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Research article

Numerical simulation of the fractal-fractional reaction diffusion equations with general nonlinear

Khaled M. Saad[1,2,*] **and Manal Alqhtani**[1]

[1] Department of Mathematics, College of Sciences and Arts, Najran University, Najran, Kingdom of Saudi Arabia

[2] Department of Mathematics, Faculty of Applied Science, Taiz University, Taiz, Yemen

* **Correspondence:** Email: khaledma_sd@hotmail.com.

Abstract: In this paper a new approach to the use of kernel operators derived from fractional order differential equations is proposed. Three different types of kernels are used, power law, exponential decay and Mittag-Leffler kernels. The kernel's fractional order and fractal dimension are the key parameters for these operators. The main objective of this paper is to study the effect of the fractal-fractional derivative order and the order of the nonlinear term, $1 < q < 2$, in the equation on the behavior of numerical solutions of fractal-fractional reaction diffusion equations (FFRDE). Iterative approximations to the solutions of these equations are constructed by applying the theory of fractional calculus with the help of Lagrange polynomial functions. In key parameter regimes, all these iterative solutions based on a power kernel, an exponential kernel and a generalized Mittag-Leffler kernel are very close. Hence, iterative solutions obtained using one of these kernels are compared with full numerical solutions of the FFRDE and excellent agreement is found. All numerical solutions in this paper were obtained using Mathematica.

Keywords: the fractal-fractional reaction diffusion equations; Lagrange polynomial interpolation; the power law; the exponential law; generalized Mittag-Leffler function
Mathematics Subject Classification: Primary 26A33, 34A08; Secondary 35A20, 35A22

1. Introduction, definitions and preliminaries

Fractional calculus is a generalization of classical calculus and many researchers have paid attention to this science as they encounter many of these issues in the real world. Most of these issues do not have analytical exact solution. Which made many researchers interest and search in numerical and approximate methods to obtain solutions using these methods. There are many of these methods, such as the homotopy analysis [1–4], He's variational iteration method [5, 6], Adomians decomposition method [7–9], Fourier spectral methods [10], finite difference schemes [11], collocation methods [12–14]. To find out more about the fractal calculus, refer to the following references [15, 16]. More recently, a new concept was introduced for the fractional operator, as this operator has two orders, the first representing the fractional order, and the second representing the fractal dimension. In our work we aim to applied the idea of fractal-fractional derivative of orders β, k to a reaction-diffusion equation with q-th nonlinear. To this end [17], we replace the derivative with respect to t by the fractal-fractional derivatives power (FFP) law, the fractal-fractional exponential(FFE) law and the fractal-fractional Mittag-Leffler (FFM) law kernels which corresponds to the [18], Caputo-Fabrizio (CF) [19] and the Atangana-Baleanu (AB) [20] fractional derivatives, respectively. This topic has attracted many researchers and has been applied to research related to the real world, such as [21–26]. Some recent developments in the area of numerical techniques can be found in [27–31].

Merkin and Needham [32] considered the reaction-diffusion travelling waves that can develop in a coupled system involving simple isothermal autocatalysis kinetics. They assumed that reactions took place in two separate and parallel regions, with, in I, the reaction being given by quadratic autocatalysis

$$F + G \rightarrow 2G(\text{rate}\, k_1 f\, g), \tag{1.1}$$

together with a linear decay step

$$G \rightarrow H(\text{rate}\, k_2\, g) \tag{1.2}$$

where f and g are the concentrations of reactant F and autocatalyst H, the $k_i (i = 1, 2)$ are the rate constants and H is some inert product of reaction. The reaction in region II was the quadratic autocatalytic step (1.1) only. The two regions were assumed to be coupled via a linear diffusive interchange of the autocatalytic species G. We shall consider a similar system as I, but with cubic autocatalysis

$$F + 2G \rightarrow 3G(\text{rate}\, k_3\, f g^2) \tag{1.3}$$

together with a linear decay step

$$G \rightarrow H(\text{rate}\, k_4 g). \tag{1.4}$$

For q-th autocatalytic, we have

$$F + qG \rightarrow (q + 1)G(\text{rate}\, k_3 f g^q), \ 1 \leq q \leq 2, \tag{1.5}$$

together with a linear decay step

$$G \rightarrow H(\text{rate}\ \ k_4 g). \tag{1.6}$$

This yields to the following system

$$\frac{\partial \eta_1}{\partial t} = \frac{\partial^2 \eta_1}{\partial \xi^2} + \nu(\eta_2 - \eta_1) - \eta_1 \zeta_1^q, \tag{1.7}$$

$$\frac{\partial \zeta_1}{\partial t} = \frac{\partial^2 \zeta_1}{\partial \xi^2} - \kappa \zeta_1 + \eta_1 \zeta_1^q, \tag{1.8}$$

$$\frac{\partial \eta_2}{\partial t} = \frac{\partial^2 \eta_2}{\partial \xi^2} + \nu(\eta_1 - \eta_2) - \eta_2 \zeta_2^q, \tag{1.9}$$

$$\frac{\partial \zeta_2}{\partial t} = \frac{\partial^2 \zeta_2}{\partial \xi^2} + \eta_2 \zeta_2^q \tag{1.10}$$

where ν represents the couple between (I) and (II) and κ represents the strength of the auto-catalyst decay. For more details see [32]. Omitting the diffusion terms in the system (1.7)–(1.10), one has the following ordinary differential equations

$$\frac{\partial \eta_1}{\partial t} = \nu(\eta_2 - \eta_1) - \eta_1 \zeta_1^q, \tag{1.11}$$

$$\frac{\partial \zeta_1}{\partial t} = -\kappa \zeta_1 + \eta_1 \zeta_1^q, \tag{1.12}$$

$$\frac{\partial \eta_2}{\partial t} = \nu(\eta_1 - \eta_2) - \eta_2 \zeta_2^q, \tag{1.13}$$

$$\frac{\partial \zeta_2}{\partial t} = \eta_2 \zeta_2^q. \tag{1.14}$$

Now we provide some basic definitions that be needed in this work. As for the theorems and proofs related to the three fractal-fractional operators, they are found in details in [17]. Thus we suffice in this work by constructing the algorithms and making the numerical simulations of the set of Eqs (1.7)–(1.10) with the three fractal-fractional operators.

Definition 1. If $\eta(t)$ is continuous and fractal differentiable on (a, b) of order k, then the fractal-fractional derivative of $\eta(t)$ of order β in Riemann Liouville sense with the power law is given by [17]:

$${}_0^{\mathrm{FFP}}D_t^{\beta,k}\eta(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt^k} \int_0^t (t-\tau)^{-\beta} \eta(\tau) d\tau, \quad (0 < \beta,\ k \leq 1), \tag{1.15}$$

and the fractal-fractional integral of $\eta(t)$ is given by

$${}^{\mathrm{FFP}}{}_0I_t^{\beta,k}\eta(t) = \frac{k}{\Gamma(\beta)} \int_0^t \tau^{k-1} (t-\tau)^{\beta-1} \eta(\tau) d\tau. \tag{1.16}$$

Definition 2. If $\eta(t)$ is continuous in the (a, b) and fractal differentiable on (a, b) with order k, then the fractal-fractional derivative of $\eta(t)$ of order β in Riemann Liouville sense with the exponential decay kernel is given by [17]:

$${}_0^{\mathrm{FFE}}D_t^{\beta,k}\eta(t) = \frac{M(\beta)}{1-\beta} \frac{d}{dt^k} \int_0^t e^{\frac{-\beta}{1-\beta}(t-\tau)} \eta(\tau) d\tau, \quad (0 < \beta,\ k \leq 1), \tag{1.17}$$

and the fractal-fractional integral of $\eta(t)$ is given by

$${}^{\mathrm{FFE}}{}_0I_t^{\beta,k}\eta(t) = \frac{(1-\beta)kt^{k-1}}{M(\beta)} \eta(t) + \frac{\beta k}{M(\beta)} \int_0^t \tau^{k-1} \eta(\tau) d\tau \tag{1.18}$$

where $M(\beta)$ is the normalization function such that $M(0) = M(1) = 1$.

Definition 3. If $\eta(t)$ is continuous in the (a, b) and fractal differentiable on (a, b) with order k, then the fractal-fractional derivative of $\eta(t)$ of order β in Riemann Liouville sense with the Mittag–Leffler type kernel is given by [17]:

$$ {}_{0}^{FFE}D_{t}^{\beta,k}\eta(t) = \frac{A(\beta)}{1-\beta}\frac{d}{dt^{k}}\int_{0}^{t} E_{\beta}\Big(\frac{-\beta}{1-\beta}(t-\tau)\Big)\eta(\tau)d\tau, \quad (0 < \beta,\ k \leq 1), \tag{1.19} $$

and the fractal-fractional integral of $\eta(t)$ is given by

$$ {}^{\mathrm{FFE}}{}_{0}I_{t}^{\beta,k}\eta(t) = \frac{(1-\beta)kt^{k-1}}{A(\beta)}\eta(t) + \frac{\beta k}{A(\beta)\Gamma(\beta)}\int_{0}^{t} \tau^{k-1}(t-\tau)^{\beta-1}\eta(\tau)d\tau, \tag{1.20} $$

$$ \frac{d\eta(t)}{dt^{k}} = lim_{\tau\to t}\frac{\eta(\tau)-\eta(t)}{\tau^{k}-t^{k}} \tag{1.21} $$

where where $A(\beta) = 1 - \beta + \dfrac{\beta}{\Gamma(\beta)}$ is a normalization function such that $A(0) = A(1) = 1$.

Our contribution to this paper is to construct the successive approximations and evaluate the numerical solutions of the FFRDE. These successive approximations allow us to study the behavior of numerical solutions based on power , exponential , and the Mittag-Leffler kernels. Also we can study the behavior of approximate solutions in the case of nonlinearity of the FFRDE in general. To our best knowledge, this is the first study of the FFRDE using fractal-fractional with these kernels. The importance of these results lies in the fact that they highlight the possibility of using these results for the benefit of chemical and physical researchers, by trying to link the numerical results of these mathematical models with the laboratory results. These results also contribute to the reliance on numerical results in the case of many models related to the real world, which often cannot find an analytical solution. The structure of this paper is summarized as follows: In sections, two, three and four, the FFRDE is presented with the three kernels that proposed in this work and construct the successive approximations. In section Five, numerical solutions for the FFRDE are discussed with a study of their behavior. Section Six the conclusion is presented.

2. Numerical scheme of FFRDE of q-th-order autocatalysis due the power law kernel

The new model is obtained by replacing the ordinary derivative with the the fractal-fractional derivative the power law kernel as [17]

$$ {}_{0}^{\mathrm{FFP}}D_{t}^{\beta}\eta_{1}(t) = \nu(\eta_{2}(t) - \eta_{1}(t)) - \eta_{1}(t)\zeta_{1}^{q}(t), \tag{2.1} $$

$$ {}_{0}^{\mathrm{FFP}}D_{t}^{\beta}\zeta_{1}(t) = -\kappa\zeta_{1}(t) + \eta_{1}(t)\zeta_{1}^{q}(t), \tag{2.2} $$

$$ {}_{0}^{\mathrm{FFP}}D_{t}^{\beta}\eta_{2}(t) = \nu(\eta_{1}(t) - \eta_{2}(t)) - \eta_{2}(t)\zeta_{2}^{q}(t), \tag{2.3} $$

$$ {}_{0}^{\mathrm{FFP}}D_{t}^{\beta}\zeta_{2}(t) = \eta_{2}(t)\zeta_{2}^{q}(t). \tag{2.4} $$

By following the procedure in [17], we can obtain the following successive approximations:

$$ \eta_{1}(t) - \eta_{1}(0) = \frac{k}{\Gamma(\beta)}\int_{0}^{t} \tau^{k-1}(t-\tau)^{\beta-1}\varphi_{1}(\eta_{1}, \zeta_{1}, \eta_{2}, \zeta_{2}, \tau)d\tau, \tag{2.5} $$

$$\zeta_1(t)-\zeta_2(0)=\frac{k}{\Gamma(\beta)}\int_0^t \tau^{k-1}(t-\tau)^{\beta-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \tag{2.6}$$

$$\eta_2(t)-\eta_3(0)=\frac{k}{\Gamma(\beta)}\int_0^t \tau^{k-1}(t-\tau)^{\beta-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \tag{2.7}$$

$$\zeta_2(t)-\zeta_2(0)=\frac{k}{\Gamma(\beta)}\int_0^t \tau^{k-1}(t-\tau)^{\beta-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau \tag{2.8}$$

where

$$\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)=\Big(\nu\left(\eta_2(\tau)-\eta_1(\tau)\right)-\eta_1(\tau)\zeta_1^q(\tau)\Big), \tag{2.9}$$

$$\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)=\Big(-\kappa\,\zeta_1(\tau)+\eta_1(\tau)\zeta_1^q(\tau)\Big), \tag{2.10}$$

$$\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)=\Big(\nu(\eta_1(\tau)-\eta_2(\tau))-\eta_2(\tau)\zeta_2^q(\tau)\Big), \tag{2.11}$$

$$\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)=\eta_2(\tau)\zeta_2^q(\tau). \tag{2.12}$$

Equation (2.5)–(2.8) can be reformulated as

$$\eta_1(t)-\eta_1(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}} \tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \tag{2.13}$$

$$\zeta_1(t)-\zeta_1(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}} \tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_2(\eta_1(\tau),\zeta_1(\tau),\eta_2(\tau),\zeta_2(\tau),\tau)d\tau, \tag{2.14}$$

$$\eta_2(t)-\eta_2(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}} \tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \tag{2.15}$$

$$\zeta_2(t)-\zeta_2(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}} \tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau. \tag{2.16}$$

Using the two-step Lagrange polynomial interpolation, we obtain

$$\eta_1(t)-\eta_1(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}} (t_{n+1}-\tau)^{\beta-1}Q_{1,m}(\tau)d\tau, \tag{2.17}$$

$$\zeta_1(t)-\zeta_1(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}} (t_{n+1}-\tau)^{\beta-1}Q_{2,m}(\tau)d\tau, \tag{2.18}$$

$$\eta_2(t)-\eta_2(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}} (t_{n+1}-\tau)^{\beta-1}Q_{3,m}(\tau)d\tau, \tag{2.19}$$

$$\zeta_2(t)-\zeta_2(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}} (t_{n+1}-\tau)^{\beta-1}Q_{4,m}(\tau)d\tau, \tag{2.20}$$

where,

$$Q_{1,m}(\tau)=\frac{\tau-t_{m-1}}{t_m-t_{m-1}}t_m^{k-1}\varphi_1(\eta_1(\tau_m),\zeta_1(\tau_m),\eta_2(\tau_m),\zeta_2(\tau_m),\tau_m)-\frac{\tau-t_m}{t_m-t_{m-1}}$$

$$\times t_{m-1}^{k-1}\varphi_1(\eta_1(\tau_{m-1}),\zeta_1(\tau_{m-1}),\eta_2(\tau_{m-1}),\zeta_2(\tau_{m-1}),\tau_{m-1}), \tag{2.21}$$

$$Q_{2,m}(\tau)=\frac{\tau-t_{m-1}}{t_m-t_{m-1}}t_m^{k-1}\varphi_2(\eta_1(\tau_m),\zeta_1(\tau_m),\eta_2(\tau_m),\zeta_2(\tau_m),\tau_m)-\frac{\tau-t_m}{t_m-t_{m-1}}$$
$$\times t_{m-1}^{k-1}\varphi_2(\eta_1(\tau_{m-1}),\zeta_1(\tau_{m-1}),\eta_2(\tau_{m-1}),\zeta_2(\tau_{m-1}),\tau_{m-1}), \tag{2.22}$$

$$Q_{3,m}(\tau)=\frac{\tau-t_{m-1}}{t_m-t_{m-1}}t_m^{k-1}\varphi_3(\eta_1(\tau_m),\zeta_1(\tau_m),\eta_2(\tau_m),\zeta_2(\tau_m),\tau_m)-\frac{\tau-t_m}{t_m-t_{m-1}}$$
$$\times t_{m-1}^{k-1}\varphi_3(\eta_1(\tau_{m-1}),\zeta_1(\tau_{m-1}),\eta_2(\tau_{m-1}),\zeta_2(\tau_{m-1}),\tau_{m-1}), \tag{2.23}$$

$$Q_{4,m}(\tau)=\frac{\tau-t_{m-1}}{t_m-t_{m-1}}t_m^{k-1}\varphi_4(\eta_4(\tau_m),\zeta_1(\tau_m),\eta_2(\tau_m),\zeta_2(\tau_m),\tau_m)-\frac{\tau-t_m}{t_m-t_{m-1}}$$
$$\times t_{m-1}^{k-1}\varphi_4(\eta_1(\tau_{m-1}),\zeta_1(\tau_{m-1}),\eta_2(\tau_{m-1}),\zeta_2(\tau_{m-1}),\tau_{m-1}). \tag{2.24}$$

These integrals are evaluated directly and the numerical solutions of (2.1)–(2.4) involving the FFP derivative are given by

$$\begin{aligned}\eta_1(t_{n+1}) &= \eta_1(0)+\frac{kh^\beta}{\Gamma(\beta+2)}\sum_{m=0}^{n}t_m^{k-1}\varphi_1(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),t_m)\Xi_1(n,m)\\ &- t_{m-1}^{k-1}\varphi_1(\eta_1(\tau_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),t_{m-1})\Xi_2(n,m)\Big),\end{aligned} \tag{2.25}$$

$$\begin{aligned}\zeta_1(t_{n+1}) &= \zeta_1(0)+\frac{kh^\beta}{\Gamma(\beta+2)}\sum_{m=0}^{n}t_m^{k-1}\varphi_2(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),t_m)\Xi_1(n,m)\\ &- t_{m-1}^{k-1}\varphi_2(\eta_1(\tau_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),t_{m-1})\Xi_2(n,m)\Big),\end{aligned} \tag{2.26}$$

$$\begin{aligned}\eta_2(t_{n+1}) &= \eta_2(0)+\frac{kh^\beta}{\Gamma(\beta+2)}\sum_{m=0}^{n}t_m^{k-1}\varphi_3(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),t_m)\Xi_1(n,m)\\ &- t_{m-1}^{k-1}\varphi_4(\eta_1(\tau_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),t_{m-1})\Xi_2(n,m)\Big),\end{aligned} \tag{2.27}$$

$$\begin{aligned}\zeta_2(t_{n+1}) &= \zeta_2(0)+\frac{kh^\beta}{\Gamma(\beta+2)}\sum_{m=0}^{n}t_m^{k-1}\varphi_4(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),t_m)\Xi_1(n,m)\\ &- t_{m-1}^{k-1}\varphi_4(\eta_1(\tau_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),t_{m-1})\Xi_2(n,m)\Big),\end{aligned} \tag{2.28}$$

$$\Xi_1(n,m)=\Big((n+1-m)^\beta(n-m+2+\beta)-(n-m)^\beta\times(n-m+2+2\beta)\Big), \tag{2.29}$$

$$\Xi_2(n,m)=\Big((n+1-m)^{\beta+1}-(n-m)^\beta(n-m+1+\beta)\Big). \tag{2.30}$$

3. Numerical scheme of FFRDE of q-th-order autocatalysis due the exponential decay kernel

Considering the FFE derivative, we have from [17]

$${}_0^{\mathrm{FFE}}D_t^\beta\eta_1(t)=\nu(\eta_2(t)-\eta_1(t))-\eta_1(t)\zeta_1^q(t), \tag{3.1}$$

$$
\begin{aligned}
{}_{0}^{\mathrm{FFE}}D_t^{\beta}\zeta_1(t) &= -\kappa\zeta_1(t) + \eta_1(t)\zeta_1^{q}(t), &(3.2)\\
{}_{0}^{\mathrm{FFE}}D_t^{\beta}\eta_2(t) &= \nu(\eta_1(t) - \eta_2(t)) - \eta_2(t)\zeta_2^{q}(t), &(3.3)\\
{}_{0}^{\mathrm{FFE}}D_t^{\beta}\zeta_2(t) &= \eta_2(t)\zeta_2^{q}(t). &(3.4)
\end{aligned}
$$

For the successive approximations of the system (3.1)–(3.4), we follow the same procedures as in [17], we obtain

$$
\begin{aligned}
\eta_1(t) - \eta_1(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, t)\\
&+ \frac{\beta}{M(\beta)}\int_0^t k\,\tau^{k-1}\varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau, \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_1(t) - \zeta_1(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1, \zeta_1, \eta_2, \zeta_2, t)\\
&+ \frac{\beta}{M(\beta)}\int_0^t k\,\tau^{k-1}\varphi_2(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau, \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
\eta_2(t) - \eta_2(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1, \zeta_1, \eta_2, \zeta_2, t)\\
&+ \frac{\beta}{M(\beta)}\int_0^t k\,\tau^{k-1}\varphi_3(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau, \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_2(t) - \zeta_2(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_4(\eta_1, \zeta_1, \eta_2, \zeta_2, t)\\
&+ \frac{\beta}{M(\beta)}\int_0^t k\,\tau^{k-1}\varphi_4(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau. \qquad (3.8)
\end{aligned}
$$

Using $t = t_{n+1}$ the following is established

$$
\begin{aligned}
\eta_1(t_{n+1}) - \eta_1(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, t_n)\\
&+ \frac{\beta}{M(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}\varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau, \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_1(t_{n+1}) - \zeta_1(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1, \zeta_1, \eta_2, \zeta_2, t_n)\\
&+ \frac{\beta}{M(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}\varphi_2(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau, \qquad (3.10)
\end{aligned}
$$

$$
\eta_2(t_{n+1}) - \eta_2(0) = \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1, \zeta_1, \eta_2, \zeta_2, t_n)
$$

$$+ \quad \frac{\beta}{M(\beta)} \int_0^{t_{n+1}} k\,\tau^{k-1} \varphi_3(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau) d\tau, \tag{3.11}$$

$$\begin{aligned} \zeta_2(t_{n+1}) - \zeta_2(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)} \varphi_4(\eta_1, \zeta_1, \eta_2, \zeta_2, t_n) \\ &+ \frac{\beta}{M(\beta)} \int_0^{t_{n+1}} k\,\tau^{k-1} \varphi_4(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau) d\tau. \end{aligned} \tag{3.12}$$

Further, we have the following:

$$\begin{aligned} \eta_1(t_{n+1}) - \eta_1(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)} \varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, t_n) \\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)} \varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, t_{n-1}) \\ &+ \frac{\beta}{M(\beta)} \int_{t_n}^{t_{n+1}} k\,\tau^{k-1} \varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau) d\tau, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \zeta_1(t_{n+1}) - \zeta_1(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)} \varphi_2(\eta_1, \zeta_1, \eta_2, \zeta_2, t_n) \\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)} \varphi_2(\eta_1, \zeta_1, \eta_2, \zeta_2, t_{n-1}) \\ &+ \frac{\beta}{M(\beta)} \int_{t_n}^{t_{n+1}} k\,\tau^{k-1} \varphi_2(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau) d\tau, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \eta_2(t_{n+1}) - \eta_2(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)} \varphi_3(\eta_1, \zeta_1, \eta_2, \zeta_2, t_n) \\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)} \varphi_3(\eta_1, \zeta_1, \eta_2, \zeta_2, t_{n-1}) \\ &+ \frac{\beta}{M(\beta)} \int_{t_n}^{t_{n+1}} k\,\tau^{k-1} \varphi_3(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau) d\tau, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \zeta_2(t_{n+1}) - \zeta_2(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)} \varphi_4(\eta_1, \zeta_1, \eta_2, \zeta_2, t_n) \\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)} \varphi_4(\eta_1, \zeta_1, \eta_2, \zeta_2, t_{n-1}) \\ &+ \frac{\beta}{M(\beta)} \int_{t_n}^{t_{n+1}} k\,\tau^{k-1} \varphi_4(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau) d\tau. \end{aligned} \tag{3.16}$$

It follows from the Lagrange polynomial interpolation and integrating the following expressions:

$$\eta_1(t_{n+1}) - \eta_1(t_n) = \frac{kt_n^{k-1}(1-\beta)}{M(\beta)} \varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, t_n)$$

$$
\begin{aligned}
& - \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}) + \frac{kh\beta}{2M(\beta)} \\
& \times \Big(3t_n^{k-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) - t_{n-1}^{k-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}),
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
\zeta_1(t_{n+1}) - \zeta_1(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\
&- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}) + \frac{kh\beta}{2M(\beta)} \\
&\times \Big(3t_n^{k-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) - t_{n-1}^{k-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}),
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
\eta_2(t_{n+1}) - \eta_2(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\
&- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}) + \frac{kh\beta}{2M(\beta)} \\
&\times \Big(3t_n^{k-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) - t_{n-1}^{k-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}),
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
\zeta_2(t_{n+1}) - \zeta_2(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\
&- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}) + \frac{kh\beta}{2M(\beta)} \\
&\times \Big(3t_n^{k-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) - t_{n-1}^{k-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}).
\end{aligned} \tag{3.20}
$$

Finally, it is appropriate to write the successive approximations of the system (3.1)–(3.4) as follows:

$$
\begin{aligned}
\eta_1(t_{n+1}) - \eta_1(t_n) &= kt_n^{k-1}\Big(\frac{(1-\beta)}{M(\beta)} + \frac{3h\beta}{2M(\beta)}\Big)\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\
&- kt_{n-1}^{k-1}\Big(\frac{(1-\beta)}{M(\beta)} + \frac{h\beta}{2M(\beta)}\Big)\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}),
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
\zeta_1(t_{n+1}) - \zeta_1(t_n) &= kt_n^{k-1}\Big(\frac{(1-\beta)}{M(\beta)} + \frac{3h\beta}{2M(\beta)}\Big)\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\
&- kt_{n-1}^{k-1}\Big(\frac{(1-\beta)}{M(\beta)} + \frac{h\beta}{2M(\beta)}\Big)\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}),
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
\eta_2(t_{n+1}) - \eta_2(t_n) &= kt_n^{k-1}\Big(\frac{(1-\beta)}{M(\beta)} + \frac{3h\beta}{2M(\beta)}\Big)\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\
&- kt_{n-1}^{k-1}\Big(\frac{(1-\beta)}{M(\beta)} + \frac{h\beta}{2M(\beta)}\Big)\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}),
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
\zeta_2(t_{n+1}) - \zeta_2(t_n) &= kt_n^{k-1}\Big(\frac{(1-\beta)}{M(\beta)} + \frac{3h\beta}{2M(\beta)}\Big)\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\
&- kt_{n-1}^{k-1}\Big(\frac{(1-\beta)}{M(\beta)} + \frac{h\beta}{2M(\beta)}\Big)\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}).
\end{aligned} \tag{3.24}
$$

4. Numerical scheme of FFRDE of q-th-order autocatalysis due the generalized Mittag Lefller kernel

Considering the FFM derivative, we have [18]

$$ {}_0^{\mathrm{FFM}}D_t^{\beta}\eta_1(t) = \nu(\eta_2(t) - \eta_1(t)) - \eta_1(t)\zeta_1^q(t), \tag{4.1}$$

$$ {}_0^{\mathrm{FFM}}D_t^{\beta}\zeta_1(t) = -\kappa\zeta_1(t) + \eta_1(t)\zeta_1^q(t), \tag{4.2}$$

$$ {}_0^{\mathrm{FFM}}D_t^{\beta}\eta_2(t) = \nu(\eta_1(t) - \eta_2(t)) - \eta_2(t)\zeta_2^q(t), \tag{4.3}$$

$$ {}_0^{\mathrm{FFM}}D_t^{\beta}\zeta_2(t) = \eta_2(t)\zeta_2^q(t). \tag{4.4}$$

Also, for this system (4.1)–(4.4), we follow the same treatment that was done in [17] to obtain the successive approximate solutions as follows:

$$\begin{aligned} \eta_1(t) - \eta_1(0) &= \frac{kt^{k-1}(1-\beta)}{A(\beta)}\varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, t) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^t k\,\tau^{k-1}(t-\tau)^{\beta-1}\varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \zeta_1(t) - \zeta_1(0) &= \frac{kt^{k-1}(1-\beta)}{A(\beta)}\varphi_2(\eta_1, \zeta_1, \eta_2, \zeta_2, t) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^t k\,\tau^{k-1}(t-\tau)^{\beta-1}\varphi_2(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \eta_2(t) - \eta_2(0) &= \frac{kt^{k-1}(1-\beta)}{A(\beta)}\varphi_3(\eta_1, \zeta_1, \eta_2, \zeta_2, t) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^t k\,\tau^{k-1}(t-\tau)^{\beta-1}\varphi_3(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \zeta_2(t) - \zeta_2(0) &= \frac{kt^{k-1}(1-\beta)}{A(\beta)}\varphi_4(\eta_1, \zeta_1, \eta_2, \zeta_2, t) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^t k\,\tau^{k-1}(t-\tau)^{\beta-1}\varphi_4(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau. \end{aligned} \tag{4.8}$$

At t_{n+1} we obtain the following

$$\begin{aligned} \eta_1(t_{n+1}) - \eta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_1(\eta_1(t_n), \zeta_1(t_n), \eta_2(t_n), \zeta_2(t_n), t_n) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_1(\eta_1, \zeta_1, \eta_2, \zeta_2, \tau)d\tau, \end{aligned} \tag{4.9}$$

$$ \zeta_1(t_{n+1}) - \zeta_1(0) = \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_2(\eta_1(t_n), \zeta_1(t_n), \eta_2(t_n), \zeta_2(t_n), t_n)$$

$$+ \quad \frac{\beta}{A(\beta)\Gamma(\beta)} \int_0^{t_{n+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \tag{4.10}$$

$$\begin{aligned} \eta_2(t_{n+1}) - \eta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_3(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)} \int_0^{t_{n+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \zeta_2(t_{n+1}) - \zeta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_4(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)} \int_0^{t_{n+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{4.12}$$

The integrals involving in (4.9)–(4.12) can be approximated as:

$$\begin{aligned} \eta_1(t_{n+1}) - \eta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_1(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)} \sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \zeta_1(t_{n+1}) - \zeta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_2(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)} \sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \eta_2(t_{n+1}) - \eta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_3(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)} \sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{4.15}$$

$$\begin{aligned} \zeta_2(t_{n+1}) - \zeta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_4(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n) \\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)} \sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau. \end{aligned} \tag{4.16}$$

The following numerical schemes after approximating the expressions $\tau^{k-1}\varphi_i(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)$, $i = 1,2,3,4$ in the interval $[t_m, t_{m+1}]$ in (4.13)–(4.16) are given by

$$\eta_1(t_{n+1}) - \eta_1(0) = \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_1(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)$$

$$
\begin{aligned}
&+\quad \frac{kh^{\beta}}{A(\beta)\Gamma(\beta+2)}\sum_{m=0}^{n}\Big[t_m^{k-1}\varphi_1(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),(t_m))\Xi_1(n,m)\\
&-\quad t_{m-1}^{k-1}\varphi_1(\eta_1(t_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),(t_{m-1}))\Xi_2(n,m)\Big],
\end{aligned}\tag{4.17}
$$

$$
\begin{aligned}
\zeta_1(t_{n+1})-\zeta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_2(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+\quad \frac{kh^{\beta}}{A(\beta)\Gamma(\beta+2)}\sum_{m=0}^{n}\Big[t_m^{k-1}\varphi_2(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),(t_m))\Xi_1(n,m)\\
&-\quad t_{m-1}^{k-1}\varphi_2(\eta_1(t_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),(t_{m-1}))\Xi_2(n,m)\Big],
\end{aligned}\tag{4.18}
$$

$$
\begin{aligned}
\eta_2(t_{n+1})-\eta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_3(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+\quad \frac{kh^{\beta}}{A(\beta)\Gamma(\beta+2)}\sum_{m=0}^{n}\Big[t_m^{k-1}\varphi_3(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),(t_m))\Xi_1(n,m)\\
&-\quad t_{m-1}^{k-1}\varphi_3(\eta_1(t_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),(t_{m-1}))\Xi_2(n,m)\Big],
\end{aligned}\tag{4.19}
$$

$$
\begin{aligned}
\zeta_2(t_{n+1})-\zeta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_4(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+\quad \frac{kh^{\beta}}{A(\beta)\Gamma(\alpha+2)}\sum_{m=0}^{n}\Big[t_m^{k-1}\varphi_4(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),(t_m))\Xi_1(n,m)\\
&-\quad t_{m-1}^{k-1}\varphi_4(\eta_1(t_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),(t_{m-1}))\Xi_2(n,m)\Big].
\end{aligned}\tag{4.20}
$$

5. Numerical results

In this section, we study in detail the effect of the non-linear term in general, as well as the effect of the fractal-fractional order on the numerical solutions that we obtained by using successive approximations in the above sections. First we begin by satisfying the effective of the numerical solutions of the proposed system when $\beta = 1$ and $k = 1$.

We compare only for the power kernel with a known numerical method which is the finite differences method. This is because all numerical solutions based on the three fractal-fractional operators that presented in this paper are very close each other when $\beta = 1$ and $k = 1$. Figure 1 illustrates the comparison between numerical solutions (2.25)–(2.28) and numerical solutions computed by using the finite differences method with k and β. The parameters that used are $\gamma = 0.4, \kappa = 0.004, h = 0.02$. From this figure we note that an excellent agreement. And the accurate is increasing as we take small h. From, Figure 1(a) and 1(c), we can see, that the profiles for η_1 and η_2 are very similar, but the profiles of ζ_1 and ζ_2 are more distinct with $\zeta_2 > \zeta_2$. For Figure 1(b), the profiles of ζ_1 and ζ_2 are very close than in Figure 1(a) and 1(c), also for ζ_1 and ζ_2. Figures 2 and 3 show that the behavior of the approximate solutions based on FFP, FFE and FFM, when the degree of the non-linear term is cubic and for different values of k and β. For the parameters γ and κ , we fixed them in all computations. The remain parameters are the same as in Figure 1. Similarly, in Figures 4 and 5, the approximate solutions are plotted in the case of a non-linear with quadratic degree and for different values of k and β. Finally in Figures 6 and 7, the approximate solutions are shown in the case of non-linear with fractional order and for different values for k and β. For the Figures 2 and 3 which the nonlinear is cubic, all the profiles are distinct. Similarly with Figures 6 and 7 when the nonlinear is quadratic. From Figures 4 and 5, we can see in the case of fraction non-linear, the profiles of η_1 and η_2 are very close to each other than the profiles of ζ_1 and ζ_2.

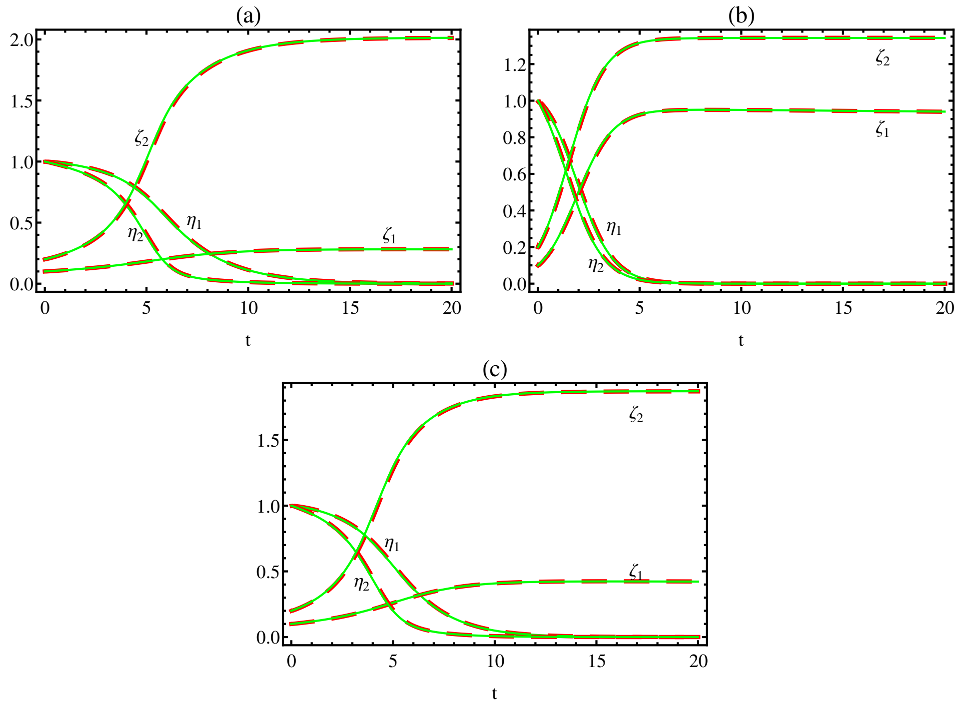

Figure 1. Comparison between the numerical solutions (2.25)–(2.28) and numerical based on finite difference methods for $\beta = 1$, $k = 1$, $\gamma = 0.4$, $\kappa = 0.001$, $h = 0.01$. $(a)\, q = 2$; $(b)\, q = 1$; $(c)\, q = 1.8$; (Green solid color: Numerical solutions (2.25)–(2.28); Red dashed color: FDM).

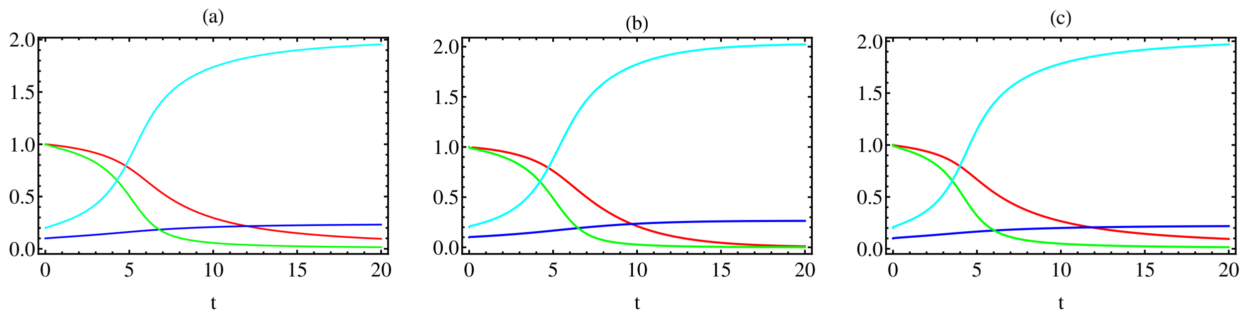

Figure 2. Graph of the numerical solutions with $q = 2$ for $\beta = 0.8, k = 1, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).

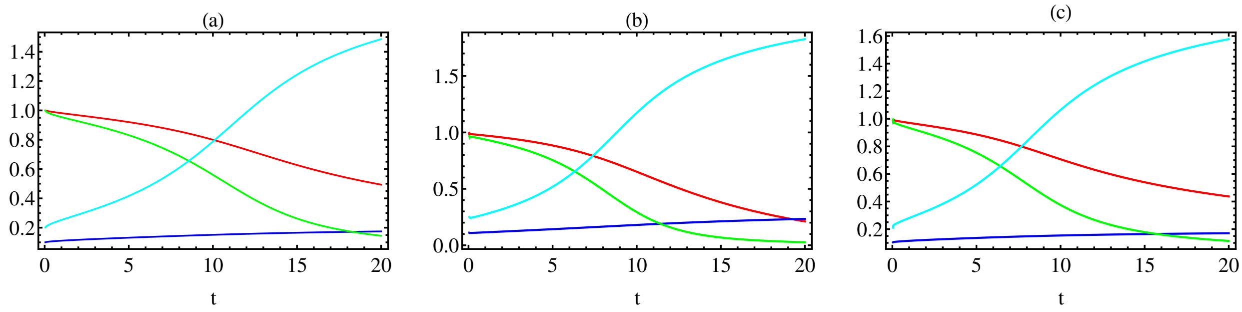

Figure 3. Graph of the numerical solutions with $q = 2$ for $\beta = 0.7, k = 0.8, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).

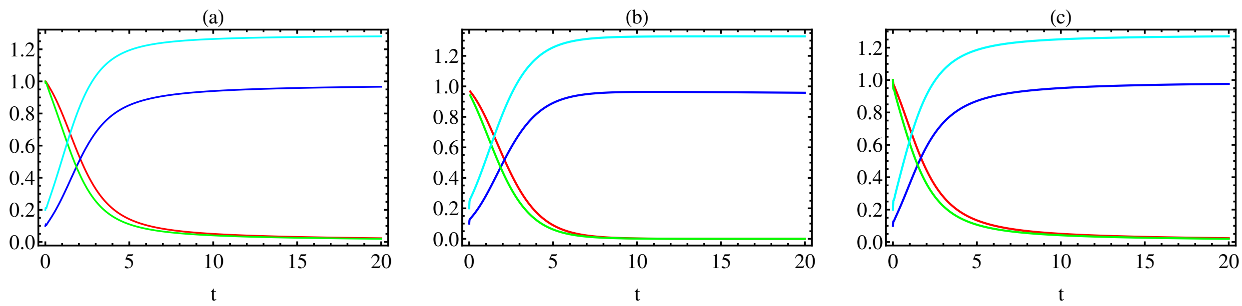

Figure 4. Graph of the numerical solutions with $q = 1$ for $\beta = 0.8, k = 1, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).

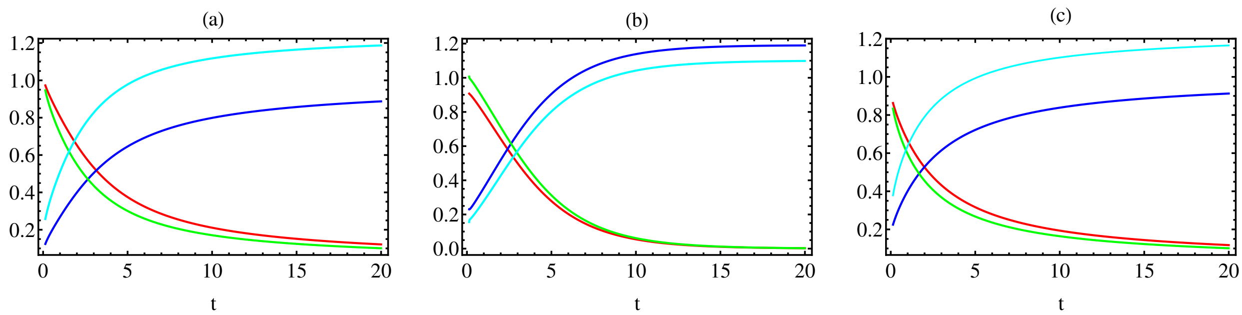

Figure 5. Graph of the numerical solutions with $q = 1$ for $\beta = 0.7, k = 0.8, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).

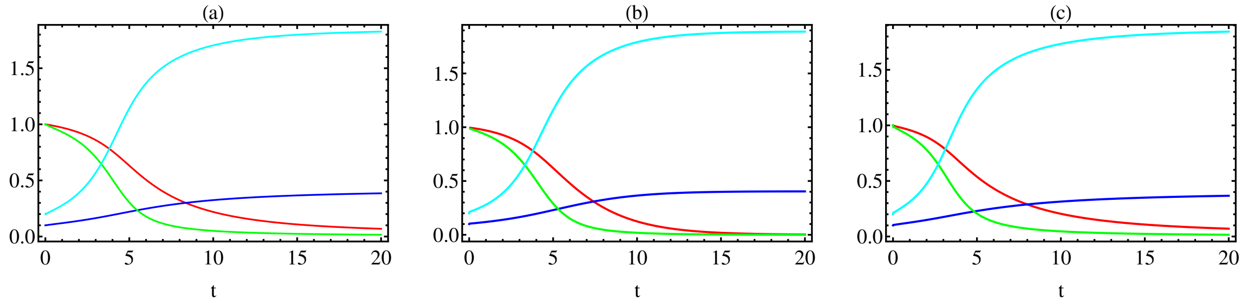

Figure 6. Graph of the numerical solutions with $q = 1.8$ for $\beta = 0.8, k = 1, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).

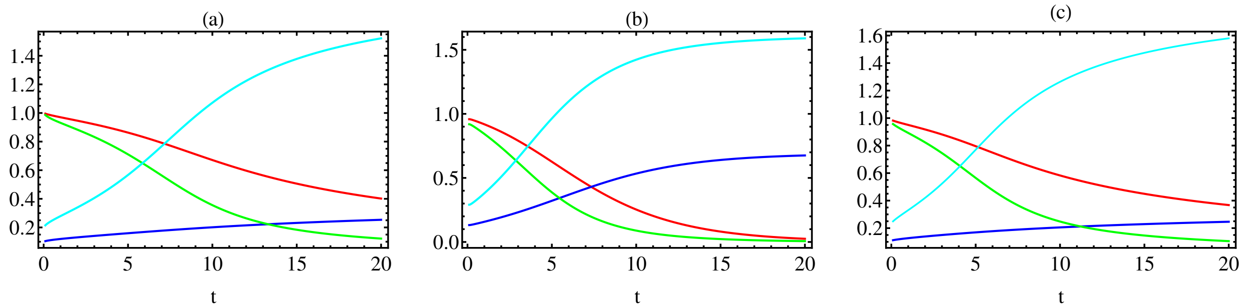

Figure 7. Graph of the numerical solutions with $q = 1.8$ for $\beta = 0.7, k = 0.8, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).

6. Conclusions

In this paper, numerical solutions of the of the fractal-fractional reaction diffusion equations with general nonlinear have been studied. We introduced the FFRDE in three instances of fractional derivatives based on power, exponential, and Mittag-Leffler kernels. After that, we used the fundamental fractional calculus with the help of Lagrange polynomial functions. We obtained the iterative and approximate formulas in the three cases. We studied the effect of the non-linear term order, in the case of cubic, quadratic, and fractional for different values of the fractal-fractional derivative order. The accuracy of the numerical solutions in the classic case of the FFRDE was tested in the case of power kernel, where all the numerical solutions in the classic case of integer order coincide to each other, and the comparison result has excellent agreement . In all calculations was used the Mathematica Program Package.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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