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Research article

Multiple solutions of Kirchhoff type equations involving Neumann conditions and critical growth

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Abstract: In this paper, we consider a Neumann problem of Kirchhoff type equation

$$\begin{cases} -\left(a+b\displaystyle\int_{\Omega}|\nabla u|^2dx\right)\Delta u+u=Q(x)|u|^4u+\lambda P(x)|u|^{q-2}u, & \text{in } \Omega,\\ \dfrac{\partial u}{\partial v}=0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with a smooth boundary, $a,b>0$, $1<q<2$, $\lambda>0$ is a real parameter, $Q(x)$ and $P(x)$ satisfy some suitable assumptions. By using the variational method and the concentration compactness principle, we obtain the existence and multiplicity of nontrivial solutions.

Keywords: Kirchhoff type equation; Neumann problem; critical growth; variation methods; nontrivial solution
Mathematics Subject Classification: 35B33, 35B35, 35J33

1. Introduction and main results

We study the following Neumann problem of Kirchhoff type equation with critical growth

$$\begin{cases} -\left(a+b\displaystyle\int_{\Omega}|\nabla u|^2dx\right)\Delta u+u=Q(x)|u|^4u+\lambda P(x)|u|^{q-2}u, & \text{in } \Omega,\\ \dfrac{\partial u}{\partial v}=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with a smooth boundary, $a,b>0$, $1<q<2$, $\lambda>0$ is a real parameter. We assume that $Q(x)$ and $P(x)$ satisfy the following conditions:
(Q_1) $Q(x)\in C(\bar{\Omega})$ is a sign-changing;

(Q_2) there exists $x_M \in \Omega$ such that $Q_M = Q(x_M) > 0$ and

$$|Q(x) - Q_M| = o(|x - x_M|) \text{ as } x \to x_M;$$

(Q_3) there exists $0 \in \partial\Omega$ such that $Q_m = Q(0) > 0$ and

$$|Q(x) - Q_m| = o(|x|) \text{ as } x \to 0;$$

(P_1) $P(x)$ is positive continuous on $\bar{\Omega}$ and $P(x_0) = \max_{x \in \bar{\Omega}} P(x)$;
(P_2) there exist $\sigma > 0$, $R > 0$ and $3 - q < \beta < \frac{6-q}{2}$ such that $P(x) \geq \sigma|x - y|^{-\beta}$ for $|x - y| \leq R$, where y is $x_M \in \Omega$ or $0 \in \partial\Omega$.

In recent years, the following Dirichlet problem of Kirchhoff type equation has been studied extensively by many researchers

$$\begin{cases} -\left(a + b\displaystyle\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x, u) \tag{1.3}$$

proposed by Kirchhoff in [13] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (1.2) and (1.3), u denotes the displacement, b is the initial tension and $f(x, u)$ stands for the external force, while a is related to the intrinsic properties of the string (such as Youngs modulus). We have to point out that such nonlocal problems appear in other fields like biological systems, such as population density, where u describes a process which depends on the average of itself (see Alves et al. [2]). After the pioneer work of Lions [18], where a functional analysis approach was proposed. The Kirchhoff type Eq (1.2) with critical growth began to call attention of researchers, we can see [1, 9, 14, 17, 23, 24, 28, 30] and so on.

Recently, the following Kirchhoff type equation has been well studied by various authors

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x, u), \text{ in } \mathbb{R}^3, \\ u > 0, u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.4}$$

There has been much research regarding the concentration behavior of the positive solutions of (1.4), we can see [10–12, 25, 33]. Many papers studied the existence of ground state solutions of (1.4), for example [5, 8, 15, 16, 21, 22, 24]. In addition, the authors established the existence of sign-changing solutions of (1.4) in [20, 31]. In papers [27, 32] proved the existence and multiplicity of nontrivial solutions of (1.4) by using mountain pass theorem.

In particular, Chabrowski in [6] studied the solvability of the Neumann problem

$$\begin{cases} -\Delta u = Q(x)|u|^{2^*-2}u + \lambda f(x, u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $2^* = \frac{2N}{N-2}(N \geq 3)$ is the critical Sobolev exponent, $\lambda > 0$ is a parameter. Assume that $Q(x) \in C(\overline{\Omega})$ is a sign-changing function and $\int_\Omega Q(x)dx < 0$, under the condition of $f(x,u)$. Using the space decomposition $H^1(\Omega) = span 1 \oplus V$, where $V = \{v \in H^1(\Omega) : \int_\Omega vdx = 0\}$, the author obtained the existence of two distinct solutions by the variational method.

In [14], Lei et al. considered the following Kirchhoff type equation with critical exponent

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = u^5 + \lambda\dfrac{u^{q-1}}{|x|^\beta}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $a, b > 0$, $1 < q < 2$, $\lambda > 0$ is a parameter. They obtained the existence of a positive ground state solution for $0 \leq \beta < 2$ and two positive solutions for $3 - q \leq \beta < 2$ by the Nehari manifold method.

In [34], Zhang obtained the existence and multiplicity of nontrivial solutions of the following equation

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u + u = \lambda|u|^{q-2}u + f(x,u) + Q(x)u^5, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is an open bounded domain in $\mathbb{R}^3$, $a, b > 0$, $1 < q < 2$, $\lambda \geq 0$ is a parameter, $f(x,u)$ and $Q(x)$ are positive continuous functions satisfying some additional assumptions. Moreover, $f(x,u) \sim |u|^{p-2}u$ with $4 < p < 6$.

Comparing with the above mentioned papers, our results are different and extend the above results to some extent. Specially, motivated by [34], we suppose $Q(x)$ changes sign on Ω and $f(x,u) \equiv 0$ for (1.5). Since (1.1) is critical growth, which leads to the cause of the lack of compactness of the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we overcome this difficulty by using P.Lions concentration compactness principle [19]. Moreover, note that $Q(x)$ changes sign on Ω, how to estimate the level of the mountain pass is another difficulty.

We define the energy functional corresponding to problem (1.1) by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 - \frac{1}{6}\int_\Omega Q(x)|u|^6 dx - \frac{\lambda}{q}\int_\Omega P(x)|u|^q dx.$$

A weak solution of problem (1.1) is a function $u \in H^1(\Omega)$ and for all $\varphi \in H^1(\Omega)$ such that

$$\int_\Omega (a\nabla u\nabla\varphi + u\varphi)dx + b\int_\Omega |\nabla u|^2 dx\int_\Omega \nabla u\nabla\varphi dx = \int_\Omega Q(x)|u|^4 u\varphi dx + \lambda\int_\Omega P(x)|u|^{q-2}u\varphi dx.$$

Our main results are the following:

Theorem 1.1. *Assume that* $1 < q < 2$ *and* $Q(x)$ *changes sign on* Ω*. Then there exists* $\Lambda_0 > 0$ *such that for every* $\lambda \in (0, \Lambda_0)$*, problem* (1.1) *has at least one nontrivial solution.*

Theorem 1.2. *Assume that* $1 < q < 2$, $3 - q < \beta < \frac{6-q}{2}$ *and* $Q(x)$ *changes sign on* Ω*, there exists* $\Lambda_* > 0$ *such that for all* $\lambda \in (0, \Lambda_*)$*. Then problem* (1.1) *has at least two nontrivial solutions.*

Throughout this paper, we make use of the following notations:

- The space $H^1(\Omega)$ is equipped with the norm $\|u\|^2_{H^1(\Omega)} = \int_\Omega (|\nabla u|^2 + u^2)dx$, the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$.
- Define $\|u\|^2 = \int_\Omega (a|\nabla u|^2 + u^2)dx$ for $u \in H^1(\Omega)$. Note that $\|\cdot\|$ is an equivalent norm on $H^1(\Omega)$ with the standard norm.
- Let $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 dx$.
- $0 < Q_M = \max_{x\in\bar{\Omega}} Q(x)$, $0 < Q_m = \max_{x\in\partial\Omega} Q(x)$.
- $\Omega^+ = \{x \in \Omega : Q(x) > 0\}$ and $\Omega^- = \{x \in \Omega : Q(x) < 0\}$.
- $C, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line.
- We denote by S_ρ (respectively, B_ρ) the sphere (respectively, the closed ball) of center zero and radius ρ, i.e. $S_\rho = \{u \in H^1(\Omega) : \|u\| = \rho\}$, $B_\rho = \{u \in H^1(\Omega) : \|u\| \leq \rho\}$.
- Let S be the best constant for Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, namely

$$S = \inf_{u\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (a|\nabla u|^2 + u^2)dx}{\left(\int_\Omega |u|^6 dx\right)^{1/3}}.$$

- Let S_0 be the best constant for Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, namely

$$S_0 = \inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{1/3}}.$$

2. Proofs of theorems

In this section, we firstly show that the functional $I_\lambda(u)$ has a mountain pass geometry.

Lemma 2.1. *There exist constants $r, \rho, \Lambda_0 > 0$ such that the functional I_λ satisfies the following conditions for each $\lambda \in (0, \Lambda_0)$:*
(i) $I_\lambda|_{u\in S_\rho} \geq r > 0$; $\inf_{u\in B_\rho} I_\lambda(u) < 0$.
(ii) *There exists $e \in H^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.*

Proof. (i) From (P_1), by the Hölder inequality and the Sobolev inequality, for all $u \in H^1(\Omega)$ one has

$$\int_\Omega P(x)|u|^q dx \leq P(x_0) \int_\Omega |u|^q dx \leq P(x_0)|\Omega|^{\frac{6-q}{6}} S^{-\frac{q}{2}} \|u\|^q, \tag{2.1}$$

and there exists a constant $C > 0$, we get

$$\left| \int_\Omega Q(x)|u|^6 dx \right| \leq C \int_\Omega |u|^6 dx \leq CS^{-3}\|u\|^6. \tag{2.2}$$

Hence, combining (2.1) and (2.2), we have the following estimate

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 - \frac{1}{6}\int_\Omega Q(x)|u|^6 dx - \frac{\lambda}{q}\int_\Omega P(x)|u|^q dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{C}{6}\int_\Omega |u|^6 dx - \frac{\lambda}{q} P(x_0)|\Omega|^{\frac{6-q}{6}} S^{-\frac{q}{2}}\|u\|^q \\ &\geq \|u\|^q \left(\frac{1}{2}\|u\|^{2-q} - \frac{C}{6} S^{-3}\|u\|^{6-q} - \frac{\lambda}{q} P(x_0)|\Omega|^{\frac{6-q}{6}} S^{-\frac{q}{2}}\right). \end{aligned}$$

Set $h(t) = \frac{1}{2}t^{2-q} - \frac{C}{6}S^{-3}t^{6-q}$ for $t > 0$, then there exists a constant $\rho = \left(\frac{3(2-q)S^3}{C(6-q)}\right)^{\frac{1}{4}} > 0$ such that $\max_{t>0} h(t) = h(\rho) > 0$. Letting $\Lambda_0 = \frac{qS^{\frac{q}{2}}}{P(x_0)|\Omega|^{\frac{6-q}{6}}}h(\rho)$, there exists a constant $r > 0$ such that $I_\lambda|_{u\in S_\rho} \geq r$ for every $\lambda \in (0, \Lambda_0)$. Moreover, for all $u \in H^1(\Omega)\backslash\{0\}$, we have

$$\lim_{t\to 0^+} \frac{I_\lambda(tu)}{t^q} = -\frac{\lambda}{q}\int_\Omega P(x)|u|^q dx < 0.$$

So we obtain $I_\lambda(tu) < 0$ for every $u \neq 0$ and t small enough. Therefore, for $\|u\|$ small enough, one has

$$m \triangleq \inf_{u\in B_\rho} I_\lambda(u) < 0.$$

(ii) Let $v \in H^1(\Omega)$ be such that supp $v \subset \Omega^+$, $v \not\equiv 0$ and $t > 0$, we have

$$I_\lambda(tv) = \frac{t^2}{2}\|v\|^2 + \frac{bt^4}{4}\left(\int_\Omega |\nabla v|^2 dx\right)^2 - \frac{t^6}{6}\int_\Omega Q(x)|v|^6 dx - \frac{\lambda t^q}{q}\int_\Omega P(x)|v|^q dx \to -\infty$$

as $t \to \infty$, which implies that $I_\lambda(tv) < 0$ for $t > 0$ large enough. Therefore, we can find $e \in H^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$. The proof is complete. □

Denote

$$\begin{cases} \Theta_1 = \dfrac{abS_0^3}{4Q_M} + \dfrac{b^3S_0^6}{24Q_M^2} + \dfrac{aS_0\sqrt{b^2S_0^4 + 4aS_0Q_M}}{6Q_M} + \dfrac{b^2S_0^4\sqrt{b^2S_0^4 + 4aS_0Q_M}}{24Q_M^2}, \\ \Theta_2 = \dfrac{abS_0^3}{16Q_m} + \dfrac{b^3S_0^6}{384Q_m^2} + \dfrac{aS_0\sqrt{b^2S_0^4 + 16aS_0Q_m}}{24Q_m} + \dfrac{b^2S_0^4\sqrt{b^2S_0^4 + 16aS_0Q_m}}{384Q_m^2}. \end{cases}$$

Then we have the following compactness result.

Lemma 2.2. *Suppose that* $1 < q < 2$. *Then the functional* I_λ *satisfies the* $(PS)_{c_\lambda}$ *condition for every* $c_\lambda < c_* = min\,\{\Theta_1 - D\lambda^{\frac{2}{2-q}}, \Theta_2 - D\lambda^{\frac{2}{2-q}}\}$, *where* $D = \frac{2-q}{3q}(\frac{6-q}{4}P(x_0)S^{-\frac{q}{2}}|\Omega|^{\frac{6-q}{6}})^{\frac{2}{2-q}}$.

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_{c_\lambda}$ sequence for

$$I_\lambda(u_n) \to c_\lambda \text{ and } I'_\lambda(u_n) \to 0 \text{ as } n \to \infty. \tag{2.3}$$

It follows from (2.1), (2.3) and the Hölder inequality that

$$\begin{aligned} c_\lambda + 1 + o(\|u_n\|) &\geq I_\lambda(u_n) - \frac{1}{6}\langle I'_\lambda(u_n), u_n\rangle \\ &\geq \frac{1}{3}\|u_n\|^2 + \frac{b}{12}\left(\int_\Omega |\nabla u_n|^2 dx\right)^2 \\ &\quad - \lambda\left(\frac{1}{q} - \frac{1}{6}\right)P(x_0)S^{-\frac{q}{2}}|\Omega|^{\frac{6-q}{6}}\|u_n\|^q \\ &\geq \frac{1}{3}\|u_n\|^2 - \frac{\lambda(6-q)}{6q}P(x_0)S^{-\frac{q}{2}}|\Omega|^{\frac{6-q}{6}}\|u_n\|^q. \end{aligned}$$

Therefore $\{u_n\}$ is bounded in $H^1(\Omega)$ for all $1 < q < 2$. Thus, we may assume up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in H^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } H^1(\Omega), \\ u_n \to u, & \text{strongly in } L^p(\Omega)\ (1 \le p < 6), \\ u_n(x) \to u(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.4}$$

as $n \to \infty$. Next, we prove that $u_n \to u$ strongly in $H^1(\Omega)$. By using the concentration compactness principle (see [19]), there exist some at most countable index set J, δ_{x_j} is the Dirac mass at $x_j \subset \bar{\Omega}$ and positive numbers $\{\nu_j\}$, $\{\mu_j\}$, $j \in J$, such that

$$|u_n|^6 dx \rightharpoonup d\nu = |u|^6 dx + \sum_{j\in J} \nu_j \delta_{x_j},$$
$$|\nabla u_n|^2 dx \rightharpoonup d\mu \ge |\nabla u|^2 dx + \sum_{j\in J} \mu_j \delta_{x_j}.$$

Moreover, numbers ν_j and μ_j satisfy the following inequalities

$$\begin{aligned} S_0 \nu_j^{\frac{1}{3}} &\le \mu_j \ \text{ if } x_j \in \Omega, \\ \frac{S_0}{2^{\frac{2}{3}}} \nu_j^{\frac{1}{3}} &\le \mu_j \ \text{ if } x_j \in \partial\Omega. \end{aligned} \tag{2.5}$$

For $\varepsilon > 0$, let $\phi_{\varepsilon,j}(x)$ be a smooth cut-off function centered at x_j such that $0 \le \phi_{\varepsilon,j} \le 1$, $|\nabla \phi_{\varepsilon,j}| \le \frac{2}{\varepsilon}$, and

$$\phi_{\varepsilon,j}(x) = \begin{cases} 1, & \text{in } B(x_j, \frac{\varepsilon}{2}) \cap \bar{\Omega}, \\ 0, & \text{in } \Omega \setminus B(x_j, \varepsilon). \end{cases}$$

There exists a constant $C > 0$ such that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_\Omega P(x)|u_n|^q \phi_{\varepsilon,j} dx \le P(x_0) \lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{B(x_j,\varepsilon)} |u_n|^q dx = 0.$$

Since $|\nabla \phi_{\varepsilon,j}| \le \frac{2}{\varepsilon}$, by using the Hölder inequality and $L^2(\Omega)$-convergence of $\{u_n\}$, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\lim_{n\to\infty} \left(a + b\int_\Omega |\nabla u_n|^2 dx\right) \int_\Omega \langle \nabla u_n, \nabla \phi_{\varepsilon,j} \rangle u_n dx \\ &\le C \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left(\int_\Omega |\nabla u_n|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |u_n|^2 |\nabla \phi_{\varepsilon,j}|^2 dx\right)^{\frac{1}{2}} \\ &\le C \lim_{\varepsilon\to 0} \left(\int_{B(x_j,\varepsilon)} |u|^6 dx\right)^{\frac{1}{6}} \left(\int_{B(x_j,\varepsilon)} |\nabla \phi_{\varepsilon,j}|^3 dx\right)^{\frac{1}{3}} \\ &\le C \lim_{\varepsilon\to 0} \left(\int_{B(x_j,\varepsilon)} |u|^6 dx\right)^{\frac{1}{6}} \left(\int_{B(x_j,\varepsilon)} \left(\frac{2}{\varepsilon}\right)^3 dx\right)^{\frac{1}{3}} \\ &\le C_1 \lim_{\varepsilon\to 0} \left(\int_{B(x_j,\varepsilon)} |u|^6 dx\right)^{\frac{1}{6}} \\ &= 0, \end{aligned}$$

where $C_1 > 0$, and we also derive that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2\phi_{\varepsilon,j}dx \geq \lim_{\varepsilon\to 0}\int_\Omega |\nabla u|^2\phi_{\varepsilon,j}dx + \mu_j = \mu_j,$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega Q(x)|u_n|^6\phi_{\varepsilon,j}dx = \lim_{\varepsilon\to 0}\int_\Omega Q(x)|u|^6\phi_{\varepsilon,j}dx + Q(x_j)\nu_j = Q(x_j)\nu_j,$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega u_n^2\phi_{\varepsilon,j}dx = \lim_{\varepsilon\to 0}\int_\Omega u^2\phi_{\varepsilon,j}dx \leq \lim_{\varepsilon\to 0}\int_{B(x_j,\varepsilon)} u^2dx = 0.$$

Noting that $u_n\phi_{\varepsilon,j}$ is bounded in $H^1(\Omega)$ uniformly for n, taking the test function $\varphi = u_n\phi_{\varepsilon,j}$ in (2.3), from the above information, one has

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle I'_\lambda(u_n), u_n\phi_{\varepsilon,j}\rangle \\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{\left(a + b\int_\Omega |\nabla u_n|^2dx\right)\int_\Omega \left\langle \nabla u_n, \nabla(u_n\phi_{\varepsilon,j})\right\rangle dx + \int_\Omega u_n^2\phi_{\varepsilon,j}dx \right. \\
&\quad \left. - \int_\Omega Q(x)|u_n|^6\phi_{\varepsilon,j}dx - \lambda\int_\Omega P(x)|u_n|^q\phi_{\varepsilon,j}dx\right\} \\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{\left(a + b\int_\Omega |\nabla u_n|^2dx\right)\int_\Omega \left(|\nabla u_n|^2\phi_{\varepsilon,j} + \langle\nabla u_n, \nabla\phi_{\varepsilon,j}\rangle u_n\right)dx \right. \\
&\quad \left. - \int_\Omega Q(x)|u_n|^6\phi_{\varepsilon,j}dx\right\} \\
&\geq \lim_{\varepsilon\to 0}\left\{\left(a + b\int_\Omega |\nabla u|^2dx + b\mu_j\right)\left(\int_\Omega |\nabla u|^2\phi_{\varepsilon,j}dx + \mu_j\right) \right. \\
&\quad \left. - \int_\Omega Q(x)|u|^6\phi_{\varepsilon,j}dx - Q(x_j)\nu_j\right\} \\
&\geq \left(a + b\mu_j\right)\mu_j - Q(x_j)\nu_j,
\end{aligned}$$

so that

$$Q(x_j)\nu_j \geq (a + b\mu_j)\mu_j,$$

which shows that $\{u_n\}$ can only concentrate at points x_j where $Q(x_j) > 0$. If $\nu_j > 0$, by (2.5) we get

$$\begin{aligned}
\nu_j^{\frac{1}{3}} &\geq \frac{bS_0^2 + \sqrt{b^2S_0^4 + 4aS_0Q_M}}{2Q_M} \quad \text{if } x_j \in \Omega, \\
\nu_j^{\frac{1}{3}} &\geq \frac{bS_0^2 + \sqrt{b^2S_0^4 + 16aS_0Q_m}}{2^{\frac{7}{3}}Q_m} \quad \text{if } x_j \in \partial\Omega.
\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned}
\mu_j &\geq \frac{bS_0^3 + \sqrt{b^2S_0^6 + 4aS_0^3Q_M}}{2Q_M} \quad \text{if } x_j \in \Omega, \\
\mu_j &\geq \frac{bS_0^3 + \sqrt{b^2S_0^6 + 16aS_0^3Q_m}}{8Q_m} \quad \text{if } x_j \in \partial\Omega.
\end{aligned} \tag{2.7}$$

To proceed further we show that (2.7) is impossible. To obtain a contradiction assume that there exists $j_0 \in J$ such that $\mu_{j_0} \geq \frac{bS_0^3+\sqrt{b^2S_0^6+4aS_0^3Q_M}}{2Q_M}$ and $x_{j_0} \in \Omega$. By (2.1), (2.3) and (2.4), one has

$$\begin{aligned} c_\lambda &= \lim_{n\to\infty}\left\{I_\lambda(u_n) - \frac{1}{6}\langle I'_\lambda(u_n), u_n\rangle\right\} \\ &= \lim_{n\to\infty}\left\{\frac{a}{3}\int_\Omega |\nabla u_n|^2dx + \frac{b}{12}\left(\int_\Omega |\nabla u_n|^2dx\right)^2 \right. \\ &\quad \left. + \frac{1}{3}\int_\Omega u_n^2dx - \lambda\frac{6-q}{6q}\int_\Omega P(x)|u_n|^q dx\right\} \\ &\geq \frac{a}{3}\left(\int_\Omega |\nabla u|^2dx + \sum_{j\in J}\mu_j\right) + \frac{b}{12}\left(\int_\Omega |\nabla u|^2dx + \sum_{j\in J}\mu_j\right)^2 \\ &\quad + \frac{1}{3}\int_\Omega u^2dx - \lambda\frac{6-q}{6q}P(x_0)S^{-\frac{q}{2}}|\Omega|^{\frac{6-q}{6}}\|u\|^q \\ &\geq \frac{a}{3}\mu_{j_0} + \frac{b}{12}\mu_{j_0}^2 + \frac{1}{3}\|u\|^2 - \lambda\frac{6-q}{6q}P(x_0)S^{-\frac{q}{2}}|\Omega|^{\frac{6-q}{6}}\|u\|^q. \end{aligned}$$

Set

$$g(t) = \frac{1}{3}t^2 - \lambda\frac{6-q}{6q}P(x_0)S^{-\frac{q}{2}}|\Omega|^{\frac{6-q}{6}}t^q, \quad t > 0,$$

then

$$g'(t) = \frac{2}{3}t - \lambda\frac{6-q}{6}P(x_0)S^{-\frac{q}{2}}|\Omega|^{\frac{6-q}{6}}t^{q-1} = 0,$$

we can deduce that $\min_{t\geq 0} g(t)$ attains at $t_0 > 0$ and

$$t_0 = \left(\lambda\frac{6-q}{4}P(x_0)S^{-\frac{q}{2}}|\Omega|^{\frac{6-q}{6}}\right)^{\frac{1}{2-q}}.$$

Consequently, we obtain

$$\begin{aligned} c_\lambda &\geq \frac{abS_0^3}{4Q_M} + \frac{b^3S_0^6}{24Q_M^2} + \frac{aS_0\sqrt{b^2S_0^4 + 4aS_0Q_M}}{6Q_M} \\ &\quad + \frac{b^2S_0^4\sqrt{b^2S_0^4 + 4aS_0Q_M}}{24Q_M^2} - D\lambda^{\frac{2}{2-q}} \\ &= \Theta_1 - D\lambda^{\frac{2}{2-q}}, \end{aligned}$$

where $D = \frac{2-q}{3q}\left(\frac{6-q}{4}P(x_0)S^{-\frac{q}{2}}|\Omega|^{\frac{6-q}{6}}\right)^{\frac{2}{2-q}}$. If $\mu_{j_0} \geq \frac{bS_0^3+\sqrt{b^2S_0^6+16aS_0^3Q_m}}{8Q_m}$ and $x_{j_0} \in \partial\Omega$, then, by the similar calculation, we also get

$$\begin{aligned} c_\lambda &\geq \frac{abS_0^3}{16Q_m} + \frac{b^3S_0^6}{384Q_m^2} + \frac{aS_0\sqrt{b^2S_0^4 + 16aS_0Q_m}}{24Q_m} \\ &\quad + \frac{b^2S_0^4\sqrt{b^2S_0^4 + 16aS_0Q_m}}{384Q_m^2} - D\lambda^{\frac{2}{2-q}} \\ &= \Theta_2 - D\lambda^{\frac{2}{2-q}}. \end{aligned}$$

Let $c_* = \min\{\Theta_1 - D\lambda^{\frac{2}{2-q}}, \Theta_2 - D\lambda^{\frac{2}{2-q}}\}$, from the above information, we deduce that $c_\lambda \geq c_*$. It contradicts our assumption, so it indicates that $\nu_j = \mu_j = 0$ for every $j \in J$, which implies that

$$\int_\Omega |u_n|^6 dx \to \int_\Omega |u|^6 dx \tag{2.8}$$

as $n \to \infty$. Now, we may assume that $\int_\Omega |\nabla u_n|^2 dx \to A^2$ and $\int_\Omega |\nabla u|^2 dx \leq A^2$, by (2.3), (2.4) and (2.8), one has

$$\begin{aligned}
0 &= \lim_{n\to\infty} \langle I'_\lambda(u_n), u_n - u\rangle \\
&= \lim_{n\to\infty} \Big[\Big(a + b\int_\Omega |\nabla u_n|^2 dx\Big)\Big(\int_\Omega |\nabla u_n|^2 dx - \int_\Omega \nabla u_n \nabla u dx\Big) \\
&\quad + \int_\Omega u_n(u_n - u)dx - \int_\Omega Q(x)|u_n|^5(u_n - u)dx - \lambda\int_\Omega P(x)|u_n|^{q-1}(u_n - u)dx\Big] \\
&= \Big(a + bA^2\Big)\Big(A^2 - \int_\Omega |\nabla u|^2 dx\Big).
\end{aligned}$$

Then, we obtain that $u_n \to u$ in $H^1(\Omega)$. The proof is complete. □

As well known, the function

$$U_{\varepsilon,y}(x) = \frac{(3\varepsilon^2)^{\frac{1}{4}}}{(\varepsilon^2 + |x-y|^2)^{\frac{1}{2}}}, \text{ for any } \varepsilon > 0,$$

satisfies

$$-\Delta U_{\varepsilon,y} = U_{\varepsilon,y}^5 \text{ in } \mathbb{R}^3,$$

and

$$\int_{\mathbb{R}^3} |\nabla U_{\varepsilon,y}|^2 dx = \int_{\mathbb{R}^3} |U_{\varepsilon,y}|^6 dx = S_0^{\frac{3}{2}}.$$

Let $\phi \in C^1(\mathbb{R}^3)$ such that $\phi(x) = 1$ on $B(x_M, \frac{R}{2})$, $\phi(x) = 0$ on $\mathbb{R}^3 - B(x_M, R)$ and $0 \leq \phi(x) \leq 1$ on $\mathbb{R}^3$, we set $v_\varepsilon(x) = \phi(x)U_{\varepsilon,x_M}(x)$. We may assume that $Q(x) > 0$ on $B(x_M, R)$ for some $R > 0$ such that $B(x_M, R) \subset \Omega$. From [4], we have

$$\begin{cases}
\|\nabla v_\varepsilon\|_2^2 = S_0^{\frac{3}{2}} + O(\varepsilon), \\
\|v_\varepsilon\|_6^6 = S_0^{\frac{3}{2}} + O(\varepsilon^3), \\
\|v_\varepsilon\|_2^2 = O(\varepsilon), \\
\|v_\varepsilon\|^2 = aS_0^{\frac{3}{2}} + O(\varepsilon).
\end{cases} \tag{2.9}$$

Moreover, by [28], we get

$$\begin{cases}
\|\nabla v_\varepsilon\|_2^4 \leq S_0^3 + O(\varepsilon), \\
\|\nabla v_\varepsilon\|_2^8 \leq S_0^6 + O(\varepsilon), \\
\|\nabla v_\varepsilon\|_2^{12} \leq S_0^9 + O(\varepsilon).
\end{cases} \tag{2.10}$$

Then we have the following Lemma.

Lemma 2.3. *Suppose that* $1 < q < 2$, $3 - q < \beta < \frac{6-q}{2}$, $Q_M > 4Q_m$, (Q_1) *and* (Q_2), *then* $\sup_{t\geq 0} I_\lambda(tv_\varepsilon) < \Theta_1 - D\lambda^{\frac{2}{2-q}}$.

Proof. By Lemma 2.1, one has $I_\lambda(tv_\varepsilon) \to -\infty$ as $t \to \infty$ and $I_\lambda(tv_\varepsilon) < 0$ as $t \to 0$, then there exists $t_\varepsilon > 0$ such that $I_\lambda(t_\varepsilon v_\varepsilon) = \sup_{t>0} I_\lambda(tv_\varepsilon) \geq r > 0$. We can assume that there exist positive constants $t_1, t_2 > 0$ and $0 < t_1 < t_\varepsilon < t_2 < +\infty$. Let $I_\lambda(t_\varepsilon v_\varepsilon) = \beta(t_\varepsilon v_\varepsilon) - \lambda\psi(t_\varepsilon v_\varepsilon)$, where

$$\beta(t_\varepsilon v_\varepsilon) = \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|^2 + \frac{bt_\varepsilon^4}{4}\|\nabla v_\varepsilon\|_2^4 - \frac{t_\varepsilon^6}{6}\int_\Omega Q(x)|v_\varepsilon|^6 dx,$$

and

$$\psi(t_\varepsilon v_\varepsilon) = \frac{t_\varepsilon^q}{q}\int_\Omega P(x)|v_\varepsilon|^q dx.$$

Now, we set

$$h(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 + \frac{bt^4}{4}\|\nabla v_\varepsilon\|_2^4 - \frac{t^6}{6}\int_\Omega Q(x)|v_\varepsilon|^6 dx.$$

It is clear that $\lim_{t\to 0} h(t) = 0$ and $\lim_{t\to\infty} h(t) = -\infty$. Therefore there exists $T_1 > 0$ such that $h(T_1) = \max_{t\geq 0} h(t)$, that is

$$h'(t)|_{T_1} = T_1\|v_\varepsilon\|^2 + bT_1^3\|\nabla v_\varepsilon\|_2^4 - T_1^5\int_\Omega Q(x)|v_\varepsilon|^6 dx = 0,$$

from which we have

$$\|v_\varepsilon\|^2 + bT_1^2\|\nabla v_\varepsilon\|_2^4 = T_1^4\int_\Omega Q(x)|v_\varepsilon|^6 dx. \tag{2.11}$$

By (2.11) we obtain

$$T_1^2 = \frac{b\|\nabla v_\varepsilon\|_2^4 + \sqrt{b^2\|\nabla v_\varepsilon\|_2^8 + 4\|v_\varepsilon\|^2\int_\Omega Q(x)|v_\varepsilon|^6 dx}}{2\int_\Omega Q(x)|v_\varepsilon|^6 dx}.$$

In addition, by (Q_2), for all $\eta > 0$, there exists $\rho > 0$ such that $|Q(x)-Q_M| < \eta|x-x_M|$ for $0 < |x-x_M| < \rho$, for $\varepsilon > 0$ small enough, we have

$$\begin{aligned}
\left|\int_\Omega Q(x)v_\varepsilon^6 dx - \int_\Omega Q_M v_\varepsilon^6 dx\right| &\leq \int_\Omega |Q(x) - Q_M|v_\varepsilon^6 dx \\
&< \int_{B(x_M,\rho)} \eta|x - x_M|\frac{(3\varepsilon^2)^{\frac{3}{2}}}{(\varepsilon^2 + |x - x_M|^2)^3}dx \\
&\quad + C\int_{\Omega\setminus B(x_M,\rho)} \frac{(3\varepsilon^2)^{\frac{3}{2}}}{(\varepsilon^2 + |x - x_M|^2)^3}dx \\
&\leq C\eta\varepsilon^3\int_0^\rho \frac{r^3}{(\varepsilon^2 + r^2)^3}dr + C\varepsilon^3\int_\rho^R \frac{r^2}{(\varepsilon^2 + r^2)^3}dr \\
&\leq C\eta\varepsilon\int_0^{\rho/\varepsilon} \frac{t^3}{(1 + t^2)^3}dt + C\int_{\rho/\varepsilon}^{R/\varepsilon} \frac{t^2}{(1 + t^2)^3}dt \\
&\leq C_1\eta\varepsilon + C_2\varepsilon^3,
\end{aligned}$$

where $C_1, C_2 > 0$ (independent of η, ε). From this we derive that

$$\limsup_{\varepsilon\to 0} \frac{|\int_\Omega Q(x)v_\varepsilon^6 dx - \int_\Omega Q_M v_\varepsilon^6 dx|}{\varepsilon} \leq C_1\eta. \tag{2.12}$$

Then from the arbitrariness of $\eta > 0$, by (2.9) and (2.12), one has

$$\int_\Omega Q(x)|v_\varepsilon|^6 dx = Q_M \int_\Omega |v_\varepsilon|^6 dx + o(\varepsilon) = Q_M S_0^{\frac{3}{2}} + o(\varepsilon). \tag{2.13}$$

Hence, it follows from (2.9), (2.10) and (2.13) that

$$\begin{aligned}
\beta(t_\varepsilon v_\varepsilon) &\leq h(T_1) \\
&= T_1^2\left(\frac{1}{3}\|v_\varepsilon\|^2 + \frac{bT_1^2}{12}\|\nabla v_\varepsilon\|_2^4\right) \\
&= \frac{b\|\nabla v_\varepsilon\|_2^4\|v_\varepsilon\|^2}{4\int_\Omega Q(x)|v_\varepsilon|^6 dx} + \frac{b^3\|\nabla v_\varepsilon\|_2^{12}}{24(\int_\Omega Q(x)|v_\varepsilon|^6 dx)^2} \\
&\quad + \frac{\|v_\varepsilon\|^2\sqrt{b^2\|\nabla v_\varepsilon\|_2^8 + 4\|v_\varepsilon\|^2\int_\Omega Q(x)|v_\varepsilon|^6 dx}}{6\int_\Omega Q(x)|v_\varepsilon|^6 dx} \\
&\quad + \frac{b^2\|\nabla v_\varepsilon\|_2^8\sqrt{b^2\|\nabla v_\varepsilon\|_2^8 + 4\|v_\varepsilon\|^2\int_\Omega Q(x)|v_\varepsilon|^6 dx}}{24(\int_\Omega Q(x)|v_\varepsilon|^6 dx)^2} \\
&\leq \frac{b(S_0^3 + O(\varepsilon))(aS_0^{\frac{3}{2}} + O(\varepsilon))}{4(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))} + \frac{b^3(S_0^9 + O(\varepsilon))}{24(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))^2} \\
&\quad + \frac{(aS_0^{\frac{3}{2}} + O(\varepsilon))\sqrt{b^2(S_0^6 + O(\varepsilon)) + 4(aS_0^{\frac{3}{2}} + O(\varepsilon))(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))}}{6(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))} \\
&\quad + \frac{b^2(S_0^6 + O(\varepsilon))\sqrt{b^2(S_0^6 + O(\varepsilon)) + 4(aS_0^{\frac{3}{2}} + O(\varepsilon))(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))}}{24(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))^2} \\
&\leq \frac{abS_0^3}{4Q_M} + \frac{b^3 S_0^6}{24Q_M^2} + \frac{aS_0\sqrt{b^2S_0^4 + 4aS_0Q_M}}{6Q_M} \\
&\quad + \frac{b^2S_0^4\sqrt{b^2S_0^4 + 4aS_0Q_M}}{24Q_M^2} + C_3\varepsilon \\
&= \Theta_1 + C_3\varepsilon,
\end{aligned}$$

where the constant $C_3 > 0$. According to the definition of v_ε, from [29], for $\frac{R}{2} > \varepsilon > 0$, there holds

$$\begin{aligned}
\psi(t_\varepsilon v_\varepsilon) &\geq \frac{1}{q}3^{\frac{q}{4}}t_1^q\int_{B(x_M,\frac{R}{2})}\frac{\sigma\varepsilon^{\frac{q}{2}}}{(\varepsilon^2 + |x - x_M|^2)^{\frac{q}{2}}|x - x_M|^\beta}dx \\
&\geq C\varepsilon^{\frac{q}{2}}\int_0^{R/2}\frac{r^2}{(\varepsilon^2 + r^2)^{\frac{q}{2}}r^\beta}dr \\
&= C\varepsilon^{\frac{6-q}{2}-\beta}\int_0^{R/2\varepsilon}\frac{t^2}{(1 + t^2)^{\frac{q}{2}}t^\beta}dt
\end{aligned} \tag{2.14}$$

$$\geq C\varepsilon^{\frac{6-q}{2}-\beta}\int_0^1 t^{2-\beta}dt$$
$$= C_4\varepsilon^{\frac{6-q}{2}-\beta},$$

where $C_4 > 0$ (independent of ε, λ). Consequently, from the above information, we obtain

$$\begin{aligned} I_\lambda(t_\varepsilon v_\varepsilon) &= \beta(t_\varepsilon v_\varepsilon) - \lambda\psi(t_\varepsilon v_\varepsilon) \\ &\leq \Theta_1 + C_3\varepsilon - C_4\lambda\varepsilon^{\frac{6-q}{2}-\beta} \\ &< \Theta_1 - D\lambda^{\frac{2}{2-q}}. \end{aligned}$$

Here we have used the fact that $\beta > 3-q$ and let $\varepsilon = \lambda^{\frac{2}{2-q}}$, $0 < \lambda < \Lambda_1 = \min\{1, (\frac{C_3+D}{C_4})^{\frac{2-q}{6-2q-2\beta}}\}$, then

$$\begin{aligned} C_3\varepsilon - C_4\lambda\varepsilon^{\frac{6-q}{2}-\beta} &= C_3\lambda^{\frac{2}{2-q}} - C_4\lambda^{\frac{8-2q-2\beta}{2-q}} \\ &= \lambda^{\frac{2}{2-q}}(C_3 - C_4\lambda^{\frac{6-2q-2\beta}{2-q}}) \\ &< -D\lambda^{\frac{2}{2-q}}. \end{aligned} \tag{2.15}$$

The proof is complete. □

We assume that $0 \in \partial\Omega$ and $Q_m = Q(0)$. Let $\varphi \in C^1(\mathbb{R}^3)$ such that $\varphi(x) = 1$ on $B(0, \frac{R}{2})$, $\varphi(x) = 0$ on $\mathbb{R}^3 - B(0,R)$ and $0 \leq \varphi(x) \leq 1$ on $\mathbb{R}^3$, we set $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$, the radius R is chosen so that $Q(x) > 0$ on $B(0,R) \cap \Omega$. If $H(0)$ denotes the mean curvature of the boundary at 0, then the following estimates hold (see [6] or [26])

$$\begin{cases} \|u_\varepsilon\|_2^2 = O(\varepsilon), \\ \frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_6^2} \leq \frac{S_0}{2^{\frac{2}{3}}} - A_3H(0)\varepsilon\log\frac{1}{\varepsilon} + O(\varepsilon), \end{cases} \tag{2.16}$$

where $A_3 > 0$ is a constant. Then we have the following lemma.

Lemma 2.4. *Suppose that* $1 < q < 2$, $3-q < \beta < \frac{6-q}{2}$, $Q_M \leq 4Q_m$, $H(0) > 0$, *Q is positive somewhere on* $\partial\Omega$, (Q_1) *and* (Q_3), *then* $\sup_{t\geq 0} I_\lambda(tu_\varepsilon) < \Theta_2 - D\lambda^{\frac{2}{2-q}}$.

Proof. Similar to the proof of Lemma 2.3, we also have by Lemma 2.1, there exists $t_\varepsilon > 0$ such that $I_\lambda(t_\varepsilon u_\varepsilon) = \sup_{t>0} I_\lambda(tu_\varepsilon) \geq r > 0$. We can assume that there exist positive constants $t_1, t_2 > 0$ such that $0 < t_1 < t_\varepsilon < t_2 < +\infty$. Let $I_\lambda(t_\varepsilon u_\varepsilon) = A(t_\varepsilon u_\varepsilon) - \lambda B(t_\varepsilon u_\varepsilon)$, where

$$A(t_\varepsilon u_\varepsilon) = \frac{t_\varepsilon^2}{2}\|u_\varepsilon\|^2 + \frac{bt_\varepsilon^4}{4}\|\nabla u_\varepsilon\|_2^4 - \frac{t_\varepsilon^6}{6}\int_\Omega Q(x)|u_\varepsilon|^6dx,$$

and

$$B(t_\varepsilon u_\varepsilon) = \frac{t_\varepsilon^q}{q}\int_\Omega P(x)|u_\varepsilon|^qdx.$$

Now, we set

$$f(t) = \frac{t^2}{2}\|u_\varepsilon\|^2 + \frac{bt^4}{4}\|\nabla u_\varepsilon\|_2^4 - \frac{t^6}{6}\int_\Omega Q(x)|u_\varepsilon|^6dx.$$

Therefore, it is easy to see that there exists $T_2 > 0$ such that $f(T_2) = \max_{f\geq 0} f(t)$, that is

$$f'(t)|_{T_2} = T_2\|u_\varepsilon\|^2 + bT_2^3\|\nabla u_\varepsilon\|_2^4 - T_2^5 \int_\Omega Q(x)|u_\varepsilon|^6 dx = 0. \tag{2.17}$$

From (2.17) we obtain

$$T_2^2 = \frac{b\|\nabla u_\varepsilon\|_2^4 + \sqrt{b^2\|\nabla u_\varepsilon\|_2^8 + 4\|u_\varepsilon\|^2 \int_\Omega Q(x)|u_\varepsilon|^6 dx}}{2\int_\Omega Q(x)|u_\varepsilon|^6 dx}.$$

By the assumption (Q_3), we have the expansion formula

$$\int_\Omega Q(x)|u_\varepsilon|^6 dx = Q_m \int_\Omega |u_\varepsilon|^6 dx + o(\varepsilon). \tag{2.18}$$

Hence, combining (2.16) and (2.18), there exists $C_5 > 0$, such that

$$\begin{aligned}
A(t_\varepsilon u_\varepsilon) &\leq f(T_2) \\
&= T_2^2\left(\frac{1}{3}\|u_\varepsilon\|^2 + \frac{bT_2^2}{12}\|\nabla u_\varepsilon\|_2^4\right) \\
&= \frac{b\|\nabla u_\varepsilon\|_2^4\|u_\varepsilon\|^2}{4\int_\Omega Q(x)|u_\varepsilon|^6 dx} + \frac{b^3\|\nabla u_\varepsilon\|_2^{12}}{24(\int_\Omega Q(x)|u_\varepsilon|^6 dx)^2} \\
&\quad + \frac{\|u_\varepsilon\|^2\sqrt{b^2\|\nabla u_\varepsilon\|_2^8 + 4\|u_\varepsilon\|^2\int_\Omega Q(x)|u_\varepsilon|^6 dx}}{6\int_\Omega Q(x)|u_\varepsilon|^6 dx} \\
&\quad + \frac{b^2\|\nabla u_\varepsilon\|_2^8\sqrt{b^2\|\nabla u_\varepsilon\|_2^8 + 4\|u_\varepsilon\|^2\int_\Omega Q(x)|u_\varepsilon|^6 dx}}{24(\int_\Omega Q(x)|u_\varepsilon|^6 dx)^2} \\
&\leq \frac{ab}{4Q_m}\left(\frac{\|\nabla u_\varepsilon\|_2^6}{\int_\Omega |u_\varepsilon|^6 dx} + O(\varepsilon)\right) + \frac{b^3}{24Q_m^2}\left(\frac{\|\nabla u_\varepsilon\|_2^{12}}{(\int_\Omega |u_\varepsilon|^6 dx)^2} + O(\varepsilon)\right) \\
&\quad + \frac{a}{6Q_m}\left(\frac{\|\nabla u_\varepsilon\|_2^2}{(\int_\Omega |u_\varepsilon|^6 dx)^{\frac{1}{3}}}\sqrt{\frac{b^2\|\nabla u_\varepsilon\|_2^8}{(\int_\Omega |u_\varepsilon|^6 dx)^{\frac{4}{3}}} + \frac{4aQ_m\|\nabla u_\varepsilon\|_2^2}{(\int_\Omega |u_\varepsilon|^6 dx)^{\frac{1}{3}}}} + O(\varepsilon)\right) \\
&\quad + \frac{b^2}{24Q_m^2}\left(\frac{\|\nabla u_\varepsilon\|_2^8}{(\int_\Omega |u_\varepsilon|^6 dx)^{\frac{4}{3}}}\sqrt{\frac{b^2\|\nabla u_\varepsilon\|_2^8}{(\int_\Omega |u_\varepsilon|^6 dx)^{\frac{4}{3}}} + \frac{4aQ_m\|\nabla u_\varepsilon\|_2^2}{(\int_\Omega |u_\varepsilon|^6 dx)^{\frac{1}{3}}}} + O(\varepsilon)\right) \\
&\leq \frac{abS_0^3}{16Q_m} + \frac{b^3S_0^6}{384Q_m^2} + \frac{aS_0\sqrt{b^2S_0^4 + 16aS_0Q_m}}{24Q_m} \\
&\quad + \frac{b^2S_0^4\sqrt{b^2S_0^4 + 16aS_0Q_m}}{384Q_m^2} + C_5\varepsilon \\
&= \Theta_2 + C_5\varepsilon.
\end{aligned}$$

Consequently, by (2.14) and (2.15), similarly, there exists $\Lambda_2 > 0$ such that $0 < \lambda < \Lambda_2$, we get

$$\begin{aligned} I_\lambda(t_\varepsilon u_\varepsilon) &= A(t_\varepsilon u_\varepsilon) - \lambda B(t_\varepsilon u_\varepsilon) \\ &\le \Theta_2 + C_5\varepsilon - C_6\lambda\varepsilon^{\frac{6-q}{2}-\beta} \\ &< \Theta_2 - D\lambda^{\frac{2}{2-q}}. \end{aligned}$$

where $C_6 > 0$ (independent of ε, λ). The proof is complete. □

Theorem 2.5. *Assume that* $0 < \lambda < \Lambda_0$ *(Λ_0 is as in Lemma 2.1) and* $1 < q < 2$. *Then problem* (1.1) *has a nontrivial solution* u_λ *with* $I_\lambda(u_\lambda) < 0$.

Proof. It follows from Lemma 2.1 that

$$m \triangleq \inf_{u\in\overline{B_\rho(0)}} I_\lambda(u) < 0.$$

By the Ekeland variational principle [7], there exists a minimizing sequence $\{u_n\} \subset \overline{B_\rho(0)}$ such that

$$I_\lambda(u_n) \le \inf_{u\in\overline{B_\rho(0)}} I_\lambda(u) + \frac{1}{n}, \quad I_\lambda(v) \ge I_\lambda(u_n) - \frac{1}{n}\|v - u_n\|, \quad v \in \overline{B_\rho(0)}.$$

Therefore, there holds $I_\lambda(u_n) \to m$ and $I'_\lambda(u_n) \to 0$. Since $\{u_n\}$ is a bounded sequence and $\overline{B_\rho(0)}$ is a closed convex set, we may assume up to a subsequence, still denoted by $\{u_n\}$, there exists $u_\lambda \in \overline{B_\rho(0)} \subset H^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_\lambda, & \text{weakly in } H^1(\Omega), \\ u_n \to u_\lambda, & \text{strongly in } L^p(\Omega),\ 1 \le p < 6, \\ u_n(x) \to u_\lambda(x), & \text{a.e. in } \Omega. \end{cases}$$

By the lower semi-continuity of the norm with respect to weak convergence, one has

$$\begin{aligned} m &\ge \liminf_{n\to\infty}\left[I_\lambda(u_n) - \frac{1}{6}\langle I'_\lambda(u_n), u_n\rangle\right] \\ &= \liminf_{n\to\infty}\left[\frac{1}{3}\int_\Omega\left(a|\nabla u_n|^2 + u_n^2\right)dx + \frac{b}{12}\left(\int_\Omega|\nabla u_n|^2dx\right)^2 \right. \\ &\quad \left. + \lambda\left(\frac{1}{6} - \frac{1}{q}\right)\int_\Omega P(x)|u_n|^q dx\right] \\ &\ge \frac{1}{3}\int_\Omega\left(a|\nabla u_\lambda|^2 + u_\lambda^2\right)dx + \frac{b}{12}\left(\int_\Omega|\nabla u_\lambda|^2dx\right)^2 \\ &\quad + \lambda\left(\frac{1}{6} - \frac{1}{q}\right)\int_\Omega P(x)|u_\lambda|^q dx \\ &= I_\lambda(u_\lambda) - \frac{1}{6}\langle I'_\lambda(u_\lambda), u_\lambda\rangle = I_\lambda(u_\lambda) = m. \end{aligned}$$

Thus $I_\lambda(u_\lambda) = m < 0$, by $m < 0 < c_\lambda$ and Lemma 2.2, we can see that $\nabla u_n \to \nabla u_\lambda$ in $L^2(\Omega)$ and $u_\lambda \not\equiv 0$. Therefore, we obtain that u_λ is a weak solution of problem (1.1). Since $I_\lambda(|u_\lambda|) = I_\lambda(u_\lambda)$, which suggests that $u_\lambda \ge 0$, then u_λ is a nontrivial solution to problem (1.1). That is, the proof of Theorem 1.1 is complete. □

Theorem 2.6. *Assume that* $0 < \lambda < \Lambda_*(\Lambda_* = \min\{\Lambda_0, \Lambda_1, \Lambda_2\})$*,* $1 < q < 2$ *and* $3 - q < \beta < \frac{6-q}{2}$*. Then the problem* (1.1) *has a nontrivial solution* $u_1 \in H^1(\Omega)$ *such that* $I_\lambda(u_1) > 0$.

Proof. Applying the mountain pass lemma [3] and Lemma 2.2, there exists a sequence $\{u_n\} \subset H^1(\Omega)$ such that

$$I_\lambda(u_n) \to c_\lambda > 0 \text{ and } I'_\lambda(u_n) \to 0 \text{ as } n \to \infty,$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \left\{\gamma \in C([0,1], H^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\right\}.$$

According to Lemma 2.2, we know that $\{u_n\} \subset H^1(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_1$ in $H^1(\Omega)$ as $n \to \infty$,

$$I_\lambda(u_1) = \lim_{n\to\infty} I_\lambda(u_n) = c_\lambda > r > 0,$$

which implies that $u_1 \not\equiv 0$. Therefore, from the continuity of I'_λ, we obtain that u_1 is a nontrivial solution of problem (1.1) with $I_\lambda(u_1) > 0$. Combining the above facts with Theorem 2.5 the proof of Theorem 1.2 is complete. □

3. Conclusions

In this paper, we consider a class of Kirchhoff type equations with Neumann conditions and critical growth. Under suitable assumptions on $Q(x)$ and $P(x)$, using the variational method and the concentration compactness principle, we proved the existence and multiplicity of nontrivial solutions.

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Conflict of interest

The authors declare no conflict of interest in this paper.

References

1. C. O. Alves, F. J. S. A. Corrêa, G. M. Figueiredo, On a class of nonlocal elliptic problems with critical growth, *Differ. Equations Appl.*, **2** (2010), 409–417.
2. C. O. Alves, F. J. S. A. Corrêa, T. F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, *Comput. Math. Appl.*, **49** (2005), 85–93.

3. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381.
4. H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, *Commun. Pure Appl. Math.*, **36** (1983), 437–477.
5. X. F. Cao, J. X. Xu, J. Wang, Multiple positive solutions for Kirchhoff type problems involving concave and convex nonlinearities in $\mathbb{R}^3$, *Electron. J. Differ. Equations*, **301** (2016), 1–16.
6. J. Chabrowski, The critical Neumann problem for semilinear elliptic equations with concave perturbations, *Ric. Mat.*, **56** (2007), 297–319.
7. I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324–353.
8. H. N. Fan, Existence of ground state solutions for Kirchhoff-type problems involving critical Sobolev exponents, *Math. Methods Appl. Sci.*, **41** (2018), 371–385.
9. G. M. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, *J. Math. Anal. Appl.*, **401** (2013), 706–713.
10. X. M. He, W. M. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$, *J. Differ. Equations*, **252** (2012), 1813–1834.
11. X. M. He, W. M. Zou, Ground states for nonlinear Kirchhoff equations with critical growth, *Ann. Mat.*, **193** (2014), 473–500.
12. Y. He, G. B. Li, S. J. Peng, Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents, *Adv. Nonlinear Stud.*, **14** (2014), 441–468.
13. G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.
14. C. Y. Lei, C. M. Chu, H. M. Suo, C. L. Tang, On Kirchhoff type problems involving critical and singular nonlinearities, *Ann. Polonici Mathematici*, **114** (2015), 269–291.
15. G. B. Li, H. Y. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, *J. Differ. Equations*, **257** (2014), 566–600.
16. Q. Q. Li, K. M. Teng, X. Wu, Ground states for Kirchhoff-type equations with critical growth, *Commun. Pure Appl. Anal.*, **17** (2018), 2623–2638.
17. J. F. Liao, P. Zhang, X. P. Wu, Existence of positive solutions for Kirchhoff problems, *Electron. J. Differ. Equations*, **280** (2015), 1–12.
18. J. L. Lions, On some questions in boundary value problems of mathematical physics, *North-Holland Math. Stud.*, **30** (1978), 284–346.
19. P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 1, *Ann. Henri Poincare (C) Nonlinear Anal.*, **2** (1984), 109–145.
20. Z. S. Liu, Y. J. Lou, J. J. Zhang, A perturbation approach to studying sign-changing solutions of Kirchhoff equations with a general nonlinearity. Available from: `https://arxiv.org/abs/1812.09240v2`.
21. Z. S. Liu, S. J. Guo, Existence and concentration of positive ground states for a Kirchhoff equation involving critical Sobolev exponent, *Z. Angew. Math. Phys.*, **66** (2015), 747–769.
22. Z. S. Liu, S. J. Guo, On ground states for the Kirchhoff-type problem with a general critical nonlinearity, *J. Math. Anal. Appl.*, **426** (2015), 267–287.

23. D. Naimen, The critical problem of Kirchhoff type elliptic equations in dimension four, *J. Differ. Equations*, **257** (2014), 1168–1193.
24. L. J. Shen, X. H. Yao, Multiple positive solutions for a class of Kirchhoff type problems involving general critical growth. Available from: `https://arxiv.org/abs/1607.01923v1`.
25. J. Wang, L. X. Tian, J. X. Xu, F. B. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, *J. Differ. Equations*, **253** (2012), 2314–2351.
26. X. J. Wang, Neumann problems of semilinear elliptic equations involving critical Sobolev exponents, *J. Differ. Equations*, **93** (1993), 283–310.
27. X. Wu, Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in $\mathbb{R}^N$, *Nonlinear Anal.: Real World Appl.*, **12** (2011), 1278–1287.
28. Q. L. Xie, X. P. Wu, C. L. Tang, Existence and multiplicity of solutions for Kirchhoff type problem with critical exponent, *Commun. Pure Appl. Anal.*, **12** (2013), 2773–2786.
29. W. Xie, H. Chen, H. Shi, Multiplicity of positive solutions for Schrödinger-Poisson systems with a critical nonlinearity in $\mathbb{R}^3$, *Bull. Malays. Mat. Sci. Soc.*, **3** (2018), 1–24.
30. W. H. Xie, H. B. Chen, Multiple positive solutions for the critical Kirchhoff type problems involving sign-changing weight functions, *J. Math. Anal. Appl.*, **479** (2019), 135–161.
31. L. P. Xu, H. B. Chen, Sign-changing solutions to Schrödinger-Kirchhoff-type equations with critical exponent, *Adv. Differ. Equations*, **121** (2016), 1–14.
32. L. Yang, Z. S. Liu, Z. S. Ouyang, Multiplicity results for the Kirchhoff type equations with critical growth, *Appl. Math. Lett.*, **63** (2017), 118–123.
33. J. Zhang, W. M. Zou, Multiplicity and concentration behavior of solutions to the critical Kirchhoff-type problem, *Z. Angew. Math. Phys.*, **68** (2017), 1–27.
34. J. Zhang, The critical Neumann problem of Kirchhoff type, *Appl. Math. Comput.*, **274** (2016), 519–530.