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Research article

Some characterizations of dual curves in dual 3-space $\mathbb{D}^3$

Rashad Abdel-Baky[1,*] **and Mohamed Khalifa Saad**[2,3]

[1] Mathematics Department, Faculty of Science, Assiut University, Assiut 71516, Egypt
[2] Mathematics Department, Faculty of Science, Sohag University, Sohag 82524, Egypt
[3] Mathematics Department, Faculty of Science, Islamic University of Madinah, Al-Madinah, KSA

* **Correspondence:** Email: rbaky@live.com; Tel: +201121157967, +201005883077.

Abstract: In this work, we prove that the ratio of torsion and curvature of any dual rectifying curve is a non-constant linear function of its dual arc length parameter. Thereafter, a dual differential equation of third order is constructed for every dual curve. Then, several well-known characterizations of dual spherical, normal and rectifying curves are consequences of this differential equation. Finally, we prove a simple new characterization of dual spherical curves in terms of the Darboux vector.

Keywords: E. Study map; Serret–Frenet formulae; rectifying dual curve; dual helices
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1. Introduction

The theory of curves is a fundamental structure of differential geometry. In the differential geometry of a regular curve in the Euclidean 3-space $\mathbb{E}^3$, it is well-known that one of the important problem is the characterization of a regular curve. Some important types of space curves are helices (characterized by τ/κ is constant), spherical curves (characterized by $\left(\sigma\rho'\right)^2 + \rho^2$ = constant, with $\kappa\rho = 1$, $\sigma\tau = 1$), and rectifying curves (characterized by τ/κ is a non–constant linear function of the arc length parameter), where τ and κ stands for the torsion and curvature of the curve, respectively. One interesting question on space curves is to find different characterizations of spherical curves, helices as well as of rectifying curves. Several interesting characterizations of spherical, and of rectifying curves in Euclidean space are obtained in [1–5].

Recently, mathematicians studied theory of curves in 3-dimensional dual space motivated by E. Study mapping. E. Study mapping is the corresponding between a dual spherical curve and a ruled surface in Euclidean 3-space (for instance, see [6–8]). Moreover, the 3-dimensional dual space $\mathbb{D}^3$ can be considered as the 6-dimensional space containing the Euclidean 3-space $\mathbb{E}^3$. Thus, the space curve in $\mathbb{D}^3$ is the natural extension of the space curve in $\mathbb{E}^3$. However, several interesting characterizations of spherical and rectifying curves are obtained in the dual space $\mathbb{D}^3$ by means of the E. Study map, but more light needs to be shed (see e.g. [9–14]).

In this work, we prove that the ratio of torsion and curvature of any dual rectifying curve is a non–constant linear function of the dual arc length parameter. Also, we prove that the tangential height dual function of every dual curve satisfies a third-order dual differential equation. Then several well-known characterizations of dual spherical, normal and rectifying curves are consequences of this differential equation. Finally, we prove a simple new characterization of dual spherical curves in term of the dual Darboux vector.

2. Preliminaries

In this section, we list some notions, formulas of dual numbers and dual vectors (see e.g. [6–8]). An oriented line L in Euclidean 3-space $\mathbb{E}^3$ can be determined by a point $\mathbf{p} \in L$ and a normalized direction vector $\mathbf{x}$ of L, i.e., $\|\mathbf{x}\| = 1$. To obtain components for L, one forms the moment vector

$$\mathbf{x}^* = \mathbf{p} \times \mathbf{x}, \tag{2.1}$$

with respect to the origin point in $\mathbb{E}^3$. If $\mathbf{p}$ is substituted by any point

$$\mathbf{y} = \mathbf{p} \ + \ v\mathbf{x}, \ v \in \mathbb{R}, \tag{2.2}$$

on L, then (2.1) implies that $\mathbf{x}^*$ is independent of $\mathbf{p}$ on L. The two vectors $\mathbf{x}$ and $\mathbf{x}^*$ are not independent of one another; they satisfy the following relationships:

$$< \mathbf{x}, \mathbf{x} >=1, \ \ < \mathbf{x}^*, \mathbf{x} >=0. \tag{2.3}$$

The six components x_i, x_i^* $(i = 1, 2, 3)$ of $\mathbf{x}$ and $\mathbf{x}^*$ are called the normalized Plücker coordinates of the line L. Hence the two vectors $\mathbf{x}$ and $\mathbf{x}^*$ determine the oriented line L.

The set of dual numbers is

$$\mathbb{D} = \{X = x + \varepsilon x^* \mid \ x, \ x^* \in \mathbb{R}, \ \varepsilon \neq 0, \ \varepsilon^2 = 0\}. \tag{2.4}$$

This set is a commutative ring under addition and multiplication. This set cannot be a field under these operations, because $0 + \varepsilon x^*$ has no multiplication inverse in $\mathbb{D}$. But this ring has a unit element according to multiplication. A dual number $X = x + \varepsilon x^*$, is called proper if $x \neq 0$.

For all pairs $(\mathbf{x}, \mathbf{x}^*) \in \mathbb{E}^3 \times \mathbb{E}^3$ the set

$$\mathbb{D}^3 = \{\mathbf{X} = \mathbf{x} + \varepsilon \mathbf{x}^*, \ \varepsilon \neq 0, \ \varepsilon^2 = 0\}, \tag{2.5}$$

together with the scalar product

$$< \mathbf{X}, \mathbf{Y} >=< \mathbf{x}, \mathbf{y} > + \varepsilon(< \mathbf{y}, \mathbf{x}^* > + < \mathbf{y}^*, \mathbf{x} >), \tag{2.6}$$

forms the dual 3-space $\mathbb{D}^3$. Thereby, a point $\mathbf{X} = (X_1, X_2, X_3)^t$ has dual coordinates $X_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$. The norm is defined by

$$< \mathbf{X}, \mathbf{X} >^{\frac{1}{2}} := \|\mathbf{X}\| = \|\mathbf{x}\| \, (1+\varepsilon \frac{< \mathbf{x}, \mathbf{x}^* >}{\|\mathbf{x}\|^2}). \tag{2.7}$$

In the dual 3-space $\mathbb{D}^3$, the set of arbitrary dual vectors

$$\mathbb{K}(\widetilde{r}) = \{\mathbf{X} \in \mathbb{D}^3 \mid \|\mathbf{X}\|^2 = X_1^2 + X_2^2 + X_3^2 = \widetilde{r}\}, \tag{2.8}$$

is a dual sphere with radius $\widetilde{r} = r + \varepsilon r^*$ and centered at the origin. Similarly, the dual unit sphere is defined by

$$\mathbb{K} = \{\mathbf{X} \in \mathbb{D}^3 \mid \|\mathbf{X}\|^2 = X_1^2 + X_2^2 + X_3^2 = 1\}. \tag{2.9}$$

Via this, the E. Study map can be stated as follows: The set of all oriented lines in the Euclidean 3-space $\mathbb{E}^3$ is one-to-one correspondence with the set of points of dual unit sphere in the dual 3-space $\mathbb{D}^3$. As a direct consequence of the E. Study's map, a differentiable curve

$$t \in \mathbb{R} \to \mathbf{X}(t) \in \mathbb{K} \tag{2.10}$$

on the dual unit sphere $\mathbb{K}$, depending on a real parameter t, represents a differentiable family of straight lines of $\mathbb{E}^3$ which is a ruled surface. The lines $\mathbf{X} = \mathbf{x} + \varepsilon \mathbf{x}^*$ are the generators of the surface [6–8].

Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, and $\alpha^*(t) = \left(\alpha_1^*(t), \alpha_2^*(t), \alpha_3^*(t)\right)$ be real valued curves in the Euclidean 3-space $\mathbb{E}^3$. Then, a differentiable curve

$$\begin{aligned} \widetilde{\alpha} &: \quad t \in \mathbb{R} \to \mathbb{D}^3 \\ \widetilde{\alpha} &= \quad (\alpha_1(t), \alpha_2(t), \alpha_3(t)) + \varepsilon\left(\alpha_1^*(t), \alpha_2^*(t), \alpha_3^*(t)\right), \end{aligned}$$

represents a curve in the dual space $\mathbb{D}^3$ and is called a dual space curve. The dual arc length of $\widetilde{\alpha}(t)$ from t_0 to t is defined by

$$\widetilde{s} = s + \varepsilon s^* = \int_{t_0}^{t} \left\|\frac{d\widetilde{\alpha}}{dt}\right\| dt = \int_{t_0}^{t} \left\|\frac{d\alpha}{dt}\right\| dt + \varepsilon \int_{t_0}^{t} < \mathbf{t}, \frac{d\alpha^*}{dt} > dt,$$

where $\mathbf{t}(t)$ is a unit tangent vector of $\alpha(t)$. From now on, we will take the arc length $\widetilde{s}$ as the parameter instead of t. Then $\widetilde{\alpha}(\widetilde{s})$ is called a dual arc-length parameter curve. From now on, we shall often not write the dual parameter $\widetilde{s}$ explicitly in our formulae.

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving dual Serret-Frenet frame along $\widetilde{\alpha}(\widetilde{s})$ in the dual space $\mathbb{D}^3$. Then, $\mathbf{t} + \varepsilon\mathbf{t}^* = \mathbf{T}(\widetilde{s})$, $\mathbf{n} + \varepsilon\mathbf{n}^* = \mathbf{N}(\widetilde{s})$, and $\mathbf{b} + \varepsilon\mathbf{b}^* = \mathbf{B}(\widetilde{s})$ are the dual unit tangent, dual unit principal normal, and dual unit binormal vectors of the curve at the point $\widetilde{\alpha}(\widetilde{s})$. The dual arc-length derivative of the dual Serret-Frenet frame is governed by the relations:

$$\begin{pmatrix} \mathbf{T}'(\widetilde{s}) \\ \mathbf{N}'(\widetilde{s}) \\ \mathbf{B}'(\widetilde{s}) \end{pmatrix} = \begin{pmatrix} 0 & \widetilde{\kappa}(\widetilde{s}) & 0 \\ -\widetilde{\kappa}(\widetilde{s}) & 0 & \widetilde{\tau}(\widetilde{s}) \\ 0 & -\widetilde{\tau}(\widetilde{s}) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(\widetilde{s}) \\ \mathbf{N}(\widetilde{s}) \\ \mathbf{B}(\widetilde{s}) \end{pmatrix}, \tag{2.11}$$

where $\widehat{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature, and $\widehat{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion. The above formulae are called the Serret-Frenet formulae of dual curve in $\mathbb{D}^3$. Here "prime" denotes the derivative with respect to the pseudo dual parameter $\widehat{s}$.

Introducing the dual vector $\mathbf{\Omega}(\widetilde{s}) = \omega + \varepsilon\omega^*$ given by $\mathbf{\Omega} = \widetilde{\tau}\mathbf{T} + \widetilde{\kappa}\mathbf{B}$, we may write

$$\begin{pmatrix} \mathbf{T}'(\widetilde{s}) \\ \mathbf{N}'(\widetilde{s}) \\ \mathbf{B}'(\widetilde{s}) \end{pmatrix} = \mathbf{\Omega} \times \begin{pmatrix} \mathbf{T}(\widetilde{s}) \\ \mathbf{N}(\widetilde{s}) \\ \mathbf{B}(\widetilde{s}) \end{pmatrix}. \tag{2.12}$$

The vector $\mathbf{\Omega}$ is called the dual Darboux vector of the Serret-Frenet formulae of dual curve in $\mathbb{D}^3$.

3. Dual curves satisfying $\widetilde{\tau}/\widetilde{\kappa}=\widetilde{a}\widetilde{s}+\widetilde{b}$

In the next theorem, we prove that the ratio of torsion and curvature of any rectifying dual curve is a non–constant linear function of the dual arc length parameter $\widetilde{s}$ (see [15], for some related results on the same topic).

Theorem 3.1. Let $\widetilde{\alpha}=\widetilde{\alpha}(\widetilde{s})$ be a unit speed dual curve with $\widetilde{\kappa}(\widetilde{s}) \neq 0$. Then, the following are equivalent:
(i) There is a point $\widetilde{\mathbf{m}}\in\mathbb{D}^3$ such that every rectifying dual plane of $\widetilde{\alpha}(\widetilde{s})$ goes through $\widetilde{\mathbf{m}}$.
(ii) $\widetilde{\tau}/\widetilde{\kappa}$ is a non-constant linear dual function $\widetilde{a}\widetilde{s}+\widetilde{b}$.
(iii) There is a point $\widetilde{\mathbf{m}}_0\in\mathbb{D}^3$ such that $\left\|\widetilde{\alpha}(\widetilde{s})-\widetilde{\mathbf{m}}_0\right\|^2=(\widetilde{s}-\widetilde{c})^2+\widetilde{d}^2$. The constants are related by

$$\widetilde{a}=\pm\frac{1}{\widetilde{d}};\ \widetilde{c}=-\frac{\widetilde{b}}{\widetilde{a}};\ \widetilde{d}\neq 0.$$

And by the uniqueness of $\widetilde{\mathbf{m}}$, $\widetilde{\mathbf{m}}$ is equal to $\widetilde{\mathbf{m}}_0$.
Proof. (i) Suppose that every rectifying dual plane of $\widetilde{\alpha}(\widetilde{s})$ goes through a fixed point $\widetilde{\mathbf{m}}\in\mathbb{D}^3$. Then, we have

$$<\widetilde{\alpha}(\widetilde{s})-\widetilde{\mathbf{m}},\ \ \mathbf{N}(\widetilde{s})>=0.$$

By differentiating this equation and using the Serret-Frenet formulae, thus obtaining

$$<\widetilde{\alpha}(\widetilde{s})-\widetilde{\mathbf{m}},\ \widetilde{\kappa}\mathbf{T}+\widetilde{\tau}\mathbf{B}>=0. \tag{3.1}$$

From the last two equations, it follows that the rectifying plane is orthogonal to both $\mathbf{N}$ and $-\widetilde{\kappa}\mathbf{T}+\widetilde{\tau}\mathbf{B}$. Hence, we can write

$$\widetilde{\alpha}(\widetilde{s})-\widetilde{\mathbf{m}}\ =\ \widetilde{\eta}(\widetilde{s})(\widetilde{\tau}\mathbf{T}+\widetilde{\kappa}\mathbf{B}), \tag{3.2}$$

for a differentiable dual function $\widetilde{\eta}=\widetilde{\eta}(\widetilde{s})$. By differentiation of (3.1), we have that:

$$-\widetilde{\kappa}+<\widetilde{\alpha}(\widetilde{s})-\widetilde{\mathbf{m}},-\widetilde{\kappa}'\mathbf{T}+\widetilde{\tau}'\mathbf{B}>=0. \tag{3.3}$$

Combing (3.2) and (3.3), implies that

$$\eta=\frac{\widetilde{\kappa}}{\widetilde{\tau}'\widetilde{\kappa}-\widetilde{\kappa}'\widetilde{\tau}}. \tag{3.4}$$

Substituting (3.4) into (3.2), we obtain

$$\alpha(\widetilde{s})-\widetilde{\mathbf{m}}\ \ =\ \frac{\widetilde{\kappa}\widetilde{\tau}}{\widetilde{\tau}'\widetilde{\kappa}-\widetilde{\kappa}'\widetilde{\tau}}\mathbf{T}+\frac{\widetilde{\kappa}^2}{\widetilde{\tau}'\widetilde{\kappa}-\widetilde{\kappa}'\widetilde{\tau}}\mathbf{B}. \tag{3.5}$$

Furthermore, one calculates

$$\frac{d\widetilde{\mathbf{m}}}{d\widetilde{s}}=(1+(\frac{\widetilde{\kappa}\widetilde{\tau}}{\widetilde{\tau}'\widetilde{\kappa}-\widetilde{\kappa}'\widetilde{\tau}})')\mathbf{T}+(\frac{\widetilde{\kappa}^2}{\widetilde{\tau}'\widetilde{\kappa}-\widetilde{\kappa}'\widetilde{\tau}})'\mathbf{B}.$$

Therefore, the coefficients vanishing identically if

$$1+(\frac{\widetilde{\kappa}\widetilde{\tau}}{\widetilde{\tau}'\widetilde{\kappa}-\widetilde{\kappa}'\widetilde{\tau}})'=0,\ (\frac{\widetilde{\kappa}^2\epsilon_0}{\widetilde{\tau}'\widetilde{\kappa}-\widetilde{\kappa}'\widetilde{\tau}})'=0,$$

whereby

$$\frac{\widetilde{\kappa\tau}}{\widetilde{\kappa'\tau}-\widetilde{\tau'\kappa}}=\widetilde{s}-\widetilde{c},\quad \frac{\widetilde{\kappa}^2}{\widetilde{\kappa'\tau}-\widetilde{\tau'\kappa}}=\widetilde{d},\ \ \widetilde{c}\in\mathbb{D}. \tag{3.6}$$

Since $\widetilde{\kappa}(\widetilde{s})\neq 0$, then $\widetilde{d}\neq 0$. From (3.5) and (3.6), we have

$$\left.\begin{array}{l}\alpha(\widetilde{s})-\widetilde{\mathbf{m}}=(\widetilde{s}-\widetilde{c})\mathbf{T}-\widetilde{d}\mathbf{B},\\ \frac{\widetilde{\tau}}{\widetilde{\kappa}}=a\widetilde{s}+\widetilde{b};\ \ \widetilde{a}=-\frac{1}{\widetilde{d}}\widetilde{b}=\frac{\widetilde{c}}{\widetilde{d}},\ \widetilde{d}\neq 0.\end{array}\right\} \tag{3.7}$$

Therefore, we can calculate that

$$\left\|\alpha(\widetilde{s})-\widetilde{\mathbf{m}}_0\right\|^2=(\widetilde{s}-\widetilde{c})^2+\widetilde{d}^2.$$

If every rectifying dual plane goes through another dual point $\widetilde{\mathbf{m}}_0$, then let $\widetilde{\gamma}(\widetilde{t})$ be the unit speed dual geodesic line through $\widetilde{\mathbf{m}}_0$, and $\widetilde{\mathbf{m}}$. Then for each $\widetilde{t}\in\mathbb{D}$, there are dual constants $\widetilde{c}(\widetilde{t})$, and $\widetilde{d}(\widetilde{t})\neq 0$ such that

$$\widetilde{\alpha}(\widetilde{s})-\widetilde{\gamma}(\widetilde{t})\ =(\widetilde{s}-\widetilde{c}(\widetilde{t}))\mathbf{T}(\widetilde{s})-\widetilde{d}(\widetilde{t})\mathbf{B}(\widetilde{s}). \tag{3.8}$$

If dot denotes to derivation with respect to $\widetilde{t}$, then from (3.8) we have

$$\dot{\widetilde{\gamma}}(\widetilde{t})\ =\ \dot{\widetilde{c}}(\widetilde{t})\mathbf{T}(\widetilde{s})+\dot{\widetilde{d}}(\widetilde{t})\mathbf{B}(\widetilde{s}).$$

Note that:

$$\frac{\widetilde{\tau}}{\widetilde{\kappa}}=\widetilde{a}(\widetilde{t})\widetilde{s}+\widetilde{b}(\widetilde{t});\ \widetilde{a}(\widetilde{t})=-\frac{1}{\widetilde{d}(\widetilde{t})},\ \widetilde{b}(\widetilde{t})=-\frac{\widetilde{c}(\widetilde{t})}{\widetilde{d}(\widetilde{t})}.$$

By differentiation with respect to $\widetilde{t}$, we have $\dot{\widetilde{a}}(\widetilde{t})\widetilde{s}+\dot{\widetilde{b}}(\widetilde{t})\ =\ 0$. Thus, we have $\dot{\widetilde{a}}(\widetilde{t})\ =\ \dot{\widetilde{b}}(\widetilde{t})\ =\ 0$. This shows that $\dot{\widetilde{\gamma}}(\widetilde{t})=0$, hence $\widetilde{\mathbf{m}}=\widetilde{\mathbf{m}}_0$. Therefore $\widetilde{\mathbf{m}}_0$ is unique. This shows (*i*) implies (*ii*) and (*iii*).

(*ii*) Suppose that $\frac{\widetilde{\tau}}{\widetilde{\kappa}}=\widetilde{a}\widetilde{s}+\widetilde{b};\ \widetilde{a}\neq 0$. If we let $\widetilde{\mathbf{m}}=\widetilde{\alpha}(\widetilde{s})-(\widetilde{s}+\frac{\widetilde{b}}{\widetilde{a}})\mathbf{T}-\frac{1}{\widetilde{a}}\mathbf{B}$, then by the assumption, we have $\widetilde{\mathbf{m}}'=\mathbf{0}$. Hence $\widetilde{\mathbf{m}}$ is a fixed point in $\mathbb{D}^3$ and

$$\widetilde{\alpha}(\widetilde{s})-\widetilde{\mathbf{m}}\ =(\widetilde{s}-\widetilde{c})\mathbf{T}-\widetilde{d}\mathbf{B},\widetilde{c}=-\frac{\widetilde{b}}{\widetilde{a}},\ \ \widetilde{a}=-\frac{1}{\widetilde{d}},\ \widetilde{d}\neq 0.$$

This shows (*ii*) implies (*i*) and (*iii*).
Now suppose that statement (*iii*) holds, then

$$<\alpha(\widetilde{s})-\widetilde{\mathbf{m}},\mathbf{T}>=(\widetilde{s}-c). \tag{3.9}$$

By differentiation of (3.9), and using Serret-Frenet formulae, we have

$$\widetilde{\kappa}(\widetilde{s})<\alpha(\widetilde{s})-\widetilde{\mathbf{m}},\mathbf{N}>=0;\ \ \widetilde{\kappa}(\widetilde{s})\neq 0\Rightarrow<\alpha(\widetilde{s})-\widetilde{\mathbf{m}},\mathbf{N}>=0,$$

which means that every rectifying dual plane of $\widetilde{\alpha}(\widetilde{s})$ goes through a fixed dual point $\widetilde{\mathbf{m}}\in\mathbb{D}^3$. This shows (*iii*) implies (*i*).

We end this section by giving a characterization of a rectifying dual curve in terms of its radial projection. Let us first assume that $\widetilde{\alpha}=\widetilde{\alpha}(\widetilde{s})$ is a unit speed curve in $\mathbb{D}^3$. Then, for a fixed point $\widetilde{\mathbf{m}}\in\mathbb{D}^3$, and by the proof of Theorem 3.1, we have

$$\widetilde{\beta}(\widetilde{s}) = \frac{1}{\widetilde{r}(s)}(\widetilde{\alpha}(\widetilde{s}) - \widetilde{\mathbf{m}}); \;\; \widetilde{r}(\widetilde{s}) = \left\|\alpha(\widetilde{s}) - \widetilde{\widetilde{\mathbf{m}}}\right\| = \sqrt{(\widetilde{s}-\widetilde{c})^2 + \widetilde{d}^2}, \tag{3.10}$$

is the radial projection of $\widetilde{\alpha}(\widetilde{s})$ into the dual unit sphere $\mathbb{K}$.

Theorem 3.2. Let $\alpha = \alpha(\widetilde{s})$ be a unit speed dual curve with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$. If $\widetilde{\mathbf{m}}\in\mathbb{D}^3$ is a fixed dual point, then $\widetilde{\alpha}(\widetilde{s_\beta}) - \widetilde{\mathbf{m}}$ is a position vector lying fully in a rectifying dual plane if and only if, up to a parametrization, $\widetilde{\alpha}(\widetilde{s_\beta}) - \widetilde{\mathbf{m}}$ is given by

$$\alpha(\widetilde{s_\beta}) - \widetilde{\mathbf{m}} = \frac{d}{\cos\widetilde{s_\beta}}\widetilde{\beta}(\widetilde{s_\beta}), \tag{3.11}$$

where $\widetilde{\beta}(\widetilde{s_\beta})$ is a unit speed dual curve lying fully in $\mathbb{K}$.

Proof. Let us first assume that $\widetilde{\beta}(\widetilde{s})$ is a unit speed dual spherical curve. A straightforward calculations show that $\left\|\widetilde{\beta}'(\widetilde{s})\right\| = \frac{\widetilde{d}}{\widetilde{r}^2}$. Then the dual arc length of $\widetilde{\beta}(\widetilde{s})$ is

$$\widetilde{s_\beta} := \int \left\|\widetilde{\beta}'(\widetilde{s})\right\| d\widetilde{s} = \tan^{-1}\left(\frac{\widetilde{s}-\widetilde{c}}{\widetilde{d}}\right). \tag{3.12}$$

From (3.12), since we have $\widetilde{s}-\widetilde{c} = \widetilde{d}\tan\widetilde{s_\beta}$, we obtain $\widetilde{r} = \widetilde{d}\sec\widetilde{s_\beta}$. Substituting this into the first Eq (3.10), we obtain the parametrization (3.11).

Conversely, assume that $\widetilde{\alpha}(s_\beta) - \widetilde{\mathbf{m}}$ is given by (3.12), where $\widetilde{\beta}(\widetilde{s_\beta})$ is unit speed dual curve lying on $\mathbb{K}$. If we calculate the derivative of (3.11), we have

$$(\alpha - \widetilde{\mathbf{m}})' = \frac{\widetilde{d}}{\cos^2\widetilde{s_\beta}}\left(\widetilde{\beta}(\widetilde{s_\beta})\sin\widetilde{s_\beta} + \frac{d\widetilde{\beta}(s_\beta)}{ds_\beta}\cos(s_\beta)\right). \tag{3.13}$$

By the assumption, we have $<\widetilde{\beta},\ \widetilde{\beta}>=<\widetilde{\beta}',\ \widetilde{\beta}'>=1$, and $<\widetilde{\beta},\ \widetilde{\beta}'>=0$. Therefore, it follows that

$$<(\alpha - \widetilde{\mathbf{m}})', \alpha - \widetilde{\mathbf{m}}> = \frac{\widetilde{d}^2\sin\widetilde{s_\beta}}{\cos^3 s_\beta}, \;\; \left\|(\alpha - \widetilde{\mathbf{m}})'\right\| = \frac{\widetilde{d}}{\cos^2 s_\beta}. \tag{3.14}$$

Let us write

$$\widetilde{\alpha} - \widetilde{\mathbf{m}} = \widetilde{\mu}(\widetilde{s_\beta})(\widetilde{\alpha} - \widetilde{\mathbf{m}})' + (\widetilde{\alpha} - \widetilde{\mathbf{m}})^\perp,$$

for dual function $\widetilde{\mu}(\widetilde{s_\beta})$, where $(\widetilde{\alpha} - \widetilde{\mathbf{m}})^\perp$ is the normal component of the position vector $\widetilde{\alpha}(s_\beta) - \widetilde{\mathbf{m}}$. Then, in view of the last equations, we easily find that

$$\widetilde{\mu}(\widetilde{s_\beta}) = \frac{<(\widetilde{\alpha} - \widetilde{\mathbf{m}})', \alpha - \widetilde{\mathbf{m}}>}{\left\|(\widetilde{\alpha} - \widetilde{\mathbf{m}})'\right\|^2} = \frac{\widetilde{d}}{\cos\widetilde{s_\beta}}.$$

Therefore, we have

$$\left\|(\widetilde{\alpha} - \widetilde{\mathbf{m}})^\perp\right\|^2 = \left\|(\alpha - \widetilde{\mathbf{m}})\right\|^2 - \mu^2(s)\left\|(\alpha - \widetilde{\mathbf{m}})'\right\|^2 = d^2 = const,$$

which means that $\alpha(s_\beta) - \widetilde{\mathbf{m}}$ is lying fully in a rectifying dual plane in $\mathbb{D}^3$.
As a result, the following Corollary can be given.

Corollary 3.1. For the dual curve $\widetilde{\beta}(\widetilde{s_\beta})$ on $\mathbb{K}$, the dual curvature $\widetilde{\kappa_\beta}$ is greater than 1. Explicitly, we have

$$\widetilde{\kappa}_\beta^2 = \frac{r^6}{d^4}\widetilde{\kappa}^2 + 1, \tag{3.15}$$

Proof. By a direct calculation, we have the following:

$$\left.\begin{array}{l} \alpha'(\widetilde{s}) = \widetilde{r}'(\widetilde{s})\widetilde{\beta}(\widetilde{s}) + \widetilde{r}(\widetilde{s})\widetilde{\beta}'(\widetilde{s}), \\ \alpha'' = \widetilde{r}''(\widetilde{s})\widetilde{\beta}(\widetilde{s}) + 2\widetilde{r}'(\widetilde{s})\widetilde{\beta}'(\widetilde{s}) + \widetilde{r}(\widetilde{s})\widetilde{\beta}''(\widetilde{s}). \end{array}\right\} \tag{3.16}$$

Therefore, we have:

$$< \widetilde{\beta}(\widetilde{s}), \widetilde{\beta}''(\widetilde{s}) >= -\frac{\widetilde{d}^2}{\widetilde{r}4}, \; < \widetilde{\beta}'(\widetilde{s}), \widetilde{\beta}''(\widetilde{s}) >= -\frac{2\widetilde{d}^2}{\widetilde{r}5}\widetilde{r}', \; \left\|\widetilde{\beta}''(\widetilde{s})\right\|^2 = \frac{4d^2r'^2}{r^6} - \frac{d^4}{r^8}\widetilde{\kappa}_\beta^2,$$

Therefore, by using the above equations, we get

$$\widetilde{\kappa}^2 := \|\alpha''(\widetilde{s})\|^2 = \frac{d^4}{r^6}(1 + \widetilde{\kappa}_\beta^2).$$

which implies the condition (3.1).

4. A differential equation for $\widetilde{\alpha}(\widetilde{s})$ in $\mathbb{D}^3$

In the next proposition, we derive a third-order differential equation satisfied for every space curves in $\mathbb{D}^3$ with $\widetilde{\kappa}$ is nowhere pure dual. For this purpose, we define a smooth dual function on $\widetilde{\alpha}(\widetilde{s})$ by

$$\widetilde{h}(\widetilde{s}) := h(\widetilde{s}) + \varepsilon h^*(\widetilde{s}) =< \widetilde{\alpha}(\widetilde{s}), \mathbf{T}(\widetilde{s}) >, \tag{4.1}$$

we call $\widetilde{h}(\widetilde{s})$ the height dual tangential function (or, tangent dual directed distance functions). From now on, we shall often not write the parameter $\widetilde{s}$.
Here, the Proposition 4.1 corresponds to Proposition 3.1 in [16], and Theorem 4.1 corresponds to Theorem 3.1 in [16].

Proposition 4.1. Let $\widetilde{\alpha}(\widetilde{s})$ be a unit speed dual curve in $\mathbb{D}^3$ with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$. Then, we have

$$\widetilde{\rho}\widetilde{\sigma}\widetilde{h}''' + \left(2\widetilde{\rho}'\widetilde{\sigma} + \widetilde{\rho}\widetilde{\sigma}'\right)\widetilde{h}'' + \left[\left(\widetilde{\sigma}\widetilde{\rho}'\right)' + \left(\frac{\widetilde{\sigma}}{\widetilde{\rho}} + \frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)\right]\widetilde{h}' + \left(\frac{\widetilde{\sigma}}{\widetilde{\rho}}\right)'\widetilde{h} = \left(\widetilde{\sigma}\widetilde{\rho}'\right)' + \frac{\widetilde{\rho}}{\widetilde{\sigma}}, \tag{4.2}$$

where

$$\widetilde{\rho}(\widetilde{s}) := \rho(\widetilde{s}) + \varepsilon\rho^*(\widetilde{s}) = \frac{1}{\widetilde{\kappa}}, \text{ and } \widetilde{\sigma} := \sigma(\widetilde{s}) + \varepsilon\sigma^*(\widetilde{s}) = \frac{1}{\widetilde{\tau}}.$$

Proof. Assume that $\widetilde{\alpha}(\widetilde{s})$ be a unit speed dual curve in $\mathbb{D}^3$ with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$. From (4.1), we have

$$\widetilde{\rho}\left(\widetilde{h}' - 1\right) =< \widetilde{\alpha}, \mathbf{N} >, \tag{4.3}$$

which, again on differentiating gives

$$\widetilde{\rho}\widetilde{\sigma}\widetilde{h}'' + \widetilde{\sigma}\widetilde{\rho}\widetilde{h}' - \widetilde{\sigma}\widetilde{\rho}' + \frac{\widetilde{\sigma}}{\widetilde{\rho}}\widetilde{h} = < \widetilde{\alpha}, \mathbf{B} > . \tag{4.4}$$

After differentiating (4.4) and applying (4.3), we obtain

$$\widetilde{\rho}\widetilde{\sigma}\widetilde{h}''' + \left(2\widetilde{\rho}'\widetilde{\sigma} + \widetilde{\rho}\widetilde{\sigma}'\right)\widetilde{h}'' + \left[\left(\widetilde{\sigma}\widetilde{\rho}'\right)' + \left(\frac{\widetilde{\sigma}}{\widetilde{\rho}} + \frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)\right]\widetilde{h}' + \left(\frac{\widetilde{\sigma}}{\widetilde{\rho}}\right)'\widetilde{h} = \left(\widetilde{\sigma}\widetilde{\rho}'\right)' + \frac{\widetilde{\rho}}{\widetilde{\sigma}},$$

which completes the proof.

4.1. Applications of Proposition 4.1

Now, we show that Proposition 4.1 implies easily several well-known characterizations of dual spherical curves, helices as well as of rectifying curves in $\mathbb{D}^3$.

Corollary 4.1. A unit speed dual curve $\widetilde{\alpha}(\widetilde{s})$ with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$ is a spherical if and only if it satisfies

$$\left(\widetilde{\sigma}\widetilde{\rho}'\right)^2 + \widetilde{\rho}^2 = \widetilde{r}^2, \tag{4.5}$$

for some dual constant $\widetilde{r} = r + \varepsilon r^* \in \mathbb{D}$.

Proof. Let $\widetilde{\alpha}(\widetilde{s})$ be a spherical dual curve lying on a dual sphere with radius $\widetilde{r}$ and centered at the origin. Then

$$< \widetilde{\alpha}(\widetilde{s}), \widetilde{\alpha}(\widetilde{s}) >= \widetilde{r}, \text{ and } h(\widetilde{s}) =< \widetilde{\alpha}(\widetilde{s}), \mathbf{T}(\widetilde{s}) >= 0. \tag{4.6}$$

Hence, the differential (4.2) reduces to

$$\left(\widetilde{\sigma}\widetilde{\rho}'\right)' + \frac{\widetilde{\rho}}{\widetilde{\sigma}} = 0.$$

By multiplying $2\widetilde{\sigma}\widetilde{\rho}'$ to this equation and integrating it give

$$\left(\widetilde{\sigma}\widetilde{\rho}'\right)^2 + \widetilde{\rho}^2 = \widetilde{r}^2. \tag{4.7}$$

Thus, $\widetilde{\alpha}(\widetilde{s})$ is a spherical curve in $\mathbb{D}^3$.

Conversely, assume $\widetilde{\alpha}(\widetilde{s})$ is a unit speed dual curve that satisfies (4.5). Then

$$\widetilde{\alpha}(\widetilde{s}) - \widetilde{\widetilde{\mathbf{m}}}(\widetilde{s}) = \widetilde{\rho}\mathbf{N} + \widetilde{\sigma}\widetilde{\rho}'\mathbf{B}, \tag{4.8}$$

for a parameterized dual curve $\widetilde{\mathbf{m}} = \widetilde{\widetilde{\mathbf{m}}}(\widetilde{s})$. Then

$$\widetilde{r}^2(\widetilde{s}) = \left\|\widetilde{\alpha} - \widetilde{\widetilde{\mathbf{m}}}\right\|^2 = \left(\widetilde{\sigma}\widetilde{\rho}'\right)^2 + \widetilde{\rho}^2, \tag{4.9}$$

If we differentiate (4.8) and (4.9) and make use of Serret-Frenet formulae, the result is $\widetilde{\mathbf{m}}' = \mathbf{0}$, and $\widetilde{r}' = 0$. Hence $\widetilde{\mathbf{m}}$ is a fixed dual point in $\mathbb{D}^3$ and $\widetilde{r}(\widetilde{s})$ is a dual constant $\widetilde{r}$. So, by Eq (4.8), $\widetilde{\alpha}(\widetilde{s})$ lies on

$\mathbb{K}(\widetilde{r})$ with center $\widetilde{\mathbf{m}}$ and radius $\widetilde{r}$.

Another easy consequence of Proposition 4.1 is the following:
Since $\widetilde{\alpha}(\widetilde{s})$ is a linear combination of the Serret-Frenet frame $\{\mathbf{T}(\widetilde{s}), \mathbf{N}(\widetilde{s}), \mathbf{B}(\widetilde{s})\}$, we put:

$$\widetilde{\alpha}(\widetilde{s}) = < \widetilde{\alpha}, \mathbf{T} > \mathbf{T} + < \widetilde{\alpha}, \mathbf{N} > \mathbf{N} + < \widetilde{\alpha}, \mathbf{B} > \mathbf{B}. \tag{4.10}$$

Corollary 4.2. Let $\widetilde{\alpha}(\widetilde{s})$ be a unit speed dual space curve in $\mathbb{D}^3$ with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$. Then

$$< \widetilde{\alpha}, \mathbf{N} >^2 + < \widetilde{\alpha}, \mathbf{B} >^2 = \widetilde{r}^2, \tag{4.11}$$

holds for a dual constant $\widetilde{r}$ if and only if $\widetilde{\alpha}(\widetilde{s})$ is either spherical or normal curve in $\mathbb{D}^3$.
Proof. Assume $\widetilde{\alpha}(\widetilde{s})$ is a unit speed dual curve that satisfies (4.11). Then, it follows from (4.1), (4.10) and (4.11) that

$$||\widetilde{\alpha}(\widetilde{s})||^2 = \widetilde{h}^2 + \widetilde{r}^2. \tag{4.12}$$

After differentiating (4.12) and using (4.1), we find

$$< \widetilde{\alpha}(\widetilde{s}), \mathbf{T} > = \widetilde{h}\widetilde{h}' + 0 \Rightarrow \widetilde{h} = \widetilde{h}\widetilde{h}' \Rightarrow \widetilde{h}(\widetilde{h}' - 1) = 0. \tag{4.13}$$

Thus, we have either $\widetilde{h} = 0$, or $\widetilde{h}' - 1 = 0$. So we have either $\widetilde{h} = 0$, or $\widetilde{h} = \widetilde{s} + \widetilde{c}$ for a dual constant $\widetilde{c}$. Hence, $\widetilde{\alpha}(\widetilde{s})$ is either spherical or normal curve. The converse is clear.

Corollary 4.3. A unit speed dual space curve in $\mathbb{D}^3$ with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$ is a rectifying curve if and only if it satisfies

$$\frac{\widetilde{\kappa}}{\widetilde{\tau}} + (\widetilde{s} + \widetilde{c})\left(\frac{\widetilde{\kappa}}{\widetilde{\tau}}\right)' = 0, \tag{4.14}$$

for some dual constant $\widetilde{c}$.
Proof. Assume that $\widetilde{\alpha}(\widetilde{s})$ is a rectifying curve in $\mathbb{D}^3$ with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual. Then, we have $\widetilde{h}(\widetilde{s}) = \widetilde{s} + \widetilde{c}$. Hence, (4.2) reduces to

$$\frac{\widetilde{\sigma}}{\widetilde{\rho}} + (\widetilde{s} + \widetilde{c})\left(\frac{\widetilde{\sigma}}{\widetilde{\rho}}\right)' = 0, \tag{4.15}$$

which implies condition (4.14).

Conversely, if condition (4.15) holds, then by integrating of (4.15), we find $\widetilde{\tau}\widetilde{c} = (\widetilde{s} + \widetilde{c})\widetilde{\kappa}$ for $\overline{c} \in \mathbb{D}$, which implies that $\widetilde{\alpha}(\widetilde{s})$ is a rectifying curve.

4.2. Characterizations of dual helices

A characterization of the dual helices by its dual curvature and dual torsion is the same as that of the helices in $\mathbb{E}^3$.

Definition 4.1. A dual unit speed curve $\widetilde{\alpha}(\widetilde{s})$ is called a dual helix if its dual unit tangent vector $\mathbf{T}$ makes a constant dual angle $\Theta = \vartheta + \varepsilon\vartheta^*$ with a fixed direction in a dual unit vector $\mathbf{U}$, that is

$$< \mathbf{U}, \mathbf{T} >= \cos\Theta = \widetilde{const}., \text{ with } < \mathbf{U}, \mathbf{U} >=1. \tag{4.16}$$

Theorem 4.1. Let $\widetilde{\alpha}(\widetilde{s})$ be a unit speed space curve in $\mathbb{D}^3$ with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$. Then $\widetilde{\alpha}(\widetilde{s})$ is a helix if and only if, with respect to a suitable arc-length parameter $\widetilde{s}$, the function $\widetilde{h}(\widetilde{s})$ satisfies

$$\left(\widetilde{\rho}\widetilde{h}'\right)' + \left(\frac{\widetilde{\sigma}}{\widetilde{\rho}} + \frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)\widetilde{\tau}\widetilde{h} - \widetilde{\rho}' - \frac{\widetilde{s}\widetilde{\rho}}{\widetilde{\sigma}^2} = 0, \tag{4.17}$$

Proof. Assume that $\widetilde{\alpha}(\widetilde{s})$ is a helix with its axis parallel to a dual unit vector $\mathbf{U}$. Then, we have

$$\left.\begin{array}{l} \mathbf{U} = \cos\Theta\mathbf{T} + \sin\Theta\mathbf{B}, \\ \widetilde{\kappa}\cos\Theta - \widetilde{\tau}\sin\Theta = 0. \end{array}\right\} \tag{4.18}$$

Since, $< \mathbf{U}, \widetilde{\alpha} >' = \cos\Theta$ holds, we have

$$< \mathbf{U}, \widetilde{\alpha} >= \widetilde{s}\cos\Theta + \overline{c}, \tag{4.19}$$

for some dual constant $\overline{c}$. Now, from (4.18) and (4.19), we find

$$< \mathbf{T}, \widetilde{\alpha} >= \widetilde{s} + C - < \widetilde{\alpha}, \mathbf{B} > \frac{\widetilde{\kappa}}{\widetilde{\tau}}, \text{ with } C = \frac{\overline{c}}{\cos\Theta}. \tag{4.20}$$

Combining this equation with (4.1) yields

$$\widetilde{h} = \widetilde{s} + C - < \widetilde{\alpha}, \mathbf{B} > \frac{\widetilde{\kappa}}{\widetilde{\tau}}. \tag{4.21}$$

By differentiating (4.21) and using (4.1), we get

$$\widetilde{\rho}\left(\widetilde{h}' - 1\right) + < \widetilde{\alpha}, \mathbf{N}) >= 0, \tag{4.22}$$

which again by differentiation leads to

$$\widetilde{\rho}\widetilde{h}'' + \widetilde{\rho}'\left(\widetilde{h}' - 1\right) - \widetilde{\kappa} < \mathbf{T}, \widetilde{\alpha} > + \widetilde{\tau} < \widetilde{\alpha}, \mathbf{B} >=0. \tag{4.23}$$

When (4.2) and (4.21) are applied to (4.23), we immediately find that:

$$\left(\widetilde{\rho}\widetilde{h}'\right)' + \left(\frac{\widetilde{\sigma}}{\widetilde{\rho}} + \frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)\widetilde{\tau}\widetilde{h} - \widetilde{\rho}' + \frac{\widetilde{\rho}}{\widetilde{\sigma}^2}(\widetilde{s} + C) = 0, \tag{4.24}$$

which, with respect to a suitable arc-length parameter $\widetilde{s}$, leads to (4.17).

Conversely, assume that $\widetilde{\alpha}(\widetilde{s})$ is a unit speed helix that satisfies (4.17). Then, by differentiating (4.24), we derive

$$\widetilde{\rho}\widetilde{\sigma}\widetilde{h}''' + \left(2\widetilde{\rho}'\widetilde{\sigma} + \widetilde{\rho}\widetilde{\sigma}'\right)\widetilde{h}'' + \left[\left(\widetilde{\sigma}\widetilde{\rho}\right)' + \left(\frac{\widetilde{\sigma}}{\widetilde{\rho}} + \frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)\right]\widetilde{h}' + \left(\frac{\widetilde{\sigma}}{\widetilde{\rho}} + \frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)'\widetilde{h}$$

$$= \left(\widetilde{\sigma}\widetilde{\rho}'\right)' + \left(\frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)'(\widetilde{s}+C) + \frac{\widetilde{\rho}}{\widetilde{\sigma}}. \tag{4.25}$$

Comparing (4.25) with (4.2) in Proposition 4.1, gives

$$\left(\frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)'\left(\widetilde{s}+C-\widetilde{h}\right) = 0. \tag{4.26}$$

If $\widetilde{s}+c=\widetilde{h}$ holds, then (4.26) reduces to

$$\left(\frac{\widetilde{\sigma}}{\widetilde{\rho}} + \frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)(\widetilde{s}+C) - \frac{\widetilde{\rho}}{\widetilde{\sigma}}(\widetilde{s}+C) = 0, \tag{4.27}$$

which implies $\widetilde{\sigma}=0$ which is impossible since $\widetilde{\alpha}(\widetilde{s})$ is assumed to be a regular unit-speed curve. Hence we get $\left(\frac{\widetilde{\rho}}{\widetilde{\sigma}}\right)' = 0$ from (4.26), which implies that $\widetilde{\alpha}(\widetilde{s})$ is a helix.

4.3. Characterizations of helix and dual spherical curve

We end this section by giving new characterizations of dual helices, dual spherical curves in terms of dual Darboux vector of $\widetilde{\alpha}(\widetilde{s})$ as follow: Let $\mathbf{\Omega}^{\perp}(\widetilde{s})$ denote the co-Darboux vector of $\widetilde{\alpha}(\widetilde{s})$ defined by $\mathbf{\Omega}^{\perp} = -\widetilde{\kappa}\mathbf{T} + \widetilde{\tau}\mathbf{B}$. It can be immediately seen that

$$< \mathbf{\Omega}', \mathbf{\Omega}^{\perp} >= -\widetilde{\kappa}\widetilde{\tau}' + \widetilde{\tau}\widetilde{\kappa}'. \tag{4.28}$$

Hence $\widetilde{\alpha}(\widetilde{s})$ is a helix if and only $\mathbf{\Omega}'$ is orthogonal to $\mathbf{\Omega}^{\perp}$.

Theorem 4.2. Let $\widetilde{\alpha}(\widetilde{s})$ be a unit speed space curve in $\mathbb{D}^3$ with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$. Then $\widetilde{\alpha}(\widetilde{s})$ is a spherical if and only if

$$\frac{\widetilde{\tau}}{\widetilde{\kappa}} = \frac{< \widetilde{\alpha}, \mathbf{\Omega} >}{< \widetilde{\alpha}, \mathbf{\Omega}^{\perp} >}, \tag{4.29}$$

holds identically.

Proof. Assume that $\widetilde{\alpha}(\widetilde{s})$ is a unit speed curve in $\mathbb{D}^3$ with $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$. Then, it follows that

$$\left.\begin{array}{l} < \widetilde{\alpha}, \mathbf{\Omega} >= \widetilde{\tau}\widetilde{h} + \widetilde{\kappa} < \widetilde{\alpha}, \mathbf{B} > \\ < \widetilde{\alpha}, \mathbf{\Omega}^{\perp} >= -\widetilde{\kappa}\widetilde{h} + \widetilde{\tau} < \widetilde{\alpha}, \mathbf{B} > . \end{array}\right\} \tag{4.30}$$

Since $\widetilde{\kappa}(\widetilde{s})$ is nowhere pure dual, and $\widetilde{\tau} \neq 0$, we have

$$\left.\begin{array}{l} < \widetilde{\alpha}, \mathbf{\Omega} > \widetilde{\tau} = \widetilde{\tau}^2 < \widetilde{\alpha}, \mathbf{B} >, \\ < \widetilde{\alpha}, \mathbf{\Omega}^{\perp} > \widetilde{\kappa} = \widetilde{\kappa}^2 < \widetilde{\alpha}, \mathbf{B} > . \end{array}\right\} \tag{4.31}$$

So, that we have

$$\frac{< \widetilde{\alpha}(\widetilde{s}), \mathbf{\Omega} >}{< \widetilde{\alpha}(\widetilde{s}), \mathbf{\Omega}^{\perp} >} = \frac{\widetilde{\tau}}{\widetilde{\kappa}}, \tag{4.32}$$

which leads to $\widetilde{\alpha}(\widetilde{s})$ is a spherical if and only if (4.29) holds identically.

5. Conclusions

Mathematical techniques based on E. Study map have been shown to be suitable for study of the characterizations of special curves. These curves are characterized by relationships between the curvatures and torsions of curves. However, well-known examples of such curves are helices, spherical curves, and rectifying curves have been studied in different spaces such as Euclidean space and Minkowski space. But there are no many studies on these curves in dual space which is a more general space than the others. In this space, a dual curve consists of two real curves. So, the characterizations of dual special curves include the characterizations of real space curves. This work simply provided a tool to investigate the characterizations of a curves in a new form.

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Conflict of interest

The authors declare no conflict of interest.

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