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Research article

On Dirac operator with boundary and transmission conditions depending Herglotz-Nevanlinna type function

Yalçın Güldü* **and Ebru Mişe**

Sivas Cumhuriyet University, Department of Mathematics Sivas, Turkey

* **Correspondence:** Email: yguldu@cumhuriyet.edu.tr; Tel: +903462191010;
Fax: +903462191186.

Abstract: In this paper, an inverse problem is considered for Dirac equations with boundary and transmission conditions eigenvalue depending as rational function of Herglotz-Nevanlinna. We give some spectral properties of the problem and also it is shown that the coefficients of the problem are uniquely determined by Weyl function and by classical spectral data made up of eigenvalues and norming constants.

Keywords: Dirac equations; transmission condition; Herglotz-Nevanlinna type function; inverse problem
Mathematics Subject Classification: 34A55, 34B24, 34L05

1. Introduction

We consider the system of Dirac equations

$$\ell y(x) := By'(x) + Q(x)y(x) = \lambda y(x),\ x \in [a, b]\,, \tag{1}$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p\,(x) \end{pmatrix}$, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $p(x)$, $q(x)$ are real valued functions in $L_2\,(a, b)$ and λ is a spectral parameter, with boundary conditions

$$U\,(y) := y_2(a) + f_1(\lambda)y_1(a) = 0 \tag{2}$$

$$V\,(y) := y_2(b) + f_2(\lambda)y_1(b) = 0 \tag{3}$$

and with transmission conditions

$$\begin{cases} y_1(w_i + 0) = \alpha_i y_1(w_i - 0) \\ y_2(w_i + 0) = \alpha_i^{-1} y_2(w_i - 0) + h_i\,(\lambda)\,y_1(w_i - 0) \end{cases} \quad (i = 1, 2) \tag{4}$$

where $f_i(\lambda)$, $h_i(\lambda)$ $(i = 1, 2)$ are rational functions of Herglotz-Nevanlinna type such that

$$f_i(\lambda) = a_i\lambda + b_i - \sum_{k=1}^{N_i} \frac{f_{ik}}{\lambda - g_{ik}} \tag{5}$$

$$h_i(\lambda) = m_i\lambda + n_i - \sum_{k=1}^{P_i} \frac{u_{ik}}{\lambda - t_{ik}} \quad (i = 1, 2) \tag{6}$$

a_i, b_i, f_{ik}, g_{ik}, m_i, n_i, u_{ik} and t_{ik} are real numbers, $a_1 < 0$, $f_{1k} < 0$, $a_2 > 0$, $f_{2k} > 0, m_i > 0$, $u_{ik} > 0$ and $g_{i1} < g_{i2} < ... < g_{iNi}$, $t_{i1} < t_{i2} < ... < t_{iPi}$, $\alpha_i > 0$ and $a < w_1 < w_2 < b$. In special case, when $f_i(\lambda) = \infty$, conditions (2) and (3) turn to Dirichlet conditions $y_1(a) = y_1(b) = 0$ respectively. Moreover, when $h_i(\lambda) = \infty$, conditions (4) turn to $y_1(w_2+0) = \alpha_2 y_1(w_2-0)$, $y_2(w_2+0) = \alpha_2^{-1} y_2(w_2-0) + h_2(\lambda) y_1(w_2-0)$ and $y_1(w_1+0) = \alpha_1 y_1(w_1-0)$, $y_2(w_1+0) = \alpha_1^{-1} y_2(w_1-0) + h_1(\lambda) y_1(w_1-0)$ according to order $i = 1, 2$.

Inverse problems of spectral analysis compose of recovering operators from their spectral data. Such problems arise in mathematics, physics, geophysics, mechanics, electronics, meteorology and other branches of natural sciences. Inverse problems also play important role in solving many equations in mathematical physics.

$R_1(\lambda) y_1(a) + R_2(\lambda) y_2(a) = 0$ is a boundary condition depending spectral parameter where $R_1(\lambda)$ and $R_2(\lambda)$ are polynomials. When $\deg R_1(\lambda) = \deg R_2(\lambda) = 1$, this equality depends on spectral parameter as linearly. On the other hand, it is more difficult to search for higher orders of polynomials $R_1(\lambda)$ and $R_2(\lambda)$. When $\dfrac{R_1(\lambda)}{R_2(\lambda)}$ is rational function of Herglotz-Nevanlinna type such that $f(\lambda) = a\lambda + b - \sum_{k=1}^{N} \dfrac{f_k}{\lambda - g_k}$ in boundary conditions, direct and inverse problems for Sturm-Liouville operator have been studied [1–11]. In this paper, direct and inverse spectral problem is studied for the system of Dirac equations with rational function of Herglotz-Nevanlinna in boundary and transmission conditions.

On the other hand, inverse problem firstly was studied by Ambarzumian in 1929 [12]. After that, G. Borg was proved the most important uniqueness theorem in 1946 [13]. In the light of these studies, we note that for the classical Sturm-Liouville operator and Dirac operator, the inverse problem has been studied fairly (see [14–20], where further references and links to applications can be found). Then, results in these studies have been extended to other inverse problems with boundary conditions depending spectral parameter and with transmission conditions. Therefore, spectral problems for differential operator with transmission conditions inside an interval and with eigenvalue dependent boundary and transmission conditions as linearly and non-linearly have been studied in so many problems of mathematics as well as in applications (see [21–43] and other works, and see [44–54] and other works cited therein respectively).

The aim of this article is to get some uniqueness theorems for mentioned above Dirac problem with eigenvalue dependent as rational function of Herglotz-Nevanlinna type in both of the boundary conditions and also transmission conditions at two different points. We take into account inverse problem for reconstruction of considered boundary value problem by Weyl function and by spectral data $\{\lambda_n, \rho_n\}_{n\in\mathbb{Z}}$ and $\{\lambda_n, \mu_n\}_{n\in\mathbb{Z}}$. Although the boundary and transmission conditions of the problem are not linearly dependent on the spectral parameter, this allows the eigenvalues to be real and to define normalizing numbers.

2. Preliminaries

Consider the space $H := L_2(a,b) \oplus L_2(a,b) \oplus \mathbb{C}^{N_1+1} \oplus \mathbb{C}^{N_2+1} \oplus \mathbb{C}^{P_1+1} \oplus \mathbb{C}^{P_2+1}$ and element Y in H is in the form of $Y = (y_1(x), y_2(x), \tau, \eta, \beta, \gamma)$, such that $\tau = (Y_1, Y_2, \ldots, Y_{N_1}, Y_{N_1+1})$, $\eta = (L_1, L_2, \ldots, L_{N_2}, L_{N_2+1})$, $\beta = (R_1, R_2, \ldots, R_{P_1}, R_{P_1+1})$, $\gamma = (V_1, V_2, \ldots, V_{P_2}, V_{P_2+1})$. H is a Hilbert space with the inner product defined by

$$
\begin{aligned}
< Y, Z > &:= \int_a^b (y_1(x)\bar{z}_1(x) + y_2(x)\bar{z}_2(x))\, dx \\
&- \frac{Y_{N_1+1}\overline{Y'_{N_1+1}}}{a_1} + \frac{L_{N_2+1}\overline{L'_{N_2+1}}}{a_2} + \frac{\alpha_1}{m_1} R_{P_1+1}\overline{R'_{P_1+1}} \\
&+ \frac{\alpha_2}{m_2} V_{P_2+1}\overline{V'_{P_2+1}} + \sum_{k=1}^{N_1} Y_k \overline{Y'_k}\left(-\frac{1}{f_{1k}}\right) \\
&+ \sum_{k=1}^{N_2} \frac{L_k \overline{L'_k}}{f_{2k}} + \sum_{k=1}^{P_1} \alpha_1 \frac{R_r \overline{R'_r}}{u_{1k}} + \sum_{k=1}^{P_2} \alpha_2 \frac{V_r \overline{V'_r}}{u_{2k}}
\end{aligned} \tag{7}
$$

for $Y = (y_1(x), y_2(x), \tau, \eta, \beta, \gamma)$ ve $Z = (z_1(x), z_2(x), \tau', \eta', \beta', \gamma')$ in H. Define the operator T on the domain

$D(T) = \{Y \in H : y_1(x), y_2(x) \in AC\,(a,b)\,,$

$ly \in L_2\,(a,b)\,, y_1(w_i^+) = \alpha_i y_1(w_i^-), i = 1,2$

$Y_{N_1+1} := -a_1 y_1(a), L_{N_2+1} := -a_2 y_1(b),$

$R_{P_1+1} := -m_1 y_1(w_1^-), V_{P_2+1} := -m_2 y_1(w_2^-)\}$

such that

$$TY := (ly, T\tau, T\eta, T\beta, T\gamma) \tag{8}$$

where

$$T\tau = TY_i = \begin{cases} g_{1i}Y_i - f_{1i}y_1(a),\ i = \overline{1, N_1} \\ y_2(a) + b_1 y_1(a) + \sum_{k=1}^{N_1} Y_k,\ i = N_1 + 1 \end{cases} \tag{9}$$

$$T\eta = TL_i = \begin{cases} g_{2i}L_i - f_{2i}y_1(b),\ i = \overline{1, N_2} \\ y_2(b) + b_2 y_1(b) + \sum_{k=1}^{N_2} L_k,\ i = N_2 + 1 \end{cases} \tag{10}$$

$$T\beta = TR_i = \begin{cases} t_{1i}R_i - u_{1i}y_1(w_1^-),\ i = \overline{1, P_1} \\ -y_2(w_1^+) + \alpha_1^{-1} y_2\left(w_1^-\right) + n_1 y_1\left(w_1^-\right) + \sum_{k=1}^{P_1} R_k,\ i = P_1 + 1 \end{cases} \tag{11}$$

$$T\gamma = TV_i = \begin{cases} t_{2i}V_i - u_{2i}y_1(w_2^-),\ i = \overline{1, P_2} \\ -y_2(w_2^+) + \alpha_2^{-1}y_2\left(w_2^-\right) + n_2y_1\left(w_2^-\right) + \sum\limits_{k=1}^{P_2} V_k,\ i = P_2 + 1. \end{cases} \tag{12}$$

Accordingly, equality $TY = \lambda Y$ corresponds to problem (1)–(4) under the domain $D(T) \subset H$.

Theorem 1. The eigenvalues of the operator T and the problem (1)–(4) coincide.

Proof. Assume that λ is an eigenvalue of T and $Y(x) = (y_1(x), y_2(x), \tau, \eta, \beta, \gamma) \in H$ is the eigenvector corresponding to λ. Since $Y \in D(T)$, it is obvious that the condition $y_1(w_i + 0) - \alpha_i y_1(w_i - 0) = 0$ and Eq (1) hold. On the other hand,boundary conditions (2)–(3) and the second condition of (4) are satisfied by the following equalities;

$$T\tau = TY_i = g_{1i} - Y_i - f_{1i}y_1(a) = \lambda Y_i,\ i = \overline{1, N_1}$$

$$TY_{N_1+1} = y_2(a) + b_1y_1(a) + \sum_{k=1}^{N_1} Y_k = -a_1y_1(a)\lambda$$

$$T\eta = TL_i = g_{2i}L_i - f_{2i}y_1(b) = \lambda L_i,\ i = \overline{1, N_2}$$

$$TL_{N_2+1} = y_2(b) + b_2y_1(b) + \sum_{k=1}^{N_2} L_k = -a_2y_1(b)\lambda$$

$$T\beta = TR_i = t_{1i}R_i - u_{1i}y_1(w_1^-),\ i = \overline{1, P_1}$$

$$TR_{P_1+1} = -y_2(w_1^+) + \alpha_1^{-1}y_2\left(w_1^-\right) + n_1y_1\left(w_1^-\right) + \sum_{k=1}^{P_1} R_k = -m_1y_1\left(w_1^-\right)\lambda$$

$$T\gamma = TV_i = t_{2i}V_i - u_{2i}y_1(w_2^-),\ i = \overline{1, P_2}$$

$$TV_{P_2+1} = -y_2(w_2^+) + \alpha_2^{-1}y_2\left(w_2^-\right) + n_2y_1\left(w_2^-\right) + \sum_{k=1}^{P_2} V_k = -m_2y_1\left(w_2^-\right)\lambda.$$

If $\lambda = g_{ik}$ ($i = 1, 2$ and $k = \{1, 2, \ldots N_i\}$) are eigenvalues of operator T, then, from above equalities and the domain of T, equalities (1), $y_1(a, g_{1k}) = 0$, $y_1(b, g_{2k}) = 0$ and (4) are satisfied.

Moreover, If $\lambda = t_{ik}$ ($i = 1, 2$ and $k = \{1, 2, \ldots P_i\}$) are eigenvalues of operator T, from above equalities and the domain of T, Eqs (1)–(3) and $y_1(w_i^-, t_{ik}) = 0 = y_1(w_i^+, t_{ik})$ are valid. In that case, λ is also an eigenvalue of L.

Conversely, let λ be an eigenvalue of L and $\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ be an eigenfunction corresponding to λ. If $\lambda \neq g_{ik}$ ($i = 1, 2$, $k = \{1, 2, \ldots N_i\}$) and $\lambda \neq t_{ik}$ ($i = 1, 2$, $k = \{1, 2, \ldots P_i\}$) then, it is clear that λ is an eigenvalue of T and the vector

$$Y = \Big(y_1(x), y_2(x), \tfrac{f_{11}}{g_{11}-\lambda}y_1(a), \tfrac{f_{12}}{g_{12}-\lambda}y_1(a), \ldots, \tfrac{f_{1N_1}}{g_{1N_1}-\lambda}y_1(a), -a_1y_1(a),$$

$$\tfrac{f_{21}}{g_{21}-\lambda}y_1(b), \tfrac{f_{22}}{g_{22}-\lambda}y_1(b), \ldots, \tfrac{f_{2N_2}}{g_{2N_2}-\lambda}y_1(b), -a_2y_1(b),$$

$$\tfrac{u_{11}}{t_{11}-\lambda}y_1(w_1^-), \tfrac{u_{12}}{t_{12}-\lambda}y_1(w_1^-), \ldots, \tfrac{u_{1P_1}}{t_{1P_1}-\lambda}y_1(w_1^-), -m_1y_1(w_1^-),$$

$$\tfrac{u_{21}}{t_{21}-\lambda}y_1(w_2^-), \tfrac{u_{22}}{t_{22}-\lambda}y_1(w_2^-), \ldots, \tfrac{u_{2P_2}}{t_{2P_2}-\lambda}y_1(w_2^-), -m_2y_1(w_2^-)\Big)$$ is the eigenvector corresponding to λ.

If $\lambda = g_{1k}$ ($k = \{1, 2, \ldots N_1\}$), then,

$Y = (y_1(x), y_2(x), Y_1, Y_2, \ldots, Y_{N_1}, 0, L_1, L_2, \ldots, L_{N_2}, L_{N_2+1}, R_1, R_2, \ldots, R_{P_1}, R_{P_1+1},\ V_1, V_2, \ldots, V_{P_2}, V_{P_2+1})$,

$Y_i = \begin{cases} 0, & i \neq k \\ -y_2(a), & i = k \end{cases}$, $i = 1, 2, \ldots, N_1$ is the eigenvector of T corresponding to g_{1k}.

If $\lambda = g_{2k}$ ($k = \{1, 2, \ldots N_2\}$), then,
$Y = (y_1(x), y_2(x), Y_1, Y_2, \ldots, Y_{N_1}, Y_{N_1+1}, L_1, L_2, \ldots, L_{N_2}, 0, R_1, R_2, \ldots, R_{P_1}, R_{P_1+1}, V_1, V_2, \ldots, V_{P_2}, V_{P_2+1})$,

$L_i = \begin{cases} 0, & i \neq k \\ -y_2(b), & i = k \end{cases}$, $i = 1, 2, \ldots, N_2$ is the eigenvector of T corresponding to g_{2k}.

Furthermore, if $\lambda = t_{1k}$ ($k = \{1, 2, \ldots P_1\}$), then,
$Y = (y_1(x), y_2(x), Y_1, Y_2, \ldots, Y_{N_1}, Y_{N_1}, L_1, L_2, \ldots, L_{N_2}, L_{N_2+1}, R_1, R_2, \ldots, R_{P_1}, 0, V_1, V_2, \ldots, V_{P_2}, V_{P_2+1})$,

$R_i = \begin{cases} 0, & i \neq k \\ y_2\left(w_1^+\right) - \alpha_1^{-1} y_2\left(w_1^-\right), & i = k \end{cases}$, $i = 1, 2, \ldots, P_1$ is the eigenvector corresponding to t_{1k}.
If $\lambda = t_{2k}$ ($k = \{1, 2, \ldots P_2\}$), then,
$Y = (y_1(x), y_2(x), Y_1, Y_2, \ldots, Y_{N_1}, Y_{N_1}, L_1, L_2, \ldots, L_{N_2}, L_{N_2+1}, R_1, R_2, \ldots, R_{P_1}, R_{P_1+1}, V_1, V_2, \ldots, V_{P_2}, 0)$,

$V_i = \begin{cases} 0, & i \neq k \\ y_2\left(w_2^+\right) - \alpha_2^{-1} y_2\left(w_2^-\right), & i = k \end{cases}$, $i = 1, 2, \ldots, P_2$ is the eigenvector corresponding to t_{2k}. □

It is possible to write $f_i(\lambda)$ as follows:

$$f_i(\lambda) = \frac{a_i(\lambda)}{b_i(\lambda)}, i = 1, 2$$

where

$$a_i(\lambda) = (a_i\lambda + b_i)\prod_{k=1}^{N_i}(\lambda - g_{ik}) - \sum_{k=1}^{N_i}\prod_{j=1(j\neq k)}^{N_i} f_{ik}\left(\lambda - g_{ij}\right)$$

$$b_i(\lambda) = \prod_{k=1}^{Ni}(\lambda - g_{ik}).$$

Assume that $a_2(\lambda)$ and $b_2(\lambda)$ do not have common zeros.

Let functions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the solutions of (1) under the initial conditions

$$\varphi(a, \lambda) = \begin{pmatrix} -b_1(\lambda) \\ a_1(\lambda) \end{pmatrix}, \psi(b, \lambda) = \begin{pmatrix} -b_2(\lambda) \\ a_2(\lambda) \end{pmatrix} \tag{13}$$

as well as the transmission conditions (4) respectively such that

$$\varphi(x, \lambda) = \begin{cases} \varphi_1(x, \lambda), x < w_1 \\ \varphi_2(x, \lambda), w_1 < x < w_2 \\ \varphi_3(x, \lambda), w_2 < x < b \end{cases} \text{ and } \psi(x, \lambda) = \begin{cases} \psi_3(x, \lambda), x < w_1 \\ \psi_2(x, \lambda), w_1 < x < w_2 \\ \psi_1(x, \lambda), w_2 < x < b \end{cases}.$$

Then it can be easily proven that $\varphi_i(x, \lambda)$ and $\psi_i(x, \lambda)$, $i = \overline{1, 3}$ are the solutions of the following integral equations;

$$\begin{aligned} &\varphi_{i+1,1}(x, \lambda) = \alpha_i \varphi_{i1}(w_i, \lambda) \cos\lambda(x - w_i) \\ &- \left[\alpha_i^{-1}\varphi_{i2}(w_i, \lambda) + h_i(\lambda)\varphi_{i1}(w_i, \lambda)\right] \sin\lambda(x - w_i) \\ &+ \int_{w_i}^{x} \left[p(t)\sin\lambda(x - t) + q(t)\cos\lambda(x - t)\right] \varphi_{i+1,1}(t, \lambda) dt \end{aligned}$$

$$+\int_{wi}^{x}\left[q(t)\sin\lambda(x-t)-p(t)\cos\lambda(x-t)\right]\varphi_{i+1,2}(t,\lambda)dt,$$

$$\varphi_{i+1,2}(x,\lambda)=\alpha_i\varphi_{i1}(w_i,\lambda)\sin\lambda\left(x-w_i\right)$$
$$+\left[\alpha_i^{-1}\varphi_{i2}(w_i,\lambda)+h_i(\lambda)\varphi_{i1}(w_i,\lambda)\right]\cos\lambda\left(x-w_i\right)$$
$$+\int_{w_i}^{x}\left[-p(t)\cos\lambda(x-t)+q(t)\sin\lambda(x-t)\right]\varphi_{i+1,1}(t,\lambda)dt$$
$$+\int_{wi}^{x}\left[-q(t)\cos\lambda(x-t)-p(t)\sin\lambda(x-t)\right]\varphi_{i+1,2}(t,\lambda)dt,\text{ for } i=1,2$$

and

$$\psi_{i1}(x,\lambda)=\alpha_i^{-1}\psi_{i+1,1}(w_i,\lambda)\cos\lambda\left(x-w_i\right)$$
$$+\left(-\alpha_i\psi_{i+1,2}(w_i,\lambda)+h_i(\lambda)\psi_{i+1,1}(w_i,\lambda)\right)\sin\lambda\left(x-w_i\right)$$
$$-\int_{x}^{w_i}\left[p(t)\sin\lambda\left(x-t\right)+q(t)\cos\lambda\left(x-t\right)\right]\psi_{i1}(t,\lambda)\,dt$$
$$+\int_{x}^{w_i}\left[-q(t)\sin\lambda\left(x-t\right)+p(t)\cos\lambda\left(x-t\right)\right]\psi_{i2}(t,\lambda)\,dt$$

$$\psi_{i2}(x,\lambda)=\alpha_i^{-1}\psi_{i+1,1}(w_i,\lambda)\sin\lambda\left(x-w_i\right)$$
$$+\left(\alpha_i\psi_{i+1,2}(w_i,\lambda)-h_i(\lambda)\psi_{i+1,1}(w_i,\lambda)\right)\cos\lambda\left(x-w_i\right)$$
$$+\int_{x}^{w_i}\left[p(t)\cos\lambda\left(x-t\right)-q(t)\sin\lambda\left(x-t\right)\right]\psi_{i1}(t,\lambda)\,dt$$
$$+\int_{x}^{w_2}\left[q(t)\cos\lambda\left(x-t\right)+p(t)\sin\lambda\left(x-t\right)\right]\psi_{i2}(t,\lambda)\,dt,\text{ for } i=2,1.$$

Lemma 1. For the solutions $\varphi_i(x,\lambda)$ and $\psi_i(x,\lambda)$, $i=\overline{1,3}$ as $|\lambda|\to\infty$, the following asymptotic estimates hold;

$$\varphi_{11}(x,\lambda)=\left\{a_1\lambda^{N_1+1}\sin\lambda(x-a)+o\left(|\lambda|^{N_1+1}\exp|\operatorname{Im}\lambda|\left[(x-a)\right]\right)\right.,$$

$$\varphi_{12}(x,\lambda)=\left\{a_1\lambda^{N_1+1}\cos\lambda(x-a)+o\left(|\lambda|^{N_1+1}\exp|\operatorname{Im}\lambda|\left[(x-a)\right]\right)\right.,$$

$$\varphi_{21}(x,\lambda)=\begin{cases} a_1m_1\lambda^{L_1+N_1+2}\sin\lambda\left(w_1-a\right)\sin\lambda\left(x-w_1\right)\\ +o\left(|\lambda|^{L_1+N_1+2}\exp|\operatorname{Im}\lambda|\left[(w_1-a)+(x-w_1)\right]\right)\end{cases}$$

$$\varphi_{22}(x,\lambda)=\begin{cases} a_1m_1\lambda^{L_1+N_1+2}\sin\lambda\left(w_1-a\right)\cos\lambda\left(x-w_1\right)\\ +o\left(|\lambda|^{L_1+N_1+2}\exp|\operatorname{Im}\lambda|\left[(w_1-a)+(x-w_1)\right]\right)\end{cases}$$

$$\varphi_{31}(x,\lambda)=\begin{cases} -m_2m_1a_1\lambda^{L_1+L_2+N_1+3}\sin\lambda\left(w_1-a\right)\sin\lambda\left(w_2-w_1\right)\sin\lambda\left(x-w_2\right)\\ +o\left(|\lambda|^{L_1+L_2+N_1+3}\exp|\operatorname{Im}\lambda|\left[(w_1-a)+(w_2-w_1)+(x-w_2)\right]\right)\end{cases}$$

$$\varphi_{32}(x,\lambda)=\begin{cases} m_2m_1a_1\lambda^{L_1+L_2+N_1+3}\sin\lambda(w_1-a)\sin\lambda(w_2-w_1)\cos\lambda(x-w_2) \\ +o\left(|\lambda|^{L_1+L_2+N_1+3}\exp|\operatorname{Im}\lambda|\left[(w_1-a)+(w_2-w_1)+(x-w_2)\right]\right)\end{cases}$$

$$\psi_{11}(x,\lambda)=\begin{cases} -a_2\lambda^{N_2+1}\sin\lambda(x-b) \\ +o\left(|\lambda|^{N_2+1}\exp|\operatorname{Im}\lambda|\left[(x-b)\right]\right)\end{cases}$$

$$\psi_{12}(x,\lambda)=\begin{cases} a_2\lambda^{N_2+1}\cos\lambda(x-b) \\ +o\left(|\lambda|^{N_2+1}\exp|\operatorname{Im}\lambda|\left[(x-b)\right]\right)\end{cases}$$

$$\psi_{21}(x,\lambda)=\begin{cases} -m_2a_2\lambda^{N_2+L_2+2}\sin\lambda(w_2-b)\sin\lambda(x-w_2) \\ +o\left(|\lambda|^{N_2+L_2+2}\exp|\operatorname{Im}\lambda|\left[(w_2-b)+(x-w_2)\right]\right)\end{cases}$$

$$\psi_{22}(x,\lambda)=\begin{cases} m_2a_2\lambda^{N_2+L_2+2}\sin\lambda(w_2-b)\cos\lambda(x-w_2) \\ +o\left(|\lambda|^{N_2+L_2+2}\exp|\operatorname{Im}\lambda|\left[(w_2-b)+(x-w_2)\right]\right)\end{cases}$$

$$\psi_{31}(x,\lambda)=\begin{cases} -m_1m_2a_2\lambda^{N_2+L_1+L_2+3}\sin\lambda(w_2-b)\sin\lambda(w_1-w_2)\sin\lambda(x-w_1) \\ \\ +o\left(|\lambda|^{N_2+L_1+L_2+3}\exp|\operatorname{Im}\lambda|\left[(w_2-b)+(w_1-w_2)+(x-w_2)\right]\right)\end{cases}$$

$$\psi_{32}(x,\lambda)=\begin{cases} m_1m_2a_2\lambda^{N_2+L_1+L_2+3}\sin\lambda(w_2-b)\sin\lambda(w_1-w_2)\cos\lambda(x-w_1) \\ +o\left(|\lambda|^{N_2+L_1+L_2+3}\exp|\operatorname{Im}\lambda|\left[(w_2-b)+(w_1-w_2)+(x-w_1)\right]\right)\end{cases}$$

Theorem 2. The eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ of problem L are real numbers.
Proof. It is enough to prove that eigenvalues of operator T are real. By using inner product (7), for Y in $D(T)$, we compute that

$$\langle TY,Y\rangle=\int_a^b l y\overline{y}dx-\frac{1}{a_1}TY_{N_1+1}\overline{Y_{N_1+1}}+\frac{1}{a_2}TL_{N_2+1}\overline{L_{N_2+1}}$$
$$+\frac{\alpha_1}{m_1}TR_{P_1+1}\overline{R_{P_1+1}}+\frac{\alpha_2}{m_2}TV_{P_2+1}\overline{V_{P_2+1}}-\sum_{k=1}^{N_1}TY_k\overline{Y_k}\left(\frac{1}{f_{1k}}\right)+\sum_{k=1}^{N_2}TL_k\overline{L_k}\left(\frac{1}{f_{2k}}\right)$$
$$+\sum_{k=1}^{P_1}\alpha_1TR_k\overline{R_k}\left(\frac{1}{u_{1k}}\right)+\sum_{k=1}^{P_2}\alpha_2TV_k\overline{V_k}\left(\frac{1}{u_{2k}}\right).$$

If necessary arrangements are made, we get

$$\langle TY,Y\rangle=\int_a^b p(x)\left(|y_1|^2-|y_2|^2\right)dx+\int_a^b q(x)\,2\operatorname{Re}(y_2\overline{y_1})\,dx+b_1|y_1(a)|+\sum_{k=1}^{N_1}2\operatorname{Re}(Y_k\overline{y_1}(a))$$
$$-b_2|y_1(b)|^2-\sum_{k=1}^{N_2}2\operatorname{Re}(L_k\overline{y_1}(b))-a_1n_1\left|y_1\left(w_1^-\right)\right|^2-\sum_{k=1}^{P_1}a_1 2\operatorname{Re}\left(R_ky_1\left(w_1^-\right)\right)$$
$$-a_2n_2\left|y_1\left(w_2^-\right)\right|^2-\sum_{k=1}^{P_2}a_2 2\operatorname{Re}\left(V_ky_1\left(w_2^-\right)\right)-\sum_{k=1}^{N_1}g_{1k}|Y_k|^2\frac{1}{f_{1k}}+\sum_{k=1}^{N_2}\frac{g_{2k}}{f_{2k}}|L_k|^2$$

$$+\sum_{k=1}^{P_1} a_1 \frac{t_{1k}}{u_{1k}} |R_k|^2 + \sum_{k=1}^{P_2} a_2 \frac{t_{2k}}{u_{2k}} |V_k|^2 - \int_a^b 2\,\mathrm{Re}\left(y_2 \overline{y_1}'\right) dx.$$

Accordingly, since $\langle TY, Y\rangle$ is real for each Y in $D(T)$, $\lambda \in \mathbb{R}$ is obtained. $\square$

Lemma 2. The equality $\|Y_n\|^2 = \rho_n$ is valid such that Y_n is eigenvector corresponding to eigenvalue λ_n of T.

Proof. Let $\lambda_n \neq g_{ik}$. When $\lambda_n = g_{ik}$, following proof is done with minor changes. By using the structure of $D(T)$ and the Eqs (8)–(12), we get

$$\begin{aligned} \|Y_n\|^2 = & \int_a^b \left(\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right) dx \\ & - \frac{\left|Y_{N_1+1}\right|^2}{a_1} + \frac{\left|L_{N_2+1}\right|^2}{a_2} + \frac{\alpha_1}{m_1}\left|R_{P_1+1}\right|^2 \\ & + \frac{\alpha_2}{m_2}\left|V_{P_2+1}\right|^2 - \sum_{k=1}^{N_1} \frac{|Y_k|^2}{f_{1k}} \\ & + \sum_{k=1}^{N_2} \frac{|L_k|^2}{f_{2k}} + \sum_{k=1}^{P_1} \frac{\alpha_1}{u_{1k}} |R_k|^2 + \sum_{k=1}^{P_2} \frac{\alpha_2}{u_{2k}} |V_k|^2 \end{aligned} \tag{14}$$

$$= \int_a^b \left(\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right) dx - a_1\varphi_1^2(a,\lambda_n) + a_2\varphi_1^2(b,\lambda_n) + m_1\alpha_1\varphi_1^2(w_1-0,\lambda_n)$$

$$+m_2\alpha_2\varphi_1^2(w_2-0,\lambda_n) - \sum_{k=1}^{N_1} \frac{f_{1k}\varphi_1^2(a,\lambda_n)}{(\lambda_n - g_{1k})^2} + \sum_{k=1}^{N_2} \frac{f_{2k}\varphi_1^2(b,\lambda_n)}{(\lambda_n - g_{2k})^2}$$

$$+\sum_{k=1}^{P_1} \frac{\alpha_1 u_{1k}\varphi_1^2(w_1-0,\lambda_n)}{(\lambda_n - t_{1k})^2} + \sum_{k=1}^{P_2} \frac{\alpha_2 u_{2k}\varphi_1^2(w_2-0,\lambda_n)}{(\lambda_n - t_{2k})^2}$$

$$= \int_a^b \left(\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right) dx - \varphi_1^2(a,\lambda_n)\left(a_1 + \sum_{k=1}^{N_1} \frac{f_{1k}}{(\lambda_n - g_{1k})^2}\right)$$

$$+\varphi_1^2(b,\lambda_n)\left(a_2 + \sum_{k=1}^{N_2} \frac{f_{2k}}{(\lambda_n - g_{2k})^2}\right) + \alpha_1\varphi_1^2(w_1-0,\lambda_n)\left(m_1 + \sum_{k=1}^{P_1} \frac{u_{1k}}{(\lambda_n - t_{1k})^2}\right)$$

$$+\alpha_2\varphi_1^2(w_2-0,\lambda_n)\left(m_2 + \sum_{k=1}^{P_2} \frac{u_{2k}}{(\lambda_n - t_{2k})^2}\right)$$

$$= \int_a^b \left(\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right) dx - \varphi_1^2(a,\lambda_n) f_1'(\lambda_n) + \varphi_1^2(b,\lambda_n) f_2'(\lambda_n)$$

$$+\alpha_1\varphi_1^2\left(w_1^-,\lambda_n\right) h_1'(\lambda_n) + \alpha_2\varphi_1^2\left(w_2^-,\lambda_n\right) h_2'(\lambda_n) = \rho_n. \quad \square$$

On the other hand, the expression

$$W(\varphi,\psi) = \varphi_1(x,\lambda)\psi_2(x,\lambda) - \varphi_2(x,\lambda)\psi_1(x,\lambda)$$

is called characteristic function of problem (1)-(4). Moreover, since solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ satisfy the problem L,
for $\forall x \in [a,b]$

$$\begin{aligned}
&\frac{\partial}{\partial x} W(\varphi,\psi) \\
&= \varphi_1'(x,\lambda)\psi_2(x,\lambda) + \psi_2'(x,\lambda)\varphi_1(x,\lambda) - \varphi_2'(x,\lambda)\psi_1(x,\lambda) - \psi_1'(x,\lambda)\varphi_2(x,\lambda) \\
&= \left[q(x)\varphi_1(x,\lambda) - p(x)\varphi_2(x,\lambda) - \lambda\varphi_2(x,\lambda)\right]\psi_2(x,\lambda) \\
&+ \left[-p(x)\psi_1(x,\lambda) - q(x)\psi_2(x,\lambda) + \lambda\psi_1(x,\lambda)\right]\varphi_1(x,\lambda) \\
&- \left[-p(x)\varphi_1(x,\lambda) - q(x)\varphi_2(x,\lambda) + \lambda\varphi_1(x,\lambda)\right]\psi_1(x,\lambda) \\
&- \left[q(x)\psi_1(x,\lambda) - p(x)\psi_2(x,\lambda) - \lambda\psi_2(x,\lambda)\right]\varphi_2(x,\lambda) = 0
\end{aligned}$$

is obtained. Furthermore, since solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ also satisfy transmission conditions (4), we get

$$\begin{aligned}
W(w_i+0) &= \varphi_1(w_i+0,\lambda)\psi_2(w_i+0,\lambda) - \varphi_2(w_i+0,\lambda)\psi_1(w_i+0,\lambda) \\
&= \alpha_i\varphi_1(w_i-0,\lambda)\left[\alpha_i^{-1}\psi_2(w_i-0,\lambda) + h_i(\lambda)\psi_1(w_i-0,\lambda)\right] \\
&- \left[\alpha_i^{-1}\varphi_2(w_i-0,\lambda) + h_i(\lambda)\varphi_1(w_i-0,\lambda)\right]\alpha_i\psi_1(w_i-0,\lambda) \\
&= \varphi_1(w_i-0,\lambda)\psi_2(w_i-0,\lambda) - \varphi_2(w_i-0,\lambda)\psi_1(w_i-0,\lambda) \\
&= W(w_i-0).
\end{aligned}$$

Therefore, since characteristic function $W(\varphi,\psi)$ is independent from x,

$$\begin{aligned}
&W\{\varphi,\psi\} := \Delta(\lambda) \\
&= \varphi_1(x,\lambda)\psi_2(x,\lambda) - \varphi_2(x,\lambda)\psi_1(x,\lambda) \\
&= a_2(\lambda)\varphi_1(b,\lambda) + b_2(\lambda)\varphi_2(b,\lambda) \\
&= -b_1(\lambda)\psi_2(a,\lambda) - a_1(\lambda)\psi_1(a,\lambda)
\end{aligned}$$

can be written.

It is clear that $\Delta(\lambda)$ is an entire function and its zeros namely $\{\lambda_n\}_{n\in\mathbb{Z}}$coincide with the eigenvalues of the problem L.

Accordingly, for each eigenvalue λ_n equality $\psi(x,\lambda_n) = s_n\varphi(x,\lambda_n)$ is valid where $s_n = \dfrac{\psi_1(a,\lambda_n)}{-b_1(\lambda_n)} = \dfrac{\psi_2(a,\lambda_n)}{a_1(\lambda_n)}$.

On the other hand, since $a_i(g_{ik}) \neq 0$ ve $b_i(g_{ik}) = 0$ for $\forall i \in \{1,2\}$ and $k = \{1,2,\ldots,N_i\}$, g_{ik} is an eigenvalue if and only if $\varphi_1(b,g_{2k}) = 0$, $\varphi_1(a,g_{1k}) = 0$ i.e., $\Delta(g_{ik}) = 0$.

At the same time, t_{ik} is an eigenvalue if and only if $\varphi_1(w_i^-,t_{ik}) = 0 = \varphi_1(w_i^+,t_{ik})$ i.e., $\Delta(t_{ik}) = 0$ such that $i = 1,2$ and $k = \{1,2,\ldots P_i\}$.

Theorem 3. Eigenvalues of problem L are simple.

Proof. Let $\lambda_n \neq g_{ik}$ and $\varphi(x,\lambda_n)$ be eigenfunction corresponds to the eigenvalue λ_n. In that case, the Eq (1) can be written for $\psi(x,\lambda)$ and $\varphi(x,\lambda_n)$ as follows;

$$\begin{aligned}
&B\psi'(x,\lambda) + Q(x)\psi(x,\lambda) = \lambda\psi(x,\lambda) \\
&B\varphi'(x,\lambda_n) + Q(x)\varphi(x,\lambda_n) = \lambda_n\varphi(x,\lambda_n).
\end{aligned}$$

If we multiply these equations by $\varphi(x,\lambda_n)$ and $\psi(x,\lambda)$ respectively and add side by side, we get the following equality;

$$\left(\psi_2(x,\lambda)\varphi_1(x,\lambda_n)-\psi_1(x,\lambda)\varphi_2(x,\lambda_n)\right)' = (\lambda-\lambda_n)\left(\psi_1(x,\lambda)\varphi_1(x,\lambda_n)+\psi_2(x,\lambda)\varphi_2(x,\lambda_n)\right).$$

Then if last equality is integrated over the interval $[a,b]$ and the initial conditions (13) and transmission conditions (4) are used to get

$$\begin{aligned}
&\int_a^b \left(\psi_1(x,\lambda)\varphi_1(x,\lambda_n)+\psi_2(x,\lambda)\varphi_2(x,\lambda_n)\right)dx\\
&+\alpha_2\varphi_1\left(w_2^-,\lambda_n\right)\psi_1\left(w_2^-,\lambda\right)\frac{h_2(\lambda)-h_2(\lambda_n)}{\lambda-\lambda_n}\\
&+\alpha_1\psi_1\left(w_1^-,\lambda\right)\varphi_1\left(w_1^-,\lambda_n\right)\frac{h_1(\lambda)-h_1(\lambda_n)}{\lambda-\lambda_n}\\
&+\psi_1(b,\lambda_n)\varphi_1(b,\lambda_n)\frac{f_2(\lambda)-f_2(\lambda_n)}{\lambda-\lambda_n}\\
&-\varphi_1(a,\lambda_n)\psi_1(a,\lambda)\frac{f_1(\lambda)-f_1(\lambda_n)}{\lambda-\lambda_n}\\
&=-\left(\frac{\Delta(\lambda)-\Delta(\lambda_n)}{(\lambda-\lambda_n)}\right).
\end{aligned}$$

Then, considering that $\psi(x,\lambda_n)=s_n\varphi(x,\lambda_n)$
if the limit is passed when $\lambda\to\lambda_n$, $s_n\rho_n=-\dot{\Delta}(\lambda_n)$ is obtained.

If g_{1k} and g_{2k} are non-simple eigenvalues then $\varphi_1(a,g_{1k})=0$, $\varphi_1(b,g_{2k})=0$ and so $\int_a^b\left(\varphi_1^2(x,\lambda_n)+\varphi_2^2(x,\lambda_n)\right)dx=-\left[\alpha_1\varphi_1^2\left(w_1^-,\lambda_n\right)h_1'(\lambda_n)+\alpha_2\varphi_1^2\left(w_2^-,\lambda_n\right)h_2'(\lambda_n)\right]$ is obtained. Since α_1, α_2 and for all λ_n, $h_1'(\lambda_n)$, $h_2'(\lambda_n)$ are positive, we have a contradiction. Therefore, eigenvalues g_{ik} are also simple. □

3. Inverse problem

Using expressions $a_2(\lambda)$, $b_2(\lambda)$ and asymptotic behaviour of solution $\varphi(x,\lambda)$, we obtain the following asymptotic of characteristic function $\Delta(\lambda)$ as $|\lambda|\to\infty$;$\Delta(\lambda)=-a_1a_2m_1m_2\lambda^{N_1+N_2+L_1+L_2+4}\sin\lambda(w_1-a)\sin\lambda(w_2-w_1)\sin\lambda(b-w_2)+o\left(|\lambda|^{N_1+N_2+L_1+L_2+4}e^{|\operatorname{Im}\lambda|(b-a)}\right)$.

Let $\Phi(x,\lambda):=\begin{pmatrix}\Phi_1(x,\lambda)\\ \Phi_2(x,\lambda)\end{pmatrix}$ be the solution of Eq (1) under the conditions $U(\Phi)=1$, $V(\Phi)=0$ as well as the transmission conditions (4).

Since $V(\Phi)=0=V(\psi)$, it can be supposed that $\Phi(x,\lambda)=k\psi(x,\lambda)$ $(k\neq 0)$ where k is a constant.

$$\begin{aligned}
W(\varphi,\Phi) &= \left.\varphi_1(x,\lambda)\Phi_2(x,\lambda)-\varphi_2(x,\lambda)\Phi_1(x,\lambda)\right|_{x=a}\\
&=-b_1(\lambda)\Phi_2(a,\lambda)-a_1(\lambda)\Phi_1(a,\lambda)\\
&=-U(\Phi)=-1.
\end{aligned}$$

By the relation $U(\Phi)=1$, we get $k\left[b_1(\lambda)\psi_2(a,\lambda)+a_1(\lambda)\psi_1(a,\lambda)\right]=1$. Since $U(\psi)=-\Delta(\lambda)$, we obtain $\Phi(x,\lambda)=k\psi(x,\lambda)=-\dfrac{\psi(x,\lambda)}{\Delta(\lambda)}$ for $\lambda\neq\lambda_n$.

Let $S(x,\lambda)=\begin{pmatrix} S_1(x,\lambda) \\ S_2(x,\lambda)\end{pmatrix}$ and $C(x,\lambda)=\begin{pmatrix} C_1(x,\lambda) \\ C_2(x,\lambda)\end{pmatrix}$ be solutions of (1) satify the conditions $S(a,\lambda)=\begin{pmatrix} 0 \\ 1\end{pmatrix}$, $C(a,\lambda)=\begin{pmatrix} 1 \\ 0\end{pmatrix}$ and transmission conditions (4).

Accordingly,the following equalities are obtained:

$$\varphi_1(x,\lambda)=-b_1(\lambda)C(x,\lambda)+a_1(\lambda)S(x,\lambda) \tag{15}$$

$$\Phi(x,\lambda)=\frac{1}{b_1(\lambda)}\left(S(x,\lambda)-\Phi_1(a,\lambda)\varphi(x,\lambda)\right). \tag{16}$$

The function $\Phi(x,\lambda)$ is called Weyl solution and the function $M(\lambda)=-\Phi_1(a,\lambda)$ is called Weyl function of problem L. Therefore, since $\Phi(x,\lambda)=-\dfrac{\psi(x,\lambda)}{\Delta(\lambda)}$, we set $M(\lambda):=\dfrac{\psi_1(a,\lambda)}{\Delta(\lambda)}$.

Consider the boundary value problem $\tilde{L}$ in the same form with L but different coefficients. Here, the expressions related to the L problem are shown with s and the ones related to $\tilde{L}$ are shown with $\tilde{s}$. According to this statement, we set the problem $\tilde{L}$ as follows:

$$\tilde{\ell}[y(x)]:=By'(x)+\tilde{Q}(x)y(x)=\lambda y(x),\ x\in[a,b]$$

$$\begin{aligned}\tilde{U}(y)&:=y_2(a)+\tilde{f}_1(\lambda)y_1(a)=0\\ \tilde{V}(y)&:=y_2(b)+\tilde{f}_2(\lambda)y_1(b)=0\\ y_1(w_i+0)&=\tilde{\alpha}_i y_1(w_i-0)\\ y_2(w_i+0)&=-\tilde{\alpha}_i^{-1}y_2(w_i-0)+\tilde{h}_i(\lambda)y_1(w_i-0)\end{aligned}$$

where $\tilde{Q}(x)=\begin{pmatrix}\tilde{p}(x) & q(x)\\ q(x) & -\tilde{p}(x)\end{pmatrix}$.

Theorem 4. If $M(\lambda)=\tilde{M}(\lambda)$, $f_1(\lambda)=\tilde{f}_1(\lambda)$, then $Q(x)=\tilde{Q}(x)$ almost everywhere in (a,b), $f_2(\lambda)=\tilde{f}_2(\lambda)$, $h_i(\lambda)=\tilde{h}_i(\lambda)$,and $\alpha_i(\lambda)=\tilde{\alpha}_i(\lambda)$ $(i=1,2)$.

Proof. Introduce a matrix $P(x,\lambda)=\left[P_{ij}(x,\lambda)\right]_{i,j=1,2}$ by the equality as follows;

$$\begin{pmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix}\begin{pmatrix}\tilde{\varphi}_1 & \tilde{\Phi}_1\\ \tilde{\varphi}_2 & \tilde{\Phi}_2\end{pmatrix}=\begin{pmatrix}\varphi_1 & \Phi_1\\ \varphi_2 & \Phi_2\end{pmatrix}.$$

According to this, we get

$$\begin{aligned}P_{11}(x,\lambda)&=-\varphi_1(x,\lambda)\tilde{\Phi}_2(x,\lambda)+\Phi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda)\\ P_{12}(x,\lambda)&=-\tilde{\varphi}_1(x,\lambda)\Phi_1(x,\lambda)+\varphi_1(x,\lambda)\tilde{\Phi}_1(x,\lambda)\\ P_{21}(x,\lambda)&=-\varphi_2(x,\lambda)\tilde{\Phi}_2(x,\lambda)+\Phi_2(x,\lambda)\tilde{\varphi}_2(x,\lambda)\\ P_{22}(x,\lambda)&=-\tilde{\varphi}_1(x,\lambda)\Phi_2(x,\lambda)+\varphi_2(x,\lambda)\tilde{\Phi}_1(x,\lambda)\end{aligned} \tag{17}$$

or by using the relation $\Phi(x,\lambda)=-\dfrac{\psi(x,\lambda)}{\Delta(\lambda)}$,

we obtain

$$P_{11}(x,\lambda)=\varphi_1(x,\lambda)\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)}-\tilde{\psi}_2(x,\lambda)\frac{\psi_1(x,\lambda)}{\Delta(\lambda)}$$

$$P_{12}(x,\lambda) = -\varphi_1(x,\lambda)\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)} + \tilde{\varphi}_1(x,\lambda)\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} \tag{18}$$
$$P_{21}(x,\lambda) = \varphi_2(x,\lambda)\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)} - \tilde{\varphi}_2(x,\lambda)\frac{\psi_2(x,\lambda)}{\Delta(\lambda)}$$
$$P_{22}(x,\lambda) = \tilde{\varphi}_1(x,\lambda)\frac{\psi_2(x,\lambda)}{\Delta(\lambda)} - \varphi_2(x,\lambda)\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}.$$

Taking into account the Eqs (15) and (16) and $M(\lambda) = \tilde{M}(\lambda)$, we can easily get

$$\begin{aligned} P_{11}(x,\lambda) &= C_1(x,\lambda)\tilde{S}_2(x,\lambda) - S_1(x,\lambda)\tilde{C}_2(x,\lambda) \\ P_{12}(x,\lambda) &= \tilde{C}_1(x,\lambda)S_1(x,\lambda) - C_1(x,\lambda)\tilde{S}_1(x,\lambda) \\ P_{21}(x,\lambda) &= C_2(x,\lambda)\tilde{S}_2(x,\lambda) - S_2(x,\lambda)\tilde{C}_2(x,\lambda) \\ P_{22}(x,\lambda) &= \tilde{C}_1(x,\lambda)S_2(x,\lambda) - C_2(x,\lambda)\tilde{S}_1(x,\lambda). \end{aligned}$$

Hence, the functions $P_{ij}(x,\lambda)$ are entire in λ.
Denote

$G_\delta := \{\lambda : |\lambda-\lambda_n| \geq \delta,\ n = 0, \pm1, \pm2, \ldots\}, \delta > 0$ and

$\tilde{G}_\delta := \left\{\lambda : \left|\lambda-\tilde{\lambda}_n\right| \geq \delta,\ n = 0, \pm1, \pm2, \ldots\right\}$ where $\delta > 0$ is sufficiently small and fixed.
Clearly, for $\lambda \in G_\delta \cap \tilde{G}_\delta$, $|\sin\lambda x| \geq C_\delta e^{|\mathrm{Im}\,\lambda| x}$, $|\lambda| \to \infty$.
Therefore, $|\Delta(\lambda)| \geq C_\delta \lambda^{N_1+N_2+L_1+L_2+4} e^{|\mathrm{Im}\,\lambda|(b-a)}$, $\lambda \in G_\delta \cap \tilde{G}_\delta$, $|\lambda| \geq \lambda^*$ for sufficiently large $\lambda^* = \lambda^*(\delta)$

and from (18) we see that $P_{ij}(x,\lambda)$ are bounded with respect to λ where $\lambda \in G_\delta \cap \tilde{G}_\delta$ and $|\lambda|$ sufficiently large. From Liouville's theorem, it is obtained that these functions do not depend on λ.

On the other hand, from (18)

$$P_{11}(x,\lambda) - 1 = \varphi_1(x,\lambda)\left(\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\right) - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\left(\tilde{\varphi}_2(x,\lambda) - \varphi_2(x,\lambda)\right)$$

$$P_{12}(x,\lambda) = \tilde{\varphi}_1(x,\lambda)\left(\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right) - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\left(\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda)\right)$$

$$P_{21}(x,\lambda) = \varphi_2(x,\lambda)\left(\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\right) - \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\left(\tilde{\varphi}_2(x,\lambda) - \varphi_2(x,\lambda)\right)$$
$$P_{22}(x,\lambda) - 1 = \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\left(\tilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda)\right) - \varphi_2(x,\lambda)\left(\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right).$$

If it is considered that $P_{ij}(x,\lambda)$ do not depend on λ and asymptotic formulas of solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, we obtain

$$\lim_{\lambda\to-\infty} \varphi_1(x,\lambda)\left(\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\right) = 0,$$
$$\lim_{\lambda\to-\infty} \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\left(\tilde{\varphi}_2(x,\lambda) - \varphi_2(x,\lambda)\right) = 0$$

for all x in $[a,b]$. Hence, $\lim\limits_{\lambda\to-\infty}[P_{11}(x,\lambda) - 1] = 0$.
Thus, $P_{11}(x,\lambda) = 1$ and similarly, $P_{22}(x,\lambda) = 1$ and $P_{12}(x,\lambda) = P_{21}(x,\lambda) = 0$.

Substitute these relations in (17), to obtain

$$\varphi_1(x,\lambda) = \tilde{\varphi}_1(x,\lambda),\ \frac{\psi_1(x,\lambda)}{\Delta(\lambda)} = \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}$$
$$\varphi_2(x,\lambda) = \tilde{\varphi}_2(x,\lambda),\ \frac{\psi_2(x,\lambda)}{\Delta(\lambda)} = \frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)} \text{ for all } x \text{ and } \lambda.$$

Taking into account these results and Eq (1), we have

$$\left(Q(x) - \tilde{Q}(x)\right)\varphi(x,\lambda) = 0.$$

Therefore, $Q(x) = \tilde{Q}(x)$ i.e., $p(x) = \tilde{p}(x)$. Moreover, it is considered that
$\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} = \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)},\ \frac{\psi_2(x,\lambda)}{\Delta(\lambda)} = \frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)}$
and
$b_2(\lambda)\psi_2(x,\lambda) + a_2(\lambda)\psi_1(x,\lambda) = 0$
$\tilde{b}_2(\lambda)\tilde{\psi}_2(x,\lambda) + \tilde{a}_2(\lambda)\tilde{\psi}_1(x,\lambda) = 0$
we get $a_2(\lambda)\tilde{b}_2(\lambda) - b_2(\lambda)\tilde{a}_2(\lambda) = 0$. As we have said above, $a_2(\lambda)$, $b_2(\lambda)$ as well as $\tilde{a}_2(\lambda)$, $\tilde{b}_2(\lambda)$ do not have common zeros. Hence, $a_2(\lambda) = \tilde{a}_2(\lambda)$, $b_2(\lambda) = \tilde{b}_2(\lambda)$, i.e., $f_2(\lambda) = \tilde{f}_2(\lambda)$.

On the other hand, substituting φ_1 and φ_2 into transmission conditions (4), we get
$\varphi_1(w_i^+,\lambda) = \alpha_i\varphi_1(w_i^-,\lambda),\ \tilde{\varphi}_1(w_i^+,\lambda) = \tilde{a}_i\tilde{\varphi}_1(w_i^-,\lambda)$
$\varphi_2(w_i^+,\lambda) = \alpha_i^{-1}\varphi_2(w_1^-,\lambda) + h_i(\lambda)\varphi_1(w_i^-,\lambda),$
$\tilde{\varphi}_2(w_i^+,\lambda) = \tilde{\alpha}_i^{-1}\tilde{\varphi}_2(w_i^-,\lambda) + \tilde{h}_i(\lambda)\tilde{\varphi}_1(w_i^-,\lambda),\ i = 1,2.$
Therefore, since $\varphi_1(x,\lambda) = \tilde{\varphi}_1(x,\lambda)$, $\varphi_2(x,\lambda) = \tilde{\varphi}_2(x,\lambda)$, these yield that $\alpha_1 = \tilde{\alpha}_1$, $\alpha_2 = \tilde{\alpha}_2$ and $h_1(\lambda) = \tilde{h}_1(\lambda)$, $h_2(\lambda) = \tilde{h}_2(\lambda)$. □

Theorem 5. If $\{\lambda_n,\rho_n\}_{n\in\mathbb{Z}} = \left\{\tilde{\lambda}_n,\tilde{\rho}_n\right\}_{n\in\mathbb{Z}}$, $f_1(\lambda) = \tilde{f}_1(\lambda)$ then $Q(x) = \tilde{Q}(x)$ almost everywhere in (a,b), $f_2(\lambda) = \tilde{f}_2(\lambda)$, $h_i(\lambda) = \tilde{h}_i(\lambda)$,and $\alpha_i(\lambda) = \tilde{\alpha}_i(\lambda)$ $(i = 1,2)$.

Proof. Since $\lambda_n = \tilde{\lambda}_n$, $\Delta(\lambda) = c\tilde{\Delta}(\lambda)$. On the other hand, also since $s_n\rho_n = -\dot{\Delta}(\lambda_n)$ and $\rho_n = \tilde{\rho}_n$, we get that $s_n = c\tilde{s}_n$. Therefore, $\psi_1(a,\lambda_n) = c\tilde{\psi}_1(a,\lambda_n)$ is obtained.

Denote $H(\lambda) := \dfrac{\psi_1(a,\lambda) - c\tilde{\psi}_1(a,\lambda)}{\Delta(\lambda)}$ which is an entire function in λ. Since $\lim_{|\lambda|\to\infty} H(\lambda) = 0$, $H(\lambda) \equiv 0$ and so $\psi_1(a,\lambda) = c\tilde{\psi}_1(a,\lambda)$. Hence, $M(\lambda) = \tilde{M}(\lambda)$. As a result, the proof of theorem is finished by Theorem 4. □

We examine the boundary value problem L_1 with the condition $y_1(a) = 0$ instead of (2) in problem L. Let $\{\mu_n\}_{n\in\mathbb{Z}}$ be eigenvalues of the problem L_1. It is clear that $\{\mu_n\}_{n\in\mathbb{Z}}$ are zeros of $\Delta_1(\mu) := -\psi_1(a,\mu)$.

Theorem 6. If $\{\lambda_n,\mu_n\}_{n\in\mathbb{Z}} = \left\{\tilde{\lambda}_n,\tilde{\mu}_n\right\}_{n\in\mathbb{Z}}$, $f_1(\lambda) = \tilde{f}_1(\lambda)$ and $K = \tilde{K}$ such that $K = a_2m_1m_2$, $\tilde{K} = \tilde{a}_2\tilde{m}_1\tilde{m}_2$ then $Q(x) = \tilde{Q}(x)$ almost everywhere in (a,b), $f_2(\lambda) = \tilde{f}_2(\lambda)$, $h_i(\lambda) = \tilde{h}_i(\lambda)$,and $\alpha_i(\lambda) = \tilde{\alpha}_i(\lambda)$ $(i = 1,2)$.

Proof. Since for all $n \in \mathbb{Z}$, $\lambda_n = \widetilde{\lambda}_n$ and $\mu_n = \widetilde{\mu}_n$, $\dfrac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)}$ and $\dfrac{\Delta_1(\mu)}{\widetilde{\Delta}_1(\mu)}$ are entire functions in λ and in μ respectively. On the other hand, taking into account the asymptotic behaviours of $\Delta(\lambda)$, $\Delta_1(\mu)$ and $K = \tilde{K}$, we obtain $\lim_{\lambda\to-\infty}\dfrac{\Delta(\lambda)}{\tilde{\Delta}(\lambda)} = 1$ and $\lim_{\mu\to-\infty}\dfrac{\Delta_1(\mu)}{\tilde{\Delta}_1(\mu)} = 1$. Therefore, since $\lambda_n = \widetilde{\lambda}_n$ and $\mu_n = \widetilde{\mu}_n$, we get $\Delta(\lambda) = \tilde{\Delta}(\lambda)$ and $\Delta_1(\mu) = \tilde{\Delta}_1(\mu)$. If we consider the case $\Delta_1(\mu) = \tilde{\Delta}_1(\mu)$, then $\psi_1(a,\mu) = \tilde{\psi}_1(a,\mu)$

is obtained. Furthermore, since $M(\lambda) = \dfrac{\psi_1(a,\lambda)}{\Delta(\lambda)}$, $M(\lambda) = \tilde{M}(\lambda)$. Hence, the proof is completed by Theorem 4. □

4. Conclusions

The purpose of this paper is to state and prove some uniqueness theorems for Dirac equations with boundary and transmission conditions depending rational function of Herglotz-Nevanlinna. Accordingly, it has been proved that while $f_1(\lambda)$ in condition (2) is known, the coefficients of the boundary value problem (1)-(4) can be determined uniquely by each of the following;

i) The Weyl function $M(\lambda)$

ii) Spectral data $\{\lambda_n, \rho_n\}$ forming eigenvalues and normalizing constants respectively

iii) Two given spectra $\{\lambda_n, \mu_n\}$

These results are the application of the classical uniqueness theorems of Marchenko, Gelfand, Levitan and Borg to such Dirac equations. Considering this study, similar studies can be made for classical Sturm-Liouville operators, the system of Dirac equations and diffusion operators with finite number of transmission conditions depending spectral parameter as Herglotz-Nevanlinna function.

Conflict of interest

There is no conflict of interest.

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