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Research article

Some generalized fractional integral inequalities with nonsingular function as a kernel

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Abstract: Integral inequalities play a key role in applied and theoretical mathematics. The purpose of inequalities is to develop mathematical techniques in analysis. The goal of this paper is to develop a fractional integral operator having a non-singular function (generalized multi-index Bessel function) as a kernel and then to obtain some significant inequalities like Hermit Hadamard Mercer inequality, exponentially $(s-m)$-preinvex inequalities, Pólya-Szegö and Chebyshev type integral inequalities with the newly developed fractional operator. These results describe in general behave and provide the extension of fractional operator theory (FOT) in inequalities.

Keywords: convexity; generalized multi-index Bessel function; inequalities and integral operators; fractional derivatives and integrals
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1. Introduction

Fractional calculus signifies the identity of the distinguished materials in the modern research field due to its integrated applications in diverse regions such as mathematical physics, fluid dynamics, mathematical biology, etc. Convex function, exponentially convex function [1–5], related inequalities like as trapezium inequality, Ostrowski's inequality and Hermite Hadamard inequality, integrals [6–10] having succeed in mathematical analysis, approximation theory due to immense applications [11, 12] have great importance in mathematics theory. Many authors established quadrature rules in numerical analysis for approximate definite integrals. Recently, Pólya-Szeg and Chebyshev inequalities occupied immense space in the field analysis. Chebyshev [13] was introduced the well-known inequality called Chebyshev inequality.

In the literature of convex function, the Jensen inequality has gained much importance which describes a connection between an integral of the convex function and the value of the convex function of an interval [14–16]. Pshtiwan and Thabet [17] considered the modified Hermite Hadamard inequality in the context of fractional calculus using the Riemann-Liouville fractional integrals. Arran and Pshtiwan [18] discussed the Hermite Hadamard inequality results with fractional integrals and derivatives using Mittag-Leffler kernel. Pshtiwan and Thabet [19] constructed a connection between the Riemann-Liouville fractional integrals of a function concerning a monotone function with nonsingular kernel and Atangana-Baleanu. Pshtiwan and Brevik [20] obtained an inequality of Hermite Hadamard type for Riemann-Liouville fractional integrals, and proved the application of obtained inequalities on modified Bessel functions and q-digamma function. In [21], Set et al. introduced Grüss type inequalities by employing generalized k-fractional integrals. Recently, Nisar et al. [22] gave some new generalized fractional integral inequalities.

Very recently, the fractional conformable and proportional fractional integral operators were given in [23, 24]. Later on, Huang et al. [25] gave Hermite–Hadamard type inequalities by using fractional conformable integrals (FCI). Qi et al. [26] investigated Čebyšev type inequalities involving FCI. The Chebyshev type inequalities and certain Minkowski's type inequalities are found in [27–29]. Nisar et al. [30] have investigated some new inequalities for a class of n $(n \in \mathbb{N})$ positive, continuous, and decreasing functions by employing FCI. Rahman et al. [31] introduced Grüss type inequalities for k-fractional conformable integrals.

Some significant inequalities are given as applications of fractional integrals [32–38]. Recently, Rahman et al. [39,40] presented fractional integral inequalities involving tempered fractional integrals. Qiang et al. [41] discussed a fractional integral containing the Mittag-Leffler function in inequality theory and contributed Hadamard type inequality, continuity, and boundedness, upper bounds of that integral. Nisar et al. [42] established weighted fractional Pólya-Szegö and Chebyshev type integral inequalities by operating the generalized weighted fractional integral involving kernel function. The dynamical approach of fractional calculus [43–49] in the field of inequalities.
Grüss inequality [50] established for two integrable function as follows

$$|T(h,l)| \leq \frac{(k-K)(s-S)}{4}, \tag{1.1}$$

where the h and l are two integrable functions which are synchronous on $[a,b]$ and satisfy:

$$s \leq h(z) \leq K, s \leq l(y_1) \leq S, \quad z, y_1 \in [a,b] \tag{1.2}$$

for some $s, k, S, K \in \mathbb{R}$.

Pólya and Szegö [51] proved the inequalities

$$\frac{\int_a^b h^2(z)dz \int_b^a l^2(z)dz}{\left(\int_b^a h(z)l(z)dz\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{KS}{ks}} + \sqrt{\frac{ks}{KS}}\right)^2. \tag{1.3}$$

Dragomir and Diamond [52], proves the inequality by using the Pólya-szegö inequality

$$|T(h,l)| \leq \frac{(S-s)(K-k)}{4(b-a)^2\sqrt{skSK}} \int_a^b h(z)l(z)dz \tag{1.4}$$

where h and l are two integrable functions which are synchronous on $[a,b]$, and

$$0 < s \leq h(z) \leq S < \infty, 0 < k \leq l(y_1) \leq K < \infty, \quad z, y_1 \in [a,b] \tag{1.5}$$

for some $s, k, S, K \in \mathbb{R}$.

The aim of this paper is to estimate a new version of Pólya-Szegö inequality, Chebyshev integral inequality, and Hermite Hadamard type integral inequality by a fractional integral operator having a nonsingular function (generalized multi-index Bessel function) as a kernel, and these established results have great contribution in the field of inequalities. The Hermite Hadamard type integral inequality provides the upper and lower estimate to find the average integral for the convex function of any defined interval.

The structure of the paper follows:

In section 2, we present some well-known definitions and mathematical preliminaries. The new generalized fractional integral with nonsingular function as a kernel is defined in section 3. In section 4, we present Hermite Hadamard type Mercer inequality of new designed fractional integral operator with nonsingular function (generalized multi-index Bessel function) as a kernel. some inequalities of $(s-m)$-preinvex function involving new designed fractional integral operator with nonsingular function (generalized multi-index Bessel function) as a kernel are presented in section 5. Here section 6 and 7, we present Pólya-Szegö and Chebyshev integral inequalities involving generalized fractional integral operator with nonsingular function as a kernel, respectively.

2. Preliminaries

Definition 2.1. *The inequality holds for the convex function if a mapping $g : K \to \mathbb{R}$ exist as*

$$g(\delta y_1 + (1-\delta)y_2) \leq \delta g(y_1) + (1-\delta)g(y_2), \tag{2.1}$$

where $\forall y_1, y_2 \in K$ and $\delta \in [0,1]$.

Definition 2.2. *The inequality derived by Hermite [53] call as Hermite Hadamard inequality*

$$g\left(\frac{y_1+y_2}{2}\right) \leq \frac{1}{y_2-y_1}\int_{y_1}^{y_2} g(t)dt \leq \frac{g(y_1)+g(y_2)}{2}, \tag{2.2}$$

where $y_1, y_2 \in I$, with $y_2 \neq y_1$, if $g : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function.

Definition 2.3. *Let $y_j \in K$ for all $j \in I_n$, $\omega_j > 0$ such that $\sum_{j\in I_n} \omega_j = 1$. Then the Jensen inequality holds*

$$g\Big(\sum_{j\in I_n} \omega_j y_j\Big) \leq \sum_{j\in I_n} \omega_j g(y_j), \tag{2.3}$$

exist if $g : k \to \mathbb{R}$ is convex function.
Mercer [54] derived the Mercer inequality by applying the Jensen inequality and properties of convex function.

Definition 2.4. *Let $y_j \in K$ for all $j \in I_n$, $\omega_j > 0$ such that $\sum_{j\in I_n} \omega_j = 1$, $m = \min_{j\in I_n}\{y_j\}$ and $n = \max_{j\in I_n}\{y_j\}$. Then the inequality holds for convex function as*

$$g\Big(m + n - \sum_{i\in I_n} \omega_j y_j\Big) \leq g(m) + g(n) - \sum_{j\in I_n} \omega_j g(y_j), \tag{2.4}$$

if $g : k \to \mathbb{R}$ is convex function.

Definition 2.5. *[55] The inequality holds for exponentially convex function, if a real valued mapping $g : K \to \mathbb{R}$ exist as*

$$g(\delta y_1 + (1-\delta)y_2) \leq \delta \frac{g(y_1)}{e^{\theta y_1}} + (1-\delta)\frac{g(y_2)}{e^{\theta y_2}}, \tag{2.5}$$

where $\forall y_1, y_2 \in K$ and $\delta \in [0,1]$ and $\theta \in \mathbb{R}$.

Suppose that $\Omega \subseteq \mathbb{R}^n$ is a set. Let $g : \Omega \to \mathbb{R}$ continuous function and let $\xi : \Omega \times \Omega \to \mathbb{R}^n$ be continuous function:

Definition 2.6. *[56] With respect to bifunction $\xi(.,.)$ a set Ω is called a invex set, if*

$$y_1 + \delta\xi(y_2, y_1), \tag{2.6}$$

where $\forall y_1, y_2 \in \Omega, \delta \in [0,1]$.

Definition 2.7. *[57] A invex set Ω and a mapping g with respect to $\xi(.,.)$ is called a preinvex function, as*

$$g(y_1 + \delta\xi(y_2, y_1)) \leq (1-\delta)g(y_1) + \delta g(y_2), \tag{2.7}$$

where $\forall\ y_1, y_2 + \xi(y_2, y_1) \in \Omega, \delta \in [0,1]$.

Definition 2.8. *A invex set Ω with real valued mapping g and respect to $\xi(.,.)$ is called a exponentially preinvex, if the inequality*

$$g(y_1 + \delta\xi(y_2, y_1)) \leq (1-\delta)\frac{g(y_1)}{e^{\theta y_1}} + \delta\frac{g(y_2)}{e^{\theta y_2}}, \tag{2.8}$$

where for all $y_1, y_2 + \xi(y_2, y_1) \in \Omega, \delta \in [0,1]$ and $\theta \in \mathbb{R}$.

Definition 2.9. *A invex set Ω with real valued mapping g and respect to $\xi(.,.)$ is called a exponentially s-preinvex, if*

$$g(y_1 + \delta\xi(y_2, y_1)) \leq (1-\delta)^s\frac{g(y_1)}{e^{\theta y_1}} + \delta^s\frac{g(y_2)}{e^{\theta y_2}}, \tag{2.9}$$

where for all $y_1, y_2 + \xi(y_2, y_1) \in \Omega, \delta \in [0,1]$, $s \in (0,1]$ and $\theta \in \mathbb{R}$.

Definition 2.10. *A invex set Ω with real valued mapping g and respect to $\xi(.,.)$ is called exponentially (s-m)-preinvex, if*

$$g(y_1 + m\delta\xi(y_2, y_1)) \leq (1-\delta)^s \frac{g(y_1)}{e^{\theta y_1}} + m\delta^s \frac{g(y_2)}{e^{\theta y_2}}, \tag{2.10}$$

where for all $y_1, y_2 + \xi(y_2, y_1) \in \Omega$, $\delta, m \in [0,1]$ and $\theta \in \mathbb{R}$.

Definition 2.11. *[58] Generalized multi-index Bessel function is defined by Choi et al as follows*

$$\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(z) = \sum_{s=0}^{\infty} \frac{(\lambda)_{\sigma s}}{\prod_{j=1}^{m} \Gamma(\xi_j s + \delta_j + 1)} \frac{(-z)^s}{s!}, \tag{2.11}$$

where $\xi_j, \delta_j, \lambda \in \mathbb{C}$, $(j = 1, \cdots, m)$, $\Re(\lambda) > 0, \Re(\delta_j) > -1, \sum_{j=1}^{m} \Re(\xi)_j > max\{0 : \Re(\sigma) - 1\}, \sigma > 0.$

Definition 2.12. *[58] Pohhammer symbol is defined for $\lambda \in \mathbb{C}$ as follows*

$$(\lambda)_s = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+s-1), & s \in \mathbb{N} \\ 1, & s = 0, \end{cases} \tag{2.12}$$

$$= \frac{\Gamma(\lambda+s)}{\Gamma(\lambda)}, \qquad (\lambda \in \mathbb{C}/\mathbb{Z}_0) \tag{2.13}$$

where Γ being the Gamma function.

3. Fractional integral operator with nonsingular function

This section presents a generalized fractional integral operator with a nonsingular function (multi-index Bessel function) as a kernel.

Definition 3.1. *Let $\xi_j, \delta_j, \lambda, \zeta \in \mathbb{C}, (j = 1, \cdots, m), \Re(\lambda) > 0, \Re(\delta_j) > -1, \sum_{j=1}^{m} \Re(\xi_j) > max\{0 : \Re(\sigma) - 1\}, \sigma > 0$. Let $g \in L\ [y_1, y_2]$ and $t \in [y_1, y_2]$. Then the corresponding left sided and right sided generalized integral operators having generalized multi-index Bessel function defined as:*

$$(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+} g)(z) = \int_{y_1}^{z} (z-t)^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j}) g(t) dt, \tag{3.1}$$

and

$$(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-} g)(z) = \int_{z}^{y_2} (t-z)^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(t-z)^{\xi_j}) g(t) dt. \tag{3.2}$$

Remark 3.1. *The special cases of generalized fractional integrals with nonsingular kernel are given below:*

1. *If set $j = m = 1$, $\sigma = 0$ and limits from $[0, z]$ in Eq (3.1), we get a fractional integral defined by Srivastava and Singh in [59] as*

$$(\text{Œ}^{\xi_1,\delta_1}_{\lambda,0,\zeta;0^+} g)(z) = \int_{0}^{z} (z-t)^{\delta_1} J^{\xi_1}_{\delta_1}(\zeta(z-t)^{\xi_1}) g(t) dt = f(z). \tag{3.3}$$

2. *If set $j = m = 1$, $\delta_1 = \delta_1 - 1$ in Eq (3.1), we have a fractional integral defined by Srivastava and Tomovski in [60] as*

$$(\text{Œ}^{\xi_1,\delta_1-1}_{\lambda,\sigma,\zeta;y_1^+} g)(z) = (\mathscr{E}^{\zeta;\lambda,\sigma}_{y_1^+;\xi-1,\delta_1} g)(z). \tag{3.4}$$

3. *If set* $j = m = 1$, $\delta_1 = \delta_1 - 1$, $\zeta = 0$ *in Eq* (3.1)*, we get a Riemann-Liouville fractional integral operator defined in [61] as*

$$(\text{Œ}^{\xi_1,\delta_1}_{\lambda,\sigma,\zeta;y_1^+}g)(z) = (I^{\delta_1}_{y_1^+}g)(z). \tag{3.5}$$

4. *If set* $j = m = 1$, $\sigma = 1$, $\delta_1 = \delta_1 - 1$, *in Eq* (3.1) *and Eq* (3.2)*, we get the fractional integral operator defined by Prabhakar in [62] as follows*

$$(\text{Œ}^{\xi_1,\delta_1-1}_{\lambda,1,\zeta;y_1^+}g)(z) = \mathfrak{E}^*(\xi_1,\delta_1;\lambda;\zeta)g(z) =^\circ g(z) \tag{3.6}$$

$$(\text{Œ}^{(\xi_1,\delta_1-1)}_{\lambda,1,\zeta;y_2^-}g)(z) = \mathfrak{E}^*(\xi_1,\delta_1;\lambda;\zeta)g(z). \tag{3.7}$$

Lemma 3.1. *From generalized fractional integral operator, we have*

$$\begin{aligned}
&(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\\
&= \int_{y_1}^{z}(z-t)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})dt\\
&= \int_{y_1}^{z}(z-t)^{\delta_j}\sum_{s=0}^{\infty}\frac{(\lambda)_{\sigma s}(-\zeta)^s}{\prod_{j=1}^{m}\Gamma(\xi_j s+\delta_j+1)}\frac{(z-t)^{\xi_j s}}{s!}dt\\
&= \sum_{s=0}^{\infty}\frac{(\lambda)_{\sigma s}(-\zeta)^s}{\prod_{j=1}^{m}\Gamma(\xi_j s+\delta_j+1)s!}\int_{y_1}^{z}(z-t)^{\xi_j s+\delta_j}dt\\
&= (z-y_1)^{\delta_j+1}\sum_{s=0}^{\infty}\frac{(\lambda)_{\sigma s}(-\zeta)^s}{\prod_{j=1}^{m}\Gamma(\xi_j s+\delta_j+1)s!}\frac{(z-y_1)^{\xi_j s}}{\xi_j s+\delta_j+1}.
\end{aligned} \tag{3.8}$$

Hence, the Eq (3.8) *becomes*

$$(\text{Œ}^{(\xi_j,\delta_j+1)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z) = (z-y_1)^{\delta_j+1}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m+1,\sigma}(\zeta(z-y_1)^{\xi_j}), \tag{3.9}$$

and similarly we have

$$(\text{Œ}^{(\xi_j,\delta_j+1)_m}_{\lambda,\sigma,\zeta;y_2^-}1)(z) = (y_2-z)^{\delta_j+1}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m+1,\sigma}(\zeta(y_2-z)^{\xi_j}). \tag{3.10}$$

4. Hadamard type Mercer inequality with fractional operators

In this section, we derive Hermite Hadamard type Mercer inequality of new designed fractional integral operator in a generalized multi-index Bessel function using a kernel.

Theorem 4.1. *Let* $g:[m,n]\to(0,\infty)$ *is convex function such that* $g\in\chi_c(m,n)$, $\forall x,y\in[m,n]$ *and the operator defined in Eq* (5.2) *in the form of left sense operator and Eq* (3.2) *in the form of right sense operator then we have*

$$\begin{aligned}
&g\Big(m+n-\frac{x+y}{2}\Big)\\
&\le g(m)+g(n)-\frac{[\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m+1,\sigma}(\zeta)]^{-1}}{2(y-x)}\Big[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;x^+}g(y)+\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y^-}g(x)\Big] &(4.1)\\
&\le g(m)+g(n)-\frac{g(x)+g(y)}{2}. &(4.2)
\end{aligned}$$

Proof. Consider the mercer inequality

$$g\Big(m+n-\frac{y_1+y_2}{2}\Big) \le g(m)+g(n)-\frac{g(y_1)+g(y_2)}{2}, \quad \forall y_1, y_2 \in [m,n]. \tag{4.3}$$

Let $x, y \in [m,n]$, $t \in [z-1,z]$, $y_1 = (z-t)x + (1-z+t)y$ and $y_2 = (1-z+t)x + (z-t)y$ then inequality (4.3) becomes

$$\begin{aligned} g\Big(m+n-\frac{y_1+y_2}{2}\Big) &\le g(m)+g(n) \\ &- \frac{g((z-t)x+(1-z+t)y)+g(1-z+t)x+(z-t)y)}{2}. \end{aligned} \tag{4.4}$$

Multiply both sides of Eq (4.4) by $(z-t)^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})$ and integrating with respect to t from $[z-1,z]$, we get

$$\begin{aligned} &\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m+1,\sigma}(\zeta) g\Big(m+n-\frac{x+y}{2}\Big) \\ &\le \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m+1,\sigma}(\zeta)[g(m)+g(n)] - \frac{1}{2}\Big[\int_{z-1}^{z} (z-t)^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j}) \\ &\quad \times [g((z-t)y_1+(1-z+t)y_2)+g(1-z+t)x+(z-t)y_2]\Big]dt \\ &= \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m+1,\sigma}(\zeta)[g(m)+g(n)] - \frac{1}{2}\Big[\int_{x}^{y} (\frac{y-u}{y-x})^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(\frac{y-u}{y-x})^{\xi_j}) \\ &\quad \times \frac{g(u)}{(y-x)}du + \int_{y}^{x} (\frac{u-x}{y-x})^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(\frac{u-x}{y-x})^{\xi_j}) \frac{g(u)}{(y-x)}du\Big] \\ &= \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m+1,\sigma}(\zeta)[g(m)+g(n)] - \frac{1}{2(y-x)}\Big[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;x^+} g(y) \\ &\quad + \text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y^-} g(x)\Big], \end{aligned}$$

we get the desired inequality, as

$$\begin{aligned} &g\Big(m+n-\frac{x+y}{2}\Big) \\ &\le g(m)+g(n) - \frac{[\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m+1,\sigma}(\zeta)]^{-1}}{2(y-x)}\Big[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;x^+} g(y) + \text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y^-} g(x)\Big]. \end{aligned} \tag{4.5}$$

Thus, we get the inequality (4.1). Let $t \in [z-1,z]$. From the convexity of function g we have

$$\begin{aligned} g\Big(\frac{x+y}{2}\Big) &= \frac{g[(z-t)x+(1-z+t)y+(1-z+t)x+(z-t)y]}{2} \\ &\le \frac{g((z-t)x+(1-z+t)y)+g((1-z+t)x+(z-t)y)}{2}. \end{aligned} \tag{4.6}$$

Both sides multiply of Eq (4.6) by $(z-t)^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})$ and integrating with respect to t from $[z-1,z]$, we obtain

$$\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta) g\Big(\frac{x+y}{2}\Big)$$

$$\leq \int_{z-1}^{z} (z-t)^{\delta_j} \mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(z-t)^{\xi_j})$$
$$\times [g((z-t)x + (1-z+t)y) + g((1-z+t)x + (z-t)y)]dt$$
$$= \frac{1}{2(y-x)}\Big[\text{Œ}_{\lambda,\sigma,\zeta;x^+}^{(\xi_j,\delta_j)_m} g(y) + \text{Œ}_{\lambda,\sigma,\zeta;y^-}^{(\xi_j,\delta_j)_m} g(x)\Big].$$

We get the inequality of negative sign

$$-g\Big(\frac{x+y}{2}\Big) \geq -\frac{[\mathrm{J}_{(\delta_j)_m+1,\sigma}^{(\xi_j)_m,\lambda}(\zeta)]^{-1}}{2(y-x)}\Big[\text{Œ}_{\lambda,\sigma,\zeta;x^+}^{(\xi_j,\delta_j)_m} g(y) + \text{Œ}_{\lambda,\sigma,\zeta;y^-}^{(\xi_j,\delta_j)_m} g(x)\Big]. \tag{4.7}$$

By adding $g(m) + g(n)$ of both sides of inequality (4.7), we have

$$g(m)+g(n) - g\Big(\frac{x+y}{2}\Big)$$
$$\geq g(m) + g(n) - \frac{[\mathrm{J}_{(\delta_j)_m+1,\sigma}^{(\xi_j)_m,\lambda}(\zeta)]^{-1}}{2(y-x)}\Big[\text{Œ}_{\lambda,\sigma,\zeta;x^+}^{(\xi_j,\delta_j)_m} g(y) + \text{Œ}_{\lambda,\sigma,\zeta;y^-}^{(\xi_j,\delta_j)_m} g(x)\Big].$$

Hence, we get the inequality (4.2). □

Theorem 4.2. *Let $g : [m,n] \to (0,\infty)$ is convex function such that $g \in \chi_c(m,n)$ then we have the following inequalities:*

$$g\Big(m+n-\frac{x+y}{2}\Big)$$
$$\leq \frac{[\mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta)]^{-1}}{2(y-x)}\Big[\text{Œ}_{\lambda,\sigma,\zeta;(m+n-y)^+}^{(\xi_j,\delta_j)_m} g(m+n-x) + \text{Œ}_{\lambda,\sigma,\zeta;(m+n-x)^-}^{(\xi_j,\delta_j)_m} g(m+n-y)\Big]. \tag{4.8}$$
$$\leq \frac{g(m+n-x)+g(m+n-y)}{2} \leq g(m)+g(n) - \frac{g(m)+g(n)}{2}. \tag{4.9}$$

Where $\forall x, y \in [m,n]$.

Proof. We see that from the convexity of g as

$$g\Big(m+n-\frac{y_1+y_2}{2}\Big) = g(\frac{m+n-y_1+m+n-y_2}{2})$$
$$\leq \frac{1}{2}[g(m+n-y_1) + g(m+n-y_2)], \qquad \forall y_1, y_2 \in [m,n]. \tag{4.10}$$

Let $x, y \in [m,n]$, $t \in [z-1,z]$, $m+n-y_1 = (z-t)(m+n-x) + (1-z+t)(m+n-y)$, $m+n-y_2 = (1-z+t)(m+n-x) + (z-t)(m+n-y)$, then inequality (4.10) gives

$$g\Big(m+n-\frac{y_1+y_2}{2}\Big)$$
$$\leq \frac{1}{2}g[(z-t)(m+n-x) + (1-z+t)(m+n-y)]$$
$$+\frac{1}{2}g[(1-z+t)(m+n-x) + (z-t)(m+n-y)], \tag{4.11}$$

multiply of both sides of inequality (4.11) by $(z-t)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})$ then integrate with respect to t from $[z-1,z]$, we get

$$\begin{aligned}
&\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta)g\Big(m+n-\frac{x+y}{2}\Big)\\
\leq&\frac{1}{2}\int_{z-1}^{z}(z-t)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})g[(z-t)(m+n-x)+(1-z+t)(m+n-y)]dt\\
+&\frac{1}{2}\int_{z-1}^{z}(z-t)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})g[(1-z+t)(m+n-x)+(z-t)(m+n-y)]dt\\
=&\frac{1}{2(y-x)}\Big[\int_{m+n-y}^{m+n-x}(\frac{u-(m+n-y)}{y-x})^{\delta_j})\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(\frac{u-(m+n-y)}{y-x})^{\xi_j})g(u)du\\
+&\int_{m+n-x}^{m+n-y}(\frac{(m+n-y)-u}{y-x})^{\delta_j})\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(\frac{(m+n-y)-u}{y-x})^{\xi_j})g(u)du\Big]\\
=&\frac{1}{2(y-x)}\Big[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(m+n-y)^+}g(m+n-x)+\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(m+n-x)^-}g(m+n-y)\Big].
\end{aligned}$$

Thus, we get the inequality (4.8)

$$\begin{aligned}
&g\Big(m+n-\frac{x+y}{2}\Big)\\
\leq&\frac{[\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta)]^{-1}}{2(y-x)}\Big[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(m+n-y)^+}g(m+n-x)+\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(m+n-x)^-}g(m+n-y)\Big].
\end{aligned}$$

From the convexity of g, we obtain

$$\begin{aligned}
&g((z-t)(m+n-x)+(1-z+t)(m+n-y))\\
&\leq(z-t)g(m+n-x)+(1-z+t)g(m+n-y),
\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}
&g((1-z+t)(m+n-x)+(z-t)(m+n-y))\\
&\leq(1-z+t)g(m+n-x)+(z-t)g(m+n-y).
\end{aligned}\tag{4.13}$$

Adding up the above inequalities and applying Jensen-Mercer inequality, we get

$$\begin{aligned}
&g((z-t)(m+n-x)+(1-z+t)(m+n-y))\\
&+g((1-z+t)(m+n-x)+(z-t)(m+n-y))\\
&\leq g(m+n-x)+g(m+n-y)\\
&\leq 2[g(m)+g(n)]-[g(x)+g(y)].
\end{aligned}\tag{4.14}$$

Multiply both sides of inequality (4.14) by $(z-t)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})$ and then integrating with respect to t from $[z-1,z]$ we obtain the two inequalities (4.9). □

5. $(s-m)$ preinvex inequalities involving fractional operators

In this section, we derive some inequalities of $(s-m)$ preinvex function involving new designed fractional integral operator $Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta} g)(z)$ having generalized multi-index Bessel function as its kernel in the form of theorems.

Theorem 5.1. *Suppose a real valued function $g : [y_1, y_1 + \xi(y_2, y_1)] \to R$ be exponentially (s-m) preinvex function, then the following fractional inequality holds:*

$$(Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+} g)(z) + (Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+} g)(z) \leq \frac{(z-y_1)}{s+1}(Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+} 1)(z)\Big[\frac{g(y_1)}{e^{\theta_1 y_1}} + m\frac{g(z)}{e^{\theta_1 z}}\Big]$$
$$+ \frac{(y_1+\xi(y_2,y_1)-z)}{s+1}(Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+} 1)(z)\Big[\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_2(y_1+\xi(y_2,y_1))}} + m\frac{g(z)}{e^{\theta_2 z}}\Big].$$

$\forall\, z \in [y_1, y_1 + \xi(y_2, y_1)]$, $\theta_1, \theta_2 \in R$.

Proof. Let $z \in [y_1, y_1+\xi(y_2, y_1)]$, and then for $t \in [y_1, z)$ and $\delta_j > -1$, we have the subsequent inequality

$$(z-t)^{\delta_j} J^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j}) \leq (z-y_1)^{\delta_j} J^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j}). \tag{5.1}$$

For g is exponentially (s-m)-preinvex function, we obtain

$$g(t) \leq \Big(\frac{z-t}{z-y_1}\Big)^s \frac{g(y_1)}{e^{\theta_1 y_1}} + m\Big(\frac{t-y_1}{z-y_1}\Big)^s \frac{g(z)}{e^{\theta_1 z}}. \tag{5.2}$$

Taking product (5.1) and (5.2), and integrating with respect to t from y_1 to z, we get

$$\int_{y_1}^{z} (z-t)^{\delta_j} J^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j}) g(t)dt \leq \int_{y_1}^{z} (z-y_1)^{\delta_j} J^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j})$$
$$\times\Big[\Big(\frac{z-t}{z-y_1}\Big)^s \frac{g(y_1)}{e^{\theta_1 y_1}} + m\Big(\frac{t-y_1}{z-y_1}\Big)^s \frac{g(z)}{e^{\theta_1 z}}\Big]dt, \tag{5.3}$$

apply definition (3.1) in Eq (5.3), we have

$$(Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+} g)(z) \leq \frac{(z-y_1)}{s+1}(Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+} 1)(z)\Big[\frac{g(y_1)}{e^{\theta_1 y_1}} + m\frac{g(z)}{e^{\theta_1 z}}\Big]. \tag{5.4}$$

Analogously for $t \in (z, y_1 + \xi(y_2, y_1)]$ and $\mu_j > -1$, we have

$$(t-z)^{\mu_j} J^{(\xi_j)_m,\lambda}_{(\mu_j)_m,\sigma}(\zeta(t-z)^{\xi_j}) \leq (y_1+\xi(y_2,y_1)-z)^{\mu_j} J^{(\xi_j)_m,\lambda}_{(\mu_j)_m,\sigma}(\zeta(y_1+\xi(y_2,y_1)-z)^{\xi_j}). \tag{5.5}$$

Further, the exponentially (s-m) convexity of g, we get

$$g(t) \leq \Big(\frac{t-z}{y_1+\xi(y_2,y_1)-z}\Big)^s \frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_2(y_1+\xi(y_2,y_1))}} + m\Big(\frac{y_1+\xi(y_2,y_1)-t}{y_1+\xi(y_2,y_1)-z}\Big)^s \frac{g(z)}{e^{\theta_2 z}}. \tag{5.6}$$

Taking product of (5.5) and (5.6) and integrating with respect to t from z to $y_1 + \xi(y_2, y_1)$, we have

$$\int_{z}^{y_1+\xi(y_2,y_1)} (t-z)^{\mu_j} J^{(\xi_j)_m,\lambda}_{(\mu_j)_m,\sigma}(\zeta(t-z)^{\xi_j}) g(t)dt$$

$$\leq \int_{z}^{y_1+\xi(y_2,y_1)} (y_1+\xi(y_2,y_1)-z)^{\mu_j} \mathrm{J}_{(\mu_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(y_1+\xi(y_2,y_1)-z)^{\xi_j})$$
$$\times\Big[\Big(\frac{t-z}{y_1+\xi(y_2,y_1)-z}\Big)^s \frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_2(y_1+\xi(y_2,y_1))}} + m\Big(\frac{y_1+\xi(y_2,y_1)-t}{y_1+\xi(y_2,y_1)-z}\Big)^s \frac{g(z)}{e^{\theta_2 z}}\Big]dt, \quad (5.7)$$

apply the definition (3.1) in inequality (5.7), we have

$$(\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} g)(z)$$
$$\leq \frac{(y_1+\xi(y_2,y_1)-z)}{s+1} (\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} 1)(z)\Big[\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_2(y_1+\xi(y_2,y_1))}} + m\frac{g(z)}{e^{\theta_2 z}}\Big]. \quad (5.8)$$

Now, add the inequalities (5.4) and (5.8), we get the result

$$(\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m} g)(z) + (\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} g)(z)$$
$$\leq \frac{(z-y_1)}{s+1}(\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m} 1)(z)\Big[\frac{g(y_1)}{e^{\theta_1 y_1}} + m\frac{g(z)}{e^{\theta_1 z}}\Big]$$
$$+ \frac{(y_1+\xi(y_2,y_1)-z)}{s+1}(\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} 1)(z)\Big[\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_2(y_1+\xi(y_2,y_1))}} + m\frac{g(z)}{e^{\theta_2 z}}\Big].$$

□

Corollary 5.1. *If* $g \in L_\infty[y_1, y_1+\xi(y_2,y_1)]$*, then under the assumption of theorem* (5.1)*, we have*

$$(\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m} g)(z) + (\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} g)(z)$$
$$\leq \frac{\|g\|_\infty}{s+1}\Big[(z-y_1)(\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m} 1)(z)\Big(\frac{1}{e^{\theta_1 y_1}} + m\frac{1}{e^{\theta_1 z}}\Big)$$
$$+ (y_1+\eta(y_2,y_1)-z)(\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} 1)(z)\Big(\frac{1}{e^{\theta_2(y_1+\xi(y_2,y_1))}} + m\frac{1}{e^{\theta_2 z}}\Big)\Big].$$

Corollary 5.2. *Setting* $m = 1$ *and* $g \in L_\infty[y_1, y_1+\xi(y_2,y_1)]$*, then under the assumption of theorem* (5.1)*, we have*

$$(\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m} g)(z) + (\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} g)(z)$$
$$\leq \frac{\|g\|_\infty}{s+1}\Big[(z-y_1)(\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m} 1)(z)\Big(\frac{1}{e^{\theta_1 y_1}} + m\frac{1}{e^{\theta_1 z}}\Big)$$
$$+ (y_1+\xi(y_2,y_1)-z)(\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} 1)(z)\Big(\frac{1}{e^{\theta_2(y_1+\xi(y_2,y_1))}} + \frac{1}{e^{\theta_2 z}}\Big)\Big].$$

Corollary 5.3. *Setting* $m = s = 1$ *and* $g \in L_\infty[y_1, y_1+\xi(y_2,y_1)]$*, then under the assumption of theorem* (5.1)*, we have*

$$(\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m} g)(z) + (\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} g)(z)$$
$$\leq \frac{\|g\|_\infty}{2}\Big[(z-y_1)(\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m} 1)(z)\Big(\frac{1}{e^{\theta_1 y_1}} + m\frac{1}{e^{\theta_1 z}}\Big)$$
$$+ (y_1+\xi(y_2,y_1)-z)(\text{Œ}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}^{(\xi_j,\mu_j)_m} 1)(z)\Big(\frac{1}{e^{\theta_2(y_1+\xi(y_2,y_1))}} + \frac{1}{e^{\theta_2 z}}\Big)\Big].$$

Corollary 5.4. *Setting $\xi(y_2, y_1) = y_2 - y_1$ and $g \in L_\infty[y_1, y_2]$, then under the assumption of theorem* (5.1)*, we have*

$$\begin{aligned}(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(z) + (\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^+}g)(z)&\\ \leq \frac{\|g\|_\infty}{s+1}\Big[(z-y_1)(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\Big(\frac{1}{e^{\theta_1 y_1}} + m\frac{1}{e^{\theta_1 z}}\Big)&\\ + (y_2 - z)(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^+}1)(z)\Big(\frac{1}{e^{\theta_2 y_2}} + \frac{1}{e^{\theta_2 z}}\Big)\Big].&\end{aligned}$$

Theorem 5.2. *Suppose a real value function $g : [y_1, y_1 + \xi(y_2, y_1)] \to R$ is differentiable and $|g|'$ is exponentially (s-m) preinvex, then the following fractional inequality for* (3.1) *and* (3.2) *holds:*

$$\begin{aligned}\Big|(\text{Œ}^{(\xi_j)_m,(\delta_j-1)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(z) + (\text{Œ}^{(\xi_j)_m,(\mu_j-1)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}g)(z) - [(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)]g(y_1)&\\ - [(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)]g(y_1+\xi(y_2,y_1))\Big| \leq \frac{(z-y_1)}{s+1}(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\Big[\frac{|g'(y_1)|}{e^{\theta_1 y_1}} + m\frac{|g'(z)|}{e^{\theta_1 z}}\Big]&\\ + \frac{((y_1+\xi(y_2,y_1)) - z)}{s+1}(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)\Big[\frac{|g'(y_1+\xi(y_2,y_1))|}{e^{\theta_1(y_1+\xi(y_2,y_1))}} + m\frac{|g'(z)|}{e^{\theta_1 z}}\Big].&\end{aligned}$$

$\forall\ z \in [y_1, y_1 + \xi(y_2, y_1)]$, $\theta_1, \theta_2 \in \mathbb{R}$.

Proof. Let $z \in [y_1, y_1 + \xi(y_2, y_1)]$, $t \in [y_1, z)$, and applying exponentially (s-m) preinvex of $|g|'$, we get

$$|g'(t)| \leq \Big(\frac{z-t}{z-y_1}\Big)^s\frac{|g'(y_1)|}{e^{\theta_1 y_1}} + m\Big(\frac{t-y_1}{z-x_1}\Big)^s\frac{|g'(z)|}{e^{\theta_1 z}}. \tag{5.9}$$

Get the inequality (5.9), we have

$$g'(t) \leq \Big(\frac{z-t}{z-y_1}\Big)^s\frac{|g'(y_1)|}{e^{\theta_1 y_1}} + m\Big(\frac{t-y_1}{z-y_1}\Big)^s\frac{|g'(x)|}{e^{\theta_1 z}}. \tag{5.10}$$

Subsequently inequality as:

$$(z-t)^{\delta_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,k}(\zeta(z-t)^{\xi_j}) \leq (z-y_1)^{\delta_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j}). \tag{5.11}$$

Conducting product of inequality (5.10) and (5.11), we have

$$\begin{aligned}(z-t)^{\delta_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})g'(t) \leq (z-y_1)^{\delta_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j})&\\ \times\Big[\Big(\frac{z-t}{z-y_1}\Big)^s\frac{|g'(y_1)|}{e^{\theta_1 y_1}} + m\Big(\frac{t-y_1}{z-y_1}\Big)^s\frac{|g'(x)|}{e^{\theta_1 z}}\Big],&\end{aligned} \tag{5.12}$$

integrating before mention inequality with respect to t from y_1 to z, we have

$$\begin{aligned}&\int_{y_1}^{z}(z-t)^{\delta_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})g'(t)dt\\ &\leq \int_{y_1}^{z}(z-y_1)^{\delta_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,k}(\zeta(z-y_1)^{\xi_j})\Big[\Big(\frac{z-t}{z-y_1}\Big)^s\frac{|g'(y_1)|}{e^{\theta_1 y_1}} + m\Big(\frac{t-y_1}{z-y_1}\Big)^s\frac{|g'(z)|}{e^{\theta_1 z}}\Big]dt\end{aligned}$$

$$= \frac{(z-y_1)}{s+1}(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\Big[\frac{|g'(y_1)|}{e^{\theta_1 y_1}} + m\frac{|g'(z)|}{e^{\theta_1 z}}\Big]. \tag{5.13}$$

Now, solving left side of (5.13) by putting $z-t=\alpha$, then we have

$$\int_{y_1}^{z}(z-t)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})g'(t)dt = \int_0^{z-y_1}\alpha^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(\alpha)^{\xi_j})g'(z-\alpha)d\alpha$$
$$= -(z-y_1)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j})g(y_1) + \int_0^{z-y_1}\alpha^{\delta_j-1}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m-1,\sigma}(\zeta(\alpha)^{\xi_j})g(z-\alpha)d\alpha.$$

Now, again subsisting $z-\alpha=t$, we get

$$\int_{y_1}^{z}(z-t)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-t)^{\xi_j})g'(t)dt$$
$$= \int_{y_1}^{z}(z-t)^{\delta_j-1}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m-1,\sigma}(\zeta(z-t)^{\xi_j})g(t)dt - (z-y_1)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j})g(y_1)$$
$$= (\text{Œ}^{(\xi_j)_m,(\delta_j-1)_m}_{\lambda,\sigma,\zeta:y_1^+}g)(z) - [(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)]g(y_1).$$

Therefore, the inequality (5.13) have the following form

$$(\text{Œ}^{(\xi_j)_m,(\delta_j-1)_m}_{\lambda,\sigma,\zeta:y_1^+}g)(x) - [(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)]g(y_1)$$
$$\leq \frac{(z-y_1)}{s+1}(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\Big[\frac{|g'(y_1)|}{e^{\theta_1 y_1}} + m\frac{|g'(z)|}{e^{\theta_1 z}}\Big]. \tag{5.14}$$

Also from (5.9), we get

$$g'(t) \geq -\Big(\frac{z-t}{z-y_1}\Big)^s\frac{|g'(y_1)|}{e^{\theta_1 y_1}} - m\Big(\frac{t-y_1}{z-y_1}\Big)^s\frac{|g'(z)|}{e^{\theta_1 z}}. \tag{5.15}$$

Adopting the same procedure as we have done for (5.10), we obtain

$$(\text{Œ}^{(\xi_j)_m,(\delta_j-1)_m}_{\lambda,\sigma,\zeta:y_1^+}g)(z) - [(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)]g(y_1)$$
$$\geq \frac{-(z-y_1)}{s+1}(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\Big[\frac{|g'(y_1)|}{e^{\theta_1 y_1}} + m\frac{|g'(z)|}{e^{\theta_1 z}}\Big]. \tag{5.16}$$

From (5.14) and (5.16), we get

$$\Big|(\text{Œ}^{(\xi_j)_m,(\delta_j-1)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(z) - [(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)]g(y_1)\Big|$$
$$\leq \frac{(z-y_1)}{s+1}(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\Big[\frac{|g'(y_1)|}{e^{\theta_1 y_1}} + m\frac{|g'(z)|}{e^{\theta_1 z}}\Big]. \tag{5.17}$$

Now, we let $z \in [y_1, y_1+\eta(y_2,y_1)]$ and $t \in (z, y_1+\xi(y_2,y_1)]$, and by exponentially (s-m) preinvex of $|g'|$, we get

$$|g'(t)| \leq \Big(\frac{t-z}{y_1+\xi(y_2,y_1)-z}\Big)^s\frac{|g'(y_1+\xi(y_2,y_1))|}{e^{\theta_2(y_1+\xi(y_2,y_1))}} + m\Big(\frac{y_1+\xi(y_2,y_1)-t}{y_1+\xi(y_2,y_1)-z}\Big)^s\frac{|g'(z)|}{e^{\theta_2 z}}, \tag{5.18}$$

repeat the same procedure from Eq (5.9) to Eq (5.17), we get

$$\Big|(Œ^{(\xi_j)_m,(\mu_j-1)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}g)(z)-[(Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)]g(y_1+\xi(y_2,y_1))\Big|$$
$$\leq\frac{((y_1+\xi(y_2,y_1)-z)}{s+1}(Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)\Big[\frac{|g'(y_1+\xi(y_2,y_1))|}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\frac{|g'(z)|}{e^{\theta_1 z}}\Big]. \tag{5.19}$$

From inequalities (5.17) and (5.19), we have

$$\Big|(Œ^{(\xi_j)_m,(\delta_j-1)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(z)+(Œ^{(\xi_j)_m,(\mu_j-1)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}g)(z)-[(Œ^{(\xi_j,\delta_j)_m}_{\lambda,k,\zeta;y_1^+}1)(z)]g(y_1)$$
$$-[(Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)]g(y_1+\xi(y_2,y_1))\Big|\leq\frac{(z-y_1)}{s+1}(Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\Big[\frac{|g'(y_1)|}{e^{\theta_1 y_1}}+m\frac{|g'(z)|}{e^{\theta_1 z}}\Big]$$
$$+\frac{((y_1+\xi(y_2,y_1)-z)}{s+1}(Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)\Big[\frac{|g'(y_1+\xi(y_2,y_1))|}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\frac{|g'(z)|}{e^{\theta_1 z}}\Big].$$

□

Corollary 5.5. *Setting $\xi(y_2,y_1)=y_2-y_1$, then under the assumption of theorem* (5.2)*, we have*

$$\Big|(Œ^{(\xi_j)_m,(\delta_j-1)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(z)+(Œ^{(\xi_j)_m,(\mu_j-1)_m}_{\lambda,\sigma,\zeta;y_2^-}g)(z)-[(Œ^{(\xi_j,\delta_j)_m}_{\lambda,k,\zeta;y_1^+}1)(z)]g(y_1)$$
$$-[(Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}1)(z)]g(y_2)\Big|\leq\frac{(z-y_1)}{s+1}(Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\Big[\frac{|g'(y_1)|}{e^{\theta_1 y_1}}+m\frac{|g'(z)|}{e^{\theta_1 z}}\Big]$$
$$+\frac{(y_2-z)}{s+1}(Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}1)(z)\Big[\frac{|g'(y_2)|}{e^{\theta_1(y_2)}}+m\frac{|g'(z)|}{e^{\theta_1 z}}\Big].$$

$\forall\ t\in[y_1,y_2]$, $\theta_1,\theta_2\in\mathbb{R}$.

Corollary 5.6. *Setting $\xi(y_2,y_1)=y_2-y_1$, along with $m=s=1$ then under the assumption of theorem* (5.2)*, we have*

$$\Big|(Œ^{(\xi_j)_m,(\delta_j-1)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(z)+(Œ^{(\xi_j)_m,(\mu_j-1)_m}_{\lambda,\sigma,\zeta;y_2^-}g)(z)-[(Œ^{(\xi_j,\delta_j)_m}_{\lambda,k,\zeta;y_1^+}1)(z)]g(y_1)$$
$$-[(Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}1)(z)]g(y_2)\Big|\leq\frac{(z-y_1)}{2}(Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(z)\Big[\frac{|g'(y_1)|}{e^{\theta_1 y_1}}+\frac{|g'(z)|}{e^{\theta_1 z}}\Big]$$
$$+\frac{(y_2-z)}{2}(Œ^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}1)(z)\Big[\frac{|g'(y_2)|}{e^{\theta_1(y_2)}}+\frac{|g'(z)|}{e^{\theta_1 z}}\Big].$$

$\forall\ t\in[y_1,y_2]$, $\theta_1,\theta_2\in\mathbb{R}$.

Definition 5.1. *Let $g:[y_1,y_1+\xi(y_2,y_1)]\to R$ is a function, and g is exponentially symmetric about $\frac{2y_1+\xi(y_2,y_1)}{2}$ if*

$$\frac{g(z)}{e^{\theta z}}=\frac{g(2y_1+\xi(y_2,y_1)-z)}{e^{\theta(2y_1+\xi(y_2,y_1)-z)}},\quad \theta\in R. \tag{5.20}$$

Lemma 5.1. *Let $g:[y_1,y_1+\xi(y_2,y_1)]\to R$ be exponentially symmetric, then*

$$g\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)\leq\frac{(1+m)g(z)}{2^s e^{\theta z}},\quad \theta\in R. \tag{5.21}$$

Proof. For g is exponentially (s-m) preinvex, therefore

$$g\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)\leq\frac{g(y_1+\delta\xi(y_{2,1}))}{2^s e^{\theta(y_1+\delta\xi(y_2,y_1))}}+m\frac{g(y_1+(1-\delta)\xi(y_2,y_1))}{2^s e^{\theta(y_1+(1-\delta)\xi(y_2,y_1))}}. \tag{5.22}$$

Let $t=y_1+\delta\xi(y_2,y_1)$, where $t\in[y_1,y_1+\xi(y_2,y_1)]$, and then $2y_1+\xi(y_2,y_1)=y_1+(1-\delta)\xi(y_2,y_1)$, we have

$$g\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)\leq\frac{g(z)}{2^s e^{\theta z}}+m\frac{g(2y_1+\xi(y_2,y_1)-z)}{2^s e^{\theta(2y_1+\xi(y_2,y_1)-z)}}. \tag{5.23}$$

applying that g is exponentially symmetric, we obtain

$$g\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)\leq\frac{(1+m)g(z)}{2^s e^{\theta z}}. \tag{5.24}$$

□

Theorem 5.3. *Suppose a real valued function* $g:[y_1,y_1+\xi(y_2,y_1)]\to R$ *is exponentially (s-m) preinvex and symmetric about exponentially* $\frac{2y_1+\xi(y_2,y_1)}{2}$, *then the following integral inequality for* (3.1) *and* (3.2) *holds:*

$$\begin{aligned}\frac{2^s}{1+m}f\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)&\Big[e^{\theta y_1}(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(y_1)+(\text{Œ}^{(\mu_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(y_1+\xi(y_2,y_1))\Big]\\&\leq(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}g)(z)+(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(y_1+\xi(y_2,y_1))\\&\leq\frac{\xi(y_2,y_1)}{s+1}\Big(\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\frac{g(y_1)}{e^{\theta_1 y_1}}\Big)\\&\times\Big[(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)+(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(y_1+\xi(y_2,y_1))\Big].\end{aligned} \tag{5.25}$$

Proof. For $z\in[y_1,y_1+\xi(y_2,y_1)]$, we have

$$(z-y_1)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j})\leq(\xi(y_2,y_1))^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(\xi(y_2,y_1))^{\xi_j}), \tag{5.26}$$

the real value function g is exponentially (s-m) preinvex, then for $z\in[y_1,y_1+\xi(y_2,y_1)]$, we get

$$g(z)\leq\Big(\frac{z-y_1}{\xi(y_2,y_1)}\Big)^s\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\Big(\frac{y_1+\xi(y_2,y_1)-z}{\xi(y_2,y_1)}\Big)^s\frac{g(y_1)}{e^{\theta_1 y_1}}. \tag{5.27}$$

Conducting product of (5.26) and (5.27), and integrating with respect to z from y_1 to y_2, we get

$$\begin{aligned}\int_{y_1}^{y_2}(z-y_1)^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j})g(z)dz&\leq\int_{y_1}^{y_2}(\xi(y_2,y_1))^{\delta_j}\mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(\xi(y_2,y_1))^{\xi_j})\\&\times\Big[\Big(\frac{z-y_1}{\xi(y_2,y_1)}\Big)^s\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\Big(\frac{y_1+\xi(y_2,y_1)-z}{\xi(y_2,y_1)}\Big)^s\frac{g(y_1)}{e^{\theta_1 y_1}}\Big]dz,\end{aligned} \tag{5.28}$$

then we have

$$(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}g)(z)$$

$$\leq (\xi(y_2,y_1))^{\delta_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(\xi(y_2,y_1))^{\xi_j})\frac{\xi(y_2,y_1)}{s+1}\Big[\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\frac{g(y_1)}{e^{\theta_1 y_1}}\Big]$$
$$= (\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)\frac{\xi(y_2,y_1)}{s+1}\Big[\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\frac{g(y_1)}{e^{\theta_1 y_1}}\Big]. \tag{5.29}$$

Analogously for $z\in[y_1,y_1+\xi(y_2,y_1)]$, we have

$$(y_1+\xi(y_2,y_1)-z)^{\mu_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\mu_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j})\leq(\xi(y_2,y_1))^{\mu_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\mu_j)_m,\sigma}(\zeta(\xi(y_2,y_1))^{\xi_j}). \tag{5.30}$$

Conducting product of (5.27) and (5.30), and integrating with respect to z from y_1 to y_2, we have

$$\int_{y_1}^{y_2}(y_1+\xi(y_2,y_1)-z)^{\mu_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\mu_j)_m,\sigma}(\zeta(z-y_1)^{\xi_j})g(z)dz$$
$$\leq\int_{y_1}^{y_2}(\xi(y_2,y_1))^{\mu_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\mu_j)_m,\sigma}(\zeta(\xi(y_2,y_1))^{\xi_j})\Big[\Big(\frac{z-y_1}{\xi(y_2,y_1)}\Big)^s\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}$$
$$+m\Big(\frac{(y_1+\xi(y_2,y_1)-z)}{\xi(y_2,y_1)}\Big)^s\frac{g(y_1)}{e^{\theta_1 y_1}}\Big]dz$$
$$=(\xi(y_2,y_1))^{\mu_j}\mathbf{J}^{(\xi_j)_m,\lambda}_{(\mu_j)_m,\sigma}(\zeta(\xi(y_2,y_1))^{\xi_j})\frac{\xi(y_2,y_1)}{s+1}\Big[\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\frac{g(y_1)}{e^{\theta_1 y_1}}\Big],$$

then

$$(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(z)$$
$$\leq(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma;(y_1+\xi(y_2,y_1))^-}1)(y_1+\xi(y_2,y_1))\frac{\xi(y_2,y_1)}{s+1}\Big[\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\frac{g(y_1)}{e^{\theta_1 y_1}}\Big]. \tag{5.31}$$

Summing (5.29) and (5.31), we obtain

$$(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}g)(z)+(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(z)\leq\frac{\xi(y_2,y_1)}{s+1}\Big(\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}$$
$$+m\frac{g(y_1)}{e^{\theta_1 y_1}}\Big)\Big[(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)+(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(y_1+\xi(y_2,y_1))\Big]. \tag{5.32}$$

Take the product of Eq (5.21) with $(z-y_1)^{\tau_j}\mathbf{J}^{(\mu_j)_m,\lambda}_{(\tau_j)_m,\sigma}(\zeta(z-y_1)^{\mu_j})$ and integrating with respect to t from y_1 to y_2, we have

$$g\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)\int_{y_1}^{y_2}(z-y_1)^{\tau_j}\mathbf{J}^{(\mu_j)_m,\lambda}_{(\tau_j)_m,\sigma}(\zeta(z-y_1)^{\mu_j})dz$$
$$\leq\frac{(1+m)}{2^s}\int_{y_1}^{y_2}(z-y_1)^{\tau_j}\mathbf{J}^{(\mu_j)_m,\lambda}_{(\tau_j)_m,\sigma}(\zeta(z-y_1)^{\mu_j})\frac{g(z)}{e^{\theta z}}dz \tag{5.33}$$

using definition (3.1), we have

$$g\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(y_1)\leq\frac{(1+m)}{2^s e^{\theta y_1}}(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}g)(z). \tag{5.34}$$

Taking product (5.21) with $(y_1+\xi(y_2,y_1)-z)^{\delta_j}\mathbf{J}^{(\mu_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(y_1+\xi(y_2,y_1)-z)^{\mu_j})$ and integrating with respect to variable z from y_1 to y_2, we have

$$g\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)(\text{Œ}^{(\mu_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(y_1+\xi(y_2,y_1))$$
$$\leq \frac{(1+m)}{2^s e^{\theta_1(y_1+\xi(y_2,y_1))}}(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(y_1+\xi(y_2,y_1)). \quad (5.35)$$

Summing up (5.34) and (5.35), we get

$$\frac{2^s}{1+m}g\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)\Big[e^{\theta y_1}(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(y_1)$$
$$+(\text{Œ}^{(\mu_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(y_1+\xi(y_2,y_1))\Big] \leq (\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}g)(z)+(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(y_1+\xi(y_2,y_1)). \quad (5.36)$$

Now, combining (5.32) and (5.36), we get inequality

$$\frac{2^s}{1+m}g\Big(\frac{2y_1+\xi(y_2,y_1)}{2}\Big)\Big[e^{\theta y_1}(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;(y_1+\eta(y_2,y_1))^-}1)(y_1)+(\text{Œ}^{(\mu_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(y_1+\xi(y_2,y_1))\Big]$$
$$\leq (\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}g)(z)+(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(y_1+\xi(y_2,y_1))$$
$$\leq \frac{\xi(y_2,y_1)}{s+1}\Big(\frac{g(y_1+\xi(y_2,y_1))}{e^{\theta_1(y_1+\xi(y_2,y_1))}}+m\frac{g(y_1)}{e^{\theta_1 y_1}}\Big)$$
$$\times\Big[(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(z)+(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;(y_1+\xi(y_2,y_1))^-}1)(y_1+\xi(y_2,y_1))\Big].$$

□

Corollary 5.7. *Setting $\xi(y_2,y_1)=y_2-y_1$, then under the assumption of theorem* (5.3)*, we have*

$$\frac{2^s}{1+m}g\Big(\frac{y_1+y_2}{2}\Big)\Big[e^{\theta y_1}(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;y_2^-}1)(y_1)+(\text{Œ}^{(\mu_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}1)(y_2)\Big]$$
$$\leq (\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;y_2^-}g)(z)+(\text{Œ}^{(\mu_j,\tau_j)_m}_{\lambda,\sigma,\zeta;y_1^+}g)(y_2)\leq \frac{(y_2-y_1)}{s+1}\Big(\frac{g(y_2-y_1)}{e^{\theta_1(y_2-y_1)}}+m\frac{g(y_1)}{e^{\theta_1 y_1}}\Big)$$
$$\times\Big[(\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}1)(z)+(\text{Œ}^{(\xi_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}1)(y_2)\Big]. \quad (5.37)$$

6. Some Pólya-Szegö type integral inequalities of fractional operators

In this section, we derive some Pólya-Szegö inequalities for four positive integrable functions having fractional operator $\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma}(z)$ in the form of theorems.

Theorem 6.1. *Let h and l are integrable functions on $[y_1,\infty)$. Suppose that there exist integrable functions θ_1,θ_2,ψ_1 and ψ_2 on $[y_1,\infty)$ such that:*

$$(R1)\ \ 0<\theta_1(b)\leq h(b)\leq\theta_2(b), 0<\psi_1(b)\leq l(b)\leq\psi_2(b)\ \ (b\in[y_1,z], z>y_1).$$

Then, for $z>y_1, y_1\geq 0$, $\xi_j,\delta_j,\lambda\in\mathbb{C},(j=1,\cdots,m),\Re(\lambda)>0,\Re(\delta_j)>-1,\sum_{j=1}^m\Re(\xi)_j>max\{0:\Re(\sigma)-1\},\sigma>0$ and $(z-b)\in\Omega$, then the following inequalities hold:

$$\frac{\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1+}[(\psi_1\psi_2)h^2](z)\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1\theta_2)l^2](z)}{[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1\psi_1+\theta_2\psi_2)hl](z)]^2}\leq\frac{1}{4}. \quad (6.1)$$

Proof. From $(R1)$, for $b \in [y_1, z]$, $z > y_1$, we have

$$\frac{h(b)}{l(b)} \leq \frac{\theta_2(b)}{\psi_1(b)}, \tag{6.2}$$

the inequality write as

$$\Big(\frac{\theta_2(b)}{\psi_1(b)} - \frac{h(b)}{l(b)}\Big) \geq 0. \tag{6.3}$$

Similarly, we get

$$\frac{\theta_1(b)}{\psi_2(b)} \leq \frac{h(b)}{l(b)}, \tag{6.4}$$

thus

$$\Big(\frac{h(b)}{l(b} - \frac{\theta_1(b)}{\psi_2(b)}\Big) \geq 0. \tag{6.5}$$

Multiplying Eq (6.3) and Eq (6.5), it follows

$$\Big(\frac{\theta_2(b)}{\psi_1(b)} - \frac{h(b)}{l(b)}\Big)\Big(\frac{h(b)}{l(b)} - \frac{\theta_1(b)}{\psi_2(b)}\Big) \geq 0, \tag{6.6}$$

i.e.

$$\Big(\frac{\theta_2(b)}{\psi_1(b)} + \frac{\theta_1(b)}{\psi_2(b)}\Big)\frac{h(b)}{l(b)} \geq \frac{h^2(b)}{l^2(b)} + \frac{\theta_1(b)\theta_2(b)}{\psi_1(b)\psi_2(b)}. \tag{6.7}$$

The last inequality can be written as

$$(\theta_1(b)\psi_1(b) + \theta_2(b)\psi_2(b))h(b)l(b) \geq \psi_1(b)\psi_2(b)h^2(b) + \theta_1(b)\theta_2(b)l^2(b). \tag{6.8}$$

Consequently, multiply both sides of (6.8) by $(y_1 - b)^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(y_1 - b)^{\xi_j})$, $(z - b) \in \Omega$ and integrating with respect to b from y_1 to z, we get

$$Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1\psi_1 + \theta_2\psi_2)hl](z) \geq Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\psi_1\psi_2 h^2](z) + Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\theta_1\theta_2 l^2](z). \tag{6.9}$$

Besides, by AM-GM (arithmetic mean- geometric mean) inequality, i.e., $a_1 + b_1 \geq 2\sqrt{a_1 b_1}\ a_1, b_1 \in \Re^+$, we get

$$Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1\psi_1 + \theta_2\psi_2)hl](x) \geq 2\sqrt{Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\psi_1\psi_2 h^2](z) + Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\theta_1\theta_2 l^2](z)}, \tag{6.10}$$

and it follows straightforward the statement of Eq (6.1). □

Corollary 6.1. *Let h and l be two integrable functions on $[0, \infty)$ and satisfying the inequality*

$$(R2)\ \ 0 < s \leq h(b) \leq S, 0 < k \leq l(b) \leq K(b \in [y_1, \tau], z > y_1). \tag{6.11}$$

For $z > y_1, y_1 \geq 0$, $\xi_j, \delta_j, \lambda \in \mathbb{C}, (j = 1, \cdots, m), \Re(\lambda) > 0, \Re(\delta_j) > -1, \sum_{j=1}^{m} \Re(\xi)_j > max\{0 : \Re(\sigma) - 1\}, \sigma > 0$ and $(z - b) \in \Omega$, then the following inequalities hold:

$$\frac{Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h^2](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[l^2](z)}{(Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[hl](z))^2} \leq \frac{1}{4}\Big(\frac{\sqrt{SK}}{\sqrt{sk}} + \frac{\sqrt{sk}}{\sqrt{SK}}\Big)^2. \tag{6.12}$$

Theorem 6.2. *Let h and l are positive integrable functions on* $[y_1, \infty)$. *Suppose that there exist integrable functions* $\theta_1, \theta_2, \psi_1$ *and* ψ_2 *on* $[y_1, \infty)$ *satisfying* $(R1)$ *on* $[y_1, \infty)$. *Then, for* $z > y_1, y_1 \geq 0$, $\xi_j, \delta_j, \lambda \in \mathbb{C}, (j = 1, \cdots, m), \Re(\lambda) > 0, \Re(\delta_j) > -1, \sum_{j=1}^{m} \Re(\xi)_j > max\{0 : \Re(\sigma) - 1\}, \sigma > 0$ *and* $(z - b), (\tau - z) \in \Omega$, *then the following inequalities hold:*

$$\frac{Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h^2](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[\psi_1\psi_2](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\theta_1\theta_2](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[l^2](z)}{[Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\theta_1 h](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[\psi_1 h](z) + Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\theta_2 h](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[\psi_2 l](z)]^2} \leq \frac{1}{4}. \tag{6.13}$$

Proof. By condition $(R1)$, it is clear that

$$\left(\frac{\theta_2(b)}{\psi_1(\alpha)} - \frac{h(b)}{l(\alpha)}\right) \geq 0, \tag{6.14}$$

and

$$\left(\frac{h(b)}{l(\alpha)} - \frac{\theta_1(b)}{\psi_2(\alpha)}\right) \geq 0, \tag{6.15}$$

these inequalities implies that

$$\left(\frac{\theta_1(b)}{\psi_2(\alpha)} + \frac{\theta_2(b)}{\psi_1(\alpha)}\right)\frac{h(b)}{l(\alpha)} \geq \frac{h^2(b)}{l^2(\alpha)} + \frac{\theta_1(b)\theta_2(b)}{\psi_1(\alpha)\psi_2(\alpha)}. \tag{6.16}$$

The Eq (6.16), multiply by $\psi_1(\alpha)\psi_2(\alpha)l^2(\alpha)$ of both sides, we have

$$\begin{aligned} &\theta_1(b)h(b)\psi_1(\alpha)l(\alpha) + \theta_2(b)h(b)\psi_2(\alpha)l(\alpha) \\ &\quad \geq \psi_1(\alpha)\psi_2(\alpha)h^2(b) + \theta_1(b)\theta_2(b)l^2(\alpha). \end{aligned} \tag{6.17}$$

Hence, the Eq (6.17) multiply both sides by

$$(z - b)^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(z - b)^{\xi_j}), (\alpha - z)^{\delta_j} \mathrm{J}^{(\xi_j)_m,\lambda}_{(\delta_j)_m,\sigma}(\zeta(\alpha - z)^{\xi_j}). \tag{6.18}$$

And integrating double with respect to b and α from y_1 to z and z to y_2 respectively, we have

$$\begin{aligned} &Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\theta_1 h](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[\psi_1 l](z) + Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\theta_2 h](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[\psi_2 l](z) \\ &\geq Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h^2](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[\psi_1\psi_2](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[\theta_1\theta_2](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[l^2](z). \end{aligned} \tag{6.19}$$

At last, we come to Eq (6.13) by using the arithmetic and geometric mean inequality to the upper inequality. □

Theorem 6.3. *Let h and l are integrable functions on* $[y_1, \infty)$. *Suppose that there exist integrable functions* $\theta_1, \theta_2, \psi_1$ *and* ψ_2 *on* $[y_1, \infty)$ *satisfying* $(R1)$ *on* $[y_1, \infty)$. *Then, for* $z > y_1, y_1 \geq 0$, $\xi_j, \delta_j, \lambda \in \mathbb{C}, (j = 1, \cdots, m), \Re(\lambda) > 0, \Re(\delta_j) > -1, \sum_{j=1}^{m} \Re(\xi)_j > max\{0 : \Re(\sigma) - 1\}, \sigma > 0$ *and* $(z - b), (\alpha - z) \in \Omega$, *then the following inequalities hold:*

$$Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h^2](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[l^2](z) \leq Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_2 hl)/\psi_1](z)Œ^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[(\psi_2 hl)/\theta_1]. \tag{6.20}$$

Proof. We have for any $(z-b),(\alpha-z)\in\Omega$, from Eq (6.2), thus

$$\int_{y_1}^{z}(z-b)^{\delta_j}\mathrm{J}_{\lambda,\sigma}^{(\xi_j,\delta_j)_m}(\zeta(z-b)^{\xi_j})h^2(b)db \leq \int_{z}^{y_1}(\alpha-z)^{\xi_j}\mathrm{J}_{\lambda,\sigma}^{(\xi_j,\delta_j)_m}(\zeta(\alpha-z)^{\xi_j})\frac{\theta_2(\alpha)}{\psi_1(\alpha)}h(\alpha)l(\alpha)d\alpha,$$

which implies

$$Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[h^2](z) \leq Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[(\theta_2 hl)/\psi_1](z). \tag{6.21}$$

and analogously, by Eq (6.4), we get

$$Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\xi_j,\delta_j)_m}[l^2](x) \leq Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\xi_j,\delta_j)_m}[(\psi_2 hl)/\theta_1](z), \tag{6.22}$$

hence, by multiplying Eq (6.21) and Eq (6.22), follow Eq (6.20). □

Corollary 6.2. *Let h and l be integrable functions on* $[y_1,\infty)$ *satisfying* $(R2)$*. Then, for* $z>y_1, y_1\geq 0,$ $\xi_j,\delta_j,\lambda\in\mathbb{C},(j=1,\cdots,m),\Re(\lambda)>0,\Re(\delta_j)>-1,\sum_{j=1}^{m}\Re(\xi)_j>max\{0:\Re(\sigma)-1\},\sigma>0$ *and* $(z-b),(\alpha-z)\in\Omega$*, we obtain*

$$\frac{Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[h^2](z)Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\xi_j,\delta_j)_m}[l^2](z)}{Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[hl](z)Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\xi_j,\delta_j)_m}[hl](z)} \leq \frac{SK}{sk}. \tag{6.23}$$

7. Chebyshev type integral inequalities of fractional operator

In this section, Chebyshev type integral inequalities established involving the fractional operator $Œ_{\lambda,\sigma}^{(\xi_j,\delta_j)_m}(z)$ and using the Pólya-Szegö fractional integral inequalities of theorem (6.1) in the form of theorem, and then discuss its corollary.

Theorem 7.1. *Let h and l be integrable functions on* $[y_1,\infty)$*, and suppose that there exist integrable functions* θ_1,θ_2,ψ_1 *and* ψ_2 *on* $[y_1,\infty)$ *satisfying* $(R1)$*. Then, for* $z>y_1,y_1\geq 0,$ $\xi_j,\delta_j,\lambda\in\mathbb{C},(j=1,\cdots,m),\Re(\lambda)>0,\Re(\delta_j)>-1,\sum_{j=1}^{m}\Re(\xi)_j>max\{0:\Re(\sigma)-1\},\sigma>0$ *and* $(z-b)(\alpha-z)\in\Omega$ *the following inequality hold:*

$$\begin{aligned}&|Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[hl](z)Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[1](z)+Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[hl](z)Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[1](z)\\&-Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[h](z)Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[l](z)-Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[l](z)Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[h](z)|\\&\leq 2[G_{y_1,y_2}(h,\theta_1,\theta_2)G_{y_1,y_2}(l,\psi_1,\psi_2)]^{\frac{1}{2}}.\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}G_{y_1,y_2}(b,y,x)(z)=&\frac{1}{8}\frac{[Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[(y+x)b](z)]^2}{Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[yx](z)}Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[1](z)\\&+\frac{1}{8}\frac{[Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[(y+x)b](z)]^2}{Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\mu_j,\nu_j)_m}[yx](z)}Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[1](z)\\&-Œ_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[b](z)Œ_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[b](z).\end{aligned}$$

Proof. For $(b,\alpha)\in(y_1,z)$ $(z>y_1)$, we defined $A(b,\alpha)=(h(b)-h(\alpha))(l(b)-l(\alpha))$ which is the same

$$A(b,\alpha)=h(b)l(b)+h(\alpha)l(\alpha)-h(b)l(\alpha)-h(\alpha)l(b). \tag{7.2}$$

Further, the Eq (7.2), multiply both sides by

$$(z-b)^{\xi_j}\mathrm{J}_{\lambda,\sigma}^{(\xi_j,\delta_j)_m}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j}), \tag{7.3}$$

and integrating double with respect to b and α from y_1 to z and z to y_2 respectively, we get

$$\begin{aligned}
&\int_{y_1}^{z}\int_{z}^{y_2}(z-b)^{\xi_j}\mathrm{J}_{\lambda,\sigma}^{(\xi_j,\delta_j)_m}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})A(b,\alpha)dbd\alpha\\
&=\int_{y_1}^{z}(z-b)^{\xi_j}\mathrm{J}_{\lambda,\sigma}^{(\xi_j,\delta_j)_m}(\zeta(z-b)^{\delta_j})h(b)l(b)db\int_{z}^{y_2}(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})d\alpha\\
&+\int_{y_1}^{z}(z-b)^{\xi_j}\mathrm{J}_{\lambda,\sigma}^{(\xi_j,\delta_j)_m}(\zeta(z-b)^{\delta_j})db\int_{z}^{y_2}(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})h(\alpha)l(\alpha)d\alpha\\
&-\int_{z}^{y_1}(z-b)^{\xi_j}\mathrm{J}_{\lambda,\sigma}^{(\xi_j,\delta_j)_m}(\zeta(z-b)^{\delta_j})h(b)db\int_{z}^{y_2}(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})h(\alpha)d\alpha\\
&-\int_{z}^{y_1}(z-b)^{\xi_j}\mathrm{J}_{\lambda,\sigma}^{(\xi_j,\delta_j)_m}(\zeta(z-b)^{\delta_j})l(b)db\int_{z}^{y_2}(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})h(\alpha)d\alpha\\
&=\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[hl](z)\text{Œ}_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[1](z)+\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[1](z)\text{Œ}_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[hl](z)\\
&-\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[h](z)\text{Œ}_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[l](z)-\text{Œ}_{\lambda,\sigma,\zeta;y_1^+}^{(\xi_j,\delta_j)_m}[l](z)\text{Œ}_{\lambda,\sigma,\zeta;y_2^-}^{(\nu_j,\mu_j)_m}[h](z).
\end{aligned}\tag{7.4}$$

Now, applying Cauchy-Schwartz inequality for integrals, we get

$$\begin{aligned}
&\Big|\int_{y_1}^{z}\int_{z}^{y_2}(z-b)^{\xi_j}\mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})A(b,\alpha)dbd\alpha\Big|\\
&\le\Big(\int_{y_1}^{z}\int_{z}^{y_2}(z-b)^{\xi_j}\mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})\alpha[h(b)]^2dbd\alpha\\
&+\int_{y_1}^{z}\int_{z}^{y_2}(z-b)^{\xi_j}\mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})[h(\alpha)]^2dbd\alpha\\
&-2\int_{y_1}^{z}\int_{z}^{y_2}(z-b)^{\xi_j}\mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})h(b)h(\alpha)dbd\alpha\Big)^{1/2}\\
&\times\Big(\int_{y_1}^{z}\int_{z}^{y_2}(z-b)^{\xi_j}\mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})\alpha[l(b)]^2dbd\alpha\\
&+\int_{y_1}^{z}\int_{z}^{y_2}(z-b)^{\xi_j}\mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})[l(\alpha)]^2dbd\alpha\\
&-2\int_{y_1}^{z}\int_{z}^{y_2}(z-b)^{\xi_j}\mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})l(b)l(\alpha)dbd\alpha\Big)^{1/2},
\end{aligned}\tag{7.5}$$

it follow as

$$\Big|\int_{y_1}^{z}\int_{z}^{y_2}(z-b)^{\xi_j}\mathrm{J}_{(\delta_j)_m,\sigma}^{(\xi_j)_m,\lambda}(\zeta(z-b)^{\delta_j})(\alpha-z)^{\nu_j}\mathrm{J}_{(\nu_j)_m,\sigma}^{(\mu_j)_m,\lambda}(\zeta(\alpha-z)^{\mu_j})A(b,\alpha)dbd\alpha\Big|$$

$$\leq 2\{1/2\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h^2](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[1](z) + 1/2\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[1](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[h^2](z)$$
$$- \text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[h](z)\}^{1/2} \times \{1/2\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[l^2](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[1](z)$$
$$+ 1/2\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[1](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[l^2](z) - \text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[l](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[l](z)\}^{1/2}. \quad (7.6)$$

By applying lemma (6.1) for $\psi_1(z) = \psi_2(z) = l(z) = 1$, we get for any $\mathrm{J}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma}(z)^{\delta_j} \in \Omega$

$$\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h^2](z) \leq \frac{1}{4}\frac{[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1+\theta_2)h](z)]^2}{\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1\theta_2)](z)}, \quad (7.7)$$

this implies

$$1/2\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h^2](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[1](z) + 1/2\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[1](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[h^2](z)$$
$$- \text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[h](z) \leq \frac{1}{8}\frac{[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1+\theta_2)h](z)]^2}{\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1\theta_2)](z)}\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[1](z)$$
$$+ \frac{1}{8}\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[1](z)\frac{[\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1+\theta_2)h](z)]^2}{\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\theta_1\theta_2)](z)} - \text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[h](z)$$
$$= G_{y_1,y_2}(h,\theta_1,\theta_2). \quad (7.8)$$

Analogously, it is clear when $\theta_1(z) = \theta_2(z) = h(z) = 1$, according to Lemma (6.1), we get

$$1/2\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[l^2](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[1](z) + 1/2\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[1](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[l^2](z)$$
$$- \text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[l](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[l](x) \leq \frac{1}{8}\frac{[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\psi_1+\psi_2)l](z)]^2}{\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\psi_1\psi_2)](z)}\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[1](z)$$
$$+ \frac{1}{8}\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[1](z)\frac{[\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\psi_1+\psi_2)l](z)]^2}{\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(\psi_1\psi_2)](z)} - \text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[l](z)\text{Œ}^{(\nu_j,\mu_j)_m}_{\lambda,\sigma,\zeta;y_2^-}[l](z)$$
$$= G_{y_1,y_2}(l,\psi_1,\psi_2). \quad (7.9)$$

Thus, by resulting Eqs (7.4), (7.6), (7.8) and (7.9), we get the desired inequality (7.1). □

Corollary 7.1. *Let h and l be integrable functions on $[y_1,\infty)$, suppose that there exist integrable functions θ_1,θ_2,ψ_1 and ψ_2 on $[y_1,\infty)$ satisfying $(R1)$. Then, for $z > y_1, y_1 \geq 0$, $\xi_j,\delta_j,\lambda \in \mathbb{C},(j=1,\cdots,m), \Re(\lambda) > 0, \Re(\delta_j) > -1, \sum_{j=1}^m \Re(\xi)_j > max\{0 : \Re(\sigma)-1\}, \sigma > 0$ and $(z-b),(\alpha-z) \in \Omega$ the following inequalities hold:*

$$|\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[hl](z)\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[1](z) - \text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[h](z)\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[l](z)|$$
$$\leq [G_{y_1,y_2}(h,\theta_1,\theta_2)G_{y_1,y_1}(l,\theta_1,\theta_2)]^{\frac{1}{2}},$$

where

$$G_{y_1,y_1}(b,y,x)(z) = \frac{1}{4}\frac{[\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[(y+x)b](z)]^2}{\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[yx](z)}\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[1] - (\text{Œ}^{(\xi_j,\delta_j)_m}_{\lambda,\sigma,\zeta;y_1^+}[b](z))^2.$$

8. Conclusions

This article analyzed the generalized fractional integral operator having nonsingular function (generalized multi-index Bessel function) as kernel and developed a new version of inequalities. We estimate some inequalities (Hermite Hadamard type Mercer inequality, exponentially $(s - m)$ preinvex inequality, Pólya-Szegö type integral inequality and the Chebyshev type inequality) with the generalized fractional integral operator in which nonsingular function as the kernel. Introducing the new version of inequalities of newly constricted operators have strengthened the idea and results.

Conflict of interest

The authors declare that they have no competing interest.

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