Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(4): 3319–3338.
DOI:10.3934/math.2021199
Received: 02 December 2020
Accepted: 12 January 2021
Published: 18 January 2021

Research article

Existence results for nonlinear fractional-order multi-term integro-multipoint boundary value problems

Ahmed Alsaedi[1], **Bashir Ahmad**[1,*], **Manal Alblewi**[1] **and Sotiris K. Ntouyas**[1,2]

[1] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[2] Department of Mathematics, University of Ioannina, 451 10, Ioannina, Greece

* **Correspondence:** Email: bashirahmad_qau@yahoo.com.

Abstract: We investigate the existence of solutions for integro-multipoint boundary value problems involving nonlinear multi-term fractional integro-differential equations. The case involving three different types of nonlinearities is also briefly described. The desired results are obtained by applying the methods of modern functional analysis and are well-illustrated with examples.

Keywords: Caputo fractional derivative; multi-term; Riemann-Liouville fractional integral; integro-multipoint boundary conditions; existence; fixed point theorems
Mathematics Subject Classification: 34A08, 34B15

1. Introduction

The nonlocal nature of fractional-order operators led to a widespread interest in the study and applications of these operators. This popularity motivated many researchers to focus on the theoretical aspects of them to facilitate their applications. For application details, for instance, see the texts [1–3], while the theoretical development can be found in [4–6]. In [7], the authors studied the existence and uniqueness of solutions for a fractional boundary value problem on a graph. The details of eigenvalue problems involving fractional differential equations can be found in [8, 9]. Nonexistence of positive solutions for a system of coupled fractional differential equations was discussed in [10]. The existence of solutions for fractional differential inclusions supplemented with sum and integral boundary conditions was proved in [11]. The authors investigated the existence of solutions for nonlocal boundary value problems involving sequential fractional integro-differential equations and inclusions in [12]. For the details on extremal solutions of generalized Caputo fractional differential equations equipped with Steiltjes-type fractional integro-initial conditions, see [13], while some results on controllability of fractional neutral integro-differential systems and hybrid integro-differential equations can respectively be found in [14] and [15]. The governing equations in the mathematical models of certain real world problems contain more than one fractional order differential operators. Examples include Bagley–Torvik [16] and Basset [17] equations. For some recent work on multi-term fractional-order boundary value problems, we refer the reader to the articles [18, 19]. In a recent work [20], the authors studied nonlinear multi-term fractional differential equations complemented with Riemann-Stieltjes integro-multipoint boundary conditions.

In this paper, we explore the existence criteria for the solutions of a nonlinear multi-term fractional integro-differential equation involving Caputo derivative operators of orders $\kappa_1 \in (1, 2], \kappa_2 \in (1, \kappa_1)$ and an integral operator of order $p > 0$:

$$\lambda_1\, {}^C D^{\kappa_1} x(t) + \lambda_2\, {}^C D^{\kappa_2} x(t) = \zeta(t, x(t)) + I^p \mu(t, x(t)), \tag{1.1}$$

complemented with nonlocal non-separated boundary conditions:

$$\begin{aligned} a_1 x(0) + a_2 x(T) =& A_1 \int_0^{\xi} x(\upsilon) d\upsilon + \sum_{i=1}^{d} \omega_i x(\eta_i), \\ a_3 x'(0) + a_4 x'(T) =& A_2 \int_\eta^{T} x(\upsilon) d\upsilon + \sum_{j=1}^{q} \gamma_j x(\xi_j), \end{aligned} \tag{1.2}$$

where $0 < t < T$, $0 < \xi < \eta < T$, λ_1, λ_2, $a_1, a_2, a_3, a_4, A_1, A_2, \omega_i, \gamma_j \in \mathbb{R}$, $0 < \eta_i, \xi_j < T$, $i = 1, 2, \ldots, d$, $j = 1, 2, \ldots, q$, $\lambda_1 \neq 0$, ${}^C D^{\kappa_1}$ and ${}^C D^{\kappa_2}$ respectively denote the Caputo fractional derivative operators of order κ_1 and κ_2, I^p denotes Riemann-Liouville fractional integral of order $p > 0$ and $\zeta, \mu : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The uniqueness result for the problem (1.1)–(1.2) is obtained by means of Banach's contraction mapping principle, while Krasnosel'skiĭ's fixed point theorem and nonlinear alternative of Leray-Schauder type are used to establish the existence results for the problem at hand.

The rest of the paper is organized as follows: In Section 2 we recall some preliminary concepts of fractional calculus and present an auxiliary result concerning a linear variant of the problem (1.1)–(1.2). The main existence and uniqueness results are proved in Section 3. The case including three types of nonlinearities is indicated in Section 4, while Section 5 is devoted to illustrative examples.

2. Preliminaries and auxiliary lemma

Here we present some auxiliary material related to the study of the problem (1.1)–(1.2).

Definition 2.1. *[4,6] The Riemann-Liouville fractional integral of order $\beta > 0$ for $y \in L_1[a, b]$, existing almost everywhere on $[a, b]$, is defined as*

$$I^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{y(\upsilon)}{(t-\upsilon)^{1-\beta}} d\upsilon, \ t \in [a, b].$$

Definition 2.2. *[4,6] For a function $y \in AC^m[a, b]$, the Caputo derivative of fractional order β, existing almost everywhere on $[a, b]$, is defined as*

$${}^C D^\beta y(t) = \frac{1}{\Gamma(m-\beta)} \int_a^t (t-\upsilon)^{m-\beta-1} y^{(m)}(\upsilon) d\upsilon, \ \beta \in (m-1, m], m \in \mathbb{N}, \ \ t \in [a, b].$$

Lemma 2.3. *(Auxiliary result) Let $\sigma \in C([0,T],\mathbb{R})$. Then the unique solution of the linear multi-term fractional differential equation:*

$$\lambda_1\, {}^C D^{\kappa_1} x(t) + \lambda_2\, {}^C D^{\kappa_2} x(t) = \sigma(t), \tag{2.1}$$

complemented with the boundary conditions (1.2) is given by the solution of the integral equation

$$\begin{aligned} x(t) =& \frac{1}{\lambda_1}\Big[I^{\kappa_1}\sigma(t) - \lambda_2 I^{\kappa_1-\kappa_2}x(t) + \rho_1(t)\Big\{ A_1\Big(I^{\kappa_1+1}\sigma(\xi) - \lambda_2 I^{\kappa_1-\kappa_2+1}x(\xi)\Big) \\ &+ \sum_{i=1}^{d}\omega_i\Big(I^{\kappa_1}\sigma(\eta_i) - \lambda_2 I^{\kappa_1-\kappa_2}x(\eta_i)\Big) - a_2\Big(I^{\kappa_1}\sigma(T) - \lambda_2 I^{\kappa_1-\kappa_2}x(T)\Big)\Big\} \\ &+ \rho_2(t)\Big\{A_2\Big(\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\sigma(u)du d\upsilon - \lambda_2\int_\eta^T\int_0^\upsilon\frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(u)dud\upsilon\Big) \\ &+ \sum_{j=1}^{q}\gamma_j\Big(I^{\kappa_1}\sigma(\xi_j) - \lambda_2 I^{\kappa_1-\kappa_2}x(\xi_j)\Big) - a_4\Big(I^{\kappa_1-1}\sigma(T) - \lambda_2 I^{\kappa_1-\kappa_2-1}x(T)\Big)\Big\}\Big], \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \rho_1(t) &= \frac{1}{2\delta}\Big\{ -(A_2(T^2-\eta^2) + 2\sum_{i=1}^{d}\gamma_j\xi_j - 2(a_3+a_4)) + 2t(A_2(T-\eta) + \sum_{j=1}^{q}\gamma_j)\Big\}, \\ \rho_2(t) &= \frac{1}{2\delta}\Big\{A_1\xi^2 + 2\sum_{i=1}^{d}\omega_i\eta_i - 2a_2T - 2t(A_1\xi + \sum_{i=1}^{d}\omega_i - (a_1+a_2))\Big\}, \end{aligned} \tag{2.3}$$

and it is assumed that

$$\begin{aligned} \delta =& \frac{1}{2}\Big[\Big((T^2-\eta^2)A_2 + 2\sum_{j=1}^{q}\gamma_j\xi_j - 2(a_3+a_4)\Big)\Big(A_1\xi + \sum_{i=1}^{d}\omega_i - (a_1+a_2)\Big) \\ &- \Big(A_1\xi^2 + 2\sum_{i=1}^{d}\omega_i\eta_i - 2a_2T\Big)\Big(A_2(T-\eta) + \sum_{j=1}^{q}\gamma_j\Big)\Big] \neq 0. \end{aligned} \tag{2.4}$$

Proof. Applying the fractional integral operator I^{κ_1} to both sides of the fractional differential equation in (2.1) we get,

$$x(t) = \frac{1}{\lambda_1}I^{\kappa_1}\sigma(t) - \frac{\lambda_2}{\lambda_1}I^{\kappa_1-\kappa_2}x(t) - c_0 - c_1 t, \tag{2.5}$$

where c_0 and c_1 are unknown arbitrary constants. From (2.5) we have

$$x'(t) = \frac{1}{\lambda_1}I^{\kappa_1-1}\sigma(t) - \frac{\lambda_2}{\lambda_1}I^{\kappa_1-\kappa_2-1}x(t) - c_1. \tag{2.6}$$

Using the boundary conditions (1.2) in (2.5) and (2.6), we get a system of equations in the unknown constants c_0 and c_1:

$$\Big\{A_1\xi + \sum_{i=1}^{d}\omega_i - (a_1+a_2)\Big\}c_0 + \Big\{\frac{A_1\xi^2 + 2\sum_{i=1}^{d}\omega_i\eta_i - 2a_2T}{2}\Big\}c_1$$

$$
\begin{aligned}
= & \frac{A_1}{\lambda_1}[I^{\kappa_1+1}\sigma(\xi) - \lambda_2 I^{\kappa_1-\kappa_2+1}x(\xi)] - \frac{a_2}{\lambda_1}[I^{\kappa_1}\sigma(T) - \lambda_2 I^{\kappa_1-\kappa_2}x(T)] \\
& + \frac{1}{\lambda_1}\sum_{i=1}^{d}\omega_i[I^{\kappa_1}\sigma(\eta_i) - \lambda_2 I^{\kappa_1-\kappa_2}x(\eta_i)],
\end{aligned}
$$

$$
\begin{aligned}
& \Big\{A_2(T-\eta) + \sum_{j=1}^{q}\gamma_j\Big\}c_0 + \Big\{\frac{((T^2-\eta^2)A_2 + 2\sum_{j=1}^{q}\gamma_j\xi_j - 2(a_3+a_4))}{2}\Big\}c_1 \qquad (2.7) \\
= & \frac{1}{\lambda_1}\sum_{j=1}^{q}\gamma_j[I^{\kappa_1}\sigma(\xi_j) - \lambda_2 I^{\kappa_1-\kappa_2}x(\xi_j)] - \frac{a_4}{\lambda_1}[I^{\kappa_1-1}\sigma(T) - \lambda_2 I^{\kappa_1-\kappa_2-1}x(T)] \\
& + \frac{A_2}{\lambda_1}\Big[\int_{\eta}^{T}\int_0^{\upsilon}\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\sigma(u)dud\upsilon - \lambda_2\int_{\eta}^{T}\int_0^{\upsilon}\frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(u)dud\upsilon\Big].
\end{aligned}
$$

Solving the system (2.7) for c_0 and c_1, we find that

$$
\begin{aligned}
c_0 = & \frac{1}{\lambda_1 2\delta}\Big[\Big(A_2(T^2-\eta^2) + 2\sum_{j=1}^{q}\gamma_j\xi_j - 2(a_3+a_4)\Big)\Big\{A_1\Big(I^{\kappa_1+1}\sigma(\xi) - \lambda_2 I^{\kappa_1-\kappa_2+1}x(\xi)\Big) \\
& + \sum_{i=1}^{d}\omega_i\Big(I^{\kappa_1}\sigma(\eta_i) - \lambda_2 I^{\kappa_1-\kappa_2}x(\eta_i)\Big) - a_2\Big(I^{\kappa_1}\sigma(T) - \lambda_2 I^{\kappa_1-\kappa_2}x(T)\Big)\Big\} \\
& -\Big(A_1\xi^2 + 2\sum_{i=1}^{d}\omega_i\eta_i - 2a_2T\Big)\Big\{A_2\Big(\int_{\eta}^{T}\int_0^{\upsilon}\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\sigma(u)dud\upsilon \\
& -\lambda_2\int_{\eta}^{T}\int_0^{\upsilon}\frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(u)dud\upsilon\Big) \\
& + \sum_{j=1}^{q}\gamma_j\Big(I^{\kappa_1}\sigma(\xi_j) - \lambda_2 I^{\kappa_1-\kappa_2}x(\xi_j)\Big) - a_4\Big(I^{\kappa_1-1}\sigma(T) - \lambda_2 I^{\kappa_1-\kappa_2-1}x(T)\Big)\Big\}\Big], \\
c_1 = & \frac{1}{\lambda_1\delta}\Big[\Big(A_2(T-\eta) + \sum_{j=1}^{q}\gamma_j\Big)\Big\{A_1\Big(I^{\kappa_1+1}\sigma(\xi) - \lambda_2 I^{\kappa_1-\kappa_2+1}x(\xi)\Big) \\
& + \sum_{i=1}^{d}\omega_i\Big(I^{\kappa_1}\sigma(\eta_i) - \lambda_2 I^{\kappa_1-\kappa_2}x(\eta_i)\Big) - a_2\Big(I^{\kappa_1}\sigma(T) - \lambda_2 I^{\kappa_1-\kappa_2}x(T)\Big)\Big\} \\
& -\Big(A_1\xi + \sum_{i=1}^{d}\omega_i - (a_1+a_2)\Big)\Big\{A_2\Big(\int_{\eta}^{T}\int_0^{\upsilon}\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\sigma(u)dud\upsilon \\
& -\lambda_2\int_{\eta}^{T}\int_0^{\upsilon}\frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(u)dud\upsilon\Big) + \sum_{j=1}^{q}\gamma_j\Big(I^{\kappa_1}\sigma(\xi_j) - \lambda_2 I^{\kappa_1-\kappa_2}x(\xi_j)\Big) \\
& -a_4\Big(I^{\kappa_1-1}\sigma(T) - \lambda_2 I^{\kappa_1-\kappa_2-1}x(T)\Big)\Big\}\Big].
\end{aligned}
$$

Inserting the values of c_0 and c_1 into (2.5) leads to the Eq (2.2). The converse of this lemma follows by direct computation. □

3. Existence and uniqueness results

Keeping in mind Lemma 2.3, we introduce an operator $\mathcal{V} : C \to C$ associated with the problem (1.1)–(1.2) by

$$\begin{aligned}
(\mathcal{V}x)(t) \;=\; & \frac{1}{\lambda_1}\Big[\int_0^t \frac{(t-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon + \int_0^t \frac{(t-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon \\
& -\lambda_2\int_0^t \frac{(t-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(\upsilon)d\upsilon + \rho_1(t)\Big\{A_1\Big\{\int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1}}{\Gamma(\kappa_1+1)}\zeta(\upsilon,x(\upsilon))d\upsilon \\
& + \int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1+p}}{\Gamma(\kappa_1+p+1)}\mu(\upsilon,x(\upsilon))d\upsilon - \lambda_2\int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1-\kappa_2}}{\Gamma(\kappa_1-\kappa_2+1)}x(\upsilon)d\upsilon\Big\} \\
& + \sum_{i=1}^d \omega_i\Big\{\int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon + \int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon \\
& -\lambda_2\int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(\upsilon)d\upsilon\Big\} - a_2\Big\{\int_0^T \frac{(T-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon \\
& + \int_0^T \frac{(T-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon - \lambda_2\int_0^T \frac{(T-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(\upsilon)d\upsilon\Big\}\Big\} \qquad (3.1) \\
& +\rho_2(t)\Big\{A_2\Big\{\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(u,x(u))dud\upsilon \\
& + \int_\eta^T\int_0^\upsilon\int_0^u \Big(\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\Big)\Big(\frac{(u-w)^{p-1}}{\Gamma(p)}\Big)\mu(w,x(w))dwdud\upsilon \\
& -\lambda_2\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(u)dud\upsilon\Big\} + \sum_{j=1}^q \gamma_j\Big\{\int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon \\
& + \int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon - \lambda_2\int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(\upsilon)d\upsilon\Big\} \\
& -a_4\Big\{\int_0^T \frac{(T-\upsilon)^{\kappa_1-2}}{\Gamma(\kappa_1-1)}\zeta(\upsilon,x(\upsilon))d\upsilon + \int_0^T \frac{(T-\upsilon)^{\kappa_1+p-2}}{\Gamma(\kappa_1+p-1)}\mu(\upsilon,x(\upsilon))d\upsilon \\
& -\lambda_2\int_0^T \frac{(T-\upsilon)^{\kappa_1-\kappa_2-2}}{\Gamma(\kappa_1-\kappa_2-1)}x(\upsilon)d\upsilon\Big\}\Big\}\Big], \; t\in[0,T],
\end{aligned}$$

where $C = C([0,T],\mathbb{R})$ is the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ equipped with the norm $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. Notice that the problem (1.1)–(1.2) will have a solution once it is shown that the operator $\mathcal{V}$ has a fixed point.

Theorem 3.1. *Assume that:*

(H_1) $\zeta,\mu : [0,T]\times\mathbb{R} \to \mathbb{R}$ *are continuous functions such that*

$$|\zeta(t,x)-\zeta(t,\overline{x})| \le L_1|x-\overline{x}|, \;\; |\mu(t,x)-\mu(t,\overline{x})| \le L_2|x-\overline{x}|, \;\; L_1,L_2>0,$$

for all $t\in[0,T]$, $x,\overline{x}\in\mathbb{R}$ *and* $|\zeta(t,0)|\le M_1<\infty$, $|\mu(t,0)|\le M_2<\infty$ *for all* $t\in[0,T]$*;*

(H_2) $(L\Omega_1+\Omega_2)<1$, *where* $L=\max\{L_1,L_2\}$,

$$\begin{aligned}
\Omega_1 &= \frac{1}{|\lambda_1|}\Big[\frac{1}{\Gamma(\kappa_1+2)}\Big\{(\kappa_1+1)\Big(T^{\kappa_1}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1}\Big)+\bar{\rho_2}\Big(\kappa_1|a_4|T^{\kappa_1-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1}\Big)\Big)\\
&\quad+\bar{\rho_1}|A_1|\xi^{\kappa_1+1}+\bar{\rho_2}|A_2||T^{\kappa_1+1}-\eta^{\kappa_1+1}|\Big\}+\frac{1}{\Gamma(\kappa_1+p+2)}\Big\{(\kappa_1+p+1)\Big(T^{\kappa_1+p}\\
&\quad+\bar{\rho_1}\Big(|a_2|T^{\kappa_1+p}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1+p}\Big)+\bar{\rho_2}\Big((\kappa_1+p)|a_4|T^{\kappa_1+p-1}\\
&\quad+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1+p}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1+p+1}+\bar{\rho_2}|A_2||T^{\kappa_1+p+1}-\eta^{\kappa_1+p+1}|\Big\}\Big],\\
\Omega_2 &= \frac{|\lambda_2|}{|\lambda_1|\Gamma(\kappa_1-\kappa_2+2)}\Big[(\kappa_1-\kappa_2+1)\Big(T^{\kappa_1-\kappa_2}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1-\kappa_2}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1-\kappa_2}\Big)\\
&\quad+\bar{\rho_2}\Big((\kappa_1-\kappa_2)|a_4|T^{\kappa_1-\kappa_2-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1-\kappa_2}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1-\kappa_2+1}\\
&\quad+\bar{\rho_2}|A_2||T^{\kappa_1-\kappa_2+1}-\eta^{\kappa_1-\kappa_2+1}|\Big],
\end{aligned}\tag{3.2}$$

where

$$\bar{\rho_i}=\sup_{t\in[0,T]}|\rho_i(t)|,\ i=1,2.$$

Then the problem (1.1)–(1.2) has a unique solution on $[0,T]$.

Proof. Consider a closed ball $B_r=\{x\in C,\|x\|\le r\}$ and show that $\mathcal{V}B_r\subset B_r$, where the operator $\mathcal{V}$ is defined by (3.1) and $r\ge M\Omega_1(1-L\Omega_1-\Omega_2)^{-1}$, where $M=\max\{M_1,M_2\}$ and Ω_i $(i=1,2)$ are given in (3.2). For any $x\in B_r$, it follows by the condition (H_1) that

$$|\zeta(t,x)|=|\zeta(t,x(t))-\zeta(t,0)+\zeta(t,0)|\le|\zeta(t,x(t))+\zeta(t,0)|+|\zeta(t,0)|\le L_1\|x\|+M_1\le L_1r+M_1.$$

In a similar manner, we can get $|\mu(t,x)|\le L_2r+M_2$. In view of the foregoing inequalities we obtain

$$\begin{aligned}
\|(\mathcal{V}x)\| &= \sup_{t\in[0,T]}|(\mathcal{V}x)(t)|\\
&\le \sup_{t\in[0,T]}\Big\{\frac{1}{|\lambda_1|}\Big\{\int_0^t\frac{(t-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon+\int_0^t\frac{(t-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))|d\upsilon\\
&\quad+|\lambda_2|\int_0^t\frac{(t-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(\upsilon)|d\upsilon+|\rho_1(t)|\Big\{|A_1|\Big\{\int_0^{\xi}\frac{(\xi-\upsilon)^{\kappa_1}}{\Gamma(\kappa_1+1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon\\
&\quad+\int_0^{\xi}\frac{(\xi-\upsilon)^{\kappa_1+p}}{\Gamma(\kappa_1+p+1)}|\mu(\upsilon,x(\upsilon))|d\upsilon+|\lambda_2|\int_0^{\xi}\frac{(\xi-\upsilon)^{\kappa_1-\kappa_2}}{\Gamma(\kappa_1-\kappa_2+1)}|x(\upsilon)|d\upsilon\Big\}\\
&\quad+\sum_{i=1}^{d}|\omega_i|\Big\{\int_0^{\eta_i}\frac{(\eta_i-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon+\int_0^{\eta_i}\frac{(\eta_i-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))|d\upsilon\\
&\quad+|\lambda_2|\int_0^{\eta_i}\frac{(\eta_i-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(\upsilon)|d\upsilon\Big\}+|a_2|\Big\{\int_0^{T}\frac{(T-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon
\end{aligned}$$

$$
\begin{aligned}
& +\int_0^T \frac{(T-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))|d\upsilon+|\lambda_2|\int_0^T \frac{(T-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(\upsilon)|d\upsilon\Big\}\Big\} \\
& +|\rho_2(t)|\Big\{|A_2|\Big\{\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(u,x(u))|dud\upsilon \\
& +\int_\eta^T\int_0^\upsilon\int_0^u \Big(\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\Big)\Big(\frac{(u-w)^{p-1}}{\Gamma(p)}\Big)|\mu(w,x(w))|dwdud\upsilon \\
& +|\lambda_2|\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(u)|dud\upsilon\Big\}+\sum_{j=1}^q|\gamma_j|\Big\{\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon \\
& +\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))|d\upsilon+|\lambda_2|\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(\upsilon)|d\upsilon\Big\} \\
& +|a_4|\Big\{\int_0^T\frac{(T-\upsilon)^{\kappa_1-2}}{\Gamma(\kappa_1-1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon+\int_0^T\frac{(T-\upsilon)^{\kappa_1+p-2}}{\Gamma(\kappa_1+p-1)}|\mu(\upsilon,x(\upsilon))|d\upsilon \\
& +|\lambda_2|\int_0^T\frac{(T-\upsilon)^{\kappa_1-\kappa_2-2}}{\Gamma(\kappa_1-\kappa_2-1)}|x(\upsilon)|d\upsilon\Big\}\Big\}\Big\}\Big\} \\
\leq\ & \frac{(L_1r+M_1)}{|\lambda_1|\Gamma(\kappa_1+2)}\Big\{(\kappa_1+1)\Big(T^{\kappa_1}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1}+\sum_{i=1}^d|\omega_i|\eta_i^{\kappa_1}\Big) \\
& +\bar{\rho_2}\Big(\kappa_1|a_4|T^{\kappa_1-1}+\sum_{j=1}^q|\gamma_j|\xi_j^{\kappa_1}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1+1}+\bar{\rho_2}|A_2||T^{\kappa_1+1}-\eta^{\kappa_1+1}|\Big\} \\
& +\frac{(L_2r+M_2)}{|\lambda_1|\Gamma(\kappa_1+p+2)}\Big\{(\kappa_1+p+1)\Big(T^{\kappa_1+p}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1+p}+\sum_{i=1}^d|\omega_i|\eta_i^{\kappa_1+p}\Big) \\
& +\bar{\rho_2}\Big((\kappa_1+p)|a_4|T^{\kappa_1+p-1}+\sum_{j=1}^q|\gamma_j|\xi_j^{\kappa_1+p}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1+p+1} \\
& +\bar{\rho_2}|A_2||T^{\kappa_1+p+1}-\eta^{\kappa_1+p+1}|\Big\}+\frac{|\lambda_2|r}{|\lambda_1|\Gamma(\kappa_1-\kappa_2+2)}\Big\{(\kappa_1-\kappa_2+1)\Big(T^{\kappa_1-\kappa_2} \\
& +\bar{\rho_1}\Big(|a_2|T^{\kappa_1-\kappa_2}+\sum_{i=1}^d|\omega_i|\eta_i^{\kappa_1-\kappa_2}\Big)+\bar{\rho_2}\Big((\kappa_1-\kappa_2)|a_4|T^{\kappa_1-\kappa_2-1}+\sum_{j=1}^q|\gamma_j|\xi_j^{\kappa_1-\kappa_2}\Big)\Big) \\
& +\bar{\rho_1}|A_1|\xi^{\kappa_1-\kappa_2+1}+\bar{\rho_2}|A_2||T^{\kappa_1-\kappa_2+1}-\eta^{\kappa_1-\kappa_2+1}|\Big\} \\
\leq\ & (Lr+M)\Omega_1+r\Omega_2\leq r,
\end{aligned}
$$

which shows that $\mathcal{V}x\in B_r$ for any $x\in B_r$. Hence $\mathcal{V}B_r\subset B_r$.

Next it will be established that the operator $\mathcal{V}$ is a contraction. For $x,y\in\mathbb{R}$, we have

$$
\begin{aligned}
& \|\mathcal{V}x-\mathcal{V}\overline{x}\| \\
=\ & \sup_{t\in[0,T]}|(\mathcal{V}x)(t)-(\mathcal{V}\overline{x})(t)| \\
\leq\ & \sup_{t\in[0,T]}\Big\{\frac{1}{|\lambda_1|}\Big\{\int_0^t\frac{(t-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))-\zeta(\upsilon,\overline{x}(\upsilon))|d\upsilon
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \frac{(t-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))-\mu(\upsilon,\overline{x}(\upsilon))|d\upsilon+|\lambda_2|\int_0^t \frac{(t-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(\upsilon)-\overline{x}(\upsilon)|d\upsilon\\
&+|\rho_1(t)|\Big\{|A_1|\Big\{\int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1}}{\Gamma(\kappa_1+1)}|\zeta(\upsilon,x(\upsilon))-\zeta(\upsilon,\overline{x}(\upsilon))|d\upsilon\\
&+\int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1+p}}{\Gamma(\kappa_1+p+1)}|\mu(\upsilon,x(\upsilon))-\mu(\upsilon,\overline{x}(\upsilon))|d\upsilon+|\lambda_2|\int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1-\kappa_2}}{\Gamma(\kappa_1-\kappa_2+1)}|x(\upsilon)-\overline{x}(\upsilon)|d\upsilon\Big\}\\
&+\sum_{i=1}^d|\omega_i|\Big\{\int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))-\zeta(\upsilon,\overline{x}(\upsilon))|d\upsilon\\
&+\int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))-\mu(\upsilon,\overline{x}(\upsilon))|d\upsilon+|\lambda_2|\int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(\upsilon)-\overline{x}(\upsilon)|d\upsilon\Big\}\\
&+|a_2|\Big\{\int_0^T \frac{(T-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))-\zeta(\upsilon,\overline{x}(\upsilon))|d\upsilon\\
&+\int_0^T \frac{(T-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))-\mu(\upsilon,\overline{x}(\upsilon))|d\upsilon+|\lambda_2|\int_0^T \frac{(T-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(\upsilon)-\overline{x}(\upsilon)|d\upsilon\Big\}\Big\}\\
&+|\rho_2(t)|\Big\{|A_2|\Big\{\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(u,x(u))-\zeta(u,\overline{x}(u))|dud\upsilon\\
&+\int_\eta^T\int_0^\upsilon\int_0^u\Big(\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\Big)\Big(\frac{(u-w)^{p-1}}{\Gamma(p)}\Big)|\mu(w,x(w))-\mu(w,\overline{x}(w))|dwdud\upsilon\\
&+|\lambda_2|\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(u)-\overline{x}(u)|dud\upsilon\Big\}\\
&+\sum_{j=1}^q|\gamma_j|\Big\{\int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))-\zeta(\upsilon,\overline{x}(\upsilon))|d\upsilon\\
&+\int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))-\mu(\upsilon,\overline{x}(\upsilon))|d\upsilon+|\lambda_2|\int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|x(\upsilon)-\overline{x}(\upsilon)|d\upsilon\Big\}\\
&+|a_4|\Big\{\int_0^T \frac{(T-\upsilon)^{\kappa_1-2}}{\Gamma(\kappa_1-1)}|\zeta(\upsilon,x(\upsilon))-\zeta(\upsilon,\overline{x}(\upsilon))|d\upsilon\\
&+\int_0^T \frac{(T-\upsilon)^{\kappa_1+p-2}}{\Gamma(\kappa_1+p-1)}|\mu(\upsilon,x(\upsilon))-\mu(\upsilon,\overline{x}(\upsilon))|d\upsilon\\
&+|\lambda_2|\int_0^T \frac{(T-\upsilon)^{\kappa_1-\kappa_2-2}}{\Gamma(\kappa_1-\kappa_2-1)}|x(\upsilon)-\overline{x}(\upsilon)|d\upsilon\Big\}\Big\}\Big\}\Big\}\\
\leq\ &\Big[\frac{L_1}{|\lambda_1|\Gamma(\kappa_1+2)}\Big\{(\kappa_1+1)\Big(T^{\kappa_1}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1}+\sum_{i=1}^d|\omega_i|\eta_i^{\kappa_1}\Big)+\bar{\rho_2}\Big(\kappa_1|a_4|T^{\kappa_1-1}+\sum_{j=1}^q|\gamma_j|\xi_j^{\kappa_1}\Big)\Big)\\
&+\bar{\rho_1}|A_1|\xi^{\kappa_1+1}+\bar{\rho_2}|A_2||T^{\kappa_1+1}-\eta^{\kappa_1+1}|\Big\}+\frac{L_2}{|\lambda_1|\Gamma(\kappa_1+p+2)}\Big\{(\kappa_1+p+1)\Big(T^{\kappa_1+p}\\
&+\bar{\rho_1}\Big(|a_2|T^{\kappa_1+p}+\sum_{i=1}^d|\omega_i|\eta_i^{\kappa_1+p}\Big)+\bar{\rho_2}\Big((\kappa_1+p)|a_4|T^{\kappa_1+p-1}+\sum_{j=1}^q|\gamma_j|\xi_j^{\kappa_1+p}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1+p+1}
\end{aligned}
$$

$$
\begin{aligned}
& +\bar{\rho_2}|A_2||T^{\kappa_1+p+1}-\eta^{\kappa_1+p+1}|\Big\} + \frac{|\lambda_2|}{|\lambda_2|\Gamma(\kappa_1-\kappa_2+2)}\Big\{(\kappa_1-\kappa_2+1)\Big(T^{\kappa_1-\kappa_2}+\bar{\rho_1}(|a_2|T^{\kappa_1-\kappa_2} \\
& +\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1-\kappa_2}\Big)+\bar{\rho_2}\Big((\kappa_1-\kappa_2)|a_4|T^{\kappa_1-\kappa_2-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1-\kappa_2}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1-\kappa_2+1} \\
& +\bar{\rho_2}|A_2||T^{\kappa_1-\kappa_2+1}-\eta^{\kappa_1-\kappa_2+1}|\Big\}\Bigg]\|x-\overline{x}\| \\
\leq\ & (L\Omega_1+\Omega_2)\|x-\overline{x}\|,
\end{aligned}
$$

which implies that the operator $\mathcal{V}$ is a contraction by the condition (H_2). Hence the operator $\mathcal{V}$ has a unique fixed point by Banach fixed point theorem. In consequence, there exists a unique solution for the problem (1.1)–(1.2) on $[0, T]$. □

Remark 3.2. *In Theorem 3.1 we used the following form of Banach fixed point theorem from [21]:* "Let X be a Banach space, $D \subset X$ be closed and $F : D \to D$ is a strict contraction, that is, $|Fx - Fy| \leq k|x - y|$ for some $k \in (0, 1)$ and all $x, y \in D$. Then F has a fixed point in D." *If we use the form of Banach fixed point theorem from [22]:* "Any contraction mapping of a complete non-empty metric space $\mathcal{M}$ into itself has a unique fixed point in $\mathcal{M}$", *then the condition* $L\Omega_1 + \Omega_2 < 1$ *can be omitted if we use the well-known Bielecki's re-norming method.*

In the following result, we apply Krasnosel'skiĭ's fixed point theorem [22, 23] to establish the existence of at least one solution for the boundary value problem (1.1)–(1.2).

Theorem 3.3. *Let* $\zeta, \mu : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be continuous functions such that*

(H_3) *we can find* $\beta_1, \beta_2 \in C([0, T], \mathbb{R}^+)$ *with* $\|\beta\| = \max\{\|\beta_1\|, \|\beta_2\|\}$ *such that* $|\zeta(t, x)| \leq \beta_1(t)$ *and* $|g(t, x)| \leq \beta_2(t)$, *for all* $(t, x) \in [0, T] \times \mathbb{R}$.

Then the problem (1.1)–(1.2) has at least one solution on $[0, T]$ *if* $\Omega_2 < 1$, *where* Ω_2 *is given in (3.2).*

Proof. Let $B_\alpha = \{x \in C : \|x\| \leq \alpha\}$ be a closed ball with $\alpha \geq \|\beta\|\Omega_1(1 - \Omega_2)^{-1}$ and define operators $\mathcal{V}_1$ and $\mathcal{V}_2$ on $K_\alpha \to C$ by

$$
\begin{aligned}
(\mathcal{V}_1x)(t) \ =\ & \frac{1}{\lambda_1}\Big\{\int_0^t\frac{(t-v)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(v,x(v))dv+\int_0^t\frac{(t-v)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(v,x(v))dv \\
& +\rho_1(t)\Big\{A_1\Big\{\int_0^\xi\frac{(\xi-v)^{\kappa_1}}{\Gamma(\kappa_1+1)}\zeta(v,x(v))dv+\int_0^\xi\frac{(\xi-v)^{\kappa_1+p}}{\Gamma(\kappa_1+p+1)}\mu(v,x(v))dv\Big\} \\
& +\sum_{i=1}^{d}\omega_i\Big\{\int_0^{\eta_i}\frac{(\eta_i-v)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(v,x(v))dv+\int_0^{\eta_i}\frac{(\eta_i-v)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(v,x(v))dv\Big\} \\
& -a_2\Big\{\int_0^T\frac{(T-v)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(v,x(v))dv+\int_0^T\frac{(T-v)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(v,x(v))dv\Big\}\Big\} \\
& +\rho_2(t)\Big\{A_2\Big\{\int_\eta^T\int_0^v\frac{(v-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(u,x(u))dudv \\
& +\int_\eta^T\int_0^v\int_0^u\Big(\frac{(v-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\Big)\Big(\frac{(u-w)^{p-1}}{\Gamma(p)}\Big)\mu(w,x(w))dwdudv\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{q}\gamma_j\Big\{\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon+\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon\Big\}\\
&-a_4\Big\{\int_0^{T}\frac{(T-\upsilon)^{\kappa_1-2}}{\Gamma(\kappa_1-1)}\zeta(\upsilon,x(\upsilon))d\upsilon+\int_0^{T}\frac{(T-\upsilon)^{\kappa_1+p-2}}{\Gamma(\kappa_1+p-1)}\mu(\upsilon,x(\upsilon))d\upsilon\Big\}\Big\}\Big\},\ t\in[0,T],\\
(\mathcal{V}_2x)(t) \ =\ & -\frac{\lambda_2}{\lambda_1}\Big\{\int_0^{t}\frac{(t-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(\upsilon)d\upsilon+\rho_1(t)\Big\{A_1\int_0^{\xi}\frac{(\xi-\upsilon)^{\kappa_1-\kappa_2}}{\Gamma(\kappa_1-\kappa_2+1)}x(\upsilon)d\upsilon\\
&+\sum_{i=1}^{d}\omega_i\int_0^{\eta_i}\frac{(\eta_i-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(\upsilon)d\upsilon-a_2\int_0^{T}\frac{(T-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(\upsilon)d\upsilon\Big\}\\
&+\rho_2(t)\Big\{A_2\int_{\eta}^{T}\int_0^{\upsilon}\frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(u)dud\upsilon+\sum_{j=1}^{q}\gamma_j\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}x(\upsilon)d\upsilon\\
&-a_4\int_0^{T}\frac{(T-\upsilon)^{\kappa_1-\kappa_2-2}}{\Gamma(\kappa_1-\kappa_2-1)}x(\upsilon)d\upsilon\Big\}\Big\},\ t\in[0,T].
\end{aligned}
$$

Observe that $\mathcal{V}=\mathcal{V}_1+\mathcal{V}_2$ on B_α. Let us now verify the hypotheses of Krasnosel'skiĭ's fixed point theorem [23].

(i) For $x,y\in K_\alpha$, we have

$$
\begin{aligned}
&\|(\mathcal{V}_1x)+(\mathcal{V}_2y)\| = \sup_{t\in[0,T]}|(\mathcal{V}_1x)(t)+(\mathcal{V}_2y)(t)|\\
\le\ & \sup_{t\in[0,T]}\Big\{\frac{1}{|\lambda_1|}\Big\{\int_0^{t}\frac{(t-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon+\int_0^{t}\frac{(t-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))|d\upsilon\\
&+|\rho_1(t)|\Big\{|A_1|\Big\{\int_0^{\xi}\frac{(\xi-\upsilon)^{\kappa_1}}{\Gamma(\kappa_1+1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon+\int_0^{\xi}\frac{(\xi-\upsilon)^{\kappa_1+p}}{\Gamma(\kappa_1+p+1)}|\mu(\upsilon,x(\upsilon))|d\upsilon\Big\}\\
&+\sum_{i=1}^{d}|\omega_i|\Big\{\int_0^{\eta_i}\frac{(\eta_i-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon+\int_0^{\eta_i}\frac{(\eta_i-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))|d\upsilon\Big\}\\
&+|a_2|\Big\{\int_0^{T}\frac{(T-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon+\int_0^{T}\frac{(T-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))|d\upsilon\Big\}\Big\}\\
&+|\rho_2(t)|\Big\{A_2\Big\{\int_{\eta}^{T}\int_0^{\upsilon}\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(u,x(u))|dud\upsilon\\
&+\int_{\eta}^{T}\int_0^{\upsilon}\int_0^{u}(\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)})(\frac{(u-w)^{p-1}}{\Gamma(p)})|\mu(w,x(w))|dwdud\upsilon\Big\}\\
&+\sum_{j=1}^{q}|\gamma_j|\Big\{\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon+\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}|\mu(\upsilon,x(\upsilon))|d\upsilon\Big\}\\
&+|a_4|\Big\{\int_0^{T}\frac{(T-\upsilon)^{\kappa_1-2}}{\Gamma(\kappa_1-1)}|\zeta(\upsilon,x(\upsilon))|d\upsilon+\int_0^{T}\frac{(T-\upsilon)^{\kappa_1+p-2}}{\Gamma(\kappa_1+p-1)}|\mu(\upsilon,x(\upsilon))|d\upsilon\Big\}\Big\}\\
&+|\lambda_2|\Big\{\int_0^{t}\frac{(t-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|y(\upsilon)|d\upsilon
\end{aligned}
$$

$$
\begin{aligned}
&+|\rho_1(t)|\Big\{|A_1|\int_0^{\xi}\frac{(\xi-\upsilon)^{\kappa_1-\kappa_2}}{\Gamma(\kappa_1-\kappa_2+1)}|y(\upsilon)|d\upsilon+\sum_{i=1}^{d}|\omega_i|\int_0^{\eta_i}\frac{(\eta_i-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|y(\upsilon)|d\upsilon\\
&+|a_2|\int_0^{T}\frac{(T-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|y(\upsilon)|d\upsilon\Big\}+|\rho_2(t)|\Big\{|A_2|\int_{\eta}^{T}\int_0^{\upsilon}\frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|y(u)|dud\upsilon\\
&+\sum_{j=1}^{q}|\gamma_j|\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}|y(\upsilon)|d\upsilon+|a_4|\int_0^{T}\frac{(T-\upsilon)^{\kappa_1-\kappa_2-2}}{\Gamma(\kappa_1-\kappa_2-1)}|y(\upsilon)|d\upsilon\Big\}\Big\}\Big\}\Big\},\\
\leq\ &\frac{\|\beta_1\|}{|\lambda_1|\Gamma(\kappa_1+2)}\Big\{(\kappa_1+1)\Big(T^{\kappa_1}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1}\Big)\\
&+\bar{\rho_2}\Big(\kappa_1|a_4|T^{\kappa_1-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1+1}+\bar{\rho_2}|A_2||T^{\kappa_1+1}-\eta^{\kappa_1+1}|\Big\}\\
&+\frac{\|\beta_2\|}{|\lambda_1|\Gamma(\kappa_1+p+2)}\Big\{(\kappa_1+p+1)\Big(T^{\kappa_1+p}+\bar{\rho_1}(|a_2|T^{\kappa_1+p}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1+p}\Big)\\
&+\bar{\rho_2}\Big((\kappa_1+p)|a_4|T^{\kappa_1+p-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1+p}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1+p+1}+\bar{\rho_2}|A_2||T^{\kappa_1+p+1}-\eta^{\kappa_1+p+1}|\Big\}\\
&+\frac{|\lambda_2|\alpha}{|\lambda_1|\Gamma(\kappa_1-\kappa_2+2)}\Big\{(\kappa_1-\kappa_2+1)\Big(T^{\kappa_1-\kappa_2}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1-\kappa_2}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1-\kappa_2}\Big)\\
&+\bar{\rho_2}\Big((\kappa_1-\kappa_2)|a_4|T^{\kappa_1-\kappa_2-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1-\kappa_2}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1-\kappa_2+1}+\bar{\rho_2}|A_2||T^{\kappa_1-\kappa_2+1}-\eta^{\kappa_1-\kappa_2+1}|\Big\}\\
\leq\ &\|\beta\|\Omega_1+\alpha\Omega_2\leq\alpha,
\end{aligned}
$$

which implies that $\mathcal{V}_1x+\mathcal{V}_2y\in B_\alpha$.

(ii) In this step we show that $\mathcal{V}_1$ is compact and continuous. Clearly continuity of ζ and μ implies that the operator $\mathcal{V}_1$ is continuous. Furthermore, $\mathcal{V}_1$ is uniformly bounded on B_α as $\|\mathcal{V}_1x\|\leq\|\beta\|\Omega_1$. Next we establish the compactness of the operator $\mathcal{V}_1$. Let us set $\sup_{(t,x)\in[0,T]\times B_\alpha}|\zeta(t,x)|=\zeta_1$ and $\sup_{(t,x)\in[0,T]\times B_\alpha}|\mu(t,x)|=\mu_1$. Then, for $\tau_1,\tau_2\in[0,T],\tau_1<\tau_2$, we have

$$
\begin{aligned}
&|(\mathcal{V}_1x)(\tau_2)-(\mathcal{V}_1x)(\tau_1)|\\
=\ &\Big|\frac{1}{\lambda_1}\Big\{\int_0^{\tau_1}\frac{((\tau_2-\upsilon)^{\kappa_1-1}-(\tau_1-\upsilon)^{\kappa_1-1})}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon\\
&+\int_0^{\tau_1}\frac{((\tau_2-\upsilon)^{\kappa_1+p-1}-(\tau_1-\upsilon)^{\kappa_1+p-1})}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon\\
&+(\rho_1(\tau_2)-\rho_1(\tau_1))\Big\{A_1\Big\{\int_0^{\xi}\frac{(\xi-\upsilon)^{\kappa_1}}{\Gamma(\kappa_1+1)}\zeta(\upsilon,x(\upsilon))d\upsilon+\int_0^{\xi}\frac{(\xi-\upsilon)^{\kappa_1+p}}{\Gamma(\kappa_1+p+1)}\mu(\upsilon,x(\upsilon))d\upsilon\Big\}\\
&+\sum_{i=1}^{d}\omega_i\Big\{\int_0^{\eta_i}\frac{(\eta_i-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon+\int_0^{\eta_i}\frac{(\eta_i-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon\Big\}\\
&-a_2\Big\{\int_0^{T}\frac{(T-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon+\int_0^{T}\frac{(T-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon\Big\}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+(\rho_2(\tau_2)-\rho_2(\tau_1))\Big\{A_2\Big\{\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(u,x(u))dud\upsilon \\
&+\int_\eta^T\int_0^\upsilon\int_0^u \Big(\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\Big)\Big(\frac{(u-w)^{p-1}}{\Gamma(p)}\Big)\mu(w,x(w))dwdud\upsilon\Big\} \\
&+\sum_{j=1}^q\gamma_j\Big\{\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon+\int_0^{\xi_j}\frac{(\xi_j-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon\Big\} \\
&-a_4\Big\{\int_0^T\frac{(T-\upsilon)^{\kappa_1-2}}{\Gamma(\kappa_1-1)}\zeta(\upsilon,x(\upsilon))d\upsilon+\int_0^T\frac{(T-\upsilon)^{\kappa_1+p-2}}{\Gamma(\kappa_1+p-1)}\mu(\upsilon,x(\upsilon))d\upsilon\Big\}\Big\}\Big\}\Big| \\
\le\;& \frac{\zeta_1}{|\lambda_1|\Gamma(\kappa_1+2)}\Big[(\kappa_1+1)(|\tau_2^{\kappa_1}-\tau_1^{\kappa_1}|+2(\tau_2-\tau_1)^{\kappa_1}) \\
&+|\rho_1(\tau_2)-\rho_1(\tau_1)|\Big\{(\kappa_1+1)\Big(|a_2|T^{\kappa_1}+\sum_{i=1}^d|\omega_i|\eta_i^{\kappa_1}\Big)+|A_1|\xi^{\kappa_1+1}\Big\} \\
&+|\rho_2(\tau_2)-\rho_2(\tau_1)|\Big\{(\kappa_1+1)\Big(\kappa_1|a_4|T^{\kappa_1-1}+\sum_{j=1}^q|\gamma_j|\xi_j^{\kappa_1}\Big)+|A_2||T^{\kappa_1+1}-\eta^{\kappa_1+1}|\Big\}\Big] \\
&+\frac{\mu_1}{|\lambda_1|\Gamma(\kappa_1+p+2)}\Big[(\kappa_1+p+1)(|\tau_2^{\kappa_1+p}-\tau_1^{\kappa_1+p}|+2(\tau_2-\tau_1)^{\kappa_1+p}) \\
&+|\rho_1(\tau_2)-\rho_1(\tau_1)|\Big\{(\kappa_1+p+1)\Big(|a_2|T^{\kappa_1+p}+\sum_{i=1}^d|\omega_i|\eta_i^{\kappa_1+p}\Big)+|A_1|\xi^{\kappa_1+p+1}\Big\} \\
&+|\rho_2(\tau_2)-\rho_2(\tau_1)|\Big\{(\kappa_1+p+1)\Big((\kappa_1+p)|a_4|T^{\kappa_1+p-1}+\sum_{j=1}^q|\gamma_j|\xi_j^{\kappa_1+p}\Big) \\
&+|A_2||T^{\kappa_1+p+1}-\eta^{\kappa_1+p+1}|\Big\}\Big]\to 0,
\end{aligned}
$$

independently of $x \in B_\alpha$ as $\tau_2-\tau_1\to 0$. Therefore, $\mathcal{V}_1$ is equicontinuous. Thus $\mathcal{V}_1$ is relatively compact on B_α by the Arzelá-Ascoli theorem.

(iii) In view of the assumption $\Omega_2 < 1$, one can easily show that the operator $\mathcal{V}_2$ is a contraction.

From the steps (i)–(iii), it is clear that the hypotheses of Krasnosel'skiĭ's fixed point theorem [23] are satisfied and Hence its conclusion implies that the boundary value problem (1.1)–(1.2) has at least one solution on $[0,T]$. The proof is completed. □

The following result is based on Leray-Schauder nonlinear alternative [24].

Theorem 3.4. *Let $\Omega_2 < 1$ where Ω_2 is given in (3.2) and $\zeta,\mu:[0,T]\times\mathbb{R}\to\mathbb{R}$ be continuous functions such that the following conditions hold:*

(H_4) $|\zeta(t,x)|\le p_1(t)\psi_1(|x|)$, $|\mu(t,x)|\le p_2(t)\psi_2(|x|)$ *for each* $(t,x)\in[0,T]\times\mathbb{R}$, *where* $\psi_i:[0,\infty)\to(0,\infty)$ *are continuous nondecreasing functions and* $p_i\in C([0,T],\mathbb{R}^+)$, $i=1,2$;

(H_5) *There exists a constant* $K>0$, *such that*

$$
\frac{(1-\Omega_2)K}{(\|p_1\|\psi_1(K)+\|p_2\|\psi_2(K))\Omega_1} > 1. \tag{3.3}
$$

Then the problem (1.1)–(1.2) has at least one solution on $[0,T]$.

Proof. Let us verify that operator $\mathcal{V}$ defined by (3.1) satisfies the hypotheses of the Leray-Schauder nonlinear alternative [24].

Step 1. We establish that operator $\mathcal{V}$ maps bounded sets (balls) into a bounded set in C. For a number $\varsigma > 0$, let $B_\varsigma = \{x \in C, \|x\| \le \varsigma\}$ be a closed ball in C. Then, for $t \in [0, T]$, we get

$$
\begin{aligned}
& \|(\mathcal{V}x)\| = \sup_{t\in[0,T]} |(\mathcal{V}x)(t)| \\
\le\ & \sup_{t\in[0,T]} \Big\{ \frac{1}{|\lambda_1|} \Big\{ \int_0^t \frac{(t-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)} |\zeta(\upsilon, x(\upsilon))| d\upsilon + \int_0^t \frac{(t-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)} |\mu(\upsilon, x(\upsilon))| d\upsilon \\
& + |\lambda_2| \int_0^t \frac{(t-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)} |x(\upsilon)| d\upsilon + |\rho_1(t)| \Big\{ |A_1| \Big\{ \int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1}}{\Gamma(\kappa_1+1)} |\zeta(\upsilon, x(\upsilon))| d\upsilon \\
& + \int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1+p}}{\Gamma(\kappa_1+p+1)} |\mu(\upsilon, x(\upsilon))| d\upsilon + |\lambda_2| \int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1-\kappa_2}}{\Gamma(\kappa_1-\kappa_2+1)} |x(\upsilon)| d\upsilon \Big\} \\
& + \sum_{i=1}^d |\omega_i| \Big\{ \int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)} |\zeta(\upsilon, x(\upsilon))| d\upsilon + \int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)} |\mu(\upsilon, x(\upsilon))| d\upsilon \\
& + |\lambda_2| \int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)} |x(\upsilon)| d\upsilon \Big\} + |a_2| \Big\{ \int_0^T \frac{(T-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)} |\zeta(\upsilon, x(\upsilon))| d\upsilon \\
& + \int_0^T \frac{(T-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)} |\mu(\upsilon, x(\upsilon))| d\upsilon + |\lambda_2| \int_0^T \frac{(T-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)} |x(\upsilon)| d\upsilon \Big\} \Big\} \\
& + |\rho_2(t)| \Big\{ |A_2| \Big\{ \int_\eta^T \int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)} |\zeta(u, x(u))| du d\upsilon \\
& + \int_\eta^T \int_0^\upsilon \int_0^u \Big(\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\Big)\Big(\frac{(u-w)^{p-1}}{\Gamma(p)}\Big) |\mu(w, x(w))| dw du d\upsilon \\
& + |\lambda_2| \int_\eta^T \int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)} |x(u)| du d\upsilon \Big\} + \sum_{j=1}^q |\gamma_j| \Big\{ \int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)} |\zeta(\upsilon, x(\upsilon))| d\upsilon \\
& + \int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)} |\mu(\upsilon, x(\upsilon))| d\upsilon + |\lambda_2| \int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)} |x(\upsilon)| d\upsilon \Big\} \\
& + |a_4| \Big\{ \int_0^T \frac{(T-\upsilon)^{\kappa_1-2}}{\Gamma(\kappa_1-1)} |\zeta(\upsilon, x(\upsilon))| d\upsilon + \int_0^T \frac{(T-\upsilon)^{\kappa_1+p-2}}{\Gamma(\kappa_1+p-1)} |\mu(\upsilon, x(\upsilon))| d\upsilon \\
& + |\lambda_2| \int_0^T \frac{(T-\upsilon)^{\kappa_1-\kappa_2-2}}{\Gamma(\kappa_1-\kappa_2-1)} |x(\upsilon)| d\upsilon \Big\} \Big\} \Big\} \Big\} \\
\le\ & (\|p_1\|\psi_1(\varsigma) + \|p_2\|\psi_2(\varsigma))\Omega_1 + \varsigma\Omega_2.
\end{aligned}
$$

Step 2. We will prove that $\mathcal{V}$ maps bounded sets into equicontinuous sets of C. Let $\varrho_1, \varrho_2 \in [0, T]$ with $\varrho_1 < \varrho_2$. Then

$$
\begin{aligned}
& |(\mathcal{V}x)(\varrho_2) - (\mathcal{V}x)(\varrho_1)| \\
\le\ & \frac{\|p_1\|\psi_1(\varsigma)}{|\lambda_1|\Gamma(\kappa_1+2)} \Big[(\kappa_1+1)(|\varrho_2^{\kappa_1} - \varrho_1^{\kappa_1}| + 2(\varrho_2-\varrho_1)^{\kappa_1}) \\
& + |\rho_1(\varrho_2) - \rho_1(\varrho_1)| \Big\{ (\kappa_1+1)\Big(|a_2| T^{\kappa_1} + \sum_{i=1}^d |\omega_i| \eta_i^{\kappa_1} \Big) + |A_1| \xi^{\kappa_1+1} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+|\rho_2(\varrho_2)-\rho_2(\varrho_1)|\Big\{(\kappa_1+1)\Big(\kappa_1|a_4|T^{\kappa_1-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1}\Big)+|A_2||T^{\kappa_1+1}-\eta^{\kappa_1+1}|\Big\}\Big]\\
&+\frac{\|p_2\|\psi_2(\varsigma)}{|\lambda_1|\Gamma(\kappa_1+p+2)}\Big[(\kappa_1+p+1)(|\varrho_2^{\kappa_1+p}-\varrho_1^{\kappa_1+p}|+2(\varrho_2-\varrho_1)^{\kappa_1+p})\\
&+|\rho_1(\varrho_2)-\rho_1(\varrho_1)|\Big\{(\kappa_1+p+1)\Big(|a_2|T^{\kappa_1+p}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1+p}\Big)+|A_1|\xi^{\kappa_1+p+1}\Big\}\\
&+|\rho_2(\varrho_2)-\rho_2(\varrho_1)|\Big\{(\kappa_1+p+1)\Big((\kappa_1+p)|a_4|T^{\kappa_1+p-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1+p}\Big)\\
&+|A_2||T^{\kappa_1+p+1}-\eta^{\kappa_1+p+1}|\Big\}\Big]+\frac{\varsigma|\lambda_2|}{|\lambda_1|\Gamma(\kappa_1-\kappa_2+2)}\Big[(\kappa_1-\kappa_2+1)(|\varrho_2^{\kappa_1-\kappa_2}-\varrho_1^{\kappa_1-\kappa_2}|\\
&+2(\varrho_2-\varrho_1)^{\kappa_1-\kappa_2})+|\rho_1(\varrho_2)-\rho_1(\varrho_1)|\Big\{(\kappa_1-\kappa_2+1)\Big(|a_2|T^{\kappa_1-\kappa_2}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1-\kappa_2}\Big)\\
&+|A_1|\xi^{\kappa_1-\kappa_2+1}\Big\}+|\rho_2(\varrho_2)-\rho_2(\varrho_1)|\Big\{(\kappa_1-\kappa_2+1)\Big((\kappa_1-\kappa_2)|a_4|T^{\kappa_1-\kappa_2-1}\\
&+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1-\kappa_2}\Big)+|A_2||T^{\kappa_1-\kappa_2+1}-\eta^{\kappa_1-\kappa_2+1}|\Big\}\Big]\to 0,
\end{aligned}
$$

independent of $x \in B_\varsigma$ as $\varrho_2-\varrho_1 \to 0$. So $\mathcal{V}$ is equicontinuous. Therefore, the operator $\mathcal{V}$ is completely continuous by the application of the Arzelá-Ascoli theorem.

Step3. We will show that the set of all solutions to equation $x = \vartheta\mathcal{V}x$ with $\vartheta \in (0,1)$ is bounded. From Step 1 we get

$$|x(t)| \le (\|p_1\|\psi_1(\|x\|)+\|p_2\|\psi_2(\|x\|))\Omega_1+\|x\|\Omega_2,\ t\in[0,T],$$

which implies

$$\frac{(1-\Omega_2)\|x\|}{(\|p_1\|\psi_1(\|x\|)+\|p_2\|\psi_2(\|x\|))\Omega_1}<1.$$

From (H_5) there exists $K>0$ satisfying $\|x\|\neq K$. We will introduce a set

$$\mathcal{U}=\{x\in C([a,b],\mathbb{R}):\|x\|<K\} \tag{3.4}$$

and $\mathcal{V}:\overline{\mathcal{U}}\to C$ is continuous and completely continuous. Thus, by choice of $\mathcal{U}$, there does not exists any $x\in\partial\mathcal{U}$ satisfying $x=\vartheta\mathcal{V}x$ for some $\vartheta\in(0,1)$. Consequently the operator $\mathcal{V}$ has a fixed-point $x\in\overline{\mathcal{U}}$ by Leray-Schauder nonlinear alternative [24], which means that the problem (1.1)–(1.2) has a solution on $[0,T]$. □

4. Mixed nonlinearities case

In this section, we consider a multi-term fractional differential equation involving three types of nonlinearities of the form:

$$\lambda_1\,{}^{C}D^{\kappa_1}x(t)+\lambda_2\,{}^{C}D^{\kappa_2}h(t,x(t))=\zeta(t,x(t))+I^{p}\mu(t,x(t)), \tag{4.1}$$

where $h : [0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, while the other quantities are the same as defined in the problem (1.1)–(1.2).

In this case, the fixed point operator $\mathcal{F} : C \to C$ associated with the Eq (4.1) complemented with the boundary conditions (1.2) is

$$
\begin{aligned}
(\mathcal{F}x)(t) \;=\; & \frac{1}{\lambda_1}\Big[\int_0^t \frac{(t-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon + \int_0^t \frac{(t-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon \\
& -\lambda_2\int_0^t \frac{(t-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}h(\upsilon,x(\upsilon))d\upsilon + \rho_1(t)\Big\{A_1\Big\{\int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1}}{\Gamma(\kappa_1+1)}\zeta(\upsilon,x(\upsilon))d\upsilon \\
& + \int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1+p}}{\Gamma(\kappa_1+p+1)}\mu(\upsilon,x(\upsilon))d\upsilon - \lambda_2\int_0^\xi \frac{(\xi-\upsilon)^{\kappa_1-\kappa_2}}{\Gamma(\kappa_1-\kappa_2+1)}h(\upsilon,x(\upsilon))d\upsilon\Big\} \\
& + \sum_{i=1}^{d}\omega_i\Big\{\int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon + \int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon \\
& -\lambda_2\int_0^{\eta_i} \frac{(\eta_i-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}h(\upsilon,x(\upsilon))d\upsilon\Big\} - a_2\Big\{\int_0^T \frac{(T-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon \\
& + \int_0^T \frac{(T-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon - \lambda_2\int_0^T \frac{(T-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}h(\upsilon,x(\upsilon))d\upsilon\Big\}\Big\} \\
& +\rho_2(t)\Big\{A_2\Big\{\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(u,x(u))dud\upsilon \\
& + \int_\eta^T\int_0^\upsilon\int_0^u \Big(\frac{(\upsilon-u)^{\kappa_1-1}}{\Gamma(\kappa_1)}\Big)\Big(\frac{(u-w)^{p-1}}{\Gamma(p)}\Big)\mu(w,x(w))dwdud\upsilon \\
& -\lambda_2\int_\eta^T\int_0^\upsilon \frac{(\upsilon-u)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}h(u,x(u))dud\upsilon\Big\} \\
& + \sum_{j=1}^{q}\gamma_j\Big\{\int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1-1}}{\Gamma(\kappa_1)}\zeta(\upsilon,x(\upsilon))d\upsilon + \int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1+p-1}}{\Gamma(\kappa_1+p)}\mu(\upsilon,x(\upsilon))d\upsilon \\
& -\lambda_2\int_0^{\xi_j} \frac{(\xi_j-\upsilon)^{\kappa_1-\kappa_2-1}}{\Gamma(\kappa_1-\kappa_2)}h(\upsilon,x(\upsilon))d\upsilon\Big\} - a_4\Big\{\int_0^T \frac{(T-\upsilon)^{\kappa_1-2}}{\Gamma(\kappa_1-1)}\zeta(\upsilon,x(\upsilon))d\upsilon \\
& + \int_0^T \frac{(T-\upsilon)^{\kappa_1+p-2}}{\Gamma(\kappa_1+p-1)}\mu(\upsilon,x(\upsilon))d\upsilon - \lambda_2\int_0^T \frac{(T-\upsilon)^{\kappa_1-\kappa_2-2}}{\Gamma(\kappa_1-\kappa_2-1)}h(\upsilon,x(\upsilon))d\upsilon\Big\}\Big\}\Big],\ t\in[0,T].
\end{aligned}
\tag{4.2}
$$

Now we present a uniqueness result for the problem consisting of the Eq (4.1) and the boundary conditions (1.2). We do not provide its proof as it can be obtained with the aid of the operator defined in (4.2) by following the procedure used to accomplish Theorem 3.1.

Theorem 4.1. *Assume that:*

$(H_1)'$ $\zeta,\mu,h : [0,T]\times\mathbb{R}\to\mathbb{R}$ *are continuous functions satisfying the conditions:*

$$|\zeta(t,x)-\zeta(t,y)| \le L_1|x-y|,\ |\mu(t,x)-\mu(t,y)| \le L_2|x-y|,\ |h(t,x)-h(t,y)| \le L_3|x-y|,$$

$L_1, L_2, L_3 > 0,$ *for all* $t\in[0,T],\ x,y\in\mathbb{R}$ *and* $|\zeta(t,0)| \le M_1 < \infty,\ |\mu(t,0)| \le M_2 < \infty,\ |h(t,0)| \le M_3 < \infty,$ *for all* $t\in[0,T]$*;*

$(H_2)'$ $L\Lambda < 1$, *where* $L = \max\{L_1, L_2, L_3\}$, *and*

$$\begin{aligned}
\Lambda \;=\; & \frac{1}{|\lambda_1|}\Big[\frac{1}{\Gamma(\kappa_1+2)}\Big\{(\kappa_1+1)\Big(T^{\kappa_1}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1}\Big) \\
& +\bar{\rho_2}\Big(\kappa_1|a_4|T^{\kappa_1-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1+1}+\bar{\rho_2}|A_2||T^{\kappa_1+1}-\eta^{\kappa_1+1}|\Big\} \\
& +\frac{1}{\Gamma(\kappa_1+p+2)}\Big\{(\kappa_1+p+1)\Big(T^{\kappa_1+p}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1+p}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1+p}\Big) \\
& +\bar{\rho_2}\Big((\kappa_1+p)|a_4|T^{\kappa_1+p-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1+p}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1+p+1}+\bar{\rho_2}|A_2||T^{\kappa_1+p+1} \\
& -\eta^{\kappa_1+p+1}|\Big\}+\frac{|\lambda_2|}{\Gamma(\kappa_1-\kappa_2+2)}\Big\{(\kappa_1-\kappa_2+1)\Big(T^{\kappa_1-\kappa_2}+\bar{\rho_1}\Big(|a_2|T^{\kappa_1-\kappa_2}+\sum_{i=1}^{d}|\omega_i|\eta_i^{\kappa_1-\kappa_2}\Big) \\
& +\bar{\rho_2}\Big((\kappa_1-\kappa_2)|a_4|T^{\kappa_1-\kappa_2-1}+\sum_{j=1}^{q}|\gamma_j|\xi_j^{\kappa_1-\kappa_2}\Big)\Big)+\bar{\rho_1}|A_1|\xi^{\kappa_1-\kappa_2+1} \\
& +\bar{\rho_2}|A_2||T^{\kappa_1-\kappa_2+1}-\eta^{\kappa_1-\kappa_2+1}|\Big\}\Big].
\end{aligned}$$

Then the Eq (4.1) complemented with the boundary conditions (1.2) has a unique solution on $[0, T]$.

5. Examples

Here we illustrate the results obtained in the previous sections by numerical examples.

Example 5.1. *Consider the following problem*

$$\begin{cases} 9\,{}^cD^{1.75}x(t)+\frac{1}{7}\,{}^cD^{1.33}x(t)=\zeta(t,x(t))+I^2\mu(t,x(t)),\ t\in[0,4], \\ 2x(0)+3x(4)=\int_0^{1.5}x(\upsilon)d\upsilon+\sum_{n=1}^{3}\frac{1}{n}x\Big(2+\frac{1}{n+1}\Big), \\ 5x'(0)+3x'(4)=2\int_2^4 x(\upsilon)d\upsilon+\sum_{n=1}^{3}\frac{n}{1+n}x\Big(1.55+\frac{1}{n}\Big), \end{cases} \tag{5.1}$$

where

$$\zeta(t,x(t))=\frac{1}{\sqrt{144+t^2}}\Big(\tan^{-1}x+\frac{1}{5}\Big),$$
$$\mu(t,x(t))=\frac{e^{-5t}}{15+t}\Big(\frac{x^2(t)}{1+|x(t)|}\Big)+\frac{2}{9}.$$

Here $\kappa_1 = 1.75,\ \kappa_2 = 1.33,\ p = 2,\ \lambda_1 = 9,\ \lambda_2 = \frac{1}{7},\ a_1 = 2,\ a_2 = 3,\ a_3 = 5,\ a_4 = 3,\ A_1 = 1, A_2 = 2,\ \eta = 2,\ \xi = 1.5$. *Using the given values, we obtain* $\Omega_1 \approx 6.476083916$, $\Omega_2 \approx 0.1380931002$ *and*

$$|\zeta(t,x)-\zeta(t,y)|\le\frac{1}{12}|x-y|,\ |\mu(t,x)-\mu(t,y)|\le\frac{1}{15}|x-y|,$$

$L_1 = \frac{1}{12}$, $L_2 = \frac{1}{15}$ *and* $L = \max\{L_1, L_2\} = \frac{1}{12}$*. Moreover, we have* $L\Omega_1 + \Omega_2 \approx 0.6777667599$*. As all the assumptions ot Theorem 3.1 are satisfied, the boundary value problem (5.1) has a unique solution on* $[0, 4]$.

Example 5.2. *Consider the following problem*

$$\begin{cases} 5\,{}^{c}D^{1.25}x(t) + {}^{c}D^{1.05}x(t) = \zeta(t, x(t)) + I^{\frac{1}{2}}\mu(t, x(t)),\ t \in [0, 5], \\ 2x(0) + 0.5x(5) = 0.2\displaystyle\int_0^{0.5} x(\upsilon)d\upsilon + \sum_{n=1}^{3} nx\Big(\frac{7n+2}{2n}\Big), \\ 4x'(0) + 3x'(5) = 5\displaystyle\int_3^{5} x(\upsilon)d\upsilon + \sum_{n=1}^{3} \frac{1}{1+n}x\Big(\frac{3n+1}{2(n+1)}\Big), \end{cases} \tag{5.2}$$

where

$$\zeta(t, x) = \frac{\sin^2 t}{t^2 + 2t + 7}(2\sin x + 2\cos x + 10) + \frac{2}{9},$$

$$\mu(t, x) = \frac{\cos^2 t}{\sqrt{1 + t^4}}\Big(4\tan^{-1} x + \cot^{-1} x + \frac{1}{2}\Big) + \frac{2}{5}.$$

Notice that

$$|\zeta(t, x)| \le \frac{14\sin^2 t}{t^2 + 2t + 7} + \frac{2}{9} = \beta_1(t),\ |\mu(t, x)| \le \frac{(6\pi + 1)\cos^2 t}{2\sqrt{1 + t^4}} + \frac{2}{5} = \beta_2(t).$$

Here $\kappa_1 = 1.25$, $\kappa_2 = 1.05$, $p = \frac{1}{2}$, $\lambda_1 = 5$, $\lambda_2 = 1$, $a_1 = 2$, $a_2 = 0.5$, $a_3 = 4$, $a_4 = 3$, $A_1 = 0.2, A_2 = 5$, $\eta = 3$, $\xi = 0.5, \|\beta_1\| = \frac{20}{9}$, $\|\beta_2\| = \frac{5(6\pi + 1) + 4}{10}$*. Using the given values, we obtain that* $\Omega_2 \approx 0.7439599002 < 1$*. Therefore, the hypotheses of Theorem 3.3 holds true and consequently its conclusion implies that the boundary value problem (5.2) has at least one solution on* $[0, 5]$.

Example 5.3. *Consider the following problem*

$$\begin{cases} 5\,{}^{c}D^{1.95}x(t) + 0.5\,{}^{c}D^{1.35}h(t, x) = \zeta(t, x(t)) + I^{8}\mu(t, x(t)),\ t \in [0, 3], \\ 0.6x(0) + 2x(3) = \displaystyle\int_0^{0.5} x(\upsilon)d\upsilon + \sum_{n=1}^{2} \frac{1}{n+1}x\Big(1.5 + \frac{1}{n+1}\Big), \\ 0.5x'(0) + 3x'(3) = 0.6\displaystyle\int_{1.5}^{3} x(\upsilon)d\upsilon + \sum_{n=1}^{2} \frac{1}{2(1+n)}x\Big(\frac{5+n}{10}\Big), \end{cases} \tag{5.3}$$

where

$$\zeta(t, x) = \frac{1}{t + 7}(\sin x + 10) + \frac{1}{4},$$

$$\mu(t, x) = \frac{e^{-4t}}{\sqrt{36 + t}}\Big(\cot^{-1} x + \frac{1}{2}\Big),$$

$$h(t, x) = \frac{1}{16 + t^4}(\tan^{-1} x + 12),$$

$\kappa_1 = 1.95,\ \kappa_2 = 1.35,\ p = 8,\ \lambda_1 = 5,\ \lambda_2 = 0.5,\ a_1 = 0.6,\ a_2 = 2,\ a_3 = 0.5,\ a_4 = 3,\ A_1 = 1, A_2 = 0.6,\ \eta = 1.5,\ \xi = 0.5$. *Using the given values, we obtain* $\Lambda \approx 3.233037708$*, and* $L = \frac{1}{6}$ *as*

$$|\zeta(t,x) - \zeta(t,y)| \le \frac{1}{7}|x-y|,\ |\mu(t,x) - \mu(t,y)| \le \frac{1}{6}|x-y|,\ |h(t,x) - h(t,y)| \le \frac{1}{16}|x-y|,$$

where $L_1 = \frac{1}{7}$, $L_2 = \frac{1}{6}$, $L_3 = \frac{1}{16}$ *and* $L = \max\{L_1, L_2, L_3\}$. *Moreover, we have* $L\Lambda \approx 0.5388396180$. *As all the assumptions of Theorem 4.1 are satisfied, the boundary value problem (5.3) has a unique solution on* $[0, 3]$.

6. Conclusions

We presented different criteria for the existence of solutions for a nonlinear multi-term fractional integro-differential equation equipped with non-separated integro-multipoint boundary conditions. We also discussed a variant of the main problem involving fractional-order, non-integral and Riemann-Liouville type integral nonlinear terms in the fractional integro-differential equation. The results obtained in this paper are of quite general nature as we we can record several interesting cases (new results) by specializing the parameters involved in the problem at hand. For instance, our results correspond the ones for initial integro-multipoint conditions if we take $a_2 = 0 = a_4$. On the other hand, by fixing $a_1 = 0 = a_3$ in the results of this paper, we obtain the ones for terminal integro-multipoint conditions. In case, we fix $A_1 = 0 = A_2$, our results become the ones associated with non-separated multipoint boundary conditions.

Acknowledgements

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-MSc-44-130-38). The authors, therefore, acknowledge with thanks DSR technical and financial support. We thank the reviewer for his/her useful comments on our work that led to its improvement.

Conflict of interest

The authors declare no conflict of interest.

References

1. R. L. Magin, *Fractional calculus in bioengineering*, USA: Begell House Publishers, 2006.
2. G. M. Zaslavsky, *Hamiltonian chaos and fractional dynamics*, Oxford: Oxford University Press, 2005.
3. H. A. Fallahgoul, S. M. Focardi, F. J. Fabozzi, *Fractional calculus and fractional processes with applications to financial economics. Theory and application*, London: Elsevier/Academic Press, 2017.

4. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier Science B. V., 2006.
5. Y. Zhou, *Basic theory of fractional differential equations*, Hackensack: World Scientific Publishing Co. Pte. Ltd., 2014.
6. K. Diethelm, *The analysis of fractional differential equations: An Application-oriented exposition using differential operators of Caputo type*, Berlin: Springer-Verlag, 2010.
7. J. R. Graef, L. Kong, M. Wang, Existence and uniqueness of solutions for a fractional boundary value problem on a graph, *Fract. Calc. Appl. Anal.*, **17** (2014), 499–510.
8. G. Wang, S. Liu, L. Zhang, Eigenvalue problem for nonlinear fractional differential equations with integral boundary conditions, *Abstr. Appl. Anal.*, **2014** (2014), 1–6.
9. J. Henderson, N. Kosmatov, Eigenvalue comparison for fractional boundary value problems with the Caputo derivative, *Fract. Calc. Appl. Anal.*, **17** (2014), 872–880.
10. J. Henderson, R. Luca, Nonexistence of positive solutions for a system of coupled fractional boundary value problems, *Bound. Value Probl.*, **2015** (2015), 138.
11. S. K. Ntouyas, S. Etemad, On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions, *Appl. Math. Comput.*, **266** (2015), 235–243.
12. B. Ahmad, R. Luca, Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions, *Appl. Math. Comput.*, **339** (2018), 516–534.
13. A. Alsaedi, B. Ahmad, M. Alghanmi, Extremal solutions for generalized Caputo fractional differential equations with Steiltjes-type fractional integro-initial conditions, *Appl. Math. Lett.*, **91** (2019), 113–120.
14. C. Ravichandran, N. Valliammal, J. J. Nieto, New results on exact controllability of a class of fractional neutral integro-differential systems with state-dependent delay in Banach spaces, *J. Franklin I.*, **356** (2019), 1535–1565.
15. D. Baleanu, S. Etemad, S. Rezapour, On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators, *Alex. Eng. J.*, **59** (2020), 3019–3027.
16. P. J. Torvik, R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, *J. Appl. Mech.*, **51** (1984), 294–298.
17. F. Mainardi, Some basic problems in continuum and statistical mechanics, In: *Fractals and fractional calculus in continuum mechanics*, Berlin: Springer, 1997, 291–348.
18. S. Stanek, Periodic problem for two-term fractional differential equations, *Fract. Calc. Appl. Anal.*, **20** (2017), 662–678.
19. Y. Liu, Boundary value problems of singular multi-term fractional differential equations with impulse effects, *Math. Nachr.*, **289** (2016), 1526–1547.
20. B. Ahmad, A. Alsaedi, Y. Alruwaily, S. K. Ntouyas, Nonlinear multi-term fractional differential equations with Riemann-Stieltjes integro-multipoint boundary conditions, *AIMS Mathematics*, **5** (2020), 1446–1461.
21. K. Deimling, *Nonlinear functional analysis*, New York: Springer-Verlag, 1985.

22. D. R. Smart, *Fixed point theorems*, New York: Cambridge University Press, 1974.

23. M. A. Krasnoselskii, Two remarks on the method of successive approximations, *Uspehi Mat. Nauk*, **10** (1955), 123–127.

24. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer-Verlag, 2003.