_0 *independent of* λ *such that*

$$||u|| \leq \tau_0 ||u||_\lambda \tag{2.1}$$

for any $u \in E_\lambda$*.*

Proof. Let

$$\Omega_1 = \{x \in \mathbb{R}^N | V(x) > M\}, \ \ \Omega_2 = \{x \in \mathbb{R}^N | V(x) \leq M\}.$$

For $\lambda \geq 1$, we have

$$\int_{\Omega_1} u^2 dx \leq \frac{1}{M} \int_{\mathbb{R}^N} \lambda V(x) u^2 dx.$$

By (V_2), the Hölder inequality and Lemma 2.1, one has

$$\int_{\Omega_2} u^2 dx \leq |\Omega_2|^{\frac{N}{2s}} \Big(\int_{\Omega_2} u^{2_s^*} dx \Big)^{\frac{2}{2_s^*}} \leq |\Omega_2|^{\frac{N}{2s}} [u]^2.$$

Consequently,

$$||u|| \leq (\frac{1}{M} + |\Omega_2|^{\frac{N}{2s}} + 1)^{\frac{1}{2}} ||u||_\lambda := \tau_0 ||u||_\lambda. \tag{2.2}$$

The proof is completed. □

Since our main aim is to find the positive solutions, without loss of generality, we assume that $f(t) = 0$ for $t \leq 0$. The corresponding energy functionals associated with equations (Q_λ) and (Q_0) are given by

$$I_\lambda(u) = \frac{1}{2}||u||_\lambda^2 - \frac{1}{2} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u))F(u) dx - \frac{1}{2_s^*} \int_{\mathbb{R}^N} |u^+|^{2_s^*} dx$$

and

$$I_0(u) = \frac{1}{2}[u]^2 - \frac{1}{2} \int_\Omega (|x|^{-\mu} * F(u))F(u) dx - \frac{1}{2_s^*} \int_\Omega |u^+|^{2_s^*} dx,$$

respectively. Clearly, $I_\lambda \in C^1(E_\lambda, \mathbb{R})$ and $I_0 \in C^1(E_0, \mathbb{R})$. Denote

$$m_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u), \qquad m_0 = \inf_{u \in \mathcal{N}_0} I_0(u),$$

where

$$\mathcal{N}_\lambda = \{u \in E_\lambda \setminus \{0\} | \langle I'_\lambda(u), u\rangle = 0\}, \quad \mathcal{N}_0 = \{u \in H^1_0(\Omega) \setminus \{0\} | \langle I'_0(u), u\rangle = 0\}.$$

Remark 2.1. *Obviously, u is a critical point of I_λ if and only if u is a solution of (Q_λ). Similarly, u is a critical point of I_0 if and only if u is a solution of (Q_0). Hence, in order to prove Theorem 1.1 and Theorem 1.2, it suffices to check that m_λ is achieved by a positive critical point of I_λ for λ large enough. Furthermore, for any sequence $\lambda_n \to \infty$, if u_{λ_n} be one of the critical points of I_λ, then there exists $u \in H^s(\mathbb{R}^N)$ such that $I'_0(u) = 0$ and $I_0(u) = m_0$. Moreover, up to a subsequence, $u_{\lambda_n} \to u$ in $H^s(\mathbb{R}^N)$.*

Lemma 2.4. *Let $c > 0$ be fixed. Assume that $\{u^\lambda_n\} \subset E_\lambda$ be a $(PS)_c$ sequence of I_λ. Then*

$$\limsup_{n\to\infty} \|u^\lambda_n\|_\lambda \le \frac{2\kappa_s c}{\kappa_s - 2}, \tag{2.3}$$

where $\kappa_s = \min\{2^*_s, 4\}$. Moreover, there exist $\delta > 0$ independent of λ such that either $u^\lambda_n \to 0$ in E_λ or $\limsup_{n\to\infty} \|u^\lambda_n\|_\lambda > \delta$.

Proof. By (f_3), $F(t) \le 2f(t)t$ for any $t \in \mathbb{R}$. Since $I'_\lambda(u^\lambda_n) = o_n(1)$ and $I_\lambda(u^\lambda_n) = c + o_n(1)$,

$$\begin{aligned}
c + o_n(1)\|u^\lambda_n\|_\lambda &= I_\lambda(u^\lambda_n) - \frac{1}{\kappa_s}\langle I'_\lambda(u^\lambda_n), u^\lambda_n\rangle \\
&= (\frac{1}{2} - \frac{1}{\kappa_s})\|u^\lambda_n\|^2_\lambda - \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(u^\lambda_n))F(u^\lambda_n)dx \\
&\quad + \frac{1}{\kappa_s}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(u^\lambda_n))f(u^\lambda_n)u^\lambda_n dx + (\frac{1}{\kappa_s} - \frac{1}{2^*_s})\int_{\mathbb{R}^N} |(u^\lambda_n)^+|^{2^*_s} dx \\
&\ge (\frac{1}{2} - \frac{1}{\kappa_s})\|u^\lambda_n\|^2_\lambda + (\frac{2}{\kappa_s} - \frac{1}{2})\int_{\mathbb{R}^N} (|x|^{-\mu} * F(u^\lambda_n))F(u^\lambda_n)dx + (\frac{1}{\kappa_s} - \frac{1}{2^*_s})\int_{\mathbb{R}^N} |(u^\lambda_n)^+|^{2^*_s} dx \\
&\ge (\frac{1}{2} - \frac{1}{\kappa_s})\|u^\lambda_n\|^2_\lambda.
\end{aligned} \tag{2.4}$$

Hence $\{u^\lambda_n\}$ is bounded in E_λ, and hence

$$c + o_n(1) \ge (\frac{1}{2} - \frac{1}{\kappa_s})\|u^\lambda_n\|^2_\lambda.$$

This leads to

$$\limsup_{n\to\infty} \|u^\lambda_n\|^2_\lambda \le \frac{2\kappa_s c}{\kappa_s - 2}.$$

For any $u \in E_\lambda$, by the Hardy-Littlewood-Sobolev inequality and Lemma 2.3, we have

$$\langle I'_\lambda(u), u\rangle \ge \frac{1}{2}\|u\|^2_\lambda - C(\|u\|^{2p_1}_\lambda + \|u\|^{p_1+p_2}_\lambda + \|u\|^{2p_2}_\lambda) - C\|u\|^{2^*_s}_\lambda. \tag{2.5}$$

Consequently, there exist $\delta > 0$ such that $u \in E_\lambda$ with $\|u\|_\lambda \le \delta$, we have

$$\langle I'_\lambda(u), u\rangle \ge \frac{1}{4}\|u\|^2_\lambda. \tag{2.6}$$

If $\limsup_{n\to\infty} \|u_n^\lambda\|_\lambda \le \delta$, without loss of generality, we may assume $\|u_n^\lambda\| \le \delta$ for all n. By (2.6), one has

$$o_n(1)\|u_n^\lambda\|_\lambda \ge \langle I'_\lambda(u_n^\lambda), u_n^\lambda\rangle \ge \frac{1}{4}\|u_n^\lambda\|_\lambda^2,$$

and hence $\|u_n^\lambda\| \to 0$ as $n \to \infty$. □

Lemma 2.5. *Let $C_0 > 0$ be fixed, $u_n^\lambda \rightharpoonup u_\lambda$ in E_λ with $I(u_n^\lambda) \in [0, C_0]$. Then for any small $\varepsilon > 0$, there exists $\Lambda_\varepsilon > 0$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n^\lambda - u_\lambda|^r dx \le \varepsilon$$

for any $\lambda > \Lambda_\varepsilon$ and $2 \le r < 2_s^*$.

Proof. Firstly, we claim that for any $\varepsilon > 0$, there exists $\Lambda_\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n^\lambda - u_\lambda|^2 dx \le \varepsilon$$

for any $\lambda > \Lambda_\varepsilon$. We argue by contradiction that there exist $\varepsilon_0 > 0$, $\lambda_k \to +\infty$ and $n_k \to +\infty$ such that

$$\int_{\mathbb{R}^N} |u_{n_k}^{\lambda_k} - u_{\lambda_k}|^2 dx \ge \varepsilon_0, \quad \forall k. \tag{2.7}$$

Let $D_R = \{x \in \mathbb{R}^N \| x| > R \text{ and } V(x) \le M\}$. In view of (V_2), $\lim_{R\to\infty} \left|D_R\right| = 0$. For k large enough, by (2.3) and the fact that $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous, one has

$$\begin{aligned}\int_{D_R} |u_{n_k}^{\lambda_k}|^2 dx &\le \left|D_R\right|^{\frac{2s}{N}} \Big(\int_{D_R} |u_{n_k}^{\lambda_k}|^{2_s^*} dx\Big)^{\frac{2}{2_s^*}} \\ &\le \left|D_R\right|^{\frac{2s}{N}} [u_{n_k}^{\lambda_k}]^2 \\ &\le C_1 |D_R\big|^{\frac{2s}{N}}.\end{aligned} \tag{2.8}$$

It follows from (2.3) that

$$\begin{aligned}\int_{B_R^c \setminus D_R} |u_{n_k}^{\lambda_k}|^2 dx &\le \frac{1}{\lambda_k M} \int_{B_R^c \setminus D_R} \lambda_k V(x) |u_{n_k}^{\lambda_k}|^2 dx \\ &\le \frac{C_1}{\lambda_k}.\end{aligned} \tag{2.9}$$

By (2.8)–(2.9), there exist $K > 0$ and $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} |u_{n_k}^{\lambda_k}|^2 dx < \frac{\varepsilon_0}{8}, \quad \forall k > K. \tag{2.10}$$

Similarly, one can check that

$$\int_{\mathbb{R}^N \setminus B_R(0)} |u_{\lambda_k}|^2 dx < \frac{\varepsilon_0}{8}, \quad \forall k > K. \tag{2.11}$$

Since $u_n^\lambda \to u_\lambda$ in $L_{loc}^r(\mathbb{R}^N)$ for $1 \le r < 2_s^*$, we may assume that

$$\int_{B_R(0)} |u_{n_k}^{\lambda_k} - u_{\lambda_k}|^2 < \frac{\varepsilon_0}{4}. \tag{2.12}$$

Combining (2.7) and (2.10)–(2.12), one has

$$\begin{aligned}\varepsilon_0 &\le \limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_{n_k}^{\lambda_k}-u_{\lambda_k}|^2dx\\ &\le 2\limsup_{n\to\infty}\int_{B_R^c(0)}|u_{n_k}^{\lambda_k}|^2dx+2\limsup_{n\to\infty}\int_{B_R^c(0)}|u_{\lambda_k}|^2dx\\ &\quad+\limsup_{n\to\infty}\int_{B_R(0)}|u_{n_k}^{\lambda_k}-u_{\lambda_k}|^2dx\\ &< \frac{3\varepsilon_0}{4},\end{aligned}$$

a contradiction. For small $\varepsilon > 0$ and $\lambda > \Lambda_\varepsilon$, by the interpolation inequality, we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_{\lambda_n}-u_\lambda|^r dx \le \varepsilon,$$

where $2 \le r < 2_s^*$. □

Lemma 2.6. *Let λ be fixed and $\{u_n^\lambda\} \subset E_\lambda$ be $(PS)_c$ of I_λ. Then, there exists $u_\lambda \in E_\lambda$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) \ge 0$. Moreover, we have*

$$I_\lambda(u_n^\lambda) - I_\lambda(v_n^\lambda) \to I_\lambda(u_\lambda) \tag{2.13}$$

and

$$I'_\lambda(u_n) - I'_\lambda(v_n) \to I'_\lambda(u_\lambda), \tag{2.14}$$

where $v_n^\lambda := u_n^\lambda - u_\lambda$.

Proof. The proof is similar to [23]. For convenience sake, we give an outline here. For the sake of simplicity of symbols, we denote u_n^λ by u_n. Lemma 2.4 implies that $\{u_n\}$ is bounded in E_λ. Up to a subsequence, we may assume that

$$u_n \rightharpoonup u_\lambda \ \text{ in } E_\lambda \quad \text{and} \quad u_n \to u_\lambda \ \text{ in } L^r_{loc}(\mathbb{R}^N) \quad \text{in } 1 \le r < 2_s^*.$$

It is easy to prove that $I'_\lambda(u_\lambda) = 0$. Similar to (2.4), one has $I_\lambda(u_\lambda) \ge 0$. As the proof of the Lemma 2.4 in [23] , we have the following nonlocal Brézis-Lieb result

$$\begin{aligned}&\int_{\mathbb{R}^N}(|x|^{-\mu} * F(u_n))F(u_n)dx - \int_{\mathbb{R}^N}(|x|^{-\mu} * F(u_\lambda))F(u_\lambda)dx\\ &\to \int_{\mathbb{R}^N}(|x|^{-\mu} * F(v_n))F(v_n)dx.\end{aligned} \tag{2.15}$$

It follows from Brézis-Lieb Lemma (see Lemma 1.32 in [29]) that

$$\int_{\mathbb{R}^N}|(u_n^\lambda)^+|^{2_s^*}dx - \int_{\mathbb{R}^N}|u_\lambda^+|^{2_s^*}dx \to \int_{\mathbb{R}^N}|(v_n^\lambda)^+|^{2_s^*}dx. \tag{2.16}$$

Combining (2.15) and (2.16), one has

$$I_\lambda(u_n) - I_\lambda(v_n) \to I_\lambda(u_\lambda). \tag{2.17}$$

Similarly, (2.14) is satisfied with some slight modifications. □

Lemma 2.7. *If* $c < \frac{s}{N}S_s^{\frac{N}{2s}}$, *then there exists* $\Lambda_0 > 0$ *such that* I_λ *satisfies the* $(PS)_c$ *condition for* $\lambda \geq \Lambda_0$.

Proof. Consider any sequence $\{u_n^\lambda\} \subset E_\lambda$ satisfying $I'_\lambda(u_n^\lambda) \to 0$ with $I_\lambda(u_n^\lambda) \to c < \frac{s}{N}S_s^{\frac{N}{2s}}$. By Lemma 2.4, $\{u_n^\lambda\}$ is bounded in E_λ. Let $v_n^\lambda = u_n^\lambda - u_\lambda$. Then

$$\begin{aligned}\int_{\mathbb{R}^N}(|x|^{-\mu}*F(u_n^\lambda))f(u_n^\lambda)u_n^\lambda dx - \int_{\mathbb{R}^N}(|x|^{-\mu} * F(u_\lambda))f(u_\lambda)u_\lambda dx \\ \to \int_{\mathbb{R}^N}(|x|^{-\mu} * F(v_n^\lambda))f(v_n^\lambda)v_n^\lambda dx.\end{aligned} \tag{2.18}$$

By (2.16), (2.18) and Lemma 2.6, one has

$$\begin{aligned}\|v_n^\lambda\|_\lambda^2 &= \|u_n^\lambda\|_\lambda^2 - \|u_\lambda\|_\lambda^2 + o_n(1) \\ &= \langle I'_\lambda(u_n^\lambda), u_n^\lambda\rangle + \int_{\mathbb{R}^N}(|x|^{-\mu} * F(u_n^\lambda))f(v_n^\lambda)u_n^\lambda dx + \int_{\mathbb{R}^N}|(u_n^\lambda)^+|^{2_s^*}dx \\ &\quad - \langle I'_\lambda(u_\lambda), u_\lambda\rangle - \int_{\mathbb{R}^N}(|x|^{-\mu} * F(u_\lambda))f(u_\lambda)u_\lambda dx - \int_{\mathbb{R}^N}|u_\lambda^+|^{2_s^*}dx + o_n(1) \\ &= \int_{\mathbb{R}^N}|(v_n^\lambda)^+|^{2_s^*}dx + \int_{\mathbb{R}^N}(|x|^{-\mu} * F(v_n^\lambda))f(v_n^\lambda)v_n^\lambda dx + o_n(1).\end{aligned}$$

Hence, up to a subsequence, we may assume

$$\lim_{n\to\infty}\|v_n^\lambda\|_\lambda^2 = \lim_{n\to\infty}\int_{\mathbb{R}^N}|(v_n^\lambda)^+|^{2_s^*}dx + \lim_{n\to\infty}\int_{\mathbb{R}^N}(|x|^{-\mu} * F(v_n^\lambda))f(v_n^\lambda)v_n^\lambda dx := \theta_\lambda \geq 0.$$

It suffices to check that there exists $\varepsilon_0 > 0$ such that $\theta_\lambda = 0$ for $\lambda > \Lambda_{\varepsilon_0}$, where Λ_ε is given in Lemma 2.5. Otherwise, without loss of generality, there exists $\lambda_k \geq \Lambda_{\frac{1}{k}} \geq 1$ such that $\theta_{\lambda_k} > 0$ for any $k \in \mathbb{Z}$. For large k and n, by Lemma 2.5 and the Hardy-Littlewood-Sobolev inequality, one has

$$\begin{aligned}\int_{\mathbb{R}^N}(|x|^{-\mu} * F(v_n^{\lambda_k}))f(v_n^{\lambda_k})v_n^\lambda dx &\leq C_2\left(\int_{\mathbb{R}^N}(|v_n^{\lambda_k}|^{p_1} + |v_n^{\lambda_k}|^{p_2})^{\frac{2N}{2N-\mu}}dx\right)^{\frac{2N-\mu}{N}} \\ &\leq C_3\left(|v_n^{\lambda_k}|_{\frac{2Np_1}{2N-\mu}}^{2p_1} + |v_n^{\lambda_k}|_{\frac{Np_1}{2N-\mu}}^{p_1}|v_n^{\lambda_k}|_{\frac{Np_2}{2N-\mu}}^{p_2} + |v_n^{\lambda_k}|_{\frac{2Np_2}{2N-\mu}}^{2p_2}\right) \\ &\leq C_3\left(\frac{1}{k^{2p_1}} + \frac{1}{k^{p_1+p_2}} + \frac{1}{k^{2p_2}}\right) \\ &\leq \frac{1}{k}.\end{aligned} \tag{2.19}$$

By Lemma 2.6, $\{v_n^{\lambda_k}\}$ be $(PS)_{c_k}$ for I_{λ_k}, where $c_k = c - I_{\lambda_k}(u_{\lambda_k})$. Since $\theta_{\lambda_k} > 0$, by Lemma 2.4, we may assume that $\theta_{\lambda_k} \geq \delta$ for all k. By the definition of S_s, there holds

$$\|v_n^{\lambda_k}\|_\lambda^2 \geq [v_n^{\lambda_k}]^2 \geq S_s|v_n^{\lambda_k}|_{2_s^*}^2 \geq S_s|(v_n^{\lambda_k})^+|_{2_s^*}^2.$$

Hence

$$\theta_{\lambda_k} \geq S_s(\theta_{\lambda_k} - \frac{1}{k})^{\frac{2}{2_s^*}} \geq S_s\theta_{\lambda_k}^{\frac{2}{2_s^*}}(1 - \frac{1}{\delta k}),$$

and hence $\theta_{\lambda_k} \geq S_s^{\frac{N}{2s}}(1-\frac{1}{\delta k})^{\frac{N}{2s}}$. For large k, by Lemma 2.6 and (2.19), one has

$$\begin{aligned}
c &= I_{\lambda_k}(v_n^{\lambda_k}) + I_{\lambda_k}(u_{\lambda_k}) + o_n(1)\\
&\geq I_{\lambda_k}(v_n^{\lambda_k}) + o_n(1)\\
&= \frac{1}{2}\|v_n^{\lambda_k}\|_{\lambda_k}^2 - \frac{1}{2}\int_{\mathbb{R}^N}(|x|^{-\mu} * F(v_n^{\lambda_k}))F(v_n^{\lambda_k})dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|(v_n^{\lambda_k})^+|^{2_s^*}dx + o_n(1)\\
&\geq \frac{1}{2}\|v_n^{\lambda_k}\|_{\lambda_k}^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|(v_n^{\lambda_k})^+|^{2_s^*}dx - \frac{1}{2k} + o_n(1)\\
&\geq \frac{s}{N}\theta_{\lambda_k} - \frac{1}{2k} + o_n(1)\\
&\geq \frac{s}{N}S_s^{\frac{N}{2s}}(1-\frac{1}{\delta k})^{\frac{N}{2s}} - \frac{1}{2k} + o_n(1).
\end{aligned}$$

This leads to $c \geq \frac{s}{N}S_s^{\frac{N}{2s}}$, which contradicts $c < \frac{s}{N}S_s^{\frac{N}{2s}}$. This completes the proof. □

Lemma 2.8. *If $p_1 \in [\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2s})$ with $p_1 > \frac{2N-\mu}{2N-4s}$, then there exists $\alpha > 0$ such that $\alpha \leq m_\lambda \leq m_0 < \frac{s}{N}S_s^{\frac{N}{2s}}$.*

Proof. Clearly, $m_\lambda \leq m_0$. Since the proof of $m_\lambda \geq \alpha$ is standard, we only need to prove that $m_0 < \frac{s}{N}S_s^{\frac{N}{2s}}$. Without loss of generality, we assume that $0 \in \Omega$. Then there exist $\delta > 0$ and $k \in \mathbb{Z}$ such that $B_\delta \subset B_{2\delta} \subset \Omega \subset B_{k\delta}$. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ in B_δ, $\eta = 0$ in $\mathbb{R}^N \setminus B_{2\delta}$. Denote

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}}u_0\Big(\frac{x}{\varepsilon|u_0|_{2_s^*}}\Big),$$

where $u_0(x) = \alpha(\beta^2 + S_s^{-\frac{1}{2s}}|x|^2)^{-\frac{N-2s}{2}}$ with $\alpha,\beta > 0$. Set

$$u_\varepsilon(x) := \eta(x)U_\varepsilon(x),$$

then $u_\varepsilon(x) \in E_0$. It follows from Proposition 21 and Proposition 22 in [25] that

$$[u_\varepsilon]^2 \leq S_s^{\frac{N}{2s}} + o(\varepsilon^{N-2s}), \qquad \int_{\mathbb{R}^N}|u_\varepsilon|^{2_s^*}dx = S_s^{\frac{N}{2s}} + o(\varepsilon^N). \tag{2.20}$$

Let

$$g_\varepsilon(t) := \frac{t^2}{2}[u_\varepsilon]^2 - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N}|u_\varepsilon|^{2_s^*}dx.$$

In view of (2.20), one has

$$\begin{aligned}
\max_{t\geq 0} g_\varepsilon(t) &= \frac{s}{N}\Big(\frac{[u_\varepsilon]^2}{|u_\varepsilon|_{2_s^*}^2}\Big)^{\frac{N}{2s}}\\
&= \frac{s}{N}\Big[\frac{S_s^{\frac{N}{2s}} + o(\varepsilon^{N-2s})}{(S_s^{\frac{N}{2s}} + o(\varepsilon^N))^{\frac{N-2s}{N}}}\Big]^{\frac{N}{2s}}\\
&\leq \frac{s}{N}S_s^{\frac{N}{2s}} + o(\varepsilon^{N-2s}).
\end{aligned} \tag{2.21}$$

Clearly, there exists $t_\varepsilon > 0$ such that $t_\varepsilon u_\varepsilon \in \mathcal{N}_0$ and $I_0(t_\varepsilon u_\varepsilon) = \max_{t\geq 0} I_0(tu_\varepsilon)$. As a consequence, $m_0 \leq I_0(t_\varepsilon u_\varepsilon)$ and

$$t_\varepsilon^2[u_\varepsilon]^2 = \int_\Omega (|x|^{-\mu} * F(t_\varepsilon u_\varepsilon)) f(t_\varepsilon u_\varepsilon) t_\varepsilon u_\varepsilon dx + t_\varepsilon^{2_s^*} \int_{\mathbb{R}^N} |u|^{2_s^*} dx. \tag{2.22}$$

Next, we prove the following claim:

Claim 2.1.

$$\frac{1}{t_\varepsilon^{2p_1} + t_\varepsilon^{2p_2}} \int_\Omega (|x|^{-\mu} * F(t_\varepsilon u_\varepsilon)) f(t_\varepsilon u_\varepsilon) t_\varepsilon u_\varepsilon dx \leq O(\varepsilon^{2N-p_2(N-2s)-\mu}). \tag{2.23}$$

.

In fact, by (f_2), for small $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{t_\varepsilon^{2p_1} + t_\varepsilon^{2p_2}} \int_\Omega (|x|^{-\mu} * F(t_\varepsilon u_\varepsilon)) f(t_\varepsilon u_\varepsilon) t_\varepsilon u_\varepsilon dx \\
&\leq \int_\Omega \int_\Omega \frac{2c_1\big(|u_\varepsilon(x)|^{p_1} + |u_\varepsilon(x)|^{p_2}\big)\big(|u_\varepsilon(y)|^{p_1} + |u_\varepsilon(y)|^{p_2}\big)}{|x-y|^\mu} dxdy \\
&\leq \int_{B_{2\delta}} \int_{B_{2\delta}} \frac{c_1 |U_\varepsilon(x)|^{p_1} |U_\varepsilon(y)|^{p_1}}{|x-y|^\mu} dxdy \\
&\quad + \int_{B_{2\delta}} \int_{B_{2\delta}} \frac{2c_1 |U_\varepsilon(x)|^{p_1} |U_\varepsilon(y)|^{p_2}}{|x-y|^\mu} dxdy \\
&\quad + \int_{B_{2\delta}} \int_{B_{2\delta}} \frac{c_1 |U_\varepsilon(x)|^{p_2} |U_\varepsilon(y)|^{p_2}}{|x-y|^\mu} dxdy \\
&\leq \int_{B_{2\delta}} \int_{B_{2\delta}} \frac{C_4 \varepsilon^{p_1(N-2s)}}{(\varepsilon^2 + |x|^2)^{\frac{p_1(N-2s)}{2}} (\varepsilon^2 + |y|^2)^{\frac{p_1(N-2s)}{2}} |x-y|^\mu} dxdy \\
&\quad + \int_{B_{2\delta}} \int_{B_{2\delta}} \frac{C_4 \varepsilon^{\frac{(N-2s)(p_1+p_2)}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{p_1(N-2s)}{2}} (\varepsilon^2 + |y|^2)^{\frac{p_2(N-2s)}{2}} |x-y|^\mu} dxdy \\
&\quad + \int_{B_{2\delta}} \int_{B_{2\delta}} \frac{C_4 \varepsilon^{p_2(N-2s)}}{(\varepsilon^2 + |x|^2)^{\frac{p_2(N-2s)}{2}} (\varepsilon^2 + |y|^2)^{\frac{p_2(N-2s)}{2}} |x-y|^\mu} dxdy \\
&\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{C_4 \varepsilon^{2N-p_1(N-2s)-\mu}}{(1 + |x|^2)^{\frac{p_1(N-2s)}{2}} (1 + |y|^2)^{\frac{p_1(N-2s)}{2}} |x-y|^\mu} dxdy \\
&\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{C_4 \varepsilon^{\frac{4N-(N-2s)(p_1+p_2)-2\mu}{2}}}{(1 + |x|^2)^{\frac{p_1(N-2s)}{2}} (1 + |y|^2)^{\frac{p_2(N-2s)}{2}} |x-y|^\mu} dxdy \\
&\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{C_4 \varepsilon^{2N-p_2(N-2s)-\mu}}{(1 + |x|^2)^{\frac{p_2(N-2s)}{2}} (1 + |y|^2)^{\frac{p_2(N-2s)}{2}} |x-y|^\mu} dxdy \\
&:= C_5(I_1 + I_2 + I_3),
\end{aligned} \tag{2.24}$$

where c_1 and c_2 are given by (f_1). Since $p_1 > \frac{2N-\mu}{2N-4s}$, $N-1-\frac{2p_1N(N-2s)}{2N-\mu} < -1$. Consequently,

$$\begin{aligned}\int_{\mathbb{R}^N}(1+|x|^2)^{-\frac{p_1N(N-2s)}{2N-\mu}}dx &= C_6\int_0^1 \frac{r^{N-1}}{(1+|r|^2)^{\frac{p_1N(N-2s)}{2N-\mu}}}dr\\ &\quad + C_6\int_1^\infty \frac{r^{N-1}}{(1+|r|^2)^{\frac{p_1N(N-2s)}{2N-\mu}}}dr\\ &\le C_7 + C_6\int_1^\infty r^{N-1-\frac{2p_1N(N-2s)}{2N-\mu}}dr\\ &< +\infty.\end{aligned} \tag{2.25}$$

By the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}I_1 &\le C_8\varepsilon^{2N-p_1(N-2s)-\mu}\Big(\int_{\mathbb{R}^N}(1+|x|^2)^{-\frac{p_1N(N-2s)}{2N-\mu}}dx\Big)^{\frac{4N}{2N-\mu}}\\ &= O(\varepsilon^{2N-p_1(N-2s)-\mu}).\end{aligned} \tag{2.26}$$

Similarly, one can check that

$$I_1 = O(\varepsilon^{\frac{4N-(N-2s)(p_1+p_2)-2\mu}{2}}) \tag{2.27}$$

and

$$I_2 = O(\varepsilon^{2N-p_2(N-2s)-\mu}). \tag{2.28}$$

Since $p_1 \le p_2$, the claim follows from (2.24), (2.26)–(2.28).

For small $\varepsilon > 0$, by (2.21) and (2.23), there exist $C_9, C_{10} > 0$ such that

$$\int_{\mathbb{R}^N}|u_\varepsilon|^{2_s^*}dx \ge C_9, \quad [u_\varepsilon]^2 \le C_{10},$$

and

$$\int_\Omega(|x|^{-\mu} * F(t_\varepsilon u_\varepsilon))f(t_\varepsilon u_\varepsilon)t_\varepsilon u_\varepsilon dx \le C_{10}(t_\varepsilon^{2p_1} + t_\varepsilon^{2p_2}).$$

According to (2.22), we have

$$C_9 \le C_{10}(t_\varepsilon^{2p_1-2} + t_\varepsilon^{2p_2-2}) + C_{10}t_\varepsilon^{2_s^*-2}.$$

Thus, for small $\varepsilon > 0$ there exists $t_0 > 0$ such that $t_\varepsilon \ge t_0$. On the other hand, by (f_2), there holds

$$\begin{aligned}\frac{q}{t_\varepsilon^{2q}}\int_\Omega(|x|^{-\mu} * F(t_\varepsilon u_\varepsilon))F(t_\varepsilon u_\varepsilon)dx &\ge c_2\int_\Omega(|x|^{-\mu} * |u_\varepsilon|^q)|u_\varepsilon|^q dx\\ &\ge \int_{B_\delta}\int_{B_\delta}\frac{c_2|u_\varepsilon(x)|^q|u_\varepsilon(y)|^q}{|x-y|^\mu}dxdy\\ &\ge \int_{B_\delta}\int_{B_\delta}\frac{C_{11}\varepsilon^{q(N-2s)}}{(\varepsilon^2+|x|^2)^{\frac{q(N-2s)}{2}}(\varepsilon^2+|y|^2)^{\frac{q(N-2s)}{2}}|x-y|^\mu}dxdy\\ &\ge \int_{B_{\frac{\delta}{\varepsilon}}}\int_{B_{\frac{\delta}{\varepsilon}}}\frac{C_{11}\varepsilon^{2N-q(N-2s)-\mu}}{(1+|x|^2)^{\frac{q(N-2s)}{2}}(1+|y|^2)^{\frac{q(N-2s)}{2}}}dxdy\\ &\ge \int_{B_\delta}\int_{B_\delta}\frac{C_{11}\varepsilon^{2N-q(N-2s)-\mu}}{(1+|x|^2)^{\frac{q(N-2s)}{2}}(1+|y|^2)^{\frac{q(N-2s)}{2}}}dxdy\\ &= C_{12}\varepsilon^{2N-q(N-2s)-\mu}.\end{aligned} \tag{2.29}$$

Hence

$$\int_{\Omega}(|x|^{-\mu}*F(t_\varepsilon u_\varepsilon))F(t_\varepsilon u_\varepsilon)dx \geq C_{13}t_\varepsilon^{2q}\varepsilon^{2N-q(N-2s)-\mu}.$$

Since $N > 2s$ and $q \geq p_1 > \frac{2N-\mu}{2N-4s}$, then $q > \frac{N+2s-\mu}{N-2s}$. Combining (2.21) and (2.29), one has

$$\begin{aligned} m_0 \leq I_0(t_\varepsilon u_\varepsilon) &\leq \max_{t\geq 0} g_\varepsilon(t) - C_{13}t_\varepsilon^{2q}\varepsilon^{2N-q(N-2s)-\mu} \\ &< S_s^{\frac{N}{2s}} + o(\varepsilon^{N-2s}) - C_{13}t_0^{2q}\varepsilon^{2N-q(N-2s)-\mu} \\ &< \frac{s}{N}S_s^{\frac{N}{2s}}. \end{aligned}$$

The proof is completed. □

3. The proof of the main results

3.1. The proof of Theorem 1.1

Proof. Assume that $\{u_n^\lambda\} \subset \mathcal{N}_\lambda$ be a minimizing sequence of m_λ. By Ekeland's Variational principle (see [17]), we may assume that $\{u_n^\lambda\}$ be a $(PS)_{m_\lambda}$ sequence for I_λ, that is $I_\lambda'(u_n^\lambda) \to 0$ and $I_\lambda(u_n^\lambda) \to m_\lambda$. In view of Lemma 2.8, $m_\lambda < \frac{s}{N}S_s^{\frac{N}{2s}}$. By lemma 2.7, there exist $\Lambda_0 > 0$, up to a subsequence, $u_n^\lambda \to u_\lambda$ in E_λ for any $\lambda > \Lambda_0$. Since $I_\lambda \in C^1(E_\lambda, \mathbb{R})$, then $I_\lambda(u_\lambda) = m_\lambda$ and $I_\lambda'(u_\lambda) = 0$. Noting that $f(t) = 0$ for $t \leq 0$ and $(t-s)(t^- - s^-) \geq |t^- - s^-|^2$ for all t , $s \in \mathbb{R}$, one has

$$\begin{aligned} \|u_\lambda^-\|_\lambda^2 &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\lambda(x)-u_\lambda(y))(u_\lambda^-(x)-u_\lambda^-(y))}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^N}\lambda V(x)u_\lambda u_\lambda^- dx \\ &= (|x|^{-\mu}*F(u_\lambda))f(u_\lambda)u_\lambda^- dx + \int_{\mathbb{R}^N}|u_\lambda^+|^{2_s^*-1}u_\lambda^- dx \\ &= 0. \end{aligned}$$

Thus $u_\lambda \geq 0$. By Lemma 2.8, we have $u_\lambda \neq 0$. In view of the Harnack inequality, $u_\lambda > 0$ and the proof is completed. □

3.2. The proof of Theorem 1.2

Proof. Suppose that $\lambda_n \to \infty$ and u_{λ_n} be one of the ground states of equation (Q_{λ_n}). That is, $I_{\lambda_n}(u_{\lambda_n}) = m_{\lambda_n}$ and $I_{\lambda_n}'(u_{\lambda_n}) = 0$. We denote u_{λ_n} by u_n for notion simplicity. Without loss of generality, we assume that $\lambda_n \geq 1$ for all n. As the proof of (2.4), one has

$$\begin{aligned} m_0 \geq m_{\lambda_n} &= I_{\lambda_n}(u_n) - \frac{1}{\kappa_s}\langle I_{\lambda_n}'(u_n), u_n\rangle \\ &\geq (\frac{1}{2}-\frac{1}{\kappa_s})([u_n]^2 + \int_{\mathbb{R}^N}\lambda_n V(x)|u_n|^2dx) \\ &\geq \frac{1}{\tau_0}(\frac{1}{2}-\frac{1}{\kappa_s})\|u_n\|^2. \end{aligned}$$

Hence $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Up to a subsequence, we may assume that

$$u_n \rightharpoonup u \text{ in } H^s(\mathbb{R}^N) \quad \text{and} \quad u_n \to u \text{ in } L^r_{loc}(\mathbb{R}^N) \quad \text{in } 1 \leq r < 2_s^*. \tag{3.1}$$

We divide into four steps to prove Theorem 1.2 as follows.

Step 1: $u(x) = 0$ a.e in $\mathbb{R}^N \setminus \Omega$.

If fact, by using the Fatou's Lemma, we get

$$\int_{\mathbb{R}^N\setminus\Omega} V(x)u^2dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)u_n^2dx \le \liminf_{n\to\infty} \frac{C_{20}}{\lambda_n} = 0,$$

which implies that $u(x) = 0$ a.e in $\mathbb{R}^N \setminus \Omega$.

Step 2: u is a critical point of I_0.

Since $I'_{\lambda_n}(u_{\lambda_n}) = 0$,

$$\langle u_n, \varphi\rangle_{\lambda_n} - \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u_n))f(u_n)\varphi dx - \int_{\mathbb{R}^N} |u_n|^{2_s^*-1}\varphi dx = 0, \quad \forall\varphi \in E_0.$$

It is clear that

$$\int_{\mathbb{R}^N} \lambda_n V(x)u_n\varphi dx = 0, \quad \forall\varphi \in E_0.$$

By (3.1), we have

$$[u_n, \varphi] \to [u_n, \varphi], \quad \forall\varphi \in E_0.$$

It is standard to prove that

$$\int_{\mathbb{R}^N} (|x|^{-\mu} * F(u_n))f(u_n)\varphi dx \to \int_{\Omega} (|x|^{-\mu} * F(u))f(u)\varphi dx, \quad \forall\varphi \in E_0,$$

and

$$\int_{\mathbb{R}^N} |u_n|^{2_s^*-1}\varphi dx \to \int_{\Omega} |u|^{2_s^*-1}\varphi dx, \quad \forall\varphi \in E_0.$$

Combining with the above results, we have $I'_0(u) = 0$.

Step 3: $u_n \to u$ in $L^s(\mathbb{R}^N)$ for $2 \le s < 2_s^*$.

Similar to (2.8) and (2.9), one has

$$\int_{D_R} |u_n|^2dx \le \left|D_R\right|^{\frac{2s}{N}} [u_n]^2 \le C_{21}\left|D_R\right|^{\frac{2s}{N}}, \tag{3.2}$$

$$\int_{B_R^c\setminus D_R} |u_n|^2dx \le \frac{C_{22}}{\lambda_n}. \tag{3.3}$$

Hence, for any $\varepsilon > 0$ there exist $R_1 = R_1(\varepsilon) > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_{R_1}(0)} |u_n|^2dx < \frac{\varepsilon}{4} + o_n(1)$$

By the decay of the Lebesgue integral, there exists $R_2 = R_2(\varepsilon) > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_{R_2}(0)} |u|^2dx < \frac{\varepsilon}{4}.$$

By (3.1), one has

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n - u|^2 dx &\le \int_{B_R(0)} |u_n - u|^2 dx + 2\int_{\mathbb{R}^N\setminus B_R(0)} |u_n|^2 dx + 2\int_{\mathbb{R}^N\setminus B_R(0)} |u|^2 dx \\ &\le o_n(1) + \varepsilon,\end{aligned}$$

where $R = \max\{R_1, R_2\}$. Consequently, $u_n \to u$ in $L^2(\mathbb{R}^N)$. By the interpolation inequality and the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^N)$, we have $u_n \to u$ in $L^r(\mathbb{R}^N)$ for $2 \le r < 2_s^*$.

Step 4: $m_0 = I_0(u)$ and $u_n \to u$ in $H^s(\mathbb{R}^N)$.

By the Hardy-Littlewood-Sobolev inequality and the Lebesgue dominant convergence theorem, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u_n)) f(u_n) u_n dx \to \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u)) f(u) u dx,$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u_n)) F(u_n) dx \to \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u)) F(u) dx.$$

It follows from the lower semicontinuity and the Fatou's Lemma that

$$\begin{aligned} m_0 \ge \liminf_{n\to\infty} m_{\lambda_n} &= \liminf_{n\to\infty} (I_{\lambda_n}(u_n) - \frac{1}{\kappa_s}\langle I'_{\lambda_n}(u_n), u_n\rangle) \\ &\ge (\frac{1}{2} - \frac{1}{\kappa_s}) \liminf_{n\to\infty} \|u_n\|_{\lambda_n}^2 + \frac{1}{\kappa_s} \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u_n)) f(u_n) u_n dx \\ &\quad - \frac{1}{2} \limsup_{n\to\infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u_n)) F(u_n) dx + (\frac{1}{\kappa_s} - \frac{1}{2_s^*}) \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx \\ &\ge (\frac{1}{2} - \frac{1}{\kappa_s})[u]^2 + \frac{1}{\kappa_s} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u)) f(u) u dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(u)) F(u) dx + (\frac{1}{\kappa_s} - \frac{1}{2_s^*}) \int_{\mathbb{R}^N} |u|^{2_s^*} dx \\ &\ge (\frac{1}{2} - \frac{1}{\kappa_s})[u]^2 + \frac{1}{\kappa_s} \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u dx \\ &\quad - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) F(u) dx + (\frac{1}{\kappa_s} - \frac{1}{2_s^*}) \int_{\Omega} |u|^{2_s^*} dx \\ &= I_0(u) - \frac{1}{\kappa_s}\langle I_0'(u), u\rangle \\ &= I_0(u) \ge m_0. \end{aligned}$$

As a consequence, $I_0(u) = m_0$ and $[u_n] \to [u]$. By Step 3, $\|u_n\| \to \|u\|$. This together with $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, we have $u_n \to u$ in $H^s(\mathbb{R}^N)$. By Lemma 2.8, $u \ge 0$ and $u \ne 0$. According to the Harnack inequality, we have $u > 0$. The proof is completed. □

3.3. The proof of Theorem 1.3

Proof. Theorem 1.3 is directly concluded by Theorem 1.1 and Theorem 1.2. □

From the proof of Theorem 1.2, we immediately get the following two Corollaries.

Corollary 3.1. *$m_\lambda \to m_0$ as $\lambda \to \infty$.*

Corollary 3.2. *Let $\{u_{\lambda_n}\}$ be a solutions of equation (Q_{λ_n}) with $\lambda_n \to \infty$ satisfying $|I_{\lambda_n}(u_n)| \leq K$. Then up to a subsequence, $u_n \to u$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$. Moreover, u is a solution of equation (Q_0).*

4. Conclusions

In this paper, we are concerned with a fractional Choquard equation with critical growth. Under some assumptions of nonlinearity, we obtain the existence and asymptotic behavior of the positive ground states to this problem by applying some analytical techniques. Several recent results of the literatures are extended and improved.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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