Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(4): 3288–3304.
DOI:10.3934/math.2021197
Received: 26 November 2020
Accepted: 30 December 2020
Published: 15 January 2021

Research article

Set-valued variational inclusion problem with fuzzy mappings involving XOR-operation

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Abstract: In this paper, we introduce a set-valued variational inclusion problem with fuzzy mappings involving XOR-operation. We define a resolvent operator involving a bi-mapping and prove resolvent operator is single-valued, comparison and Lipschitz-type continuous. Based on resolvent operator we proposed an iterative algorithm to find the approximate solution of our problem. An existence and convergence result is proved for set-valued variational inclusion problem with fuzzy mappings involving XOR-operation without using the properties of a normal cone. Examples are constructed for illustration.

Keywords: algorithm; fuzzy mapping; inclusion; operation, set-valued
Mathematics Subject Classification: 47J22, 47J25, 47S40

1. Introduction

In mathematical terms, a variational inequality is an inequality involving a functional, which has to be solved for all possible values of a given variable, belonging usually to a convex set. The general frame work of variational inequality theory provides us powerful tools to deal with the problems arising in elasticity, structured analysis, physical and engineering sciences, etc., see for example [1,2,8,29–31]. Variational inclusions are important and applicable generalization of classical variational inequalities studied by Hassouni and Moudafi [16]. Chang et al. [11] and Ansari [7] simultaneously introduced the concept of variational inequalities for fuzzy mappings in abstract spaces. A lot of literature is available related to variational inequalities (inclusions) with fuzzy mappings, see for example [3, 9, 10, 12–15, 17–19, 26, 28, 33] and references therein. It is worth to mention that the fuzzy set theory due to Zadeh [32] specifically modeled to mathematically represent uncertainly and vagueness. Moreover, this theory provides formalized tools for dealing with imprecision intrinsic to many problems.

Generalized nonlinear ordered variational inequalities(ordered equation) have ample applications in mathematics, physics, economics, optimization, nonlinear programming, engineering sciences. Recently Li et al. [20–23] introduced XOR and XNOR operations and studied some properties of these operations in ordered sapces. XOR and XNOR operations depicts interesting facts and observations and forms various real time applications that is data encryption, error detection in digital communication, image processing and in neural networks. For related work, we refer to [4–6].

Inspired and motivated by the above research works in this paper, we study a set-valued variational inclusion problem with fuzzy mappings involving XOR-operation in ordered Hilbert spaces. For solving this problem, we used the fixed point iteration technique. We define a resolvent operator of the type $[H\oplus\lambda M(\cdot,z)]^{-1}$ and proved that resolvent operator is single-valued, comparison and Lipschitz-type continuous. By using the definition of resolvent operator fixed point lemma is obtained and proposed an iterative algorithm based on it. An existence and convergence result is proved without using the properties of a normal cone. Examples are constructed for illustration.

2. Preliminaries

Throughout this paper, we assume that $\mathcal{H}$ is a real ordered Hilbert space equipped with the usual norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $C \subseteq \mathcal{H}$ be a cone with partial ordering " $\leq$ ".

For the set $\{x, y\}$, where x and y are arbitrary elements of $\mathcal{H}$, we denote the least upper bound by $lub\{x, y\}$ and greatest lower bound by $glb\{x, y\}$, also we suppose that they exist. The operation $\oplus$ is called XOR-operation if $x\oplus y = lub\{x-y, y-x\}$ and $\odot$ is called $XNOR$-operation if $x\odot y = glb\{x-y, y-x\}$. x and y are said to be comparable to each other if either $x \leq y$ or $y \leq x$ holds and is denoted by $x \propto y$, see [27].

Let $\mathcal{F}(\mathcal{H})$ be the collection of all fuzzy sets over $\mathcal{H}$. A mapping $F : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ is called a fuzzy mapping on $\mathcal{H}$. For each $x \in \mathcal{H}$, $F(x)$ (denoted by F_x in the sequel) is a fuzzy set on $\mathcal{H}$ and $F_x(y)$ is the membership function of y in F_x.

A fuzzy mapping $F : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ is said to be closed if for each $x \in \mathcal{H}$, the function $y \to F_x(y)$ is upper semi-continuous, that is, for any given net $\{y_\alpha\} \subset \mathcal{H}$ satisfying $y_\alpha \to y_0 \in \mathcal{H}$, we have

$$\limsup_{\alpha} F_x(y_\alpha) \leq F_x(y_0).$$

For $B \in \mathcal{F}(\mathcal{H})$ and $\lambda \in [0, 1]$, the set $(B)_\lambda = \{x \in \mathcal{H} : B(x) \geq \lambda\}$ is called λ-cut set of B. Let $F : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ be a closed fuzzy mapping satisfying the following condition:

Condition(f): If there exists a function $a : \mathcal{H} \to [0, 1]$ such that for each $x \in \mathcal{H}$, the set $(F_x)_{a(x)} = \{y \in \mathcal{H} : F_x(y) \geq a(x)\}$ is a nonempty bounded subset of $\mathcal{H}$.

If F is a closed fuzzy mapping satisfying the condition (f), then for each $x \in \mathcal{H}$, $(F_x)_{a(x)} \in CB(\mathcal{H})$. In fact, let $\{y_\alpha\} \subset (F_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in \mathcal{H}$, then $(F_x)_{a(x)} \geq a(x)$, for each α.
Since F is closed, we have

$$F_x(y_0) \geq \limsup_{\alpha} F_x(y_\alpha) \geq a(x),$$

which implies that $y_0 \in (F_x)_{a(x)}$ and so $(F_x)_{a(x)} \in CB(\mathcal{H})$.

We mention some known concepts, results and their extensions to prove the main result of this paper.

Proposition 2.1. *Let $\oplus$ be an XOR-operation and $\odot$ be an XNOR-operation. Then the following relations hold:*

(*i*) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;
(*ii*) *if* $x \propto 0$, *then* $-x \oplus 0 \leq x \leq x \oplus 0$;
(*iii*) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
(*iv*) $0 \leq x \oplus y$, *if* $x \propto y$;
(*v*) *if* $x \propto y$, *then* $x \oplus y = 0$ *if and only if* $x = y$.

Proposition 2.2. *Let C be a cone in $\mathcal{H}$, then for all $x, y \in \mathcal{H}$, the following relations hold:*

(*i*) $\|0 \oplus 0\| = \|0\| = 0$;
(*ii*) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$;
(*iii*) $\|x \oplus y| \leq \|x - y\|$;
(*iv*) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 2.1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping, then*

(*i*) *A is said to be comparison mapping, if for each $x, y \in \mathcal{H}$, $x \propto y$ then $A(x) \propto A(y)$, $x \propto A(x)$ and $y \propto A(y)$.*
(*ii*) *A is said to be strongly comparison mapping, if A is a comparison mapping and $A(x) \propto A(y)$ if and only if $x \propto y$, for all $x, y \in \mathcal{H}$.*

Definition 2.2. *A set-valued mapping $F : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be*

(*i*) *relaxed Lipschitz continuous with respect to a mapping $P : \mathcal{H} \to \mathcal{H}$, if there exists a constant $k \geq 0$ such that*

$$\langle P(u_1) - P(u_2), x_1 - x_2 \rangle \leq -k \|x_1 - x_2\|^2, \ \ for\ all\ x_i \in \mathcal{H},\ u_i \in F(x_i),\ i = 1, 2.$$

(*ii*) *relaxed monotone with respect to a mapping $f : \mathcal{H} \to \mathcal{H}$, if there exists a constant $c > 0$ such that*

$$\langle f(v_1) - f(v_2), x_1 - x_2 \rangle \geq -c \|x_1 - x_2\|^2, \ \ for\ all\ x_i \in \mathcal{H},\ u_i \in F(x_i),\ i = 1, 2.$$

Definition 2.3. *Let $A : \mathcal{H} \to \mathcal{H}$ be a single-valued comparison mapping and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued comparison mapping, then*

(*i*) *the mapping A is said to be β-ordered compression mapping, if*

$$A(x) \oplus A(y) \leq \beta(x \oplus y),\ for\ 0 < \beta < 1,$$

(*ii*) *the mapping M is said to be θ-ordered rectangular, if there exists a constant $\theta > 0$, for any $x, y \in \mathcal{H}$, there exist $v_x \in M(x)$ and $v_y \in M(y)$ such that*

$$\langle v_x \odot v_y, -(x \oplus y) \rangle \geq \theta \|x \oplus y\|^2, \quad for\ all\ \ x, y \in \mathcal{H},$$

holds.

(*iii*) *the mapping M is said to be λ-XNOR-ordered strongly monotone compression mapping, if $x \propto y$, then there exist a constant $\lambda > 0$ such that*

$$\lambda(v_x \oplus v_y) \geq x \oplus y, \quad for\ all\ \ x, y \in \mathcal{H}, v_x \in M(x), v_y \in M(y).$$

Let $T : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ be a closed fuzzy mapping satisfying condition (f). Let $\tilde{T}$ be the set-valued mapping induced by the fuzzy mapping T such that $\tilde{T}(x) = (T_x)_{c(x)}$, for all $x \in \mathcal{H}$, where $c : \mathcal{H} \to [0, 1]$ is a mapping. Suppose that $M : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping and $A : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping. Then, we have the following new definitions:

Definition 2.4. *The mapping $A : \mathcal{H} \to \mathcal{H}$ is said to be β-ordered compression mapping with respect to $\tilde{T}$, if A is a comparison mapping and*

$$A(x) \oplus A(y) \leq \beta[(x, z) \oplus (y, z)], \quad for\ all\ x, y \in \mathcal{H}\ and\ z \in \tilde{T}(x).$$

Definition 2.5. *The set-valued mapping $M : \mathcal{H}\times\mathcal{H} \to 2^{\mathcal{H}}$ is said to be θ-ordered rectangular mapping with respect to $\tilde{T}$, if*

$$\theta\|(x, z) \oplus (y, z)\|^2 \leq \langle v_{x_z} \odot v_{y_z}, -[(x, z) \oplus (y, z)]\rangle,$$

for all $x, y, \in \mathcal{H}$, $z \in \tilde{T}(x)$, and $v_{x_z} \in M(x, z), v_{y_z} \in M(y, z)$.

Definition 2.6. *The set-valued mapping $M : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ is said to be λ-XOR-ordered strongly monotone with respect to $\tilde{T}$, if*

$$(x, z) \oplus (y, z) \leq \lambda(v_{x_z} \oplus v_{y_z}),$$

for all $x, y \in \mathcal{H}, z \in \tilde{T}(x), v_{x_z} \in M(x, z), v_{y_z} \in M(y, z)$.

Similarly, we can extend the definitions of ordered compression mapping, ordered rectangular mapping and ordered strongly monotone mapping with respect to $J^H_{\lambda,M(\cdot,z)}$.

Definition 2.7. *Let $H : \mathcal{H} \to \mathcal{H}$ be β-ordered compression mapping with respect to $\tilde{T}$. Then a set-valued mapping $M : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ is said to be ORSM-mapping with respect to $\tilde{T}$ if M is θ-ordered rectangular mapping with respect to $\tilde{T}$, λ-XOR-ordered strongly monotone with respect to $\tilde{T}$ and*

$$[H \oplus \lambda M(\cdot, z)](\mathcal{H}) = \mathcal{H}, \quad for\ all\ \ \beta, \theta, \lambda > 0,\ \ x \in \mathcal{H}\ and\ z \in \tilde{T}(x).$$

Based on Definition 2.7, we define the following resolvent operator.

Definition 2.8. *The resolvent operator $J^H_{\lambda,M(\cdot,z)}$ associated with $H, M, \tilde{T}$, that is, $J^H_{\lambda,M(\cdot,z)} : \mathcal{H} \to \mathcal{H}$ is defined as*

$$J^H_{\lambda,M(\cdot,z)}(x) = [H \oplus \lambda M(\cdot, z)]^{-1}(x), \quad for\ all\ x \in \mathcal{H}, z \in \tilde{T}\ and\ \lambda > 0. \tag{2.1}$$

We show some of the properties of the resolvent operator defined by (2.1).

Proposition 2.3. *Let $H : \mathcal{H} \to \mathcal{H}$ be β-ordered compression mapping with respect to $\tilde{T}$ and $M : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ be θ-ordered rectangular mapping with respect to $\tilde{T}$. Then the resolvent operator $J^H_{\lambda,M(\cdot,z)} : \mathcal{H} \to \mathcal{H}$ is single-valued, for $\theta\lambda > \beta$, where $\theta, \lambda, \beta > 0$.*

Proof. For any $u \in \mathcal{H}$ and a constant $\lambda > 0$, let $x, y \in [H \oplus M(\cdot, z)]^{-1}(u)$, then

$$v_{x_z} = \frac{1}{\lambda}[u \oplus H(x)] \in M(x, z),$$

and

$$v_{y_z} = \frac{1}{\lambda}[u \oplus H(y)] \in M(y, z), \quad where \;\; z \in \tilde{T}(x).$$

Using (i) and (ii) of Proposition 2.1, we have

$$\begin{aligned} v_{x_z} \odot v_{y_z} &= \frac{1}{\lambda}[(u \oplus H(x)) \odot (u \oplus H(y))] \\ &= -\frac{1}{\lambda}[(u \oplus H(x)) \oplus (u \oplus H(y))] \\ &= -\frac{1}{\lambda}[(u \oplus u) \oplus (H(x) \oplus H(y))] \\ &= -\frac{1}{\lambda}[0 \oplus (H(x) \oplus H(y))] \\ &\leq -\frac{1}{\lambda}[H(x) \oplus H(y))], \end{aligned}$$

thus, we have

$$v_{x_z} \odot v_{y_z} \leq -\frac{1}{\lambda}[H(x) \oplus H(y))]. \tag{2.2}$$

Since M is θ-ordered rectangular mapping with respect to $\tilde{T}$, H is β-ordered compression mapping with respect to $\tilde{T}$ and using (2.2), we have

$$\begin{aligned} \theta\|(x, z) \oplus (y, z)\|^2 &\leq \langle v_{x_z} \odot v_{y_z}, -[(x, z) \oplus (y, z)]\rangle \\ &\leq \langle -\frac{1}{\lambda}[H(x) \oplus H(y)], -[(x, z) \oplus (y, z)]\rangle \\ &\leq \frac{1}{\lambda}\langle H(x) \oplus H(y), [(x, z) \oplus (y, z)]\rangle \\ &\leq \frac{\beta}{\lambda}\langle [(x, z) \oplus (y, z)], [(x, z) \oplus (y, z)]\rangle \\ &\leq \frac{\beta}{\lambda}\|(x, z) \oplus (y, z)\|^2, \end{aligned}$$

i.e.,

$$\left(\theta - \frac{\beta}{\lambda}\right)\|(x, z) \oplus (y, z)\|^2 \leq 0, \quad for \;\; \theta\lambda > \beta.$$

It follows that

$$\|(x, z) \oplus (y, z)\|^2 = 0,$$

or

$$(x, z) \oplus (y, z) = 0,$$

implies

$$(x, z) = (y, z),$$

thus

$$x = y.$$

Hence, the resolvent operator $J^H_{\lambda,M(\cdot,z)}$ is single-valued, for $\theta\lambda > \beta$. □

Proposition 2.4. *Let the mapping $M : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ be λ-XOR-ordered strongly monotone with respect to $J^H_{\lambda,M(\cdot,z)}$ and the mapping $H : \mathcal{H} \to \mathcal{H}$ be strongly compression mapping with respect to $J^H_{\lambda,M(\cdot,z)}$. Suppose that $(x,z) \oplus (y,z) \propto (x \oplus y)$ and $0 \propto \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right]$. Then, the resolvent operator $J^H_{\lambda,M(\cdot,z)} : \mathcal{H} \to \mathcal{H}$ is a comparison mapping, for all $x, y \in \mathcal{H}$ and $z \in \tilde{T}(x)$.*

Proof. For any $x, y \in \mathcal{H}$, let

$$v^*_{x_z} = \frac{1}{\lambda}\left[x \oplus H\left(J^H_{\lambda,M(\cdot,z)}(x)\right)\right] \in M\left(J^H_{\lambda,M(\cdot,z)}(x), J^H_{\lambda,M(\cdot,z)}(z)\right), \tag{2.3}$$

$$v^*_{y_z} = \frac{1}{\lambda}\left[y \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right] \in M\left(J^H_{\lambda,M(\cdot,z)}(y), J^H_{\lambda,M(\cdot,z)}(z)\right). \tag{2.4}$$

As M is λ-XOR-ordered strongly monotone with respect to $J^H_{\lambda,M(\cdot,z)}$, using (2.3) and (2.4), we have

$$\begin{aligned} (x,z) \oplus (y,z) &\leq \lambda(v^*_{x_z} \oplus v^*_{y,z}) \\ (x,z) \oplus (y,z) &\leq \left[\left[x \oplus H\left(J^H_{\lambda,M(\cdot,z)}(x)\right)\right] \oplus \left[y \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right]\right] \\ &= (x \oplus y) \oplus \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right]. \end{aligned}$$

Thus,

$$\begin{aligned} [(x,z) \oplus (y,z)] \oplus [(x,z) \oplus (y,z)] &\leq \left[[(x,z) \oplus (y,z)] \oplus (x \oplus y)\right] \\ &\quad \oplus \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right] \\ 0 &\leq \left[[(x,z) \oplus (y,z)] \oplus (x \oplus y)\right] \\ &\quad \oplus \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right]. \end{aligned} \tag{2.5}$$

Since $(x,z) \oplus (y,z) \propto (x \oplus y)$ and $0 \propto H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)$, from (2.5) we have

$$0 \leq (x \oplus y) \oplus (x \oplus y) \oplus \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right]$$

$$\begin{aligned} 0 &\leq 0 \oplus \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right] \\ 0 &\leq \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right] \\ 0 &\leq \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) - H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right] \vee \left[H\left(J^H_{\lambda,M(\cdot,z)}(y)\right) - H\left(J^H_{\lambda,M(\cdot,z)}(x)\right)\right], \end{aligned}$$

which implies either

$$0 \leq \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) - H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right]$$

or

$$0 \leq \left[H\left(J^H_{\lambda,M(\cdot,z)}(y)\right) - H\left(J^H_{\lambda,M(\cdot,z)}(x)\right)\right].$$

It follows that

$$H\left(J^{H}_{\lambda,M(\cdot,z)}(x)\right) \propto H\left(J^{H}_{\lambda,M(\cdot,z)}(y)\right).$$

Since H is strongly comparison mapping with respect to $J^{H}_{\lambda,M(\cdot,z)}$. Therefore $J^{H}_{\lambda,M(\cdot,z)}(x) \propto J^{H}_{\lambda,M(\cdot,z)}(y)$, i.e., the resolvent operator $J^{H}_{\lambda,M(\cdot,z)}$ is a comparison mapping. □

Proposition 2.5. *Let $H : \mathcal{H} \to \mathcal{H}$ be β'-ordered compression mapping with respect to $J^{H}_{\lambda,M(\cdot,z)}$ and $M : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ be θ'-ordered rectangular mapping with respect to $J^{H}_{\lambda,M(\cdot,z)}$. Let $\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \propto J^{H}_{\lambda,M(\cdot,z)}(x) \oplus J^{H}_{\lambda,M(\cdot,z)}(y)$, then the resolvent operator $J^{H}_{\lambda,M(\cdot,z)} : \mathcal{H} \to \mathcal{H}$ is $\left(\dfrac{1}{\lambda\theta' - \beta'}\right)$-Lipschitz-type continuous. That is*

$$\|J^{H}_{\lambda,M(\cdot,z)}(x) \oplus J^{H}_{\lambda,M(\cdot,z)}(y)\| \leq \left(\frac{1}{\lambda\theta' - \beta'}\right) \|x \oplus y\|,$$

for all $x, y \in \mathcal{H}$ and $z \in \tilde{T}(x)$.

Proof. Let $v^*_{x_z}$ and $v^*_{y_z}$ are same as in (2.3) and (2.4). Then

$$v^*_{x_z} \oplus v^*_{y_z} = \frac{1}{\lambda}\left[(x \oplus y) \oplus \left(H\left(J^{H}_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^{H}_{\lambda,M(\cdot,z)}(y)\right)\right)\right]. \tag{2.6}$$

Since H is β'-ordered compression mapping with respect to $J^{H}_{\lambda,M(\cdot,z)}$ and using (2.6), we have

$$\begin{aligned} v^*_{x_z} \oplus v^*_{y_z} &= \frac{1}{\lambda}\left[(x \oplus y) \oplus \left(H\left(J^{H}_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^{H}_{\lambda,M(\cdot,z)}(y)\right)\right)\right] \\ &\leq \frac{1}{\lambda}\Big[(x \oplus y) \oplus \beta'\Big[\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \\ &\quad \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right)\Big]\Big]. \end{aligned} \tag{2.7}$$

As M is θ'-ordered rectangular mapping with respect to $J^{H}_{\lambda,M(\cdot,z)}$ and using (2.7), we have

$$\begin{aligned} &\theta' \left\|\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right)\right\|^2 \\ \leq\ & \left\langle v^*_{x_z} \odot v^*_{y_z}, -\left[\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right)\right]\right\rangle \\ =\ & \left\langle v^*_{x_z} \oplus v^*_{y_z}, \left[\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right)\right]\right\rangle \\ \leq\ & \frac{1}{\lambda}\Big\langle (x \oplus y) \oplus \beta'\left[\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right)\right], \\ & \left[\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right)\right] \Big\rangle \\ \leq\ & \frac{1}{\lambda}\left[\left\|(x \oplus y) \oplus \beta'\left[\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right)\right]\right\|\right] \\ & \left\|\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right)\right\| \\ \leq\ & \frac{1}{\lambda}\left[\|(x \oplus y)\| \left\|\left(J^{H}_{\lambda,M(\cdot,z)}(x), J^{H}_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^{H}_{\lambda,M(\cdot,z)}(y), J^{H}_{\lambda,M(\cdot,z)}(z)\right)\right\|\right] \end{aligned}$$

$$+\frac{\beta'}{\lambda}\left\|\left(J^H_{\lambda,M(\cdot,z)}(x), J^H_{\lambda,M(\cdot,z)}(z)\right)\oplus\left(J^H_{\lambda,M(\cdot,z)}(y), J^H_{\lambda,M(\cdot,z)}(z)\right)\right\|^2,$$

which implies that

$$\begin{aligned}&\left(\theta'-\frac{\beta'}{\lambda}\right)\left\|\left(J^H_{\lambda,M(\cdot,z)}(x), J^H_{\lambda,M(\cdot,z)}(z)\right)\oplus\left(J^H_{\lambda,M(\cdot,z)}(y), J^H_{\lambda,M(\cdot,z)}(z)\right)\right\|^2\\ \leq\ &\frac{1}{\lambda}\left[\|(x\oplus y)\|\left\|\left(J^H_{\lambda,M(\cdot,z)}(x), J^H_{\lambda,M(\cdot,z)}(z)\right)\oplus\left(J^H_{\lambda,M(\cdot,z)}(y), J^H_{\lambda,M(\cdot,z)}(z)\right)\right\|\right],\end{aligned}\tag{2.8}$$

i.e.,

$$(\lambda\theta'-\beta')\left\|\left(J^H_{\lambda,M(\cdot,z)}(x), J^H_{\lambda,M(\cdot,z)}(z)\right)\oplus\left(J^H_{\lambda,M(\cdot,z)}(y), J^H_{\lambda,M(\cdot,z)}(z)\right)\right\|\leq\|x\oplus y\|.$$

As

$$\left(J^H_{\lambda,M(\cdot,z)}(x), J^H_{\lambda,M(\cdot,z)}(z)\right)\oplus\left(J^H_{\lambda,M(\cdot,z)}(y), J^H_{\lambda,M(\cdot,z)}(z)\right)\ \propto\ J^H_{\lambda,M(\cdot,z)}(x)\oplus J^H_{\lambda,M(\cdot,z)}(y),$$

we have

$$J^H_{\lambda,M(\cdot,z)}(x)\oplus J^H_{\lambda,M(\cdot,z)}(y)\ \leq\ \left(J^H_{\lambda,M(\cdot,z)}(x), J^H_{\lambda,M(\cdot,z)}(z)\right)\oplus\left(J^H_{\lambda,M(\cdot,z)}(y), J^H_{\lambda,M(\cdot,z)}(z)\right).$$

Thus from (2.8), it follows that

$$\begin{aligned}(\lambda\theta'-\beta')\left\|J^H_{\lambda,M(\cdot,z)}(x)\oplus J^H_{\lambda,M(\cdot,z)}(y)\right\|\ &\leq\ (\lambda\theta'-\beta')\Big\|\left(J^H_{\lambda,M(\cdot,z)}(x), J^H_{\lambda,M(\cdot,z)}(z)\right)\\ &\qquad\oplus\left(J^H_{\lambda,M(\cdot,z)}(y), J^H_{\lambda,M(\cdot,z)}(z)\right)\Big\|\\ &\leq\ \|x\oplus y\|.\end{aligned}$$

That is,

$$\left\|J^H_{\lambda,M(\cdot,z)}(x)\oplus J^H_{\lambda,M(\cdot,z)}(y)\right\|\leq\left(\frac{1}{\lambda\theta'-\beta'}\right)\|x\oplus y\|,\quad for\ \ \lambda\theta'>\beta'.$$

Thus, the resolvent operator $J^H_{\lambda,M(\cdot,z)}$ is $\dfrac{1}{(\lambda\theta'-\beta')}$-Lipschitz-type continuous. □

3. Formulation of the problem and iterative algorithm

Let $F, S, T : \mathcal{H}\to\mathcal{F}(\mathcal{H})$ be the closed fuzzy mappings satisfying the condition (f). Then, there exist mappings $a, b, c : \mathcal{H}\to[0,1]$ such that for each $x\in\mathcal{H}, (F_x)_{a(x)}\in CB(\mathcal{H}), (S_x)_{b(x)}\in CB(\mathcal{H})$ and $(T_x)_{c(x)}\in CB(\mathcal{H})$. We define the set-valued mappings induced by the fuzzy mappings F, S and T, respectively, by

$$\tilde{F}(x)=(F_x)_{a(x)},\ \ \tilde{S}(x)=(S_x)_{b(x)}\ \ and\ \ \tilde{T}(x)=(T_x)_{c(x)},\ \ for\ \ all\ \ x\in\mathcal{H}.$$

Suppose that $P, f : \mathcal{H}\to\mathcal{H}$ are the single-valued mappings and $M : \mathcal{H}\times\mathcal{H}\to 2^{\mathcal{H}}$ is a set-valued mapping. We consider the following problem:

Find $x\in\mathcal{H}$, $u\in(F_x)_{a(x)}$, $v\in(S_x)_{b(x)}$ and $z\in(T_x)_{c(x)}$ such that

$$0\in P(u)-f(v)\oplus M(x,z)\tag{3.1}$$

Problem (3.1) is called set-valued variational inclusion problem with fuzzy mappings involving XOR-operation.

For suitable choices of operators involved in the formulation of problem (3.1), one can obtain many previously studied problems by Li et al. [20–23] and Ahmad et al. [4–6], etc..

In support of our problem (3.1), we provide the following example.

Example 3.1. *Let* $\mathcal{H} = C = [0,1]$ *and we define the closed fuzzy mappings* $F, S, T : \mathcal{H} \to F(\mathcal{H})$*, for* $u, v, z \in [0,1]$ *as*

$$F_x(u) = \begin{cases} \dfrac{x+u}{2}, & \text{if } x \in [0, \frac{1}{2})\,; \\ x, & \text{if } x \in [\frac{1}{2}, 1]\,. \end{cases}$$

$$S_x(v) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2})\,; \\ \dfrac{x+v}{3}, & \text{if } x \in [\frac{1}{2}, 1]\,. \end{cases}$$

and

$$T_x(z) = \begin{cases} \dfrac{x}{2} + \dfrac{z}{3}, & \text{if } x \in [0, \frac{1}{2})\,; \\ 0, & \text{if } x \in [\frac{1}{2}, 1]\,. \end{cases}$$

We define the mapping $a, b, c : \mathcal{H} \to [0,1]$ *by*

$$a(x) = \begin{cases} \dfrac{x}{2}, & \text{if } x \in [0, \frac{1}{2})\,; \\ 0, & \text{if } x \in [\frac{1}{2}, 1]\,. \end{cases}$$

$$b(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2})\,; \\ \dfrac{x}{3}, & \text{if } x \in [\frac{1}{2}, 1]\,. \end{cases}$$

and

$$c(x) = \begin{cases} \dfrac{x}{6}, & \text{if } x \in [0, \frac{1}{2})\,; \\ 0, & \text{if } x \in [\frac{1}{2}, 1]\,. \end{cases}$$

Clearly, $F_x(u) \geq a(x), S_x(v) \geq b(x)$ *and* $T_x(z) \geq c(x)$ *for all* $x \in [0,1]$*, that is,* $u \in (F_x)_{a(x)}, v \in (S_x)_{b(x)}$ *and* $z \in (T_x)_{c(x)}$.

Now, we define the mapping $P : \mathcal{H} \to \mathcal{H}$ *by*

$$P(u) = \frac{u}{2},$$

mapping $f : \mathcal{H} \to \mathcal{H}$ *by*

$$f(v) = \frac{v}{3},$$

and mapping $M : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ *by*

$$M(x,z) = \{x + z : \ x \in [0,1] \text{ and } z \in (T_x)_{c(x)}\}.$$

Now, we evaluate

$$P(u) - f(v) \oplus M(x,z) = \frac{u}{2} - \frac{v}{3} \oplus (x+z)$$

In view of above, $u, v, x, z \in [0,1]$*. Particularly taking* $u, v, x, z = 0 \implies 0 - 0 \oplus 0 = 0$*. Hence,* $0 \in P(u) - f(v) \oplus M(x,z)$*, that is, problem* (3.1) *is satisfied.*

The following example shows that fuzzy capacity game can be obtain from set-valued variational inclusion problem with fuzzy mappings involving XOR-operation (3.1).

Example 3.2. *The characteristic function of cooperative games is a function, $v : L(N) \to \mathbb{R}$ such that $v(\theta) = 0$, where N is the player set and $L(N)$ is the set of fuzzy coalitions in N.*

For every random fuzzy coalition $\tilde{S} \in L(N)$ with non-negative variables $s = \{s_1, s_2, \cdots, s_n\}$, the fuzzy capacity game with concave integral is defined as:

$$v^{Cav}(\tilde{S}) = \int^{Cav} sdv = \max\left\{\alpha_T v(T), \sum_{T\subseteq N} \alpha_T 1_T = S, \alpha_T \geq 0\right\},$$

where $v(T) \neq 0$, 1_T is an indicator of $T \subseteq N$ and $\int^{Cav}$ denotes the concave integral. For more details, see [25].

If we take $\mathcal{H} = \mathbb{R}$ and define $P : \mathcal{H} \to \mathcal{H}$ by

$$p(u) = \int^{Cav} sdv,$$

and all other functions involved in the formulation of problem (3.1) are zero. Then, we can obtain fuzzy capacity game from set-valued variational inclusion problem with fuzzy mappings involving XOR-operation (3.1).

The following Lemma is a fixed point formulation of set-valued variational inclusion problem with fuzzy mappings involving XOR-operation (3.1).

Lemma 3.1. *Let $x \in \mathcal{H}$, $u \in (F_x)_{a(x)}$, $v \in (S_x)_{b(x)}$ and $z \in (T_x)_{c(x)}$ is a solution of set-valued XOR-variational inclusion problem (3.1) if and only if (x, u, v, z) satisfying the following equation:*

$$x = J^H_{\lambda,M(\cdot,z)}[\lambda(P(u) - f(v)) \oplus H(x)], \tag{3.2}$$

where $\lambda > 0$ is a constant.

Proof. It can be proved easily by using the definition of resolvent operator $J^H_{\lambda,M(\cdot,z)}$. □

Based on Lemma 3.1, we construct the following iterative algorithm for solving set-valued variational inclusion problem with fuzzy mappings involving XOR-operation (3.1).

Iterative Algorithm 3.1.
Step 1. *Choose an arbitrary initial point $x_0 \in \mathcal{H}$, $u_0 \in (F_{x_0})_{a(x_0)}$, $v_0 \in (S_{x_0})_{b(x_0)}$ and $z_0 \in (T_{x_0})_{c(x_0)}$.*
Step 2. *Let*

$$x_1 = (1 - \alpha)x_0 + \alpha J^H_{\lambda,M(\cdot,z_0)}[\lambda(P(u_0) - f(v_0)) \oplus H(x_0)]. \tag{3.3}$$

Since $u_0 \in (F_{x_0})_{a(x_0)} \in CB(\mathcal{H}), v_0 \in (S_{x_0})_{b(x_0)} \in CB(\mathcal{H})$ and $z_0 \in (T_{x_0})_{c(x_0)} \in CB(\mathcal{H})$, by Nadler's theorem [24], there exists $u_1 \in (F_{x_1})_{a(x_1)}$, $v_1 \in (S_{x_1})_{b(x_1)}$ and $z_1 \in (T_{x_1})_{c(x_1)}$ and using Proposition 2.2, we have

$$\|u_0 \oplus u_1\| \leq \|u_0 - u_1\| \leq (1 + 1)D((F_{x_0})_{a(x_0)}, (F_{x_1})_{a(x_1)}), \tag{3.4}$$
$$\|v_0 \oplus v_1\| \leq \|v_0 - v_1\| \leq (1 + 1)D((S_{x_0})_{b(x_0)}, (S_{x_1})_{b(x_1)}), \tag{3.5}$$

$$\|z_0 \oplus z_1\| \leq \|z_0 - z_1\| \leq (1+1)D((T_{x_0})_{c(x_0)}, (T_{x_1})_{c(x_1)}), \tag{3.6}$$

where D is the Hausdörff metric on $CB(\mathcal{H})$.

Step 3. *For*

$$x_2 = (1-\alpha)x_1 + \alpha J^H_{\lambda,M(\cdot,z_1)}[\lambda(P(u_1) - f(v_1)) \oplus H(x_1)], \tag{3.7}$$

and in a similar manner for $x_3, x_4, \cdots$ etc., continuing the above process inductively, we compute the sequences $\{x_n\}, \{u_n\}, \{v_n\}$ and $\{z_n\}$ by the following iterative scheme:

$$x_{n+1} = (1-\alpha)x_n + \alpha J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_n) - f(v_n)) \oplus H(x_n)]. \tag{3.8}$$

Since $u_{n+1} \in (F_{x_{n+1}})_{a(x_{n+1})}$, $v_{n+1} \in (S_{x_{n+1}})_{b(x_{n+1})}$ and $z_{n+1} \in (T_{x_{n+1}})_{c(x_{n+1})}$ such that

$$\|u_n \oplus u_{n+1}\| \leq \|u_n - u_{n+1}\| \leq (1+(n+1)^{-1})D((F_{x_n})_{a(x_n)}, (F_{x_{n+1}})_{a(x_{n+1})}), \tag{3.9}$$

$$\|v_n \oplus v_{n+1}\| \leq \|v_n - v_{n+1}\| \leq (1+(n+1)^{-1})D((S_{x_n})_{b(x_n)}, (S_{x_{n+1}})_{b(x_{n+1})}), \tag{3.10}$$

$$\|z_n \oplus z_{n+1}\| \leq \|z_n - z_{n+1}\| \leq (1+(n+1)^{-1})D((T_{x_n})_{c(x_n)}, (T_{x_{n+1}})_{c(x_{n+1})}), \tag{3.11}$$

where $\alpha \in [0,1], n = 0, 1, 2, \cdots$.

Step 4. If the sequences $\{x_n\}, \{u_n\}, \{v_n\}$ and $\{z_n\}$ satisfy (3.8), (3.9), (3.10) and (3.11), respectively, to an amount of accuracy, stop. Otherwise, set $n = n+1$ and repeat step 3.

Theorem 3.1. *Let $\mathcal{H}$ be a real ordered Hilbert space and $C \subseteq \mathcal{H}$ be a cone. Let $P, f, H : \mathcal{H} \to \mathcal{H}$ be the single-valued mappings such that P and f are Lipschitz continuous mappings with corresponding constant ξ and η, respectively; H is β-ordered compression mapping with respect to $\tilde{T}$, strongly comparison mapping with respect to $J^H_{\lambda,M(\cdot,z)}$ and β'-ordered compression mapping with respect to $J^H_{\lambda,M(\cdot,z)}$.*

Let $\tilde{F}, \tilde{S}, \tilde{T}$ be the set-valued mappings induced by the fuzzy mappings F, S and T, respectively, such that $\tilde{F}$ is relaxed Lipschitz continuous with respect to P with corresponding constant k, $\tilde{S}$ is relaxed monotone with respect to f with corresponding constant c and $\tilde{F}, \tilde{S}, \tilde{T}$ are D-Lipschitz continuous mappings with corresponding constant h, d and r, respectively. Suppose that $M : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping such that M is θ-ordered rectangular mapping with respect to $\tilde{T}$ and ORSM mapping with respect to $J^H_{\lambda,M(\cdot,z)}$, that is, it is θ'-ordered rectangular mapping with respect to $J^H_{\lambda,M(\cdot,z)}$, λ-XOR ordered strongly monotone mapping with respect to $J^H_{\lambda,M(\cdot,z)}$. If $x_{n+1} \propto x_n$; $(x_n, z_n) \oplus (x_{n-1}, z_n) \propto (x_n \oplus x_{n-1})$; $0 \propto \left[H\left(J^H_{\lambda,M(\cdot,z)}(x)\right) \oplus H\left(J^H_{\lambda,M(\cdot,z)}(y)\right)\right]$; $\left(J^H_{\lambda,M(\cdot,z)}(x), J^H_{\lambda,M(\cdot,z)}(z)\right) \oplus \left(J^H_{\lambda,M(\cdot,z)}(y), J^H_{\lambda,M(\cdot,z)}(z)\right) \propto J^H_{\lambda,M(\cdot,z)}(x) \oplus J^H_{\lambda,M(\cdot,z)}(y)$ and the following conditions are satisfied:

$$\left\|J^H_{\lambda,M(\cdot,z_n)}(x) \oplus J^H_{\lambda,M(\cdot,z_{n-1})}(x)\right\| \leq \mu\|z_n \oplus z_{n-1}\|, \tag{3.12}$$

where $x \in \mathcal{H}$, $z_n \in \tilde{T}(x_n)$, $z_{n-1} \in \tilde{T}(x_{n-1})$ and $\mu > 0$, and

$$\left|\lambda - \frac{2(k-c)}{(\xi h + \eta d)^2}\right| < \frac{\sqrt{4(k-c)^2 - (\xi h + \eta d)^2[(1-\mu\gamma)(\alpha\theta' - \beta') - \beta](2+\beta) - (1-\mu\gamma)(\alpha\theta' - \beta')}}{(\xi h + \eta d)^2}, \tag{3.13}$$

where $k > c$, $\mu\gamma < 1$, $\alpha\theta' > \beta'$ *and* $(1-\mu\gamma)(\alpha\theta'-\beta') > \beta$.

Then, the set-valued variational inclusion problem with fuzzy mappings involving XOR-operation (3.1) *admits a solution* $x \in \mathcal{H}, u \in (F_x)_{a(x)}, v \in (S_x)_{b(x)}, z \in (T_x)_{c(x)}$ *and the iterative sequences* $\{x_n\}, \{u_n\}, \{v_n\}$ *and* $\{z_n\}$ *generated by Algorithm 3.1 converge strongly to* x, u, v *and* z*, respectively, the solution of set-valued variational inclusion problem with fuzzy mappings involving XOR-operation* (3.1).

Proof. Using Algorithm 3.1, Proposition 2.2, Lipschitz-type continuity of the resolvent operator $J^H_{\lambda,M(\cdot,z)}$ and condition (3.12), we evaluate

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &= \Big\| \Big[(1-\alpha)x_n + \alpha J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_n) - f(v_n)) \oplus H(x_n)]\Big] \\
&\quad \oplus \Big[(1-\alpha)x_{n-1} + \alpha J^H_{\lambda,M(\cdot,z_{n-1})}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})]\Big] \Big\| \\
&= \Big\|(1-\alpha)(x_n \oplus x_{n-1}) + \alpha J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_n) - f(v_n)) \oplus H(x_n)] \\
&\quad \oplus \alpha J^H_{\lambda,M(\cdot,z_{n-1})}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \Big\| \\
&= \Big\|(1-\alpha)(x_n \oplus x_{n-1}) + \Big(\alpha J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_n) - f(v_n)) \oplus H(x_n)] \\
&\quad \oplus \alpha J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \Big) \\
&\quad \oplus \Big(\alpha J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \\
&\quad \oplus \alpha J^H_{\lambda,M(\cdot,z_{n-1})}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \Big)\Big\| \\
&\leq \Big\|(1-\alpha)(x_n \oplus x_{n-1}) + \Big(\alpha J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_n) - f(v_n)) \oplus H(x_n)] \\
&\quad \oplus \alpha J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \Big) \\
&\quad - \Big(\alpha J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \\
&\quad \oplus \alpha J^H_{\lambda,M(\cdot,z_{n-1})}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \Big)\Big\| \\
&\leq (1-\alpha)\|x_n \oplus x_{n-1}\| + \alpha \Big\| J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_n) - f(v_n)) \oplus H(x_n)] \\
&\quad \oplus J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \Big\| \\
&\quad + \alpha \Big\| J^H_{\lambda,M(\cdot,z_n)}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \\
&\quad \oplus J^H_{\lambda,M(\cdot,z_{n-1})}[\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \Big\| \\
&\leq (1-\alpha)\|x_n \oplus x_{n-1}\| + \alpha \left(\frac{1}{\lambda\theta' - \beta'}\right) \Big\| [\lambda(P(u_n) - f(v_n)) \oplus H(x_n)] \\
&\quad \oplus [\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \Big\| + \alpha\mu\|z_n \oplus z_{n-1}\| \\
&\leq (1-\alpha)\|x_n \oplus x_{n-1}\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right) \Big\| [\lambda(P(u_n) - f(v_n)) \oplus H(x_n)] \\
&\quad - [\lambda(P(u_{n-1}) - f(v_{n-1})) \oplus H(x_{n-1})] \Big\| + \alpha\mu\|z_n \oplus z_{n-1}\| \\
&= (1-\alpha)\|x_n \oplus x_{n-1}\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right) \Big\| [\lambda(P(u_n) - f(v_n)) - \lambda(P(u_{n-1}) - f(v_{n-1}))]
\end{aligned}$$

$$
\begin{aligned}
& \oplus H(x_n) \oplus H(x_{n-1})\Big\| + \alpha\mu\|z_n \oplus z_{n-1}\| \\
\leq\ & (1-\alpha)\|x_n \oplus x_{n-1}\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\Big\| [\lambda(P(u_n) - P(u_{n-1})) - \lambda(f(v_n) - f(v_{n-1}))] \\
& - [H(x_n) \oplus H(x_{n-1})]\Big\| + \alpha\mu\|z_n \oplus z_{n-1}\| \\
\leq\ & (1-\alpha)\|x_n \oplus x_{n-1}\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\Big\|\lambda(P(u_n) - P(u_{n-1})) - \lambda(f(v_n) - f(v_{n-1}))\Big\| \\
& + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\Big\|H(x_n) \oplus H(x_{n-1})\Big\| + \alpha\mu\|z_n \oplus z_{n-1}\| \\
\leq\ & (1-\alpha)\|x_n \oplus x_{n-1}\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\Big\|(x_n - x_{n-1}) + \lambda(P(u_n) - P(u_{n-1})) - \lambda(f(v_n) - f(v_{n-1})) \\
& -(x_n - x_{n-1})\Big\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\Big\|H(x_n) \oplus H(x_{n-1})\Big\| + \alpha\mu\|z_n \oplus z_{n-1}\| \\
\leq\ & (1-\alpha)\|x_n \oplus x_{n-1}\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\Big\|x_n - x_{n-1} + \lambda(P(u_n) - P(u_{n-1})) - \lambda(f(v_n) - f(v_{n-1}))\Big\| \\
& + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\Big\|x_n - x_{n-1}\Big\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\Big\|H(x_n) \oplus H(x_{n-1})\Big\| + \alpha\mu\|z_n \oplus z_{n-1}\|. \qquad (3.14)
\end{aligned}
$$

Since $\tilde{F}$ and $\tilde{S}$ are D-Lipschitz continuous, P and f are Lipschitz continuous and using (3.9) and (3.10), we have

$$
\begin{aligned}
\|P(u_n) - P(u_{n-1})\| &\leq \xi\|u_n - u_{n-1}\| \leq \xi\left(1 + n^{-1}\right) D((F_{x_n})_{a(x_n)}, (F_{x_{n-1}})_{a(x_{n-1})}) \\
&\leq \xi h\left(1 + n^{-1}\right)\|x_n - x_{n-1}\|, \qquad (3.15) \\
\|f(v_n) - f(v_{n-1})\| &\leq \eta\|v_n - v_{n-1}\| \leq \eta\left(1 + n^{-1}\right) D((S_{x_n})_{b(x_n)}, (S_{x_{n-1}})_{b(x_{n-1})}) \\
&\leq \eta d\left(1 + n^{-1}\right)\|x_n - x_{n-1}\|. \qquad (3.16)
\end{aligned}
$$

Further, since $\tilde{F}$ is relaxed Lipschitz continuous with respect to P and $\tilde{S}$ is relaxed monotone with respect to f and using (3.15), (3.16), we have

$$
\begin{aligned}
& \|x_n - x_{n-1} + \lambda(P(u_n) - P(u_{n-1})) - \lambda(f(v_n) - f(v_{n-1}))\|^2 \\
=\ & \|x_n - x_{n-1}\|^2 + 2\lambda\langle P(u_n) - P(u_{n-1}), x_n - x_{n-1}\rangle - 2\lambda\langle f(v_n) - f(v_{n-1}), x_n - x_{n-1}\rangle \\
& + \lambda^2\|(P(u_n) - P(u_{n-1})) - (f(v_n) - f(v_{n-1}))\| \\
\leq\ & [1 - 2\lambda(k-c) + \lambda^2\left(1 + n^{-1}\right)^2(\xi h + \eta d)^2]\|x_n - x_{n-1}\|^2. \qquad (3.17)
\end{aligned}
$$

As H is β-ordered compression mapping with respect to $\tilde{T}$ and $[(x_n, z_n) \oplus (x_n, z_n)] \propto (x_n \oplus x_{n-1})$, we have

$$
\begin{aligned}
\|H(x_n) \oplus H(x_{n-1})\| &\leq \beta\|(x_n, z_n) \oplus (x_{n-1}, z_n)\|, \\
&\leq \beta\|x_n \oplus x_{n-1}\|, \qquad (3.18)
\end{aligned}
$$

for all $x_n, y_n \in \mathcal{H}$, and $z_n \in \tilde{T}(x_n)_{c(x_n)}$.

As $\tilde{T}$ is D-Lipschitz continuous and using (3.11), we have

$$\begin{aligned} \|z_n \oplus z_{n-1}\| &\leq \|z_n - z_{n-1}\| \leq \left(1+n^{-1}\right) D((T_{x_n})_{c(x_n)}, (T_{x_{n-1}})_{c(x_{n-1})}) \\ &\leq \left(1+n^{-1}\right) r\|x_n - x_{n-1}\|. \end{aligned} \tag{3.19}$$

Using (3.17), (3.18) and (3.19), (3.14) becomes

$$\begin{aligned} \|x_{n+1} \oplus x_n\| &\leq (1-\alpha)\|x_n \oplus x_{n-1}\| \\ &+ \left(\frac{\alpha}{\lambda\theta' - \beta'}\right) \sqrt{\left[1 - 2\lambda(k-c) + \lambda^2 \left(1+n^{-1}\right)^2 (\xi h + \eta d)^2\right]} \|x_n - x_{n-1}\| \\ &+ \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\|x_n - x_{n-1}\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\beta\|x_n \oplus x_{n-1}\| \\ &+ \alpha\mu\left(1+n^{-1}\right) r\|x_n - x_{n-1}\|. \end{aligned} \tag{3.20}$$

Since, $x_{n+1} \propto x_n$, for all $n = 1, 2, 3, \cdots$, from (3.20), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1-\alpha)\|x_n - x_{n-1}\| \\ &+ \left(\frac{\alpha}{\lambda\theta' - \beta'}\right) \sqrt{\left[1 - 2\lambda(k-c) + \lambda^2 \left(1+n^{-1}\right)^2 (\xi h + \eta d)^2\right]} \|x_n - x_{n-1}\| \\ &+ \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\|x_n - x_{n-1}\| + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\beta\|x_n - x_{n-1}\| \\ &+ \alpha\mu\left(1+n^{-1}\right) r\|x_n - x_{n-1}\|. \end{aligned}$$

Thus

$$\|x_{n+1} - x_n\| \leq \sigma_n(\theta)\|x_n - x_{n-1}\|,$$

where

$$\begin{aligned} \sigma_n(\theta) &= (1-\alpha) + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right) \sqrt{\left[1 - 2\lambda(k-c) + \lambda^2 \left(1+n^{-1}\right)^2 (\xi h + \eta d)^2\right]} \\ &+ \left(\frac{\alpha}{\lambda\theta' - \beta'}\right) + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\beta + \alpha\mu\left(1+n^{-1}\right) r. \end{aligned}$$

Let

$$\begin{aligned} \sigma(\theta) &= (1-\alpha) + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right) \sqrt{[1 - 2\lambda(k-c) + \lambda^2(\xi h + \eta d)^2]} \\ &+ \left(\frac{\alpha}{\lambda\theta' - \beta'}\right) + \left(\frac{\alpha}{\lambda\theta' - \beta'}\right)\beta + \alpha\mu r. \end{aligned}$$

By condition (3.13), it follows that $0 < \sigma(\theta) < 1$, thus $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}$ and since $\mathcal{H}$ is complete, there exists an $x \in \mathcal{H}$, such that $x_n \to x$ as $n \to \infty$.

It is clear from step 3 of Algorithm 3.1 and D-Lipschitz continuity of $\tilde{F}$, $\tilde{S}$ and $\tilde{T}$ that $\{u_n\}, \{v_n\}$ and $\{z_n\}$ are also Cauchy sequences in $\mathcal{H}$, thus, there exist u, v and z in $\mathcal{H}$ such that $u_n \to u$, $v_n \to v$ and $z_n \to z$, as $n \to \infty$. By using the techniques of Ahmad et al. [6], one can show that $u \in (F_x)_{a(x)}$, $v \in (S_x)_{b(x)}$ and $z \in (T_x)_{c(x)}$. By Lemma 3.1, we conclude that (x, u, v, z) is a solution of set-valued variational inclusion problem with fuzzy mappings involving XOR-operation (3.1). □

4. Conclusion

Variational inclusions are useful to study many problems related to DC programming, prox-regularity, multicommodity network, image restoring processing, optimization etc..

Zadeh's possibility and fuzzy set theory is an extension of the usual model semantics. Necessity is not a duplicate of possibility, when we know the possibility of an event, we can not directly deduce its necessity. Possibility and necessity are clearly distinct from probability. Fuzzy sets and fuzzy logic are useful mathematical tools for modeling many real-world problems.
On the other hand, XOR-operation has wide applications as we had discussed in introduction section.

Keeping in mind the interesting application of all the above discussed concepts, in this paper, we introduce and solve a set-valued variational inclusion problem with fuzzy mappings involving XOR-operation. Further, we remark that our problem may be studied in higher-dimensional spaces.

Acknowledgments

The authors are highly thankful to anonymous referees for their valuable suggestions and comments which improve the manuscript a lot.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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