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Research article

Estimates of upper bound for differentiable mappings related to Katugampola fractional integrals and p-convex mappings

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Abstract: We use the definition of a fractional integral operators, recently introduced by Katugampola, to establish a parameterized identity associated with differentiable mappings. The identity is then used to derive the estimates of upper bound for mappings whose first derivatives absolute values are p-convex mappings. Four examples are also provided to illustrate the obtained results.

Keywords: Riemann–Liouville fractional integrals; Hadamard fractional integrals; p-convex mappings
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1. Introduction

In 1938, Ostrowski proved an important integral inequality which provides an upper bound for difference between the value $F(x)$ and mean value of F for mappings whose derivatives' absolute values are bounded, which can be seen in [31] as the following statement.

Theorem 1. *Let $F : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ ($\mathcal{I}^\circ$ be the interior of $\mathcal{I}$) and $a, b \in \mathcal{I}$ with $a < b$. If $|F'(x)| \leq M$, then, for all $x \in [a, b]$, the following inequality holds:*

$$\left|F(x) - \frac{1}{b-a}\int_a^b F(\tau)\mathrm{d}\tau\right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right]. \tag{1.1}$$

Here, $\frac{1}{4}$ is the best possible constant.

We now start our discussion by evoking certain important concepts and the related results.

Definition 1. *A mapping* $F : \emptyset \neq \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ *is named convex on* $\mathcal{I}$*, if the following inequality*

$$F(t\tau_1 + (1-t)\tau_2) \leq tF(\tau_1) + (1-t)F(\tau_2)$$

holds for all $\tau_1, \tau_2 \in \mathcal{I}$ *and* $t \in [0,1]$.

Definition 2. [16] *Let* $\mathcal{I} \subseteq (0,\infty)$ *be a real interval and* $p \in \mathbb{R}\backslash\{0\}$. *A mapping* $F : \mathcal{I} \to \mathbb{R}$ *is said to be a p-convex mapping, if the following inequality*

$$F\Big(\big[t\tau_1^p + (1-t)\tau_2^p\big]^{\frac{1}{p}}\Big) \leq tF(\tau_1) + (1-t)F(\tau_2)$$

holds for all $\tau_1, \tau_2 \in \mathcal{I}$ *and* $t \in [0,1]$.

Many inequalities and properties for p-convex mappings have been worked by a lot of researchers. For example, Zhang and Wan [44] gave certain properties involving p-convex mappings. Noor et al. [29] presented several Hermite–Hadamard's inequalities by means of p-convexity. İşcan et al. [18] provided some Hermite–Hadamard's inequalities via p-quasi-convexity. Further inequalities of the Hermite–Hadamard type associated with p-convexity in question with applications to fractional integrals can be found in [22, 23, 39]. For more results related to p-convex mappings, please see, for example, [24, 28, 30] and the references cited therein.

Also, we recall the following fractional integral operators, which are essential to our current work.

Definition 3. [32] *Let* $F \in L^1([a,b])$. *The Riemann–Liouville integrals* $\mathcal{J}^{\mu}_{a^+}F$ *and* $\mathcal{J}^{\mu}_{b^-}F$ *of order* $\mu > 0$ *are defined by*

$$\mathcal{J}^{\mu}_{a^+}F(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}F(t)\mathrm{d}t$$

and

$$\mathcal{J}^{\mu}_{b^-}F(x) = \frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1}F(t)\mathrm{d}t$$

with $a < x < b$ *and* $\Gamma(\mu) = \int_0^\infty e^{-t}t^{\mu-1}\mathrm{d}t$. *It is to be noted that* $\mathcal{J}^0_{a^+}F(x) = \mathcal{J}^0_{b^-}F(x) = F(x)$.

Definition 4. [33] *Let* $\alpha > 0$ *with* $a < x < b$. *The left-hand side and right-hand side Hadamard fractional integral operators of order* α *of function* F *are given by*

$$\mathcal{H}^{\alpha}_{a^+}F(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1}\frac{F(t)}{t}\mathrm{d}t$$

and

$$\mathcal{H}^{\alpha}_{b^-}F(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1}\frac{F(t)}{t}\mathrm{d}t.$$

For recent results associated with Riemann–Liouville fractional integrals and Hadamard fractional integrals, the interested reader is referred, for example, to [1, 10, 42] and references therein.

In what follows, we review the space of all complex-valued Lebesgue measurable functions, which will be used subsequently.

Let $\chi^p_c(a,b)$ ($c \in \mathbb{R}, 1 \leq p \leq \infty$) be the space of all complex-valued Lebesgue measurable functions F on $[a,b]$ for which $\|F\|_{\chi^p_c} < \infty$, where the norm $\|\cdot\|_{\chi^p_c}$ is defined with the following expression:

$$\|F\|_{\chi^p_c} = \left(\int_a^b |t^cF(t)|^p\frac{\mathrm{d}t}{t}\right)^{1/p}, \quad (1 \leq p < \infty)$$

and

$$\|F\|_{\chi_c^\infty} = \operatorname*{ess\,sup}_{a<t<b}[t^c|F(t)|], \quad p = \infty,$$

where ess sup stands for essential supremum.

Definition 5. [20] *Let* $[a,b] \subset \mathbb{R}$ *be a finite interval. Then the left-hand side and right-hand side Katugampola fractional integrals of order* $\alpha > 0$ *of* $F \in \chi_c^p(a,b)$ *are defined by*

$$({}^\rho I_{a^+}^\alpha F)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^\rho - t^\rho)^{\alpha-1} t^{\rho-1} F(t)\mathrm{d}t$$

and

$$({}^\rho I_{b^-}^\alpha F)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^\rho - x^\rho)^{\alpha-1} t^{\rho-1} F(t)\mathrm{d}t$$

with $a < x < b$ *and* $\rho > 0$*, if the integrals exist.*

Theorem 2. [20] *Let* $\alpha > 0$ *and* $\rho > 0$*. Then, for* $x < b$*, we have*
(i) $\lim_{\rho\to 1}({}^\rho I_{b^-}^\alpha F)(x) = \mathcal{J}_{b^-}^\alpha F(x)$;
(ii) $\lim_{\rho\to 0^+}({}^\rho I_{b^-}^\alpha F)(x) = \mathcal{H}_{b^-}^\alpha F(x)$.
Similar results are also valid for left-sided operators.

It is undeniable that the Katugampola fractional integral operators has a great influence on pure science and applied science. Recently, the study of some well-known integral inequalities for the Katugampola fractional integrals has been set up by some authors, including Chen and Katugampola [8] and Jleli et al. [19] in the study of Hermite–Hadamard type inequalities for convex mappings, Kermausuor [21] in the study of the generalized Ostrowski type inequalities for strong (s,m)-convex mappings, Mumcu et al. [27] in the Hermite–Hadamard type inequalities for harmonically convex mappings, Sousa and Capelas de Oliveira [37] in the study of a generalization of the reverse Minkowski's inequality. In addition, some applications related to Katugampola fractional integrals can be found in [25] and [43]. For more results related to the Katugampola fractional integral operators, the interested reader is directed to [14, 15, 26, 38] and the references cited therein.

To obtain Ostrowski-type integral inequalities, many authors proved some interesting identities of Ostrowski type. Let's collate them as follows:

Lemma 1. [4] *Let* $F : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $\mathcal{I}^\circ$ *where* $a, b \in \mathcal{I}$ *with* $a < b$*. If* $F' \in L^1([a,b])$*, then, for each* $x \in [a,b]$*, we have the following equality:*

$$F(x) - \frac{1}{b-a}\int_a^b F(\tau)\mathrm{d}\tau = \frac{(x-a)^2}{b-a}\int_0^1 tF'(tx+(1-t)a)\mathrm{d}t - \frac{(b-x)^2}{b-a}\int_0^1 tF'(tx+(1-t)b)\mathrm{d}t. \quad (1.2)$$

Lemma 2. [36] *Let* $F : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$*. If* $F' \in L^1([a,b])$*, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}F(x) - \frac{\Gamma(\alpha+1)}{b-a}\left[\mathcal{J}_{b-}^\alpha F(a) + \mathcal{J}_{a+}^\alpha F(b)\right]\\ =&\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha F'(tx+(1-t)a)\mathrm{d}t - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha F'(tx+(1-t)b)\mathrm{d}t.\end{aligned} \quad (1.3)$$

Lemma 3. [17] *Let $F : \mathcal{I} \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ such that $F' \in L^1([a,b])$, where $a, b \in \mathcal{I}$ with $a < b$. Then, for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$, the following identity holds:*

$$\begin{aligned}\Delta_F(x,\lambda,\alpha,a,b) =& a\left(\ln\frac{x}{a}\right)^{\alpha+1}\int_0^1 (t^\alpha-\lambda)\left(\frac{x}{a}\right)^t F'\left(x^t a^{1-t}\right)\mathrm{d}t \\ &- b\left(\ln\frac{b}{x}\right)^{\alpha+1}\int_0^1 (t^\alpha-\lambda)\left(\frac{x}{b}\right)^t F'\left(x^t b^{1-t}\right)\mathrm{d}t,\end{aligned} \tag{1.4}$$

where

$$\begin{aligned}\Delta_F(x,\lambda,\alpha,a,b) = (1-\lambda)\Big[\ln^\alpha\frac{x}{a}+\ln^\alpha\frac{b}{x}\Big]F(x)+\lambda\Big[F(a)\ln^\alpha\frac{x}{a}+F(b)\ln^\alpha\frac{b}{x}\Big] \\ -\Gamma(\alpha+1)\left[\mathcal{H}^\alpha_{x^-}F(a)+\mathcal{H}^\alpha_{x^+}F(b)\right],\end{aligned}$$

$a, b \in \mathcal{I}$ with $a < b$, $x \in [a,b]$, $\lambda \in [0,1]$, $\alpha > 0$ and Γ is the Euler gamma function.

Lemma 4. [38] *Let $F : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $p > 0$. If $F' \in L^1([a,b])$, then the following equality holds:*

$$\begin{aligned}&-\frac{(x^p-a^p)^\alpha F(a)+(b^p-x^p)^\alpha F(b)}{p^\alpha(b-a)}+\frac{\Gamma(\alpha+1)}{b-a}\left[\left({}^pI^\alpha_{a+}F\right)(x)+\left({}^pI^\alpha_{b-}F\right)(x)\right] \\ &=\frac{(x^p-a^p)^{\alpha+1}}{p^{1+\alpha}(b-a)}\int_0^1 t^\alpha\left(ta^p+(1-t)x^p\right)^{\frac{1-p}{p}}F'\left(\sqrt[p]{ta^p+(1-t)x^p}\right)\mathrm{d}t \\ &\quad-\frac{(b^p-x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)}\int_0^1 t^\alpha\left(tb^p+(1-t)x^p\right)^{\frac{1-p}{p}}F'\left(\sqrt[p]{tb^p+(1-t)x^p}\right)\mathrm{d}t.\end{aligned} \tag{1.5}$$

Lemma 5. [13] *Let F be defined from interval $\mathcal{I}$ which consists of positive real numbers to $\mathbb{R}$ as a differentiable mapping on $\mathcal{I}^\circ$, where $a, b \in \mathcal{I}$ with $a < b$ and $F' \in L^1([a,b])$. Then, for all $x \in [a,b]$, $p > 0$ and $\alpha > 0$, the following equality holds:*

$$\begin{aligned}&\frac{p}{b-a}\Big[(x^p-a^p)^\alpha F(a)+(b^p-x^p)^\alpha F(b)\Big]-\frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}\Big[\left({}^pI^\alpha_{x^-}F\right)(a)+\left({}^pI^\alpha_{x^+}F\right)(b)\Big] \\ &=\frac{(x^p-a^p)^{\alpha+1}}{b-a}\int_0^1\frac{(t^\alpha-1)F'\left([tx^p+(1-t)a^p]^{\frac{1}{p}}\right)}{\left(tx^p+(1-t)a^p\right)^{1-\frac{1}{p}}}\mathrm{d}t \\ &\quad+\frac{(b^p-x^p)^{\alpha+1}}{b-a}\int_0^1\frac{(1-t^\alpha)F'\left([tx^p+(1-t)b^p]^{\frac{1}{p}}\right)}{\left(tx^p+(1-t)b^p\right)^{1-\frac{1}{p}}}\mathrm{d}t.\end{aligned} \tag{1.6}$$

We will also need the following Lemma and special functions in proving our results.

Lemma 6. [41] *For $\mathcal{A} \geq 0$, $\mathcal{B} \geq 0$, we have*

$$(\mathcal{A}+\mathcal{B})^\sigma \leq 2^{\sigma-1}(\mathcal{A}^\sigma+\mathcal{B}^\sigma),\ \sigma \geq 1 \tag{1.7}$$

and

$$(\mathcal{A}+\mathcal{B})^\sigma \leq \mathcal{A}^\sigma+\mathcal{B}^\sigma,\ 0<\sigma\leq 1. \tag{1.8}$$

(I) The beta function:

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,\quad x,y>0.$$

(II) The incomplete beta function:

$$\beta(a;x,y)=\int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t,\ \ 0<a<1,\ \ x,y>0.$$

(III) The hypergeometric function:

$${}_2\mathcal{F}_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t,\ \ c>b>0,|z|<1.$$

To see more studies pertaining to Ostrowski-type inequalities, the interested readers can read a series of works, such as [2, 3, 5–7, 11, 12, 34, 35, 40] and references therein. Also, in [4] and [16], Ostrowski-type inequalities using integer order integrals, in [17], Ostrowski-type inequalities by means of Hadamard fractional integrals, and in [36], Ostrowski-type inequalities in terms of Riemann–Liouville integral operator were obtained. Here, the results in this work are derived by using Katugampola fractional integrals, which shows more general results than inequalities utilizeing integer order integral, Hadamard fractional integrals or Riemann–Liouville fractional integrals.

Motivated by the results in the papers above, especially the results developed in [13, 36, 38], this work aims to investigate estimates of upper bound for differentiable mappings, which are related to the famous Ostrowski type inequality. For this purpose, we first establish a general integral identity concerning Katugampola fractional integrals. We then apply the identity to derive certain estimates of the upper bound for differentiable mappings involving Katugampola fractional integrals via p-convex mappings.

2. Main results

To prove our main results, we need the following important lemma.

Lemma 7. *Suppose that $F:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) satisfying $F'\in L^1([a,b])$, where $0<a<b$. Then, for $\lambda\in[0,1], x\in(a,b), \alpha>0$ and certain fixed $p>0$, the following equality holds:*

$$\begin{aligned}&(1-\lambda)p\frac{(x^p-a^p)^\alpha+(b^p-x^p)^\alpha}{b-a}F(x)+\lambda p\frac{(x^p-a^p)^\alpha F(a)+(b^p-x^p)^\alpha F(b)}{b-a}\\&-\frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}\Big[({}^pI^\alpha_{x^-}F)(a)+({}^pI^\alpha_{x^+}F)(b)\Big]\\&=\frac{(x^p-a^p)^{\alpha+1}}{b-a}\int_0^1(t^\alpha-\lambda)(tx^p+(1-t)a^p)^{\frac{1}{p}-1}F'(\sqrt[p]{tx^p+(1-t)a^p})\mathrm{d}t\\&-\frac{(b^p-x^p)^{\alpha+1}}{b-a}\int_0^1(t^\alpha-\lambda)(tx^p+(1-t)b^p)^{\frac{1}{p}-1}F'(\sqrt[p]{tx^p+(1-t)b^p})\mathrm{d}t.\end{aligned}\tag{2.1}$$

Proof. Integrating by parts, we have that

$$\begin{aligned}
&\int_0^1 (t^\alpha-\lambda)(tx^p+(1-t)a^p)^{\frac{1}{p}-1}F'(\sqrt[p]{tx^p+(1-t)a^p})\mathrm{d}t\\
&=\frac{p(t^\alpha-\lambda)F(\sqrt[p]{tx^p+(1-t)a^p})}{x^p-a^p}\bigg|_0^1-\int_0^1\frac{p\alpha}{x^p-a^p}t^{\alpha-1}F(\sqrt[p]{tx^p+(1-t)a^p})\mathrm{d}t\\
&=\frac{p}{x^p-a^p}\Big[(1-\lambda)F(x)+\lambda F(a)\Big]-\frac{p\alpha}{x^p-a^p}\int_0^1 t^{\alpha-1}F(\sqrt[p]{tx^p+(1-t)a^p})\mathrm{d}t\\
&=\frac{p}{x^p-a^p}\Big[(1-\lambda)F(x)+\lambda F(a)\Big]-\frac{p\alpha}{x^p-a^p}\int_a^x\Big(\frac{u^p-a^p}{x^p-a^p}\Big)^{\alpha-1}F(u)\frac{pu^{p-1}}{x^p-a^p}\mathrm{d}u\\
&=\frac{p}{x^p-a^p}\Big[(1-\lambda)F(x)+\lambda F(a)\Big]-\frac{p^2\alpha}{(x^p-a^p)^{\alpha+1}}\int_a^x(u^p-a^p)^{\alpha-1}F(u)u^{p-1}\mathrm{d}u\\
&=\frac{p}{x^p-a^p}\Big[(1-\lambda)F(x)+\lambda F(a)\Big]-\frac{p^2\alpha}{(x^p-a^p)^{\alpha+1}}\frac{\Gamma(\alpha)}{p^{1-\alpha}}\frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_a^x(u^p-a^p)^{\alpha-1}F(u)u^{p-1}\mathrm{d}u\\
&=\frac{p}{x^p-a^p}\Big[(1-\lambda)F(x)+\lambda F(a)\Big]-\frac{p^{\alpha+1}\Gamma(\alpha+1)}{(x^p-a^p)^{\alpha+1}}({}^pI^\alpha_{x^-}F)(a).
\end{aligned}\tag{2.2}$$

By a similar way, we have that

$$\begin{aligned}
&\int_0^1 (t^\alpha-\lambda)(tx^p+(1-t)b^p)^{\frac{1}{p}-1}F'(\sqrt[p]{tx^p+(1-t)b^p})\mathrm{d}t\\
&=\frac{p(t^\alpha-\lambda)F(\sqrt[p]{tx^p+(1-t)a^p})}{x^p-b^p}\bigg|_0^1-\int_0^1\frac{p\alpha}{x^p-b^p}t^{\alpha-1}F(\sqrt[p]{tx^p+(1-t)b^p})\mathrm{d}t\\
&=\frac{p}{x^p-b^p}\Big[(1-\lambda)F(x)+\lambda F(b)\Big]-\frac{p\alpha}{x^p-b^p}\int_0^1 t^{\alpha-1}F(\sqrt[p]{tx^p+(1-t)b^p})\mathrm{d}t\\
&=\frac{p}{x^p-b^p}\Big[(1-\lambda)F(x)+\lambda F(b)\Big]-\frac{p\alpha}{x^p-b^p}\int_b^x\Big(\frac{b^p-u^p}{b^p-x^p}\Big)^{\alpha-1}F(u)\frac{-pu^{p-1}}{b^p-x^p}\mathrm{d}u\\
&=\frac{p}{x^p-b^p}\Big[(1-\lambda)F(x)+\lambda F(b)\Big]+\frac{p^2\alpha}{(b^p-x^p)^{\alpha+1}}\int_x^b(b^p-u^p)^{\alpha-1}F(u)u^{p-1}\mathrm{d}u\\
&=\frac{p}{x^p-b^p}\Big[(1-\lambda)F(x)+\lambda F(b)\Big]+\frac{p^2\alpha}{(b^p-x^p)^{\alpha+1}}\frac{\Gamma(\alpha)}{p^{1-\alpha}}\frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_x^b(b^p-u^p)^{\alpha-1}F(u)u^{p-1}\mathrm{d}u\\
&=-\frac{p}{b^p-x^p}\Big[(1-\lambda)F(x)+\lambda F(b)\Big]+\frac{p^{\alpha+1}\Gamma(\alpha+1)}{(b^p-x^p)^{\alpha+1}}({}^pI^\alpha_{x^+}F)(b).
\end{aligned}\tag{2.3}$$

Multiplying both sides of (2.2) and (2.3) by $\frac{(x^p-a^p)^{\alpha+1}}{b-a}$ and $\frac{(b^p-x^p)^{\alpha+1}}{b-a}$, respectively, we have that

$$\begin{aligned}
&\frac{(x^p-a^p)^{\alpha+1}}{b-a}\int_0^1(t^\alpha-\lambda)(tx^p+(1-t)a^p)^{\frac{1}{p}-1}F'(\sqrt[p]{tx^p+(1-t)a^p})\mathrm{d}t\\
&=\frac{(x^p-a^p)^\alpha p}{b-a}(1-\lambda)F(x)+\frac{(x^p-a^p)^\alpha p}{b-a}\lambda F(a)-\frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}({}^pI^\alpha_{x^-}F)(a)
\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}
&\frac{(b^p-x^p)^{\alpha+1}}{b-a}\int_0^1(t^\alpha-\lambda)(tx^p+(1-t)b^p)^{\frac{1}{p}-1}F'(\sqrt[p]{tx^p+(1-t)b^p})\mathrm{d}t\\
&=-\frac{(b^p-x^p)^\alpha p}{b-a}(1-\lambda)F(x)-\frac{(b^p-x^p)^\alpha p}{b-a}\lambda F(b)+\frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}({}^pI^\alpha_{x^+}F)(b).
\end{aligned}\tag{2.5}$$

Combining (2.4) with (2.5) yields the desired result. The proof is completed.

Remark 1. *Consider Lemma 7.*
(i) *Taking $\lambda=1$, we have Lemma 5.*
(ii) *Taking $\lambda=0$ and $p=1$, we have Lemma 2.*
(iii) *Taking $\lambda=0$ and $p=1=\alpha$, we have Lemma 1.*

In the rest of this article, for the sake of simplicity, we denote

$$\begin{aligned}
\mathcal{T}_F(\alpha,p,\lambda;a,b):=&(1-\lambda)p\frac{(x^p-a^p)^\alpha+(b^p-x^p)^\alpha}{b-a}F(x)+\lambda p\frac{(x^p-a^p)^\alpha F(a)+(b^p-x^p)^\alpha F(b)}{b-a}\\
&-\frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}\Big[({}^pI^\alpha_{x^-}F)(a)+({}^pI^\alpha_{x^+}F)(b)\Big],
\end{aligned}$$

unless otherwise specified.

The following calculations of definite integrals are needed in Theorem 3.

For $\alpha>0$, $\left(\frac{1}{2}\right)^\alpha<\lambda\le 1$, $0<p\le 1$ and $\theta\in\{a,b\}$, we have

$$\begin{aligned}
A(\theta):=&\int_0^1 t\Big|t^\alpha-\lambda\Big|(t^{\frac{1}{p}-1}x^{1-p}+(1-t)^{\frac{1}{p}-1}\theta^{1-p})\mathrm{d}t\\
=&\int_0^{\lambda^{\frac{1}{\alpha}}} t(\lambda-t^\alpha)(t^{\frac{1}{p}-1}x^{1-p}+(1-t)^{\frac{1}{p}-1}\theta^{1-p})\mathrm{d}t\\
&+\int_{\lambda^{\frac{1}{\alpha}}}^1 t(t^\alpha-\lambda)(t^{\frac{1}{p}-1}x^{1-p}+(1-t)^{\frac{1}{p}-1}\theta^{1-p})\mathrm{d}t\\
=&\,x^{1-p}\frac{1}{\alpha+\frac{1}{p}+1}\Big[1-2\lambda^{1+\frac{1}{\alpha p}+\frac{1}{\alpha}}\Big]-\lambda x^{1-p}\frac{1}{\frac{1}{p}+1}\Big[1-2\lambda^{\frac{1}{\alpha p}+\frac{1}{\alpha}}\Big]\\
&-\theta^{1-p}\beta(\lambda^{\frac{1}{\alpha}};\alpha+2,\frac{1}{p})+\lambda\theta^{1-p}\beta(\lambda^{\frac{1}{\alpha}};2,\frac{1}{p})\\
&+\theta^{1-p}\lambda^{1+\frac{1}{\alpha}}(1-\lambda^{\frac{1}{\alpha}})^{\frac{1}{p}}\beta(1,\frac{1}{p})\,{}_2\mathcal{F}_1\Big(-\alpha-1,1;\frac{1}{p}+1;\frac{\lambda^{\frac{1}{\alpha}}-1}{\lambda^{\frac{1}{\alpha}}}\Big)\\
&-\lambda\theta^{1-p}\Big\{(1-\lambda^{\frac{1}{\alpha}})^{\frac{1}{p}+1}\beta\Big(2,\frac{1}{p}\Big)+(1-\lambda^{\frac{1}{\alpha}})^{\frac{1}{p}}\lambda^{\frac{1}{\alpha}}p\Big\}
\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}
B(\theta) &:= \int_0^1 \Big|t^\alpha - \lambda\Big|(t^{\frac{1}{p}-1}x^{1-p} + (1-t)^{\frac{1}{p}-1}\theta^{1-p})\mathrm{d}t \\
&= \int_0^{\lambda^{\frac{1}{\alpha}}} (\lambda - t^\alpha)(t^{\frac{1}{p}-1}x^{1-p} + (1-t)^{\frac{1}{p}-1}\theta^{1-p})\mathrm{d}t \\
&\quad + \int_{\lambda^{\frac{1}{\alpha}}}^1 (t^\alpha - \lambda)(t^{\frac{1}{p}-1}x^{1-p} + (1-t)^{\frac{1}{p}-1}\theta^{1-p})\mathrm{d}t \\
&= x^{1-p}\frac{1}{\alpha + \frac{1}{p}}\Big[1 - 2\lambda^{1+\frac{1}{\alpha p}}\Big] - \lambda x^{1-p}p\Big[1 - 2\lambda^{\frac{1}{\alpha p}}\Big] + \lambda\theta^{1-p}p\Big[1 - 2(1-\lambda^{\frac{1}{\alpha}})^{\frac{1}{p}}\Big] \\
&\quad - \theta^{1-p}\beta\Big(\lambda^{\frac{1}{\alpha}}; \alpha+1, \frac{1}{p}\Big) + \theta^{1-p}\lambda(1-\lambda^{\frac{1}{\alpha}})^{\frac{1}{p}}\beta\Big(1, \frac{1}{p}\Big){}_2\mathcal{F}_1\Big(-\alpha, 1; \frac{1}{p}+1; \frac{\lambda^{\frac{1}{\alpha}} - 1}{\lambda^{\frac{1}{\alpha}}}\Big).
\end{aligned} \tag{2.7}$$

Theorem 3. *Let $F : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $F' \in L^1([a,b])$. Suppose that $|F'|$ is a p-convex mapping on $[a,b]$ for $0 < p \leq 1$, $\alpha > 0$, $\left(\frac{1}{2}\right)^\alpha < \lambda \leq 1$ and $x \in (a,b)$. Then we have the following results.*
(i) For $p \in (0, \frac{1}{2}]$, the following inequality is true:

$$\begin{aligned}
&\Big|\mathcal{T}_F(\alpha, p, \lambda; a, b)\Big| \\
&\leq 2^{\frac{1}{p}-2}\Big\{\frac{(x^p - a^p)^{\alpha+1}}{b-a}[A(a)|F'(x)| + (B(a) - A(a))|F'(a)|] \\
&\quad + \frac{(b^p - x^p)^{\alpha+1}}{b-a}[A(b)|F'(x)| + (B(b) - A(b))|F'(b)|]\Big\}.
\end{aligned} \tag{2.8}$$

(ii) For $p \in (\frac{1}{2}, 1]$, the following inequality is true:

$$\begin{aligned}
&\Big|\mathcal{T}_F(\alpha, p, \lambda; a, b)\Big| \\
&\leq \frac{(x^p - a^p)^{\alpha+1}}{b-a}[A(a)|F'(x)| + (B(a) - A(a))|F'(a)|] \\
&\quad + \frac{(b^p - x^p)^{\alpha+1}}{b-a}[A(b)|F'(x)| + (B(b) - A(b))|F'(b)|].
\end{aligned} \tag{2.9}$$

Here, $A(\theta)$ and $B(\theta)$ are defined by (2.6) and (2.7), respectively.

Proof. (i) Suppose that $0 < p \leq \frac{1}{2}$. By using Lemma 7, we have that

$$\begin{aligned}
&\Big|\mathcal{T}_F(\alpha, p, \lambda; a, b)\Big| \\
&\leq \frac{(x^p - a^p)^{\alpha+1}}{b-a}\int_0^1 \Big|t^\alpha - \lambda\Big|(tx^p + (1-t)a^p)^{\frac{1}{p}-1}\Big|F'(\sqrt[p]{tx^p + (1-t)a^p})\Big|\mathrm{d}t \\
&\quad + \frac{(b^p - x^p)^{\alpha+1}}{b-a}\int_0^1 \Big|t^\alpha - \lambda\Big|(tx^p + (1-t)b^p)^{\frac{1}{p}-1}\Big|F'(\sqrt[p]{tx^p + (1-t)b^p})\Big|\mathrm{d}t.
\end{aligned}$$

Using p-convexity of $|F'|$, we have that

$$
\begin{aligned}
&\left|\mathcal{T}_F(\alpha, p, \lambda; a, b)\right| \\
&\leq \frac{(x^p - a^p)^{\alpha+1}}{b-a} \\
&\quad \times \left\{ \begin{array}{c} |F'(x)| \int_0^1 t\left|t^\alpha - \lambda\right|(tx^p + (1-t)a^p)^{\frac{1}{p}-1}\mathrm{d}t \\ +|F'(a)| \int_0^1 (1-t)\left|t^\alpha - \lambda\right|(tx^p + (1-t)a^p)^{\frac{1}{p}-1}\mathrm{d}t \end{array} \right\} \\
&\quad + \frac{(b^p - x^p)^{\alpha+1}}{b-a} \\
&\quad \times \left\{ \begin{array}{c} |F'(x)| \int_0^1 t\left|t^\alpha - \lambda\right|(tx^p + (1-t)b^p)^{\frac{1}{p}-1}\mathrm{d}t \\ +|F'(b)| \int_0^1 (1-t)\left|t^\alpha - \lambda\right|(tx^p + (1-t)b^p)^{\frac{1}{p}-1}\mathrm{d}t \end{array} \right\}.
\end{aligned} \tag{2.10}
$$

Since $0 < p \leq \frac{1}{2}$, by using Lemma 6, we have that

$$
(tx^p + (1-t)\theta^p)^{\frac{1}{p}-1} \leq 2^{\frac{1}{p}-2}(t^{\frac{1}{p}-1}x^{1-p} + (1-t)^{\frac{1}{p}-1}\theta^{1-p})
$$

for all $t \in [0, 1], \theta \in \{a, b\}$.
Therefore, we have

$$
\begin{aligned}
&\int_0^1 t\left|t^\alpha - \lambda\right|(tx^p + (1-t)\theta^p)^{\frac{1}{p}-1}\mathrm{d}t \\
&\leq 2^{\frac{1}{p}-2} \int_0^1 t\left|t^\alpha - \lambda\right|(t^{\frac{1}{p}-1}x^{1-p} + (1-t)^{\frac{1}{p}-1}\theta^{1-p})\mathrm{d}t \\
&= 2^{\frac{1}{p}-2}A(\theta),
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
&\int_0^1 (1-t)\left|t^\alpha - \lambda\right|(tx^p + (1-t)\theta^p)^{\frac{1}{p}-1}\mathrm{d}t \\
&\leq 2^{\frac{1}{p}-2} \int_0^1 (1-t)\left|t^\alpha - \lambda\right|(t^{\frac{1}{p}-1}x^{1-p} + (1-t)^{\frac{1}{p}-1}\theta^{1-p})\mathrm{d}t \\
&= 2^{\frac{1}{p}-2}[B(\theta) - A(\theta)].
\end{aligned} \tag{2.12}
$$

Using (2.11) and (2.12) in (2.10), we get the desired inequality in (2.8). This completes the proof for case $p \in (0, \frac{1}{2}]$.

To prove (ii), suppose that $p \in (\frac{1}{2}, 1]$, then we obtain the required inequality in (2.9) by applying the fact that

$$
(tx^p + (1-t)\theta^p)^{\frac{1}{p}-1} \leq t^{\frac{1}{p}-1}x^{1-p} + (1-t)^{\frac{1}{p}-1}\theta^{1-p}, \quad t \in [0, 1], \ \theta \in \{a, b\}. \tag{2.13}
$$

Corollary 1. *If $|F'| \leq M$ in Theorem 3, then we obtain the following results.*
(i) *For $p \in (0, \frac{1}{2}]$, we have*

$$
\begin{aligned}
&\left|\mathcal{T}_F(\alpha, p, \lambda; a, b)\right| \\
&\leq \frac{2^{\frac{1}{p}-2}M}{b-a}\left\{(x^p - a^p)^{\alpha+1}B(a) + (b^p - x^p)^{\alpha+1}B(b)\right\}.
\end{aligned}
$$

(*ii*) *For* $p \in (\frac{1}{2}, 1]$*, we have*

$$\begin{aligned}&\left|\mathcal{T}_F(\alpha, p, \lambda; a, b)\right|\\&\leq \frac{M}{b-a}\Big\{(x^p - a^p)^{\alpha+1}B(a) + (b^p - x^p)^{\alpha+1}B(b)\Big\}.\end{aligned}$$

Theorem 4. *Let* $F : [a, b] \subset (0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with* $a < b$ *such that* $F' \in L^1([a, b])$*. Assume that* $|F'|^q$ *is a p-convex mapping on* $[a, b]$ *for certain fixed* $p > 0$*. If* $x \in (a, b), \lambda \in [0, 1], \alpha > 0$ *and* $r > 1, q > 1$ *such that* $\frac{1}{r} + \frac{1}{q} = 1$*, then we have*

$$\begin{aligned}&\left|\mathcal{T}_F(\alpha, p, \lambda; a, b)\right|\\&\leq \frac{(x^p - a^p)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(a)\Big(C(q)|F'(x)|^q + D(q)|F'(a)|^q\Big)^{\frac{1}{q}}\\&\quad+ \frac{(b^p - x^p)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(b)\Big(C(q)|F'(x)|^q + D(q)|F'(b)|^q\Big)^{\frac{1}{q}},\end{aligned} \tag{2.14}$$

where

$$K(\theta) = \frac{p}{x^p - \theta^p}\frac{x^{(r-1)(1-p)+1} - \theta^{(r-1)(1-p)+1}}{(r-1)(1-p)+1},\ \theta \in \{a, b\},$$

$$C(q) = \frac{1}{\alpha q + 2} - \frac{1}{2}\lambda^q + \frac{\alpha q}{\alpha q + 2}\lambda^{q+\frac{2}{\alpha}}$$

and

$$D(q) = \frac{1}{(\alpha q + 1)(\alpha q + 2)} - \frac{1}{2}\lambda^q + \frac{2\alpha q}{\alpha q + 1}\lambda^{q+\frac{1}{\alpha}} - \frac{\alpha q}{\alpha q + 2}\lambda^{q+\frac{2}{\alpha}}.$$

Proof. With the help of Lemma 7 and properties of modulus, one can write

$$\begin{aligned}&\left|\mathcal{T}_F(\alpha, p, \lambda; a, b)\right|\\&\leq \frac{(x^p - a^p)^{\alpha+1}}{b-a}\int_0^1 \left|t^\alpha - \lambda\right|\left(tx^p + (1-t)a^p\right)^{\frac{1}{p}-1}\left|F'(\sqrt[p]{tx^p + (1-t)a^p})\right|\mathrm{d}t\\&\quad+ \frac{(b^p - x^p)^{\alpha+1}}{b-a}\int_0^1 \left|t^\alpha - \lambda\right|\left(tx^p + (1-t)b^p\right)^{\frac{1}{p}-1}\left|F'(\sqrt[p]{tx^p + (1-t)b^p})\right|\mathrm{d}t.\end{aligned}$$

By using Hölder inequality, it can be written as

$$\begin{aligned}&\left|\mathcal{T}_F(\alpha, p, \lambda; a, b)\right|\\&\leq \frac{(x^p - a^p)^{\alpha+1}}{b-a}\left(\int_0^1 \left(\left(tx^p + (1-t)a^p\right)^{\frac{1}{p}-1}\right)^r\mathrm{d}t\right)^{\frac{1}{r}}\\&\quad\times\left(\int_0^1 \left|t^\alpha - \lambda\right|^q\left|F'(\sqrt[p]{tx^p + (1-t)a^p})\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\&\quad+ \frac{(b^p - x^p)^{\alpha+1}}{b-a}\left(\int_0^1 \left(\left(tx^p + (1-t)b^p\right)^{\frac{1}{p}-1}\right)^r\mathrm{d}t\right)^{\frac{1}{r}}\\&\quad\times\left(\int_0^1 \left|t^\alpha - \lambda\right|^q\left|F'(\sqrt[p]{tx^p + (1-t)b^p})\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

Using the p-convexity of $|F'|^q$, it follows that

$$\begin{aligned}
&\left|\mathcal{T}_F(\alpha,p,\lambda;a,b)\right|\\
&\le \frac{(x^p-a^p)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(a)\\
&\quad\times\left(\int_0^1\left|t^\alpha-\lambda\right|^q[t|F'(x)|^q+(1-t)|F'(a)|^q]\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\frac{(b^p-x^p)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(b)\\
&\quad\times\left(\int_0^1\left|t^\alpha-\lambda\right|^q[t|F'(x)|^q+(1-t)|F'(b)|^q]\mathrm{d}t\right)^{\frac{1}{q}}\\
&=\frac{(x^p-a^p)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(a)\\
&\quad\times\left(|F'(x)|^q\int_0^1t\left|t^\alpha-\lambda\right|^q\mathrm{d}t+|F'(a)|^q\int_0^1(1-t)\left|t^\alpha-\lambda\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\frac{(b^p-x^p)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(b)\\
&\quad\times\left(|F'(x)|^q\int_0^1t\left|t^\alpha-\lambda\right|^q\mathrm{d}t+|F'(b)|^q\int_0^1(1-t)\left|t^\alpha-\lambda\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}$$

Using the inequality $(\mathcal{A}-\mathcal{B})^\tau\le\mathcal{A}^\tau-\mathcal{B}^\tau$ for any $\mathcal{A}>\mathcal{B}\ge0$ and $\tau\ge1$, it yields that

$$\begin{aligned}
&\int_0^1t\left|t^\alpha-\lambda\right|^q\mathrm{d}t\\
&=\int_0^{\lambda^{\frac{1}{\alpha}}}t(\lambda-t^\alpha)^q\mathrm{d}t+\int_{\lambda^{\frac{1}{\alpha}}}^1t(t^\alpha-\lambda)^q\mathrm{d}t\\
&\le\int_0^{\lambda^{\frac{1}{\alpha}}}t(\lambda^q-t^{\alpha q})\mathrm{d}t+\int_{\lambda^{\frac{1}{\alpha}}}^1t(t^{\alpha q}-\lambda^q)\mathrm{d}t\\
&=\frac{1}{\alpha q+2}-\frac{1}{2}\lambda^q+\frac{\alpha q}{\alpha q+2}\lambda^{q+\frac{2}{\alpha}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1(1-t)\left|t^\alpha-\lambda\right|^q\mathrm{d}t\\
&\le\frac{1}{(\alpha q+1)(\alpha q+2)}-\frac{1}{2}\lambda^q+\frac{2\alpha q}{\alpha q+1}\lambda^{q+\frac{1}{\alpha}}-\frac{\alpha q}{\alpha q+2}\lambda^{q+\frac{2}{\alpha}}.
\end{aligned}$$

The desired inequality is given by noting that

$$\begin{aligned}
K(\theta)&=\int_0^1\left((tx^p+(1-t)\theta^p)^{\frac{1}{p}-1}\right)^r\mathrm{d}t\\
&=\frac{p}{x^p-\theta^p}\frac{x^{(r-1)(1-p)+1}-\theta^{(r-1)(1-p)+1}}{(r-1)(1-p)+1},\ \theta\in\{a,b\}.
\end{aligned}$$

This completes the proof.

Corollary 2. *If $|F'| \leq M$ in Theorem 4, then it is easy to see that*

$$\begin{aligned}
&\Big|\mathcal{T}_F(\alpha, p, \lambda; a, b)\Big| \\
&\leq \frac{M}{b-a}\Big(C(q) + D(q)\Big)^{\frac{1}{q}}\Big\{(x^p - a^p)^{\alpha+1}K^{\frac{1}{r}}(a) + (b^p - x^p)^{\alpha+1}K^{\frac{1}{r}}(b)\Big\}.
\end{aligned}$$

Theorem 5. *Let $F : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $F' \in L^1([a, b])$. Suppose that $|F'|^q$ is a p-convex mapping on $[a, b]$ for $0 < p \leq 1, q > 1$, $\alpha > 0$, $\left(\frac{1}{2}\right)^{\alpha} < \lambda \leq 1$ and $x \in (a, b)$. Then we have the following results.*
(i) For $p \in (0, \frac{1}{2}]$, the following inequality is true:

$$\begin{aligned}
&\Big|\mathcal{T}_F(\alpha, p, \lambda; a, b)\Big| \\
&\leq 2^{\frac{1}{p}-2}\Big\{B^{1-\frac{1}{q}}(a)\frac{(x^p - a^p)^{\alpha+1}}{b-a}\big[A(a)|F'(x)|^q + \big(B(a) - A(a)\big)|F'(a)|^q\big]^{\frac{1}{q}} \\
&\quad + B^{1-\frac{1}{q}}(b)\frac{(b^p - x^p)^{\alpha+1}}{b-a}\big[A(b)|F'(x)|^q + \big(B(b) - A(b)\big)|F'(b)|^q\big]^{\frac{1}{q}}\Big\}.
\end{aligned} \tag{2.15}$$

(ii) For $p \in (\frac{1}{2}, 1]$, the following inequality is true:

$$\begin{aligned}
&\Big|\mathcal{T}_F(\alpha, p, \lambda; a, b)\Big| \\
&\leq B^{1-\frac{1}{q}}(a)\frac{(x^p - a^p)^{\alpha+1}}{b-a}\big[A(a)|F'(x)|^q + \big(B(a) - A(a)\big)|F'(a)|^q\big]^{\frac{1}{q}} \\
&\quad + B^{1-\frac{1}{q}}(b)\frac{(b^p - x^p)^{\alpha+1}}{b-a}\big[A(b)|F'(x)|^q + \big(B(b) - A(b)\big)|F'(b)|^q\big]^{\frac{1}{q}}.
\end{aligned} \tag{2.16}$$

Here, $A(\theta)$ and $B(\theta)$ are defined by the same expressions as described in (2.6) and (2.7).

Proof. (i) Suppose that $0 < p \leq \frac{1}{2}$. Using Lemma 7, we have that

$$\begin{aligned}
&\Big|\mathcal{T}_F(\alpha, p, \lambda; a, b)\Big| \\
&\leq \frac{(x^p - a^p)^{\alpha+1}}{b-a}\int_0^1 \Big|t^{\alpha} - \lambda\Big|(tx^p + (1-t)a^p)^{\frac{1}{p}-1}\Big|F'(\sqrt[p]{tx^p + (1-t)a^p})\Big|\mathrm{d}t \\
&\quad + \frac{(b^p - x^p)^{\alpha+1}}{b-a}\int_0^1 \Big|t^{\alpha} - \lambda\Big|(tx^p + (1-t)b^p)^{\frac{1}{p}-1}\Big|F'(\sqrt[p]{tx^p + (1-t)b^p})\Big|\mathrm{d}t.
\end{aligned}$$

Making use of power-mean inequality, we have that

$$\begin{aligned}
&\Big|\mathcal{T}_F(\alpha, p, \lambda; a, b)\Big| \\
&\leq \frac{(x^p - a^p)^{\alpha+1}}{b-a}\Big(\int_0^1 \Big|t^{\alpha} - \lambda\Big|(tx^p + (1-t)a^p)^{\frac{1}{p}-1}\mathrm{d}t\Big)^{1-\frac{1}{q}} \\
&\quad \times\Big(\int_0^1 \Big|t^{\alpha} - \lambda\Big|(tx^p + (1-t)a^p)^{\frac{1}{p}-1}\Big|F'(\sqrt[p]{tx^p + (1-t)a^p})\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \frac{(b^p - x^p)^{\alpha+1}}{b-a}\Big(\int_0^1 \Big|t^{\alpha} - \lambda\Big|(tx^p + (1-t)b^p)^{\frac{1}{p}-1}\mathrm{d}t\Big)^{1-\frac{1}{q}} \\
&\quad \times\Big(\int_0^1 \Big|t^{\alpha} - \lambda\Big|(tx^p + (1-t)b^p)^{\frac{1}{p}-1}\Big|F'(\sqrt[p]{tx^p + (1-t)b^p})\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}.
\end{aligned}$$

Using p-convexity of $|F'|^q$, we have that

$$\begin{aligned}
&\left|\mathcal{T}_F(\alpha,p,\lambda;a,b)\right| \\
&\le \frac{(x^p-a^p)^{\alpha+1}}{b-a}\left(\int_0^1 \left|t^\alpha-\lambda\right|(tx^p+(1-t)a^p)^{\frac{1}{p}-1}\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad\times\left\{\begin{array}{c} |F'(x)|^q\int_0^1 t\left|t^\alpha-\lambda\right|(tx^p+(1-t)a^p)^{\frac{1}{p}-1}]\mathrm{d}t \\ +|F'(a)|^q\int_0^1(1-t)\left|t^\alpha-\lambda\right|(tx^p+(1-t)a^p)^{\frac{1}{p}-1}]\mathrm{d}t \end{array}\right\}^{\frac{1}{q}} \\
&\quad+\frac{(b^p-x^p)^{\alpha+1}}{b-a}\left(\int_0^1 \left|t^\alpha-\lambda\right|(tx^p+(1-t)b^p)^{\frac{1}{p}-1}\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad\times\left\{\begin{array}{c} |F'(x)|^q\int_0^1 t\left|t^\alpha-\lambda\right|(tx^p+(1-t)b^p)^{\frac{1}{p}-1}]\mathrm{d}t \\ +|F'(b)|^q\int_0^1(1-t)\left|t^\alpha-\lambda\right|(tx^p+(1-t)b^p)^{\frac{1}{p}-1}]\mathrm{d}t \end{array}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{2.17}$$

By using (2.11) and (2.12) in (2.17), we get the desired inequality in (2.15).

To prove (ii), suppose that $p \in (\frac{1}{2},1]$, then we obtain the required inequality in (2.16) by applying inequality (2.13). This ends the proof.

Corollary 3. *If $|F'| \le M$ in Theorem 5, then we obtain the following results.*
(i) *For $p \in (0,\frac{1}{2}]$, we have*

$$\begin{aligned}
&\left|\mathcal{T}_F(\alpha,p,\lambda;a,b)\right| \\
&\le \frac{2^{\frac{1}{p}-2}M}{b-a}\left\{(x^p-a^p)^{\alpha+1}B(a)+(b^p-x^p)^{\alpha+1}B(b)\right\}.
\end{aligned}$$

(ii) *For $p \in (\frac{1}{2},1]$, we have*

$$\begin{aligned}
&\left|\mathcal{T}_F(\alpha,p,\lambda;a,b)\right| \\
&\le \frac{M}{b-a}\left\{(x^p-a^p)^{\alpha+1}B(a)+(b^p-x^p)^{\alpha+1}B(b)\right\}.
\end{aligned}$$

Theorem 6. *Let $F:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $F'\in L^1([a,b])$. Assume that $|F'|^q$ is a p-convex mapping on $[a,b]$ for certain fixed $0<p\le 1$, $\alpha>0,\left(\frac{1}{2}\right)^\alpha<\lambda\le 1$, $x\in(a,b)$ and $r>1, q>1$ such that $\frac{1}{r}+\frac{1}{q}=1$. Then we have the following results.*
(i) *For $p \in (0,\frac{1}{1+\frac{1}{r}}]$, the following inequality is true:*

$$\begin{aligned}
&\left|\mathcal{T}_F(\alpha,p,\lambda;a,b)\right| \\
&\le 2^{\frac{1}{p}-1-\frac{1}{r}}\left\{E^{\frac{1}{r}}(a)\frac{(x^p-a^p)^{\alpha+1}}{b-a}\left[C(1)|F'(x)|^q+D(1)|F'(a)|^q\right]^{\frac{1}{q}}\right. \\
&\quad\left.+E^{\frac{1}{r}}(b)\frac{(b^p-x^p)^{\alpha+1}}{b-a}\left[C(1)|F'(x)|^q+D(1)|F'(b)|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{2.18}$$

(*ii*) *For* $p \in (\frac{1}{1+\frac{1}{r}}, 1]$*, the following inequality is true:*

$$
\begin{aligned}
&\left|\mathcal{T}_F(\alpha, p, \lambda; a, b)\right| \\
&\leq E^{\frac{1}{r}}(a)\frac{(x^p - a^p)^{\alpha+1}}{b-a}\Big[C(1)|F'(x)|^q + D(1)|F'(a)|^q\Big]^{\frac{1}{q}} \\
&\quad + E^{\frac{1}{r}}(b)\frac{(b^p - x^p)^{\alpha+1}}{b-a}\Big[C(1)|F'(x)|^q + D(1)|F'(b)|^q\Big]^{\frac{1}{q}},
\end{aligned}
\tag{2.19}
$$

where

$$C(1) = \frac{1}{\alpha+2} - \frac{1}{2}\lambda + \frac{\alpha}{\alpha+2}\lambda^{1+\frac{2}{\alpha}},$$

$$D(1) = \frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{2}\lambda + \frac{2\alpha}{\alpha+1}\lambda^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2}\lambda^{1+\frac{2}{\alpha}},$$

and

$$
\begin{aligned}
&E(\theta) \\
&= x^{r-rp}\frac{1}{\alpha+\frac{r}{p}-r+1}\Big[1 - 2\lambda^{1+\frac{r}{\alpha p}-\frac{r}{\alpha}+\frac{1}{\alpha}}\Big] - \lambda x^{r-rp}\frac{1}{\frac{r}{p}-r+1}\Big[1 - 2\lambda^{\frac{r}{\alpha p}-\frac{r}{\alpha}+\frac{1}{\alpha}}\Big] \\
&\quad + \lambda\theta^{r-rp}\frac{1}{\frac{r}{p}-r+1}\Big[1 - 2(1-\lambda^{\frac{1}{\alpha}})^{\frac{r}{p}-r+1}\Big] - \theta^{r-rp}\beta\Big(\lambda^{\frac{1}{\alpha}}; \alpha+1, \frac{r}{p}-r+1\Big) \\
&\quad + \theta^{r-rp}\lambda(1-\lambda^{\frac{1}{\alpha}})^{\frac{r}{p}-r+1}\beta\Big(1, \frac{r}{p}-r+1\Big){}_2\mathcal{F}_1\Big(-\alpha, 1; \frac{r}{p}-r+2; \frac{\lambda^{\frac{1}{\alpha}}-1}{\lambda^{\frac{1}{\alpha}}}\Big), \ \theta \in \{a, b\}.
\end{aligned}
$$

Proof. (i) Suppose that $0 < p \leq \frac{1}{1+\frac{1}{r}}$, i.e., $\frac{r}{p} - r \geq 1$. By using Lemma 7, we have that

$$
\begin{aligned}
&\left|\mathcal{T}_F(\alpha, p, \lambda; a, b)\right| \\
&\leq \frac{(x^p - a^p)^{\alpha+1}}{b-a}\int_0^1 \left|t^\alpha - \lambda\right|\left(tx^p + (1-t)a^p\right)^{\frac{1}{p}-1}\left|F'(\sqrt[p]{tx^p + (1-t)a^p})\right| \mathrm{d}t \\
&\quad + \frac{(b^p - x^p)^{\alpha+1}}{b-a}\int_0^1 \left|t^\alpha - \lambda\right|\left(tx^p + (1-t)b^p\right)^{\frac{1}{p}-1}\left|F'(\sqrt[p]{tx^p + (1-t)b^p})\right| \mathrm{d}t.
\end{aligned}
$$

Now, considering the weighted version of Hölder's inequality (see [9]),

$$\left|\int_{\mathcal{I}} F(s)G(s)H(s)\mathrm{d}s\right| \leq \Big(\int_{\mathcal{I}} |F(s)|^r H(s)\mathrm{d}s\Big)^{\frac{1}{r}}\Big(\int_{\mathcal{I}} |G(s)|^q H(s)\mathrm{d}s\Big)^{\frac{1}{q}}$$

for $q > 1$, $r^{-1} + q^{-1} = 1$, and H is non-negative on $\mathcal{I}$ and provided all the other integrals exist and are finite, we have that

$$
\begin{aligned}
&\Big|\mathcal{T}_F(\alpha,p,\lambda;a,b)\Big|\\
&\leq \frac{(x^p-a^p)^{\alpha+1}}{b-a}\Big(\int_0^1\Big[(tx^p+(1-t)a^p)^{\frac{1}{p}-1}\Big]^r\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{r}}\\
&\quad\times\Big(\int_0^1\Big|F'(\sqrt[p]{tx^p+(1-t)a^p})\Big|^q\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{q}}\\
&\quad+\frac{(b^p-x^p)^{\alpha+1}}{b-a}\Big(\int_0^1\Big[(tx^p+(1-t)b^p)^{\frac{1}{p}-1}\Big]^r\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{r}}\\
&\quad\times\Big(\int_0^1\Big|F'(\sqrt[p]{tx^p+(1-t)b^p})\Big|^q\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Using the p-convexity of $|F'|^q$, it follows that

$$
\begin{aligned}
&\Big|\mathcal{T}_F(\alpha,p,\lambda;a,b)\Big|\\
&\leq \frac{(x^p-a^p)^{\alpha+1}}{b-a}\Big(\int_0^1\Big[(tx^p+(1-t)a^p)^{\frac{1}{p}-1}\Big]^r\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{r}}\\
&\quad\times\Big(\int_0^1\Big|F'(\sqrt[p]{tx^p+(1-t)a^p})\Big|^q\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{q}}\\
&\quad+\frac{(b^p-x^p)^{\alpha+1}}{b-a}\Big(\int_0^1\Big[(tx^p+(1-t)b^p)^{\frac{1}{p}-1}\Big]^r\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{r}}\\
&\quad\times\Big(\int_0^1\Big|F'(\sqrt[p]{tx^p+(1-t)b^p})\Big|^q\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{q}}\\
&\leq \frac{(x^p-a^p)^{\alpha+1}}{b-a}\Big(\int_0^1\Big[(tx^p+(1-t)a^p)^{\frac{1}{p}-1}\Big]^r\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{r}}\\
&\quad\times\left\{\begin{array}{c}|F'(x)|^q\int_0^1 t\Big|t^\alpha-\lambda\Big|\mathrm{d}t\\ +|F'(a)|^q\int_0^1(1-t)\Big|t^\alpha-\lambda\Big|\mathrm{d}t\end{array}\right\}^{\frac{1}{q}}\\
&\quad+\frac{(b^p-x^p)^{\alpha+1}}{b-a}\Big(\int_0^1\Big[(tx^p+(1-t)b^p)^{\frac{1}{p}-1}\Big]^r\Big|t^\alpha-\lambda\Big|\mathrm{d}t\Big)^{\frac{1}{r}}\\
&\quad\times\left\{\begin{array}{c}|F'(x)|^q\int_0^1 t\Big|t^\alpha-\lambda\Big|\mathrm{d}t\\ +|F'(b)|^q\int_0^1(1-t)\Big|t^\alpha-\lambda\Big|\mathrm{d}t\end{array}\right\}^{\frac{1}{q}}.
\end{aligned}
\tag{2.20}
$$

Also

$$
\begin{aligned}
&\int_0^1 t\Big|t^\alpha-\lambda\Big|\mathrm{d}t\\
&=\frac{1}{\alpha+2}-\frac{1}{2}\lambda+\frac{\alpha}{\alpha+2}\lambda^{1+\frac{2}{\alpha}}
\end{aligned}
$$

and

$$\int_0^1 (1-t)\Big|t^\alpha - \lambda\Big|\mathrm{d}t = \frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{2}\lambda + \frac{2\alpha}{\alpha+1}\lambda^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2}\lambda^{1+\frac{2}{\alpha}}.$$

Since $0 < p \le \frac{1}{1+\frac{1}{r}}$, by using Lemma 6, we have that

$$(tx^p + (1-t)\theta^p)^{\frac{r}{p}-r} \le 2^{\frac{r}{p}-r-1}(t^{\frac{r}{p}-r}x^{r-rp} + (1-t)^{\frac{r}{p}-r}\theta^{r-rp}) \tag{2.21}$$

for all $t \in [0,1]$ and $\theta \in \{a,b\}$.
By using (2.21), we have that

$$\begin{aligned}&\int_0^1 \Big[(tx^p + (1-t)\theta^p)^{\frac{1}{p}-1}\Big]^r\Big|t^\alpha - \lambda\Big|\mathrm{d}t \\ &\le 2^{\frac{r}{p}-r-1}\int_0^1 |t^\alpha - \lambda|(t^{\frac{r}{p}-r}x^{r-rp} + (1-t)^{\frac{r}{p}-r}\theta^{r-rp})\mathrm{d}t,\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}&\int_0^1 |t^\alpha - \lambda|(t^{\frac{r}{p}-r}x^{r-rp} + (1-t)^{\frac{r}{p}-r}\theta^{r-rp})\mathrm{d}t \\ &= x^{r-rp}\frac{1}{\alpha+\frac{r}{p}-r+1}\Big[1-2\lambda^{1+\frac{r}{\alpha p}-\frac{r}{\alpha}+\frac{1}{\alpha}}\Big] - \lambda x^{r-rp}\frac{1}{\frac{r}{p}-r+1}\Big[1-2\lambda^{\frac{r}{\alpha p}-\frac{r}{\alpha}+\frac{1}{\alpha}}\Big] \\ &\quad + \lambda\theta^{r-rp}\frac{1}{\frac{r}{p}-r+1}\Big[1-2(1-\lambda^{\frac{1}{\alpha}})^{\frac{r}{p}-r+1}\Big] - \theta^{r-rp}\beta\Big(\lambda^{\frac{1}{\alpha}};\alpha+1,\frac{r}{p}-r+1\Big) \\ &\quad + \theta^{r-rp}\lambda(1-\lambda^{\frac{1}{\alpha}})^{\frac{r}{p}-r+1}\beta\Big(1,\frac{r}{p}-r+1\Big){}_2\mathcal{F}_1\Big(-\alpha,1;\frac{r}{p}-r+2;\frac{\lambda^{\frac{1}{\alpha}}-1}{\lambda^{\frac{1}{\alpha}}}\Big).\end{aligned}$$

Using (2.22) in (2.20), we get the desired inequality in (2.18). This completes the proof for case $p \in (0, \frac{1}{1+\frac{1}{r}}]$.

To prove (ii), suppose that $p \in (\frac{1}{1+\frac{1}{r}}, 1]$, then we obtain the required inequality in (2.19) by applying the fact that

$$(tx^p + (1-t)\theta^p)^{\frac{r}{p}-r} \le (t^{\frac{r}{p}-r}x^{r-rp} + (1-t)^{\frac{r}{p}-r}\theta^{r-rp}), \ \ t \in [0,1], \ \theta \in \{a,b\}.$$

Corollary 4. *If $|F'| \le M$ in Theorem 6, then we obtain the following results.*
(i) *For $p \in (0, \frac{1}{1+\frac{1}{r}}]$, we have*

$$\begin{aligned}&\Big|\mathcal{T}_F(\alpha,p,\lambda;a,b)\Big| \\ &\le \frac{2^{\frac{1}{p}-1-\frac{1}{r}}M}{b-a}\Big[C(1)+D(1)\Big]^{\frac{1}{q}}\Big\{(x^p-a^p)^{\alpha+1}E^{\frac{1}{r}}(a) + (b^p-x^p)^{\alpha+1}E^{\frac{1}{r}}(b)\Big\}.\end{aligned}$$

(ii) For $p \in (\frac{1}{1+\frac{1}{r}}, 1]$, *we have*

$$\left|\mathcal{T}_F(\alpha, p, \lambda; a, b)\right| \\ \leq \frac{M}{b-a}\Big[C(1) + D(1)\Big]^{\frac{1}{q}}\Big\{(x^p - a^p)^{\alpha+1}E^{\frac{1}{r}}(a) + (b^p - x^p)^{\alpha+1}E^{\frac{1}{r}}(b)\Big\}.$$

3. Examples

In this section, we present four examples to illustrate our main results.

Example 1. For $p \in (0, \frac{1}{2}]$, let $F(x) = \frac{1}{1-p}x^{1-p}$ for $x \in (0, \infty)$. Then $|F'(x)| = x^{-p}$ is a p-convex mapping. If we take $a = 2$, $b = 4$, $x = 3$, $\alpha = 1$, $\lambda = \frac{3}{4}$ and $p = \frac{1}{3}$, then all the assumptions in Theorem 3 are satisfied.
The left-hand side term of (2.8) is:

$$\Bigg|(1-\lambda)p\frac{(x^p - a^p)^\alpha + (b^p - x^p)^\alpha}{b-a}F(x) + \lambda p\frac{(x^p - a^p)^\alpha F(a) + (b^p - x^p)^\alpha F(b)}{b-a} \\ - \frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}\Big[({}^pI^\alpha_{x^-}F)(a) + ({}^pI^\alpha_{x^+}F)(b)\Big]\Bigg| \approx 0.0012.$$

The right-hand side term of (2.8) is:

$$2^{\frac{1}{p}-2}\Bigg\{\frac{(x^p - a^p)^{\alpha+1}}{b-a}\big[A(a)|F'(x)| + \big(B(a) - A(a)\big)|F'(a)|\big] \\ + \frac{(b^p - x^p)^{\alpha+1}}{b-a}\big[A(b)|F'(x)| + \big(B(b) - A(b)\big)|F'(b)|\big]\Bigg\} \approx 0.0167.$$

It is clear that $0.0012 < 0.0167$, which demonstrates the first result described in Theorem 3.

Example 2. For $p > 0$, let $F(x) = \frac{1}{2}x^2$ for $x \in (0, \infty)$. Then $|F'(x)| = x$ is a p-convex mapping. If we take $a = 2$, $b = 4$, $x = 3$, $\alpha = 1.3$, $\lambda = \frac{1}{2}$, $r = 2$, $p = \frac{1}{4}$ and $q = 2$, then all the assumptions in Theorem 4 are satisfied.
The left-hand side term of (2.14) is:

$$\Bigg|(1-\lambda)p\frac{(x^p - a^p)^\alpha + (b^p - x^p)^\alpha}{b-a}F(x) + \lambda p\frac{(x^p - a^p)^\alpha F(a) + (b^p - x^p)^\alpha F(b)}{b-a} \\ - \frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}\Big[({}^pI^\alpha_{x^-}F)(a) + ({}^pI^\alpha_{x^+}F)(b)\Big]\Bigg| \approx 0.0022.$$

The right-hand side term of (2.14) is:

$$\frac{(x^p - a^p)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(a)\Big(C(q)|F'(x)|^q + D(q)|F'(a)|^q\Big)^{\frac{1}{q}} \\ + \frac{(b^p - x^p)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(b)\Big(C(q)|F'(x)|^q + D(q)|F'(b)|^q\Big)^{\frac{1}{q}} \approx 0.0280.$$

It is clear that $0.0022 < 0.0280$, which demonstrates the result described in Theorem 4.

Example 3. For $p \in (0, \frac{1}{2}]$, let $F(x) = \frac{1}{1+\frac{1}{q}} x^{1+\frac{1}{q}}$ for $x \in (0, \infty)$. Then $|F'(x)|^q = x$ is a p-convex mapping. If we take $a = 2$, $b = 4$, $x = 3$, $\alpha = 2$, $\lambda = \frac{3}{4}$, $q = 2$ and $p = \frac{1}{4}$, then all the assumptions in Theorem 5 are satisfied.
The left-hand side term of (2.15) is:

$$\Bigg|(1-\lambda)p\frac{(x^p-a^p)^\alpha+(b^p-x^p)^\alpha}{b-a}F(x)+\lambda p\frac{(x^p-a^p)^\alpha F(a)+(b^p-x^p)^\alpha F(b)}{b-a} - \frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}\Big[({}^pI^\alpha_{x^-}F)(a)+({}^pI^\alpha_{x^+}F)(b)\Big]\Bigg| \approx 0.00018390.$$

The right-hand side term of (2.15) is:

$$2^{\frac{1}{p}-2}\Bigg\{B^{1-\frac{1}{q}}(a)\frac{(x^p-a^p)^{\alpha+1}}{b-a}[A(a)|F'(x)|^q+(B(a)-A(a))|F'(a)|^q]^{\frac{1}{q}} +B^{1-\frac{1}{q}}(b)\frac{(b^p-x^p)^{\alpha+1}}{b-a}[A(b)|F'(x)|^q+(B(b)-A(b))|F'(b)|^q]^{\frac{1}{q}}\Bigg\} \approx 0.00466554.$$

It is clear that 0.00018390 < 0.00466554, which demonstrates the first result described in Theorem 5.

Example 4. For $p \in (0, \frac{1}{1+\frac{1}{r}}]$, let $F(x) = \int_0^x (t^p+1)^{\frac{1}{q}} dt$ for $x \in (0, \infty)$. Then $|F'(x)|^q = x^p + 1$ is a p-convex mapping. If we take $a = 2$, $b = 4$, $x = 3$, $\alpha = 1.3$, $\lambda = \frac{1}{2}$, $r = 2$, $p = \frac{1}{4}$ and $q = 2$, then all the assumptions in Theorem 6 are satisfied.
The left-hand side term of (2.18) is:

$$\Bigg|(1-\lambda)p\frac{(x^p-a^p)^\alpha+(b^p-x^p)^\alpha}{b-a}F(x)+\lambda p\frac{(x^p-a^p)^\alpha F(a)+(b^p-x^p)^\alpha F(b)}{b-a} - \frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}\Big[({}^pI^\alpha_{x^-}F)(a)+({}^pI^\alpha_{x^+}F)(b)\Big]\Bigg| \approx 0.00033174.$$

The right-hand side term of (2.18) is:

$$2^{\frac{1}{p}-1-\frac{1}{r}}\Bigg\{E^{\frac{1}{r}}(a)\frac{(x^p-a^p)^{\alpha+1}}{b-a}\Big[C(1)|F'(x)|^q+D(1)|F'(a)|^q\Big]^{\frac{1}{q}} + E^{\frac{1}{r}}(b)\frac{(b^p-x^p)^{\alpha+1}}{b-a}\Big[C(1)|F'(x)|^q+D(1)|F'(b)|^q\Big]^{\frac{1}{q}}\Bigg\} \approx 0.02170969.$$

It is clear that 0.00033174 < 0.02170969, which demonstrates the first result described in Theorem 6.

Remark 2. *Theorems 3–6 provide an upper bound for the approximation involving the fractional integrals* $\frac{p^{\alpha+1}\Gamma(\alpha+1)}{b-a}[({}^pI^\alpha_{x^-}F)(a)+({}^pI^\alpha_{x^+}F)(b)]$. *Generally, the direct calculation of this kind of fractional integral is difficult because some of the integral functions are too complicated to be estimated. Moreover, some integral functions cannot even be expressed in term of elementary functions. Therefore, Theorems 3–6 provide powerful tools to deal with such integral functions. The numerical examples presented above show the validation of these theorems.*

4. Conclusions

In this paper, we establish a parameterized identity involving differentiable mappings and the Katugampola fractional integrals. By using it, we give four theorems in section 2, and each theorem provides the estimates of the upper bound for mappings whose first derivatives absolute values are p-convex mappings. More interesting estimates results can be derived by choosing different values for the parameters λ, p and α. What we want to emphasize here is that it is convenient to estimate these upper bounds from the perspective of numerical calculation. It is worth mentioning that our results contain, as special cases, the estimates of the upper bound of mappings for the classical Riemann–Liouville fractional integrals and Hadamard fractional integrals associated with p-convex mappings, respectively. With these ideas and techniques developed in this work, the interested readers can be inspired to explore this fascinating field of Katugampola fractional integral operators, which involve other related classes of mappings.

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Conflict of interest

The authors declare no conflict of interest.

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