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Research article

Essential norm of generalized Hilbert matrix from Bloch type spaces to BMOA and Bloch space

Songxiao Li[1] **and Jizhen Zhou**[2,*]

[1] Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, 610054, Chengdu, Sichuan, P. R. China

[2] School of Sciences, Anhui University of Science and Technology, 232001, Huainan, Anhui, P. R. China

* **Correspondence:** Email: hope189@163.com.

Abstract: Let μ be a positive Borel measure on the interval $[0, 1)$. The Hankel matrix $\mathcal{H}_\mu = (\mu_{n+k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$ induces the operator

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n$$

on the space of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disk $\mathbb{D}$. In this paper, we characterize the boundedness and compactness of $\mathcal{H}_\mu$ from Bloch type spaces to the BMOA and the Bloch space. Moreover we obtain the essential norm of $\mathcal{H}_\mu$ from $\mathcal{B}^\alpha$ to $\mathcal{B}$ and BMOA.

Keywords: Bloch type space; BMOA space; Carleson measure; Hilbert operator; essential norm
Mathematics Subject Classification: 47B38, 30H30

1. Introduction

Denote by $H(\mathbb{D})$ the space of all analytic functions on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane. For $0 < p \leq \infty$, we let H^p denote the classical Hardy space. If $f \in H(\mathbb{D})$ and

$$\|f\|_{BMOA} = |f(0)| + \sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2} < \infty,$$

we say that $f \in BMOA$. Here $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, $a \in \mathbb{D}$, is a Möbius transformation of $\mathbb{D}$. Fefferman's duality theorem says that $BMOA = (H^1)^*$. We refer to [10] about the theory of $BMOA$.

Let $0 < \alpha < \infty$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch type space (or called the $\alpha-$Bloch space), denoted by $\mathcal{B}^\alpha$, if

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty.$$

The classical Bloch space $\mathcal{B}$ is just $\mathcal{B}^1$. It is clear that $\mathcal{B}^\alpha$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha}$. See [21] for the theory of Bloch type spaces.

For a subarc $I \subset \partial\mathbb{D}$, let $S(I)$ be the Carleson box based on I with

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \leq |z| < 1 \ \ and\ \frac{z}{|z|} \in I\}.$$

Here $|I| = (2\pi)^{-1}\int_I |d\xi|$ is the normalized length of the arc I. If $I = \partial\mathbb{D}$, let $S(I) = \mathbb{D}$. For $0 < s < \infty$, we say that a positive Borel measure μ is an $s-$Carleson measure on $\mathbb{D}$ if (see [7])

$$\|\mu\| = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty.$$

We say that a positive Borel measure μ is a vanishing $s-$Carleson measure on $\mathbb{D}$ if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0.$$

Here and henceforth $\sup_{I\subset\partial\mathbb{D}}$ indicates the supremum taken over all subarcs I of $\partial\mathbb{D}$. When $s = 1$, μ is called a Carleson measure on $\mathbb{D}$. It is well known that, for any $f \in H^p (0 < p < \infty)$,

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq \|f\|^p_{H^p}$$

if and only if μ is a Carleson measure. See, for example, [8].

A positive Borel measure μ on $[0, 1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with measure $\widetilde{\mu}$ defined by

$$\widetilde{\mu}(E) = \mu(E \cap [0, 1))$$

for any Borel subset E of $\mathbb{D}$. Then a positive Borel measure μ on [0,1) is an s-Carleson measure if there exists a constant $C > 0$ such that (see [11])

$$\mu([t, 1)) \leq C(1 - t)^s.$$

A vanishing $s-$Carleson measure on $[0, 1)$ can be defined similarly.

Let μ be a finite positive measure on $[0, 1)$ and $n = 0, 1, 2, \cdots$. Denote μ_n the moment of order n of μ, that is, $\mu_n = \int_{[0,1)} t^n d\mu(t)$. Let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix $\mathcal{H}_\mu$ induces an operator, denoted also by $\mathcal{H}_\mu$, on $H(\mathbb{D})$ by its action on the Taylor coefficient:

$$a_n \to \sum_{k=0}^{\infty} \mu_{n,k} a_k, n = 0, 1, 2, \cdots.$$

More precisely, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$, then

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

As in [9], to obtain an integral representation of $\mathcal{H}_\mu$, we write

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz} d\mu(t), \tag{1.1}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$, then the matrix $\mathcal{H}_\mu$ is just the classical Hilbert matrix $H = (\frac{1}{n+k+1})_{n,k\geq 0}$, which induces the classical Hilbert operator H. The Hilbert operator H was studied in [1, 2, 4–6, 14]. A generalized Hilbert operator was studied in [11, 12, 14, 15].

The operator $\mathcal{H}_\mu$ acting on analytic functions spaces has been studied by many authors. Galanopoulos and Peláez [9] obtained a characterization that $\mathcal{H}_\mu$ is bounded or compact on H^1. Chatzifountas, Girela and Peláez [3] described the measure μ for which $\mathcal{H}_\mu$ is bounded (compact) operator from H^p into $H^q, 0 < p, q < \infty$. See [13] about the Hankel matrix acting on the Dirichlet space.

Let X and Y be two Banach spaces. The essential norm of a continuous linear operator T between normed linear spaces X and Y is the distance to the set of compact operators K, that is, $\|T\|_e^{X\to Y} = \inf\{\|T-K\| : K \text{ is compact}\}$, where $\|\cdot\|$ is the operator norm. It is easy to see that $\|T\|_e^{X\to Y} = 0$ if and only if T is compact. See [16, 19] for the study of essential norm of some operators.

In [11, 12], Girela and Merchán studied the operator $\mathcal{H}_\mu$ acting on spaces of analytic functions on $\mathbb{D}$ such as the Bloch space, BMOA, the Besov space and Hardy spaces. The paper generalizes some results of [11]. Moreover we also characterize the essential norm of $\mathcal{H}_\mu$ from $\mathcal{B}^\alpha$ to $\mathcal{B}$ and BMOA. We first acknowledge that the proof of part result are suggested by the technique of [11].

In this paper, C denotes a constant which may be different in each case.

2. The operator $\mathcal{H}_\mu : \mathcal{B}^\alpha \to BMOA$ ($\mathcal{B}$), $0 < \alpha < 1$

In this section, we characterize the boundedness of $\mathcal{H}_\mu$ from $\mathcal{B}^\alpha$ into the $BMOA$ and the Bloch space when $0 < \alpha < 1$. For this purpose, we need some auxiliary results.

Lemma 2.1. *[21] If $0 < \alpha < 1$, then $f \in \mathcal{B}^\alpha$ are bounded. If $\alpha > 1$, then $f \in \mathcal{B}^\alpha$ if and only if there exists some constant C such that*

$$|f(z)| \leq \frac{C}{(1-|z|^2)^{\alpha-1}}.$$

The following lemma can be found in [18] (see Corollary 3.3.1 in [18]).

Lemma 2.2. *If $a_n \downarrow 0$, then $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$ if and only if $\sup_n na_n < \infty$.*

Theorem 2.3. *Let μ be a positive measure on $[0,1)$ and $0 < \alpha < 1$. Then the following statements are equivalent.*

(1) The operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ into $\mathcal{B}$.
(2) The operator $\mathcal{H}_\mu$ is compact from $\mathcal{B}^\alpha$ into $\mathcal{B}$.
(3) The operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ into $BMOA$.
(4) The operator $\mathcal{H}_\mu$ is compact from $\mathcal{B}^\alpha$ into $BMOA$.

(5) The measure μ is a Carleson measure.

Proof. (1)⇒ (5). Assume that the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ into $\mathcal{B}$. Let $f(z) = 1 \in \mathcal{B}^\alpha$. Then

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n = \sum_{n=0}^{\infty}\mu_{n,0}z^n \in \mathcal{B}.$$

Note that $\mu_{n,0}$ is positive and decreasing. For any $0 < \lambda < 1$, we choose n such that $1-\frac{1}{n} \le \lambda < 1-\frac{1}{n+1}$. Lemma 2.2 gives that

$$\infty > n\mu_{n,0} = n\int_0^1 t^n d\mu(t) \ge n\lambda^n \int_\lambda^1 d\mu(t) \ge \frac{\mu([\lambda, 1))}{e(1-\lambda)}.$$

The above estimate gives that μ is a Carleson measure.

(5)⇒(3). Assume that μ is a Carleson measure. Lemma 2.1 implies that $\mathcal{B}^\alpha$ is a subspace of H^1 for $0 < \alpha < 1$. Then $\mathcal{H}_\mu(f)$ is an analytic function for any $f \in \mathcal{B}^\alpha$ by Proposition 1 in [9]. Moreover, $\mathcal{H}_\mu(f) = I_\mu(f)$ for any $f \in \mathcal{B}^\alpha$.

For any given $f \in \mathcal{B}^\alpha$,

$$\int_{[0,1)}|f(t)|d\mu(t) \le \|f\|_{\mathcal{B}^\alpha}\int_{[0,1)} d\mu(t) < \infty.$$

Then we have

$$\int_0^{2\pi}\int_{[0.1)}\left|\frac{f(t)g(e^{i\theta})}{1-rte^{i\theta}}\right| d\mu(t)d\theta < \infty$$

for any $f \in \mathcal{B}^\alpha, g \in H^1$ and $0 < r < 1$. It is easy to obtain that

$$\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}d\theta = \int_{[0.1)} f(t)\overline{g(rt)}d\mu(t) \tag{2.1}$$

whenever $f \in \mathcal{B}^\alpha$ and $g \in H^1$. The reader can refer to the proof of Theorem 2.2 in [11]. Using (2.1), we have

$$\begin{aligned}\left|\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}d\theta\right| =& \left|\int_{[0.1)} f(t)\overline{g(rt)}d\mu(t)\right| \\ \le& \|f\|_{\mathcal{B}^\alpha}\int_{[0.1)}|g(rt)|d\mu(t) \\ \le& \|\mu\|\|f\|_{\mathcal{B}^\alpha}\int_0^{2\pi}|g(re^{i\theta})|d\theta \\ \le& \|\mu\|\|f\|_{\mathcal{B}^\alpha}\|g\|_{H^1}.\end{aligned}$$

We obtain $\mathcal{H}_\mu(f) = I_\mu(f) \in BMOA$ for any $f \in \mathcal{B}^\alpha$ by Fefferman's duality Theorem.

(5)⇒(4). Assume that μ is a Carleson measure. Then $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ to $BMOA$ and $\mathcal{H}_\mu(f) = I_\mu(f)$ for any $f \in \mathcal{B}^\alpha, 0 < \alpha < 1$. Let $\{f_n\}$ be any sequence with $\sup_n \|f_n\|_{\mathcal{B}^\alpha} \le 1$ and $\lim_{n\to\infty} f_n(z) = 0$ on any compact subset of $\mathbb{D}$. Then we have $\sup_{z\in\mathbb{D}}|f_n(z)| \to 0$ as $n \to \infty$ by Lemma

3.2 in [20]. Applying (2.1) again, we have

$$\begin{aligned}\left|\int_0^{2\pi} I_\mu(f_n)(re^{i\theta})\overline{g(e^{i\theta})}d\theta\right| &= \left|\int_{[0.1)} f_n(t)\overline{g(rt)}d\mu(t)\right| \\ &\le \sup_{0<t<1}|f_n(t)|\int_{[0.1)}|g(rt)|d\mu(t) \\ &\le \sup_{0<t<1}|f_n(t)|\|\mu\|\|g\|_{H^1}.\end{aligned}$$

Then

$$\lim_{n\to\infty}\int_0^{2\pi} I_\mu(f_n)(re^{i\theta})\overline{g(e^{i\theta})}d\theta = 0.$$

This prove that $\lim_{n\to\infty}\mathcal{H}_\mu(f_n) = \lim_{n\to\infty} I_\mu(f_n) = 0$. So $\mathcal{H}_\mu$ is compact.

The other cases are trivial. The proof is complete. □

Corollary 2.4. *Let μ be a positive Borel measure on $[0,1)$. If $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}$ for any $0<\alpha<1$, then*

$$\|\mathcal{H}_\mu\|_e^{\mathcal{B}^\alpha\to\mathcal{B}} = \|\mathcal{H}_\mu\|_e^{\mathcal{B}^\alpha\to BMOA} = 0.$$

3. The operator $\mathcal{H}_\mu : \mathcal{B}^\alpha \to BMOA$ ($\mathcal{B}$), $\alpha > 1$

In this section, we will give the essential norm of the operator $\mathcal{H}_\mu$ from $\mathcal{B}^\alpha$ to $BMOA$ and $\mathcal{B}$ when $\alpha > 1$. The following lemma will be needed in the proof of the main results.

Lemma 3.1. *Let μ be a positive Borel measure on $[0,1)$ and $\alpha > 1$. Then the following conditions are equivalent.*

(1) $\int_{[0,1)}(1-t)^{1-\alpha}d\mu(t) < \infty$.

(2) For any given $f\in\mathcal{B}^\alpha$, the integral in (1.1) converges for all $z\in\mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic on $\mathbb{D}$.

Proof. (1)⇒(2). We assume that (1) holds. Lemma 2.1 gives

$$\int_{[0,1)}|f(t)|d\mu(t) \le C\|f\|_{\mathcal{B}^\alpha}\int_{[0,1)}(1-t^2)^{1-\alpha}d\mu(t) \le C\|f\|_{\mathcal{B}^\alpha}. \tag{3.1}$$

This implies that

$$\int_{[0,1)}\frac{|f(t)|}{|1-tz|}d\mu(t) \le C\frac{\|f\|_{\mathcal{B}^\alpha}}{1-|z|}$$

for any $f\in\mathcal{B}^\alpha$ and $z\in\mathbb{D}$. By (3.1) we have

$$\sup_{n\ge 0}\left|\int_{[0,1)} t^n f(t)d\mu(t)\right| < \infty. \tag{3.2}$$

(3.2) and Fubini's Theorem give that the integral $\int_{[0,1)}\frac{f(t)}{1-tz}d\mu(t)$ converges absolutely for any fixed $z\in\mathbb{D}$. Then we have

$$\int_{[0,1)}\frac{f(t)}{1-tz}d\mu(t) = \sum_{n=0}^\infty\left(\int_{[0,1)} t^n f(t)d\mu(t)\right)z^n, \quad z\in\mathbb{D}.$$

Hence $I_\mu(f)$ is a well defined analytic function in $\mathbb{D}$ and

$$I_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\int_{[0,1)} t^n f(t)d\mu(t)\right)z^n, \quad z \in \mathbb{D}.$$

(2)⇒(1). Let $f(z) = (1-z)^{1-\alpha}$. Then f belongs to $\mathcal{B}^\alpha$. So $I_\mu(f)$ is well defined for every $z \in \mathbb{D}$. In particular,

$$I_\mu(f)(0) = \int_{[0,1)} (1-t)^{1-\alpha}d\mu(t)$$

is a complex number. Since μ is a positive Borel measure on $[0,1)$, we get the desired result. The proof is complete. □

Lemma 3.2. *Let μ be a positive measure on $[0,1)$ and $\alpha > 1$. Let ν be the positive measure on $[0,1)$ defined by*

$$d\nu(t) = (1-t)^{1-\alpha}d\mu(t).$$

Then the following conditions are equivalent.

(1) μ is an α-Carleson measure.
(2) ν is a Carleson measure.

Proof. (2)⇒(1) Note that $\nu([t,1) \lesssim (1-t)$ and $d\mu(t) = (1-t)^{\alpha-1}d\nu(t)$. We have

$$\mu([t,1)) = \int_t^1 (1-s)^{\alpha-1}d\nu(s) \le (1-t)^{\alpha-1}\int_t^1 d\nu(s) \lesssim (1-t)^\alpha.$$

(1)⇒(2) Note that $\mu([t,1)) \lesssim (1-t)^\alpha$. Integrating by parts, we obtain

$$\begin{aligned}\nu([t,1)) =& \int_t^1 (1-s)^{1-\alpha}d\mu(s)\\ =&(1-t)^{1-\alpha}\mu([t,1)) + (\alpha-1)\int_t^1 (1-s)^{-\alpha}\mu([s,1))ds\\ \lesssim&(1-t) + (\alpha-1)\int_t^1 ds\\ \lesssim&(1-t).\end{aligned}$$

The proof is complete. □

Lemma 3.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}^\alpha$ for any $\alpha > 0$. Then*

$$\sup_n \sum_{k=2^n+1}^{2^{n+1}} \left|\frac{a_k}{k^{\alpha-1}}\right|^2 < C\|f\|^2_{\mathcal{B}^\alpha}. \tag{3.3}$$

Proof. For any $0 < r < 1$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}^\alpha$, we have

$$(1-r)^{2\alpha}\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \le \|f\|^2_{\mathcal{B}^\alpha}.$$

This gives that

$$\sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k} \le \|f\|^2_{\mathcal{B}^\alpha}(1-r)^{-2\alpha}.$$

Choosing $r = 1 - 2^{-n}$ for any fixed n, we obtain

$$\sum_{k=2^n+1}^{2^{n+1}} k^2|a_k|^2(1-2^{-n})^{2k} \le \|f\|^2_{\mathcal{B}^\alpha} 2^{2\alpha n}. \tag{3.4}$$

Then (3.3) follows by (3.4). □

A complex sequence $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier from $l(2,\infty)$ to l^1 if and only if there exists a positive constant C such that whenever $\{a_n\}_{n=0}^{\infty} \in l(2,\infty)$, we have $\sum_{n=0}^{\infty} |\lambda_n a_n| \le C\|\{a_n\}\|_{l(2,\infty)}$. $l(2,\infty)$ consists all the sequences $\{b_k\}_{k=0}^{\infty}$ for which

$$\left\{\left(\sum_{k=2^n+1}^{2^{n+1}} |b_k|^2\right)^{1/2}\right\}_{n=0}^{\infty} \in l^\infty.$$

The following result can be found in [17].

Lemma 3.4. *A complex sequence $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier from $l(2,\infty)$ to l^1 if and only if*

$$\sum_{n=1}^{\infty}\left(\sum_{k=2^n+1}^{2^{n+1}} |\lambda_k|^2\right)^{1/2} < \infty.$$

Theorem 3.5. *Let μ be a positive measure on $[0,1)$ and $\alpha > 1$. Then the following statements are equivalent.*

(1) The measure μ is an α-Carleson measure.
(2) The operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ into $\mathcal{B}$.
(3) The operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ into BMOA.

Proof. (3)⇒ (2). It is trivial.

(2)⇒(1). We suppose that $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ into $\mathcal{B}$ for $\alpha > 1$. For any $0 < \lambda < 1$, let

$$f_\lambda(z) = \frac{1-\lambda^2}{(1-\lambda z)^\alpha} = \sum_{k=0}^{\infty} a_{k,\lambda} z^n, \tag{3.5}$$

where $a_{k,\lambda} = O((1-r^2)k^{\alpha-1}\lambda^k)$. It is easy to see that $f_\lambda \in \mathcal{B}^\alpha$. Then

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \mu_{n,k} a_k\right) z^n \in \mathcal{B}.$$

Lemma 2.2 gives that

$$\begin{aligned}\infty &> \sup_n n\sum_{k=0}^{\infty}\mu_{n,k}a_{k,\lambda}\\&=\sup_n n(1-\lambda^2)\sum_{k=0}^{\infty}k^{\alpha-1}\lambda^k\int_0^1 t^{n+k}d\mu(t)\\&\geq \sup_n n(1-\lambda^2)\sum_{k=0}^{\infty}k^{\alpha-1}\lambda^k\int_\lambda^1 t^{n+k}d\mu(t)\\&\geq \sup_n n(1-\lambda^2)\lambda^n\mu([\lambda,1))\sum_{k=0}^{\infty}k^{\alpha-1}\lambda^{2k}\\&=\sup_n n\lambda^n\frac{1-\lambda^2}{(1-\lambda^2)^{\alpha}}\mu([\lambda,1)).\end{aligned}$$

We choose n such that $1-\frac{1}{n}\leq\lambda<1-\frac{1}{n+1}$. We have

$$\infty > \frac{1}{e(1-\lambda^2)^{\alpha}}\mu([\lambda,1)). \tag{3.6}$$

So μ is an $\alpha-$Carleson measure.

(1)⇒(3). Assume that the condition (1) holds. Lemma 3.1 shows that $I_\mu(f)$ is analytic on $\mathbb{D}$. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{B}^{\alpha}$. By Lemma 3.3 we have that the sequence $\{a_k/k^{\alpha-1}\}\in l(2,\infty)$. Since μ is an α-Carleson measure, we have $\mu_k\leq\frac{C}{k^{\alpha}}$ by Lemma 2.7 in [11]. There exists a constant C such that

$$\sum_{n=1}^{\infty}\left(\sum_{k=2^n+1}^{2^{n+1}}(\mu_kk^{\alpha-1})^2\right)^{1/2}\lesssim\sum_{n=1}^{\infty}\left(\sum_{k=2^n+1}^{2^{n+1}}\frac{1}{k^2}\right)^{1/2}\lesssim\sum_{n=1}^{\infty}\frac{1}{2^{n/2}}<\infty.$$

This shows that the sequence $\{\mu_kk^{\alpha-1}\}$ is a multiplier from $l(2,\infty)$ to l^1 by Lemma 3.4. Note that $\{\mu_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive numbers. Given any $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{B}^{\alpha}$ for $\alpha>1$, we have

$$\begin{aligned}\sum_{k=1}^{\infty}|\mu_{n+k}a_k|\leq\sum_{k=1}^{\infty}|\mu_ka_k|&\leq\sum_{k=1}^{\infty}\frac{\mu_k}{k^{1-\alpha}}\frac{|a_k|}{k^{\alpha-1}}\\&\leq C\sup_n\left(\sum_{k=2^n}^{2^{n+1}-1}\frac{|a_k|^2}{k^{2(\alpha-1)}}\right)^{1/2}<C\|f\|_{\mathcal{B}^{\alpha}}.\end{aligned}$$

This implies that $\mathcal{H}_\mu(f)(z)$ is well defined for all $z\in\mathbb{D}$ and $\mathcal{H}_\mu(f)$ is an analytic function in $\mathbb{D}$. Applying

Fubini's Theorem, we get

$$\begin{aligned}
\mathcal{H}_\mu(f)(z) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n+k}a_k\right)z^n = \sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}\mu_{n+k}z^n\right)\\
&= \sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}\int_{[0,1)}t^{n+k}z^n d\mu(t)\right)\\
&= \sum_{k=0}^{\infty}\int_{[0,1)}\left(\sum_{n=0}^{\infty}t^n z^n\right)a_k t^k d\mu(t)\\
&= \int_{[0,1)}\sum_{k=0}^{\infty}\frac{a_k t^k}{1-tz}d\mu(t) = I_\mu(f)(z).
\end{aligned}$$

Note that $|f(t)| \lesssim (1-t)^{1-\alpha}$ by Lemma 2.1. Applying (2.1) and Lemma 3.2, we have

$$\begin{aligned}
\left|\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}d\theta\right| &= \left|\int_{[0.1)} f(t)\overline{g(rt)}d\mu(t)\right|\\
&\le \|f\|_{\mathcal{B}^\alpha}\int_{[0.1)}|g(rt)|(1-t)^{1-\alpha}d\mu(t)\\
&\le \|\mu\|\|f\|_{\mathcal{B}^\alpha}\int_0^{2\pi}|g(re^{i\theta})|d\theta\\
&\le \|\mu\|\|f\|_{\mathcal{B}^\alpha}\|g\|_{H^1}.
\end{aligned}$$

We obtain $\mathcal{H}_\mu(f) = I_\mu(f) \in BMOA$ by Fefferman's duality Theorem for any $f \in \mathcal{B}^\alpha$. The proof is complete. □

Theorem 3.6. *Let μ be a positive measure on $[0,1)$ and $\alpha > 1$. Then the following statements are equivalent.*

(1) The measure μ is a vanishing α-Carleson measure.
(2) The operator $\mathcal{H}_\mu$ is compact from $\mathcal{B}^\alpha$ spaces into $\mathcal{B}$.
(3) The operator $\mathcal{H}_\mu$ is compact from $\mathcal{B}^\alpha$ spaces into $BMOA$.

Proof. (3)⇒(2). It is trivial.

(2)⇒(1). Suppose that $\mathcal{H}_\mu : \mathcal{B}^\alpha \to \mathcal{B}$ is compact. Let f_λ be defined by (3.5). Then $\{f_\lambda\}$ is a bounded sequence in $\mathcal{B}^\alpha$ and $\lim_{r\to 1} f_\lambda(z) = 0$ on any compact subset of $\mathbb{D}$. Then we have

$$\lim_{\lambda\to 1}\|\mathcal{H}_\mu(f_\lambda)\|_{\mathcal{B}^\alpha} = 0.$$

The proof of Theorem 3.5 gives that

$$\|\mathcal{H}_\mu(f_{\lambda_n})\|_{\mathcal{B}^\alpha} \ge \frac{\mu([\lambda,1))}{e(1-\lambda^2)^\alpha}.$$

Consequently, μ is a vanishing α-Carleson measure.

(1)⇒(3). Assume that μ is a vanishing α-Carleson measure. The proof of the sufficiency for the boundedness gives that $\mathcal{H}_\mu(f) = I_\mu(f)$ and

$$\left|\int_0^{2\pi} \mathcal{H}_\mu(f)(e^{i\theta})\overline{g(re^{i\theta})}d\theta\right| \leq \int_{[0.1)} |f(t)g(rt)|d\mu(t)$$

for all $f \in \mathcal{B}^\alpha$ and $g \in H^1$. Let $\{f_n\}$ be any sequence with $\sup_n \|f_n\|_{\mathcal{B}^\alpha} \leq 1$ and $\lim_{n\to\infty} f_n(z) = 0$ on any compact subset of $\mathbb{D}$. Then we have

$$\lim_{n\to\infty} \int_{[0,r)} |f_n(t)g(rt)|d\mu(t) = 0. \tag{3.7}$$

Since ν is a vanishing Carleson measure, where ν is defined by $d\nu(t) = (1-t)^{1-\alpha}d\mu(t)$. We obtain

$$\int_{[r,1)} |f_n(t)g(rt)|d\mu(t) \leq \int_{[0,1)} |g(rt)|d\nu_r(t) < \|\nu - \nu_r\|\|g\|_{H^1}, \tag{3.8}$$

where $d\nu_r(t) = \chi_{0<t<r}d\nu(t)$. It is well known that ν is a vanishing Carleson measure if and only if

$$\|\nu - \nu_r\| \to 0, r \to 1.$$

See p. 283 of [22]. Combining (3.7) and (3.8), then

$$\lim_{n\to\infty}\left(\lim_{r\to 1} \int_{[0,1)} |f_n(t)g(rt)|d\mu(t)\right) = 0.$$

This prove that $\lim_{n\to\infty} \mathcal{H}_\mu(f_n) = 0$. So $\mathcal{H}_\mu$ is compact. The proof is complete. □

Theorem 3.7. *Let μ be a positive measure on* $[0, 1)$*. If $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}$ for any $\alpha > 1$, then*

$$\|\mathcal{H}_\mu\|_e^{\mathcal{B}^\alpha\to\mathcal{B}} \approx \|\mathcal{H}_\mu\|_e^{\mathcal{B}^\alpha\to BMOA} \approx \limsup_{r\to 1^-} \frac{\mu([r,1))}{(1-r)^\alpha}. \tag{3.9}$$

Proof. For any $f \in \mathcal{B}^\alpha$, we have

$$\|\mathcal{H}_\mu(f)\|^{\mathcal{B}^\alpha\to\mathcal{B}} \lesssim \|\mathcal{H}_\mu(f)\|^{\mathcal{B}^\alpha\to BMOA}.$$

This gives that

$$\|\mathcal{H}_\mu\|_e^{\mathcal{B}^\alpha\to\mathcal{B}} \lesssim \|\mathcal{H}_\mu\|_e^{\mathcal{B}^\alpha\to BMOA}.$$

We now give the upper estimate of $\mathcal{H}_\mu$ from $\mathcal{B}^\alpha$ to BMOA. Since $\mathcal{H}_\mu$ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}$, then the operator $\mathcal{H}_\mu$ from $\mathcal{B}^\alpha$ to BMOA is bounded and μ is an $\alpha-$Carleson measure by Theorem 3.5. For any $0 < r < 1$, the positive measure μ_r is defined by

$$\mu_r(t) = \begin{cases} \mu(t), & 0 \leq t \leq r, \\ 0, & r < t < 1. \end{cases} \tag{3.10}$$

It is easy to check that μ_r is a vanishing α-Carleson measure. We have that $\mathcal{H}_{\mu_r}$ is compact from $\mathcal{B}^\alpha$ to BMOA by Theorem 3.6. Then

$$\|\mathcal{H}_\mu - \mathcal{H}_{\mu_r}\|^{\mathcal{B}^\alpha\to BMOA} = \inf_{\|f\|_{\mathcal{B}^\alpha}=1} \|\mathcal{H}_{\mu-\mu_r}(f)\|_{BMOA}. \tag{3.11}$$

By (2.1) we have

$$\left|\int_0^{2\pi} \mathcal{H}_{\mu-\mu_r}(f)(re^{i\theta})\overline{g(e^{i\theta})}d\theta\right| \le \int_{[0.1)} \left|\overline{g(rt)}\right| (1-t)^{1-\alpha} d(\mu-\mu_r)(t)$$
$$\le \|\nu-\nu_r\| \|g\|_{H^1},$$

for any $g \in H^1$, where $d\nu(t) = (1-t)^{1-\alpha}d\mu(t)$ and $d\nu_r(t) = (1-t)^{1-\alpha}d\mu_r(t)$. The above estimate gives

$$\|\mathcal{H}_\mu\|_e^{\mathcal{B}^\alpha \to BMOA} \lesssim \limsup_{r\to 1^-} \frac{\mu([r,1))}{(1-r)^\alpha}.$$

We now give the lower estimate of $\mathcal{H}_\mu$ from $\mathcal{B}^\alpha$ to $\mathcal{B}$. For any $0<\lambda<1$, let f_λ be defined by (3.5). Then $f_\lambda \in \mathcal{B}^\alpha$. Since $f_\lambda \to 0$ weakly in $\mathcal{B}^\alpha$, we have that $\|Kf_\lambda\| \to 0$ as $\lambda \to 1$ for any compact operator K on $\mathcal{B}^\alpha$. Moreover

$$\|\mathcal{H}_\mu - K\|^{\mathcal{B}^\alpha \to \mathcal{B}} \ge \|(\mathcal{H}_\mu - K)f_\lambda\|_{\mathcal{B}} \ge \|\mathcal{H}_\mu f_\lambda\|_{\mathcal{B}} - \|Kf_\lambda\|_{\mathcal{B}}.$$

By the proof of Theorem 3.5 we have

$$\|\mathcal{H}_\mu(f_\lambda)\|_{\mathcal{B}} \ge \sup_n n \sum_{k=0}^{\infty} \mu_{n,k} a_{k,\lambda} \ge \sup_n nr^n \frac{1-\lambda^2}{(1-r\lambda)^\alpha}\mu([r,1)).$$

Let $r=\lambda$ and we choose n such that $1-\frac{1}{n+1} \le \lambda < 1-\frac{1}{n}$. We have

$$\|\mathcal{H}_\mu(f_\lambda)\|_{\mathcal{B}} > \frac{1}{e(1-\lambda^2)^\alpha}\mu([\lambda,1)). \tag{3.12}$$

Then

$$\|\mathcal{H}_\mu\|_e^{\mathcal{B}^\alpha \to \mathcal{B}} \ge \limsup_{\lambda \to 1^-} \|\mathcal{H}_\mu f_\lambda\|_{\mathcal{B}} \gtrsim \limsup_{r\to 1^-} \frac{\mu([r,1))}{(1-r)^\alpha}.$$

The proof is complete. □

4. Essential norm of $\mathcal{H}_\mu$ on $\mathcal{B}$

The reader can refer to [11, 12] for the results of $\mathcal{H}_\mu : \mathcal{B} \to BMOA$ and $\mathcal{H}_\mu : \mathcal{B} \to \mathcal{B}$. In this section, we characterize the essential of norm of $\mathcal{H}_\mu$ on $\mathcal{B}$. The following results will be needed in the proof of the main result.

Lemma 4.1. *[11] Let μ be a positive Borel measure on* $[0,1)$. *Let ν be the Borel measure on* $[0,1)$ *defined by*

$$d\nu(t) = \log\frac{e}{1-t}d\mu(t)$$

Then the following statements are equivalent.

(1) ν is a Carleson measure.
(2) μ is a $1-$*logarithmic* $1-$*Carleson measure.*

Lemma 4.2. *[11] Let μ be a positive Borel measure on $[0,1)$. Then the following statements are equivalent.*

(1) The measure μ is a vanishing $1-$logarithmic $1-$Carleson measure.
(2) The operator $\mathcal{H}_\mu$ is compact on $\mathcal{B}$.
(3) The operator $\mathcal{H}_\mu$ is compact from $\mathcal{B}$ to BMOA.

Theorem 4.3. *Let μ be an $1-$logarithmic $1-$Carleson measure on $[0,1)$. Then*

$$\|\mathcal{H}_\mu\|_e^{\mathcal{B}\to\mathcal{B}} \approx \|\mathcal{H}_\mu\|_e^{\mathcal{B}\to BMOA} \approx \limsup_{r\to 1^-} \frac{\mu([r,1))\log\frac{e}{1-r}}{1-r}. \tag{4.1}$$

Proof. For any $f\in\mathcal{B}$, we have

$$\|\mathcal{H}_\mu(f)\|^{\mathcal{B}\to\mathcal{B}} \lesssim \|\mathcal{H}_\mu(f)\|^{\mathcal{B}\to BMOA}.$$

This gives that

$$\|\mathcal{H}_\mu\|_e^{\mathcal{B}\to\mathcal{B}} \lesssim \|\mathcal{H}_\mu\|_e^{\mathcal{B}\to BMOA}.$$

We now give the upper estimate of $\mathcal{H}_\mu$ from $\mathcal{B}$ to BMOA. Since μ is an $1-$logarithmic $1-$Carleson measure on $[0,1)$, the operator $\mathcal{H}_\mu$ from $\mathcal{B}$ to BMOA is bounded by Theorem 2.8 of [11]. For any $0<r<1$, let the positive measure μ_r defined by (3.10). It is easy to check that μ_r is a vanishing $1-$logarithmic $1-$Carleson measure. We have that $\mathcal{H}_{\mu_r}$ is compact from $\mathcal{B}$ to BMOA by Lemma 4.2. Then

$$\|\mathcal{H}_\mu\|_e^{\mathcal{B}\to BMOA} \le \|\mathcal{H}_\mu - \mathcal{H}_{\mu_r}\|^{\mathcal{B}\to BMOA} = \inf_{\|f\|_{\mathcal{B}}=1} \|\mathcal{H}_{\mu-\mu_r}(f)\|_{BMOA}. \tag{4.2}$$

By (2.1) we have

$$\begin{aligned}\left|\int_0^{2\pi} \mathcal{H}_{\mu-\mu_r}(f)(re^{i\theta})\overline{g(e^{i\theta})}d\theta\right| &\le \int_{[0.1)} \left|f(t)\overline{g(rt)}\right| d(\mu-\mu_r)(t)\\ &\le \int_{[0.1)} \left|\overline{g(rt)}\right| \log\frac{e}{1-t} d(\mu-\mu_r)(t)\\ &\lesssim \|\nu-\nu_r\|\|g\|_{H^1},\end{aligned}$$

where $d\nu(t)=\log\frac{e}{1-t}d\mu(t)$ and $d\nu_r(t)=\log\frac{e}{1-t}d\mu_r(t)$. The positive measure $\nu-\nu_r$ is a Carleson measure by Lemma 4.1. The above estimate gives

$$\|\mathcal{H}_\mu\|_e^{\mathcal{B}\to BMOA} \lesssim \limsup_{\lambda\to 1^-} \frac{\mu([\lambda,1))\log\frac{e}{1-\lambda}}{1-\lambda}.$$

We will give the lower estimate for $\mathcal{H}_\mu$. Let $0<\lambda<1$ and

$$f_\lambda(z) = \beta_\lambda \log^2\frac{e}{1-\lambda z}, \tag{4.3}$$

where $\beta_\lambda = \log^{-1}\frac{e}{1-\lambda^2}$. Then $\{f_\lambda\}$ is a bounded sequence in $\mathcal{B}$ and $\lim_{\lambda\to 1^-} f_\lambda(z)=0$ on any compact subset of $\mathbb{D}$. Since $f_\lambda\to 0$ weakly in $\mathcal{B}$, we have that $\|Kf_\lambda\|\to 0$ as $\lambda\to 1$ for any compact operator K on $\mathcal{B}$. Moreover

$$\|\mathcal{H}_\mu - K\|^{\mathcal{B}\to\mathcal{B}} \ \ge\ \|(\mathcal{H}_\mu - K)f_\lambda\|_{\mathcal{B}} \ge \|\mathcal{H}_\mu f_\lambda\|_{\mathcal{B}} - \|Kf_\lambda\|_{\mathcal{B}}.$$

Note that $\mathcal{H}_\mu(f_\lambda) = I_\mu(f_\lambda)$. We have

$$\begin{aligned}\|\mathcal{H}_\mu(f_\lambda)\|_{\mathcal{B}} \geq &(1-\lambda^2)\left|\left(I_\mu(f_\lambda)\right)'(\lambda)\right| \\ \geq &(1-\lambda^2)\int_\lambda^1 \frac{f_\lambda(t)}{(1-t\lambda)^2}d\mu(t) \\ \geq &\log\frac{e}{1-\lambda^2}\frac{\mu([\lambda,1))}{1-\lambda^2}.\end{aligned}$$

The above estimate shows that

$$\|\mathcal{H}_\mu - K\|_e^{\mathcal{B}\to\mathcal{B}} \geq \limsup_{\lambda\to 1^-}\|\mathcal{H}_\mu f_\lambda\|_{\mathcal{B}} \gtrsim \limsup_{\lambda\to 1^-}\frac{\mu([\lambda,1))\log\frac{e}{1-\lambda}}{1-\lambda}.$$

The proof is complete. □

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Conflict of interest

The authors declare that they have no conflict of interest.

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