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Research article

Global dynamic analysis of periodic solution for discrete-time inertial neural networks with delays

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Abstract: This paper is devoted to studying global dynamic behaviours of periodic solutions of discrete-time inertial neural networks with delays by applying Mawhin's continuation theorem and some innovative mathematical analysis techniques. Finally, an numerical example is given to illustrate our theoretical results.

Keywords: periodic solution; discrete-time; inertial neural networks; stability
Mathematics Subject Classification:34C25, 34C13

1. Introduction

Inertial neural networks (INNs) was firstly introduced by Wheeler and Schieve [1] in 1997. After that, lots of results for INNs have been gained. Jian and Duan [2] considered the finite-time synchronization for fuzzy neutral-type inertial neural networks with time-varying coefficients and proportional delays. Some novel delay-independent criteria about finite-time synchronization were obtained by using finite-time stability theory and combining with inequality techniques and some analysis methods. Long etc. [3] investigated finite-time stabilization of state-based switched chaotic inertial neural networks with distributed delays by the theory of finite-time control and non-smooth analysis. In [4], the global exponential stabilization (GES) of inertial memristive neural networks with discrete and distributed time-varying delays was studied. Using the generalized Halanay inequality, matrix measure and matrix-norm inequality, the authors [5] investigated the global dissipativity for INNs with delays and parameter uncertainties. For more research contents about INNs, see e.g. [6, 7, 8, 9, 10] and related references.

Recent years, periodic solution problems of INNs have been studied by some authors. Aouiti etc. [11] studied the exponential stability of piecewise pseudo almost periodic solutions for neutral-type inertial neural networks with mixed delays and impulses by using inequality techniques and Lyapunov method. Huang and Zhang [12] considered a class of non-autonomous inertial neural networks with proportional delays and time-varying coefficients by combining Lyapunov function method with differential inequality approach. For more results of periodic solutions of neural network systems, see e. g. [13, 14, 15, 16, 17, 18, 19, 20].

Classic INNs with multiple time-varying delays which can be described by

$$\frac{dx_i^2(t)}{dt^2} = -a_i(t)\frac{dx_i(t)}{dt} - b_i(t)x_i(t) + \sum_{j=1}^{n} c_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} d_{ij}(t)f_j(x_j(t-\tau_j(t))) + I_i(t), \tag{1.1}$$

where $t \geq 0$, $i = 1, \cdots, n$, $x_i(t)$ denotes the state of ith neuron at time t, $a_i(t) > 0$ is the damping coefficient, $b_i(t) > 0$ denotes the strength of different neuron at time t, $c_{ij}(t)$ and $d_{ij}(t)$ are the neuron connection weights at time t, $f_j(\cdot)$ is the activation function which is a continuous function, $\tau_j(t)$ is a delay function, $I_i(t)$ is an external input of ith neuron at time t. For system (1.1) and its generalization, there exist lots of results, see e.g. [21, 22]

To our best knowledge, there are few results reported on the research of discrete-time INNs with multiple time-varying delays. Motivated by the above work, in this paper, we study the periodic solutions problem for a discrete-time inertial neural networks with multiple time-varying delays as follows:

$$\begin{aligned} \Delta^2 x_i(n) = &-a_i(n)\Delta x_i(n) - b_i(n)x_i(n) \\ &+ \sum_{j=1}^{m} c_{ij}(n)f_j(x_j(n)) + \sum_{j=1}^{m} d_{ij}(n)f_j(x_j(n-\tau_j(n))) + I_i(n) \end{aligned} \tag{1.2}$$

which initial conditions are given by

$$\begin{cases} x_i(s) = \phi_i(s), & s \in (-\tau, 0]_{\mathbb{Z}}, \\ \Delta x_i(s) = \psi_i(s), & s \in (-\tau, 0]_{\mathbb{Z}}, \end{cases} \tag{1.3}$$

where τ is defined by (1.4), $n \in \mathbb{Z}_0^+ = \{n \in \mathbb{Z} : n \geq 0\}$, $i = 1, 2, \cdots, m$, $a_i(n) > 0$ is a $N-$periodic function, $\tau_j(n)$ is non-negative $N-$periodic function, $b_i(n), c_{ij}(n), d_{ij}(n)$ and $I_i(n)$ are $N-$periodic functions. Let

$$\tau = \max_{n \in I_N}\{\tau_j(n),\ j = 1, 2, \cdots, m\},\ I_N = \{0, 1, 2, \cdots, N-1\}, \tag{1.4}$$

N is a positive integer. For a periodic function $f(n)$ on $\mathbb{Z}_0^+$, let

$$f^- = \min_{n \in I_N}\{|f(n)|\}, \quad f^+ = \max_{n \in I_N}\{|f(n)|\}.$$

Denote

$$[a, b]_{\mathbb{Z}} = \{a, a+1, \cdots, b-1, b\} \text{ for } a, b \in \mathbb{Z} \text{ and } a \leq b.$$

The highlights of this paper are threefold:

(1) The discrete-time delayed INNs as shown in system (1.2) is established, which is different from the existing continuous INNs, see e.g. [1, 2, 7, 8].

(2) For discrete-time INNs, Lyapunov-Krasovskii functional is no longer applicable for studying stability problems. In this paper, we develop innovative mathematical analysis for the stability of

discrete-time INNs.
(3) Discretization is needed in the implementation of continuous-time neural networks. Hence, the research of discrete-time INNs has important theoretical and practical values.

The following sections are organized as follows: In Section 2, sufficient conditions are established for existence and uniqueness of periodic solution to system (1.2). The exponential stability is given in Sections 3. In Section 4, an numerical example is given to show the feasibility of our results. Finally, some conclusions and discussions are given about this paper.

2. Existence and uniqueness of periodic solution

In this section , we need the following assumptions.
(H_1) There exists non-negative constant p_j such that

$$|f_j(x_j)| \leq p_j,\ \ j = 1, 2, \cdots, m.$$

(H_2) There exist non-negative constants q_j and e_j such that

$$|f_j(x_j)| \leq q_j|x_j| + e_j,\ \ j = 1, 2, \cdots, m.$$

(H_3) There exists constants $L_j \geq 0$ such that

$$|f_j(x) - f_j(y)| \leq L_j|x - y|,\ \ j = 1, 2, \cdots, m,\ \forall x, y \in \mathbb{R}.$$

Lemma 2.1 [23] Assume that $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces, and $L : D(L) \subset \mathcal{X} \rightarrow \mathcal{Y}$, is a Fredholm operator with index zero. Furthermore, $\Omega \subset \mathcal{X}$ is an open bounded set and $N : \bar{\Omega} \rightarrow \mathcal{Y}$ is L-compact on $\bar{\Omega}$. if all the following conditions hold:

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \forall \lambda \in (0, 1)$,
(2) $Nx \notin ImL, \forall x \in \partial\Omega \cap KerL$,
(3) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$,

where $J : ImQ \rightarrow KerL$ is an isomorphism. Then equation $Lx = Nx$ has a solution on $\bar{\Omega} \cap D(L)$. Let

$$y_i(n) = \Delta x_i(n) + \xi_i x_i(n),\ \ i = 1, 2, \cdots, m, \tag{2.1}$$

where $\xi_i > 0$ is a constant. Then system (1.2) is changed into the following system:

$$\begin{cases} \Delta x_i(n) = -\xi_i x_i(n) + y_i(n), \\ \Delta y_i(n) = -(a_i(n) - \xi_i)y_i(n) + [(a_i(n) - \xi_i)\xi_i - b_i(n)]x_i(n) \\ + \sum_{j=1}^{m} c_{ij}(n)f_j(x_j(n)) + \sum_{j=1}^{m} d_{ij}(n)f_j(x_j(n - \tau_j(n))) + I_i(n). \end{cases} \tag{2.2}$$

Theorem 2.1 Suppose that assumption (H_1) holds. Then system (1.2) has at least one N−periodic solution, provide that the following conditions hold:

$$\xi_i < 1,$$
$$1 - (a_i^+ - \xi_i) > 0,\ \ a_i^- - \xi_i > 0,$$

$$(a_i^- - \xi_i)\xi_i - [(a_i(n) - \xi_i)\xi_i - b_i(n)]^+ > 0,$$

$$\xi_i\widetilde{m} \neq \pm(M+1) \text{ or } \xi_i(M+1) \neq \pm\widetilde{m},$$

where $i = 1, 2, \cdots, m$, M and $\widetilde{m}$ are defined by (2.15) and (2.16).

Proof Let

$$l_{2m} = \{w(n) = (w_1(n), w_2(n), \cdots, w_{2m}(n))^\top \in \mathbb{R}^{2m},\ n \in \mathbb{Z}\}.$$

Let

$$l_N = \{w(n) \in l_{2m} :\ \ w(n+N) = w(n),\ n \in \mathbb{Z},\ N \in \mathbb{Z}^+\}$$

equipped with the norm

$$\|w\| = \max_{n \in I_N} |w_i(n)|,\ \ w \in l_N,\ i = 1, 2, \cdots, 2m.$$

Then l_N is a Banach space. Let

$$l_N^0 = \{y(n) \in l_N : \sum_{n=0}^{N-1} y(n) = 0\},\ \ l_N^c = \{x(n) \in l_N : x(n) = \text{constant},\ n \in \mathbb{Z}\}.$$

Obviously, l_N^0 and l_N^c are both closed linear subspaces of l_N, and $l_N = l_N^0 \oplus l_N^c$, $\dim l_N^c = 2m$. Define a linear operator

$$L : D(L) \subset l_N \to l_N,\ (Lw)(n) = \Delta w(n) = (\Delta x(n), \Delta y(n))^\top,\ n \in \mathbb{Z}_0^+,$$

$$(Lw)_i(n) = \Delta x_i(n),\ i = 1, 2, \cdots, m,\ n \in \mathbb{Z}_0^+, \tag{2.3}$$

and

$$(Lw)_{m+i}(n) = \Delta y_i(n),\ i = 1, 2, \cdots, m,\ n \in \mathbb{Z}_0^+. \tag{2.4}$$

Let $N : l_N \to l_N$ with

$$(Nw)_i(n) = -\xi_i x_i(n) + y_i(n)\ i = 1, 2, \cdots, m,\ n \in \mathbb{Z}_0^+, \tag{2.5}$$

and

$$\begin{aligned}(Nw)_{m+i}(n) = &-(a_i(n) - \xi_i)y_i(n) + [(a_i(n) - \xi_i)\xi_i - b_i(n)]x_i(n) \\ &+ \sum_{j=1}^{m} c_{ij}(n) f_j(x_j(n)) + \sum_{j=1}^{m} d_{ij}(n) f_j(x_j(n - \tau_j(n))) + I_i(n),\ i = 1, 2, \cdots, m,\ n \in \mathbb{Z}_0^+.\end{aligned} \tag{2.6}$$

Then, $KerL = l_N^c$ and $ImL = l_N^0$. Hence, L is a Fredholm mapping of index zero. Define continuous projectors $P,\ Q$ by

$$P : l_N \to \text{Ker}L,\ (Pw)(n) = \frac{1}{N}\sum_{n=0}^{N-1} w(n)$$

and

$$Q : l_N \to l_N/\text{Im}L,\ Qw = \frac{1}{N}\sum_{n=0}^{N-1} w(n).$$

Let

$$L_P = L|_{D(L)\cap \text{Ker}P} : D(L) \cap \text{Ker}P \to \text{Im}L,$$

then

$$L_P^{-1} = K_p : \text{Im}L \to D(L) \cap \text{Ker}P.$$

Since $\text{Im}L \subset l_N$ and $D(L) \cap \text{Ker}P \subset l_N$, then K_p is an embedding operator and is a completely operator in $\text{Im}L$. Let $\Omega \subset l_N$. In view of the definitions of Q and N, we know that $QN(\bar{\Omega})$ is bounded on $\bar{\Omega}$. Hence nonlinear operator N is L-compact on $\overline{\Omega}$. Let

$$\Omega_1 = \{w : w \in D(L),\ Lw = \lambda Nw,\ \lambda \in (0, 1)\},$$

where L and N are defined by (2.3)-(2.6). $\forall x \in \Omega_1$, it follows that

$$\Delta x_i(n) = \lambda[-\xi_i x_i(n) + y_i(n)], \tag{2.7}$$

$$\begin{aligned}\Delta y_i(n) = \lambda\Big[&- (a_i(n) - \xi_i)y_i(n) + [(a_i(n) - \xi_i)\xi_i - b_i(n)]x_i(n) \\ &+ \sum_{j=1}^{m} c_{ij}(n) f_j(x_j(n)) + \sum_{j=1}^{m} d_{ij}(n) f_j(x_j(n - \tau_j(n))) + I_i(n)\Big].\end{aligned} \tag{2.8}$$

By (2.7), we have

$$x_i(n + 1) - x_i(n) = \lambda[-\xi_i x_i(n) + y_i(n)]$$

and

$$x_i(n + 1) = x_i(n) + \lambda[-\xi_i x_i(n) + y_i(n)].$$

Using $\xi_i < 1$, we gain

$$\begin{aligned}\max_{n\in I_N} |x_i(n)| &= \max_{n\in I_N} |x_i(n + 1)| \\ &\le (1 - \lambda\xi_i) \max_{n\in I_N} |x_i(n)| + \lambda \max_{n\in I_N} |y_i(n)|,\end{aligned}$$

i.e.,

$$\max_{n\in I_N} |x_i(n)| \le \frac{1}{\xi_i} \max_{n\in I_N} |y_i(n)|. \tag{2.9}$$

By (2.8), we have

$$\begin{aligned}y_i(n + 1) - y_i(n) = \lambda\Big[&- (a_i(n) - \xi_i)y_i(n) + [(a_i(n) - \xi_i)\xi_i - b_i(n)]x_i(n) \\ &+ \sum_{j=1}^{m} c_{ij}(n) f_j(x_j(n)) + \sum_{j=1}^{m} d_{ij}(n) f_j(x_j(n - \tau_j(n))) + I_i(n)\Big]\end{aligned}$$

and

$$\begin{aligned}y_i(n + 1) = y_i(n) + \lambda\Big[&- (a_i(n) - \xi_i)y_i(n) + [(a_i(n) - \xi_i)\xi_i - b_i(n)]x_i(n) \\ &+ \sum_{j=1}^{m} c_{ij}(n) f_j(x_j(n)) + \sum_{j=1}^{m} d_{ij}(n) f_j(x_j(n - \tau_j(n))) + I_i(n)\Big].\end{aligned}$$

Using $1-(a_i^+-\xi_i)>0$ and assumption (H$_1$), we have

$$\begin{aligned}\max_{n\in I_N}|y_i(n)| &= \max_{n\in I_N}|y_i(n+1)|\\ &\le [1-\lambda(a_i(n)-\xi_i)]\max_{n\in I_N}|y_i(n)|+\lambda[(a_i(n)-\xi_i)\xi_i-b_i(n)]^+\max_{n\in I_N}|x_i(n)|\\ &+\lambda\sum_{j=1}^{m}c_{ij}^+p_j+\lambda\sum_{j=1}^{m}d_{ij}^+p_j+\lambda I_i^+.\end{aligned}\tag{2.10}$$

From $a_i^- - \xi_i > 0$ and (2.10), we have

$$\max_{n\in I_N}|y_i(n)|\le\frac{[(a_i(n)-\xi_i)\xi_i-b_i(n)]^+}{a_i^- -\xi_i}\max_{n\in I_N}|x_i(n)|+\frac{\sum_{j=1}^m c_{ij}^+p_j+\sum_{j=1}^m d_{ij}^+p_j+I_i^+}{a_i^- -\xi_i}.\tag{2.11}$$

From $(a_i^- -\xi_i)\xi_i-[(a_i(n)-\xi_i)\xi_i-b_i(n)]^+>0$, (2.9) and (2.11), we gain

$$\max_{n\in I_N}|y_i(n)|\le\frac{\xi_i(\sum_{j=1}^m c_{ij}^+p_j+\sum_{j=1}^m d_{ij}^+p_j+I_i^+)}{(a_i^- -\xi_i)\xi_i-[(a_i(n)-\xi_i)\xi_i-b_i(n)]^+}:=B_i.\tag{2.12}$$

By (2.9) and (2.12), we get

$$\max_{n\in I_N}|x_i(n)|\le\frac{B_i}{\xi_i}:=A_i.\tag{2.13}$$

Hence, Ω_1 is a bounded set and condition (1) of Lemma 2.1 holds. In view of (2.12) and (2.13), let

$$\|w\|=\max\left\{\max_{i=1,\cdots,m}A_i,\ \max_{i=1,\cdots,m}B_i\right\}:=M.$$

Let $\Omega_2=\{w\in l_N:\|w\|<M+1\}$. We claim that

$$QNw\neq\mathbf{0}\quad\forall w\in\partial\Omega_2\cap KerL.\tag{2.14}$$

Assume that (2.14) does not hold. In fact, $\forall w\in\partial\Omega_2\cap KerL$, then $w\in\mathbb{R}^{2m}$ is a constant vector, and there exists at least one $i\in\{1,2,\cdots,m\}$ such that

$$|y_i|=M+1\text{ and }|x_i|=\widetilde{m}<M+1\tag{2.15}$$

or

$$|x_i|=M+1\text{ and }|y_i|=\widetilde{m}<M+1\tag{2.16}$$

Case 1: If (2.15) holds, let $y_i=M+1,\ x_i=\pm\widetilde{m}$, then by (2.5),

$$QNw_i=\frac{1}{N}(\pm\xi_i\widetilde{m}+M+1)=0,$$

i.e.,

$$\xi_i\widetilde{m}=\pm(M+1)$$

which is contract to $\xi_i\widetilde{m}\neq\pm(M+1)$. If $y_i=-(M+1),\ x_i=\pm\widetilde{m}$, we also obtain the similar contraction.
Case 2: If (2.16) holds, let $y_i=\widetilde{m},\ x_i=\pm(M+1)$, then by (2.5),

$$QNw_i=\frac{1}{N}(\pm\xi_i(M+1)+\widetilde{m})=0,$$

i.e.,

$$\xi_i(M+1) = \pm\widetilde{m}$$

which is contract to $\xi_i(M+1) \neq \pm\widetilde{m}$. If $y_i = -\widetilde{m},\ x_i = \pm(M+1)$, we also obtain the similar contraction. Hence, condition (2) of Lemma 2.1 holds. We will show that condition (3) of Lemma 2.1 holds. Take the homotopy

$$H(w,\mu) = -\mu w + (1-\mu)QNw,\ w \in \overline{\Omega_2} \cap KerL,\ \mu \in [0,1].$$

We claim $H(w,\mu) \neq \mathbf{0}$ for all $w \in \partial\Omega_2 \cap KerL$. If this is not true, then

$$\mu x_i = \frac{1-\mu}{N}\Sigma_{n=0}^{N-1}\Big[-\xi_i x_i(n) + y_i(n)\Big] \tag{2.17}$$

and

$$\begin{aligned}\mu y_i = \frac{1-\mu}{N}\Sigma_{n=0}^{N-1}\Big[&-(a_i(n)-\xi_i)y_i(n) + [(a_i(n)-\xi_i)\xi_i - b_i(n)]x_i(n)\\ &+\sum_{j=1}^{m} c_{ij}(n)f_j(x_j(n)) + \sum_{j=1}^{m} d_{ij}(n)f_j(x_j(n-\tau_j(n))) + I_i(n)\Big].\end{aligned} \tag{2.18}$$

By (2.17), we have

$$x_i = \frac{1-\mu}{\mu+(1-\mu)\xi_i}y_i.$$

Thus,

$$|x_i| \leq \frac{1}{\xi_i}|y_i|. \tag{2.19}$$

In view of (2.18) and (2.19), we have

$$\Big[\mu\xi_i + (1-\mu)\xi_i(a_i-\xi_i) - (1-\mu)\xi_i[(a_i(n)-\xi_i)\xi_i - b_i(n)]^+\Big]\mu|y_i| \leq (1-\mu)\sum_{j=1}^{m}[b_{ij}^+p_j + d_{ij}^+p_j + I_i^+]$$

and

$$\begin{aligned}\max_{n\in I_N}|y_i(n)| &\leq \frac{\xi_i(\sum_{j=1}^{m} c_{ij}^+p_j + \sum_{j=1}^{m} d_{ij}^+p_j + I_i^+)}{(a_i^- - \xi_i)\xi_i - [(a_i(n)-\xi_i)\xi_i - b_i(n)]^+}\\ &= B_i < M+1\end{aligned}$$

which is a contradiction. And then by the degree theory,

$$\begin{aligned}deg\{QN,\Omega_2\cap KerL,0\} &= deg\{H(\cdot,0),\Omega_2\cap KerL,0\}\\ &= deg\{H(\cdot,1),\Omega_2\cap KerL,0\}\\ &= deg\{-I,\Omega_2\cap KerL,0\} \neq 0.\end{aligned}$$

Applying Lemma 2.1, we reach the conclusion.

Theorem 2.2 Suppose that assumption (H_2) holds. Then system (1.2) has at least one $N-$periodic solution, provide that the following conditions hold:

$$\xi_i < 1,\ \ i = 1,2,\cdots,m,$$

$$1-(a_i^+ - \xi_i) > 0,\ \ i = 1,2,\cdots,m,$$

$$a_i^- - \xi_i - \frac{[(a_i(n)-\xi_i)\xi_i - b_i(n)]^+}{\xi_i} > 0, \quad i = 1, 2, \cdots, m,$$

$$\rho_1 - \frac{m\rho_2}{\check{\xi}} > 0,$$

where $\check{\xi} = \min_{i=1,2,\cdots,m} \xi_i$,

$$\rho_1 = \min_{i=1,2,\cdots,m}\left[a_i^- - \xi_i - \frac{[(a_i(n)-\xi_i)\xi_i - b_i(n)]^+}{\xi_i}\right] > 0,\ \rho_2 = \max_{i,j=1,2,\cdots,m}(c_{ij}^+ + d_{ij}^+)q_j,$$

$$\xi_i\widetilde{m} \neq \pm(M+1) \text{ or } \xi_i(M+1) \neq \pm\widetilde{m}, \quad i = 1, 2, \cdots, m,$$

where M is defined by (2.25), $\widetilde{m} < M$ is a positive constant.

Proof We only prove that Ω_1 is bounded, other proofs are similar to the proofs of Theorem 2.1. In fact, $\forall w \in \Omega_1$, by $1 - (a_i^+ - \xi_i) > 0$, by assumption (H_2) we have

$$\begin{aligned}\max_{n\in I_N}|y_i(n)| &= \max_{n\in I_N}|y_i(n+1)| \\ &\le [1 - \lambda(a_i(n)-\xi_i)]\max_{n\in I_N}|y_i(n)| + \lambda[(a_i(n)-\xi_i)\xi_i - b_i(n)]^+\max_{n\in I_N}|x_i(n)| \\ &+ \lambda\sum_{j=1}^m(c_{ij}^+ + d_{ij}^+)q_j|x_j| + \lambda\sum_{j=1}^m(c_{ij}^+ + d_{ij}^+)q_je_j + \lambda\sum_{j=1}^m d_{ij}^+q_j|\phi_j| + \lambda I_i^+.\end{aligned} \tag{2.20}$$

From (2.20) and $a_i^- - \xi_i - \frac{[(a_i(n)-\xi_i)\xi_i - b_i(n)]^+}{\xi_i} > 0$, we get

$$\rho_1\max_{n\in I_N}|y_i(n)| \le \rho_2\sum_{j=1}^m|x_j| + \sum_{j=1}^m(c_{ij}^+ + d_{ij}^+)q_je_j + \sum_{j=1}^m d_{ij}^+q_j|\phi_j| + I_i^+$$

and

$$\rho_1\sum_{i=1}^m\max_{n\in I_N}|y_i(n)| \le m\rho_2\sum_{i=1}^m\max_{n\in I_N}|x_i| + \sum_{i=1}^m\sum_{j=1}^m(c_{ij}^+ + d_{ij}^+)q_je_j + \sum_{i=1}^m\sum_{j=1}^m d_{ij}^+q_j|\phi_j| + \sum_{i=1}^m I_i^+, \tag{2.21}$$

Using $\xi_i < 1$, similar to the proof of Theorem 2.1, we have

$$\max_{n\in I_N}|x_i(n)| \le \frac{1}{\xi_i}\max_{n\in I_N}|y_i(n)|. \tag{2.22}$$

From $\rho_1 - \frac{m\rho_2}{\check{\xi}} > 0$, (2.21) and (2.22), we have

$$\left[\rho_1 - \frac{m\rho_2}{\check{\xi}}\right]\sum_{i=1}^m\max_{n\in I_N}|y_i(n)| \le \sum_{i=1}^m\sum_{j=1}^m(c_{ij}^+ + d_{ij}^+)q_je_j + \sum_{i=1}^m I_i^+.$$

Hence, there exists $C_i > 0$ such that

$$\max_{n\in I_N}|y_i(n)| \le C_i, \ i = 1, 2, \cdots, m. \tag{2.23}$$

In view of (2.22) and (2.23), we get

$$\max_{n\in I_N}|x_i(n)| \le \frac{C_i}{\xi_i} := D_i,\ i = 1, 2, \cdots, m. \tag{2.24}$$

In view of (2.23) and (2.24), let

$$\|w\| = \max\left\{\max_{i=1,\cdots,m} C_i,\ \max_{i=1,\cdots,m} D_i\right\} := M. \tag{2.25}$$

Due to the assumption (H_3), the term $f_j(x_j),\ j = 1, 2, \cdots, m$ in system (1.2) satisfies Lipschiz condition on $\mathbb{R}$. Thus, by basic results of functional differential equation, we have the following theorems for the unique existence of periodic solution to system (1.2).

Theorem 2.3 Suppose all the conditions of Theorem 2.1 and assumption (H_3) hold. Then system (1.2) has unique $N-$periodic solution.

Theorem 2.4 Suppose all the conditions of Theorem 2.2 and assumption (H_3) hold. Then system (1.2) has unique $N-$periodic solution.

3. Exponential stability of periodic solution

Since system (1.2) is equivalent to system (2.2) under the transformation (2.1), then we will consider the exponential stability problems of system (2.2).

Definition 3.1 If $w^*(n) = (x_1^*(n), \cdots, x_m^*(n), y_1^*(n), \cdots, y_m^*(n))^\top$ is a periodic solution of system (2.2) and $w(n) = (x_1(n), \cdots, x_m(n), y_1(n), \cdots, y_m(n))^\top$ is any solution of system (2.2) satisfying

$$|w_i(n) - w_i^*(n)| \le L\|\phi_i - \phi_i^*\|e^{-n},\ n \in \mathbb{Z}_0^+,\ i = 1, 2\cdots, 2m,$$

then $w^*(n)$ is globally asymptotic stable, where $L > 0$ is a constant, ϕ is initial condition of $w(n)$, ϕ^* is initial condition of $w^*(n)$.

Theorem 3.1 Under conditions of Theorem 2.3, system (2.2) has unique $T-$periodic solution $w^*(n) = (x_1^*(n), \cdots, x_n^*(n), y_1^*(n), \cdots, y_n^*(n))^\top$ which is exponential stable, provided that

$$a_i^- - [(a_i(n) - \xi_i)\xi_i - b_i(n)]^+ - \sum_{j=1}^m c_{ij}^+ L_j - \sum_{j=1}^m d_{ij}^+ L_j > 0,\ \ i = 1, 2, \cdots, m. \tag{3.1}$$

Proof By (2.2), we have

$$x_i(n+1) - x_i^*(n+1) = (1 - \xi_i)(x_i(n) - x_i^*(n)) + (y_i(n) - y_i^*(n)) \tag{3.2}$$

and

$$\begin{aligned} y_i(n+1) - y_i^*(n+1) = {} & [1 - (a_i(n) - \xi_i)](y_i(n) - y_i^*(n)) + [(a_i(n) - \xi_i)\xi_i - b_i(n)](x_i(n) - x_i^*(n)) \\ & + \sum_{j=1}^m c_{ij}(n)[f_j(x_j(n)) - f_j(x_j^*(n))] \\ & + \sum_{j=1}^m d_{ij}(n)[f_j(x_j(n - \tau_j(n))) - f_j(x_j^*(n - \tau_j(n)))]. \end{aligned} \tag{3.3}$$

For $i = 1, 2, \cdots, m$, define function:

$$F_i(\alpha) = 1 + \xi_i - (1 + \xi_i)\alpha + \alpha\Big[a_i^- - [(a_i(n) - \xi_i)\xi_i - b_i(n)]^+ - \sum_{j=1}^{m} c_{ij}^+ L_j - \sum_{j=1}^{m} d_{ij}^+ L_j \alpha^\tau\Big].$$

In view of condition (3.1), we get $F_i(1) > 0$. Hence, there exists a constant $\alpha_0 > 1$ such that

$$F_i(\alpha_0) > 0, \quad i = 1, 2, \cdots, m. \tag{3.4}$$

By (3.2), we have

$$|x_i(n+1) - x_i^*(n+1)| = (1 - \xi_i)|(x_i(n) - x_i^*(n))| + |(y_i(n) - y_i^*(n))|. \tag{3.5}$$

By (3.3), we have

$$\begin{aligned} |y_i(n+1) - y_i^*(n+1)| \le &(1 + \xi_i - a_i^-)|y_i(n) - y_i^*(n)| + [(a_i(n) - \xi_i)\xi_i - b_i(n)]^+|(x_i(n) - x_i^*(n))| \\ &+ \sum_{j=1}^{m} c_{ij}^+ L_j |x_j(n) - x_j^*(n)| \\ &+ \sum_{j=1}^{m} d_{ij}^+ L_j |x_j(n - \tau_j(n)) - x_j^*(n - \tau_j(n))|. \end{aligned} \tag{3.6}$$

Define

$$u_i(n) = \alpha_0^n |x_i(n) - x_i^*(n)|, \quad n \in [-\tau, +\infty)_{\mathbb{Z}},$$
$$v_i(n) = \alpha_0^n |y_i(n) - y_i^*(n)|, \quad n \in [-\tau, +\infty)_{\mathbb{Z}},$$

where α_0 is defined by (3.4). By (3.5), we have

$$\begin{aligned} u_i(n+1) &= \alpha_0(1 - \xi_i)u_i(n) + \alpha_0 v_i(n) \\ &\le \alpha_0(u_i(n) + v_i(n)). \end{aligned} \tag{3.7}$$

By (3.6), we have

$$\begin{aligned} v_i(n+1) \le &(1 + \xi_i - a_i^-)\alpha_0 v_i(n) \\ &+ [(a_i(n) - \xi_i)\xi_i - b_i(n)]^+ \alpha_0 u_i(n) + \sum_{j=1}^{m} c_{ij}^+ L_j \alpha_0 u_j(n) \\ &+ \sum_{j=1}^{m} d_{ij}^+ L_j \alpha_0^{\tau_j(n)+1} u_j(n - \tau_j(n)). \end{aligned} \tag{3.8}$$

Assume that $K = \max_{s \in [-\tau, 0]_{\mathbb{Z}}} |\phi_i(s) - \phi_i^*(s)|,\ i = 1, 2, \cdots, 2m$. Then we claim that

$$u_i(n) \le K \text{ and } v_i(n) \le K,\ n \in \mathbb{Z}_0^+,\ i = 1, 2, \cdots, m. \tag{3.9}$$

Otherwise, there exist integer $i_0 \in \{1, 2, \cdots, m\}$ and $n_0 \in \mathbb{Z}_0^+$ such that

$$u_i(n) \le K,\ n \in [-\tau, n_0]_{\mathbb{Z}},\ i \ne i_0,\ u_{i_0}(n) \le K,\ n \in [-\tau, n_0 - 1]_{\mathbb{Z}},\ u_{i_0}(n_0) > K \tag{3.10}$$

and

$$v_i(n) \le K,\ n \in [-\tau, n_0]_{\mathbb{Z}},\ i \ne i_0,\ v_{i_0}(n) \le K,\ n \in [-\tau, n_0 - 1]_{\mathbb{Z}},\ v_{i_0}(n_0) > K. \tag{3.11}$$

If (3.10) and (3.11) hold, by (3.7) we have

$$\begin{aligned} K < u_{i_0}(n_0) &\le \alpha_0(u_{i_0}(n_0-1) + v_{i_0}(n_0-1)) \\ &\le 2K\alpha_0, \end{aligned}$$

thus, $\alpha_0 > \frac{1}{2}$ which is contradict to $\alpha_0 > 1$. On the other hand, if (3.10) and (3.11) hold, by (3.8) and (3.4) we have

$$\begin{aligned} K < v_{i_0}(n_0) &\le (1 + \xi_i - a_i^-)\alpha_0 v_{i_0}(n_0-1) + [(a_i(n) - \xi_i)\xi_i - b_i(n)]^+ \alpha_0 u_{i_0}(n_0-1) \\ &+ \sum_{j=1}^{n} c_{ij}^+ L_j \alpha_0 u_j(n_0-1) \\ &+ \sum_{j=1}^{n} d_{ij}^+ L_j \alpha_0^{\tau_j(n_0-1)+1} u_j(n_0 - 1 - \tau_j(n_0-1)) \\ &\le (1+\xi_i)\alpha_0 K - K\Big[a_i^-\alpha_0 - [(a_i(n) - \xi_i)\xi_i - b_i(n)]^+\alpha_0 - \sum_{j=1}^{m} c_{ij}^+ L_j \alpha_0 - \sum_{j=1}^{m} d_{ij}^+ L_j \alpha_0^{\tau+1}\Big] \\ &< K \end{aligned}$$

which is a contradiction. Hence, (3.9) holds, i.e.,

$$|x_i(n) - x_i^*(n)| \le \alpha_0^{-n}\|\phi_i - \phi_i^*\|e^{-n},\ n \in \mathbb{Z}_0^+,\ i = 1, 2\cdots, m$$

and

$$|y_i(n) - y_i^*(n)| \le \alpha_0^{-n}\|\phi_{m+i} - \phi_{m+i}^*\|e^{-n},\ n \in \mathbb{Z}_0^+,\ i = 1, 2\cdots, m.$$

Hence, periodic solution of system (2.2) is exponentially stable, i.e., periodic solution of system (1.2) is exponentially stable.

4. Numerical example

This section presents an example that demonstrates the validity of our theoretical results as follows:

$$\begin{cases} \Delta x_1(n) = -\xi_1 x_1(n) + y_1(n), \\ \Delta y_1(n) = -(a_1(n) - \xi_1)y_1(n) + [(a_1(n) - \xi_1)\xi_1 - b_1(n)]x_1(n) \\ +c_{11}(n)f_1(x_1(n)) + d_{11}(n)f_1(x_1(n - \tau_1(n))) + I_1(n), \end{cases} \tag{4.1}$$

where

$$\xi_1 = 0.2,\ a_1(n) = 0.9,\ b_1(n) = 0.12,\ c_{11}(n) = d_{11}(n) = \sin\frac{n\pi}{2},$$
$$\tau_1(n) = 1 + \cos\frac{n\pi}{2},\ f_1(u) = 0.2\sin u,\ I_1(n) = \sin\frac{n\pi}{2}.$$

Obviously, $p_1 = 0.2$ and assumption (H_2) holds. Furthermore, $L_1 = 0.2$ and assumption (H_3) holds. By simple calculating, we have

$$1 - (a_1^+ - \xi_1) = 0.3 > 0,\quad a_1^- - \xi_1 = 0.7 > 0,$$

$$(a_1^- - \xi_1)\xi_1 - [(a_1(n) - \xi_1)\xi_1 - b_1(n)]^+ = 0.12 > 0,$$

$$a_1^- - [(a_1(n) - \xi_1)\xi_1 - b_1(n)]^+ - c_{11}^+ L_1 - d_{11}^+ L_1 = 0.68 > 0.$$

Thus, all assumptions of Theorem 3.1 hold and system (4.1) exists unique periodic solution which is globally exponentially stable. The corresponding numerical simulations are presented in Figures 1 and 2 with random initial conditions. Figure 1 shows that system (4.1) exists at least one periodic solution. Figure 2 shows that system (4.1) exist stable periodic solutions.

Remark 4.1 For all we know, the periodic solution problems of discrete-time INNs with delays are studied in the present paper for the first time. Using Mawhin's continuation theorem and some innovative mathematical analysis techniques, we get some brand new results on the existence, uniqueness and exponential stability of periodic solution of discrete-time INNs. We can confirm the truth of the proposed methods, for example, in [8, 9, 10, 11] cannot be generalized to the problems studied in this article. There are a large number of periodic phenomena in nature and society. One of the important trends in the investigations of inertial neural networks is related to the periodic solutions of these systems. Hence, studying periodic solution problems of system (1.2) has important theoretical and practical values.

Remark 4.2 In this paper, we obtain stability results of INNs which can be extended to INNs with distributed delays, see [24]. In the future work, we will study global stability problem of INNs with distributed delays.

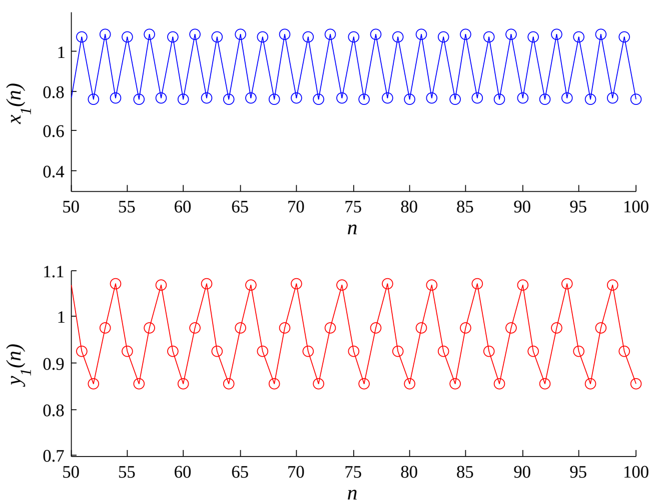

Figure 1. Periodic solution $((x_1(n), y_1(n))$ of system (4.1).

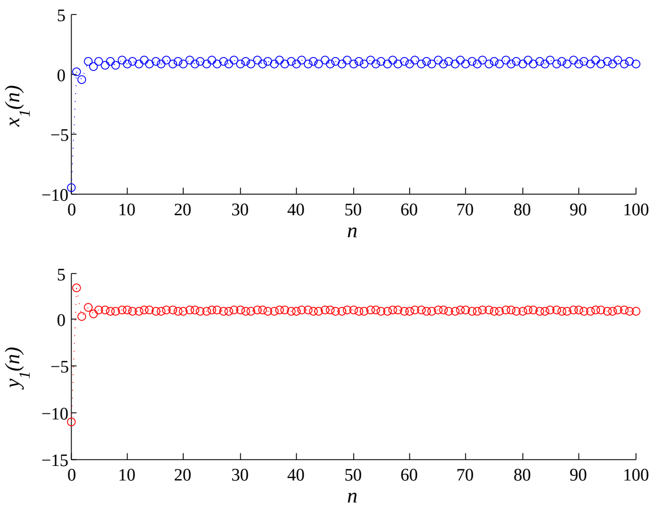

Figure 2. Stable periodic solution $((x_1(n), y_1(n))$ of system (4.1).

5. Conclusions and discussions

In this paper we study the problems of periodic solutions for discrete-time inertial neural networks with multiple delays. First, by applying Mawhin's continuous theorem to the system, we get a set of sufficient conditions for guaranteeing the existence and uniqueness of periodic solutions to the considered system. Then, on the basis of existence and uniqueness, we obtain globally exponential stability of periodic solutions. The efficacy of the obtained results has been demonstrated by an numerical example. It is important to note that the practical implementation of INNs is typically encountered with certain type of uncertainties such as interval parameters. Extending the results of this paper to discrete-time INNs with interval uncertainties proves to be an interesting problem. In addition, it is also interesting and challenging to extend the approach presented in this paper to discrete-time neural network-based problems with mixed delays such as state estimation and approximation, fault isolation and diagnosis, or filter/observer design. These issues require further investigations in the future works.

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Conflict of interest

The authors confirm that they have no conflict of interest in this paper.

References

1. D. Wheeler, W. Schieve, Stability and chaos in an inertial two-neuron system, *Physica D*, **105** (1997), 267–284.
2. J. Jian, L. Duan, Finite-time synchronization for fuzzy neutral-type inertial neural networks with time-varying coefficients and proportional delays, *Fuzzy Set. Syst.*, **381** (2020), 51–67.
3. C. Long, G. Zhang, Z. Zeng, Novel results on finite-time stabilization of state-based switched chaotic inertial neural networks with distributed delays, *Neural Networks*, **129** (2020), 193–202.
4. Y. Sheng, T. Huang, Z. Zeng, P. Li, Exponential Stabilization of Inertial Memristive Neural Networks With Multiple Time Delays, *IEEE T. Cybernetics*, **12** (2019), 1–10.
5. Z. Tu, J. Cao, T. Hayat, Matrix measure based dissipativity analysis for inertial delayed uncertain neural networks, *Neural Networks*, **75** (2016), 47–55.
6. K. Babcock, R. Westervelt, Stability and dynamics of simple electronic neural networks with added inertia, *Physica D*, **23** (1986), 464–469.
7. S. Hu, J. Wang, Global stability of a class of discrete-time recurrent neural networks, *IEEE T. Circuits-I*, **49** (2002), 1104–1117.
8. P. Wan, J. Jian, Global convergence analysis of impulsive inertial neural networks with time-varying delays, *Neurocomputing*, **245** (2017), 68–76.
9. Z. Tu, J. Cao, T. Hayat, Global exponential stability in Lagrange sense for inertial neural networks with time-varying delays, *Neurocomputing*, **171** (2016), 524–531.
10. J. Wang, L. Tian, Global Lagrange stability for inertial neural networks with mixed time-varying delays, *Neurocomputing*, **235** (2017), 140–146.
11. C. Aouiti, E. A. Assali, I. B. Gharbia, Y. El Foutayeni, Existence and exponential stability of piecewise pseudo almost periodic solution of neutral-type inertial neural networks with mixed delay and impulsive perturbations, *Neurocomputing*, **357** (2019), 292–309.
12. C. Huang, H. Zhang, Periodicity of non-autonomous inertial neural networks involving proportional delays and non-reduced order method, *Int. J. Biomath.*, **12** (2019), 1950016.
13. K. Wang, Z. Teng, H. Jiang, Adaptive synchronization in an array of linearly coupled neural networks with reaction-diffusion terms and time delays, *Commun. Nonlinear Sci.*, **17** (2012), 3866–3875.
14. F. Zheng, Periodic wave solutions of a non-Newtonian filtration equation with an indefinite singularity, *AIMS Mathematics*, **6** (2021), 1209–1222.
15. J. Wang, H. Wu, Synchronization and adaptive control of an array of linearly coupled reaction-diffusion neural networks with hybrid coupling, *IEEE T. Cybernetics*, **44** (2017), 1350–1361.
16. B. Liu, L. Huang, Existence and exponential stability of periodic solutions for a class of Cohen-Grossberg neural networks with time-varying delays, *Chaos, Solitons and Fractals*, **32** (2007), 617–627.
17. H. Yin, B. Du, Stochastic patch structure Nicholsonis blowfies system with mixed delays, *Adv. Differ. Equ.*, **2020** (2020), 1–11.

18. M. Xu, B. Du, Dynamic behaviors for reaction-diffusion neural networks with mixed delays, *AIMS Mathematics*, **5** (2020), 6841–6855.

19. H. Yin, B. Du, Q. Yang, F. Duan, Existence of homoclinic orbits for a singular differential equation involving p-Laplacian, *J. Funct. Spaces*, **2020** (2020), 1–7.

20. T. Zhou, Y. Liu, Y. Liu, Existence and global exponential stability of periodic solution for discrete-time BAM neural networks, *Appl. Math. Comput.*, **182** (2006), 1341–1354.

21. L. Hien, L. Hai-An, Positive solutions and exponential stability of positive equilibrium of inertial neural networks with multiple time-varying delays, *Neural Comput. Appl.*, **31** (2019), 6933–6943.

22. J. Yogambigai, M. Syed Ali, H. Alsulami, M. S. Alhodaly, Global Lagrange stability for neutral-type inertial neural networks with discrete and distributed time delays, *Chinese J. Phys.*, **65** (2020), 513–525.

23. R. Gaines, J. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer, Berlin, 1977.

24. G. Zhang, Z. Zeng, J. Hu, New results on global exponential dissipativity analysis of memristive inertial neural networks with distributed time-varying delays, *Neural Networks*, **97** (2018), 183–191.

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