



Research article

Stability analysis of stochastic fractional-order competitive neural networks with leakage delay

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Abstract: This article, we explore the stability analysis of stochastic fractional-order competitive neural networks with leakage delay. The main objective of this paper is to establish a new set of sufficient conditions, which is for the uniform stability in mean square of such stochastic fractional-order neural networks with leakage. Specifically, the presence and uniqueness of arrangements and stability in mean square for a class of stochastic fractional-order neural systems with delays are concentrated by using Cauchy-Schwartz inequality, Burkholder-Davis-Gundy inequality, Banach fixed point principle and stochastic analysis theory, respectively. Finally, four numerical recreations are given to confirm the hypothetical discoveries.

Keywords: fractional order; stochastic; competitive neural networks; leakage

Mathematics Subject Classification: 93D05

1. Introduction

The fresh concept of fractional order calculus and differential equations has three hundred years old of branch. For long period, the theory of fractional calculus is developed only on pure mathematics. In 1695, the establishment of non-integer order math, which is a speculation integer order differential and integrals, was most importantly talked about through Guillaume de Leibnitz and Gottfried Wilhelm Leibnitz, furthermore, its advancement were continuous for extensive stretch [1]. Owing to lack of solution methods, the development of fractional order calculus has not much attracted more mathematicians in those periods. In recent years, fractional order dynamical system has aroused

interest of many researchers in the field of nonlinear science and technology. Differential condition and dynamic framework displaying have gotten significant research themes in normal science and building innovation [2–6]. In recent years, fractional-order differential equations are thought of as a recent topic. A geometric interpretation of fractional integral and derivative is given in [7]. Although there are very many possible generalizations of $\frac{d^n}{dt^n} f(t)$, the most commonly used definitions are Riemann-Liouville and Caputo fractional derivatives. A strong motivation for studying fractional differential equations comes from the fact that have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, and economics. For more interesting theoretical results of fractional differential equations, see [8–12]. In most of these techniques, either the solutions of integer order differential equation versions of the given fractional differential equations or the series expansions in the neighborhood of the initial conditions are used.

Recently, because of the introduction of fractional calculus, fractional-order competitive network is attracting more and more attentions. At present, various kinds of neural network model exist, which include competitive type-neural networks [13], Cohen-Gross berg-type neural networks [14], cellular-type neural networks [15], recurrent type neural networks [16], BAM-type neural networks [17, 18] and so on. In the genuine neural frameworks model, there are two kinds of state factors: the short-term memory (STM) variable depicting the quick neural movement, and the long-term memory(LTM) variable portraying the moderate unaided synaptic adjustments. STM describes the rapidly changing behavior of neuronal dynamics, whereas LTM describes the slow behavior of unsupervised neuronal synapses. Therefore, there are two time scales in the competitive neural networks model, one of which corresponds to the quick difference in the state, and the other to the moderate difference in the neural connection by outside improvements. However, considerable attention paid on the study of FCNN has just started with a few literatures reported [19, 20]. As far as it goes, there are few literatures on incommensurate FCNN with different time scales, which is more general and less conservative than common one.

As is well known, the applications of fractional-order neural networks heavily depend on the stability of networks. In recent years, the fractional-order dynamic behaviors plays a crucial role in stability of neural networks and the research of the fractional-order dynamical system has been a hot research topic. As we know, there are various types of stability results have been demonstrated in the literature, for example, Robust stability [21], Exponential stability [22], Finite time stability [23] and so on. It is pointed that many of the researchers mainly targeted fractional-order dynamic behavior of other types of neural network model. In recent several years, there have been a lot of excellent works on the stability analysis of fractional-order BAM neural networks [24, 25].

Actually, the synaptic transmission in real neural networks can be viewed as a noisy process introduced by random fluctuations from the release of neurotransmitters and other probabilistic causes. Hence, noise is unavoidable and should be taken into consideration in modeling. Moreover, it is important to check the stability issue of neural networks with stochastic disturbance. Practically, noises are omnipresent each in nature and in artificial systems. So, the stochastic influence exists doubtless. Therefore, the study of stochastic neural networks aren't solely vital however additionally necessary. It is acknowledge that in real system, there exists the abrupt phenomena such as abrupt environments changes, and conjugation transmission could be rip-roaring method brought on by random probabilistic causes, and it should degrade the stability of the neural systems. Hence, considerable attention has been paid on the study of stochastic neural networks theory and various interesting results have been

reported in [26–34]. A large number of stochastic financial models appeared in the literature, see, for example, [35–40] and the references therein. Also, the important effect of noise disturbances should be taken into account in studying the dynamics of a financial system by means of the neural network approach.

Stochastic differential equations becomes a extraordinary interest due to its applications in characterizing numerous issues in physics, biology, mechanics, etc. Qualitative properties such as existence, uniqueness, controllability and stability for various stochastic differential systems have been extensively studied by many researchers, see for instance [41–48] and the references therein. Around 1960, for obvious mathematical reasons, systems of ordinary stochastic differential equations of Ito type [49–51], stochastic partial differential equations [52, 53], stochastic fractional differential equations [54, 55]. The effects of random environmental fluctuations are characterized by normalized Wiener process [56]. So it is characteristic and important to explore dynamical properties of the solutions for SDEs to discover the impacts of random perturbations in the relating realistic systems. The numerical models got have been extraordinarily produced for SDEs under an irregular disturbance of the Gaussian white noise, namely, the examinations concerning SDEs driven by Brownian movement have been extremely plentiful up to now.

To the best of our knowledge, stability analysis of stochastic fractional-order competitive neural networks with leakage delay has not been fully investigated, and there is still much room left for further investigation. Motivated by the above discussions, this paper devotes to presenting a sufficient criterion for stability analysis of stochastic fractional-order competitive neural networks with leakage delay model. Meanwhile, the existence, uniqueness, and uniform stability in mean square are proved.

The main aim and contribution of our paper are highlighted as follows:

- (1) We get stochastic fractional-order competitive neural networks with leakage delay model by use fractional-order instead of integer-order.
- (2) Our main theme of our paper is to design both the stability analysis of stochastic fractional-order competitive neural networks with leakage delay by using Cauchy-Schwartz inequality, Burkholder-Davis-Gundy inequality, some sufficient condition for guarantee the stability.
- (3) We establish a new set of sufficient criterion ensuring the uniform stability in mean square of the system and existence and uniqueness of solutions also proved by using contraction mapping principle.
- (4) Various lemma's and fractional-order theory are applied to derive the main results.

This paper is organized as follows. In Section 2, we introduce the definitions and lemmas and stochastic fractional-order competitive neural networks with leakage delay model. In Section 3, we shall establish a new set of sufficient criterion ensuring the uniform stability in mean square of the system and the existence, uniqueness, and uniform stability in mean square. In Section 4, we give a numerical examples which confirm the theoretical results. Finally, the paper is concluded in Section 5.

Notations: The Caputo fractional derivative operator D^p is chosen for fractional-order derivative with order p ; R^n and $R^{n \times n}$ denote the n -dimensional Euclidean space and the set of all $n \times n$ real matrices, respectively; C the complex number set; R , R^+ and Z^+ are the set of all real numbers, the set of all nonnegative numbers and the set of all nonnegative integer numbers, respectively; $E(\cdot)$ stands for the mathematical expectation with some probability measure; $\Omega = (L^2_{F_0}([0, T], R^n), \|\cdot\|)$; for any $z =$

$(z_1, \dots, z_n)^T \in \mathbb{R}^n$, we define the vector.

$$\|z(t)\| = \sqrt{\sum_{i=1}^n \sup_{t \in (0, T]} \{e^{-2Nt} |z_i^2(t)|\}}. \quad (1.1)$$

For any $\phi = (\phi_1(t), \dots, \phi_n(t))^T \in L_{F_0}^2([-\tau, 0], \mathbb{R}^n)$, we define the vector norm

$$\|\phi(t)\| = \sqrt{\sum_{i=1}^n \sup_{t \in [-\tau, 0]} \{e^{-2Nt} |\phi_i(t)|^2\}}. \quad (1.2)$$

2. Preliminaries

In this section we present some definitions, lemma and recall the well-known results about fractional differential equations.

Definition 2.1. [57] The Caputo fractional-order derivative with order p for a differential function $z(t)$ is defined as

$$D^p z(t) = \frac{1}{\Gamma(m-p)} \int_0^t \frac{z^m(s)}{(t-s)^{p+1-m}} ds, \quad (2.1)$$

where $t \geq 0$ and $m-1 < p < m \in \mathbb{Z}^+$. Peculiarly, when $p \in (0, 1)$,

$$D^p z(t) = \frac{1}{\Gamma(1-p)} \int_0^t \frac{z'(s)}{(t-s)^p} ds.$$

Definition 2.2. [57] The e Riemann-Liouville fractional integral of order $p \in (0, 1)$ for a function $z(t)$ is defined as

$$I^p z(t) = \frac{1}{\Gamma(p)} \int_0^t \frac{z(s)}{(t-s)^{1-p}} ds, \quad (2.2)$$

where $\Gamma(\cdot)$ is the gamma function,

$$\Gamma(s) = N^s \int_0^\infty t^{s-1} e^{-Nt} dt. \quad (2.3)$$

Definition 2.3. The solution of system Eq (3.1) is said to be stable if for any $\epsilon > 0$ there exists $\delta(t_0, \epsilon) > 0$ such that $t \geq t_0 \geq 0$, $E\|\psi(t) - \phi(t)\|^2 < \delta$ imply $E\|u(t, t_0, \psi) - v(t, t_0, \phi)\|^2 < \epsilon$ for any two solutions $u(t, t_0, \psi)$ and $v(t, t_0, \phi)$. It is uniformly stable in mean square if the above δ is independent of t_0 .

Lemma 2.4. [58] Let m be a positive integer such that $m-1 < p < m$. If $g(t) \in C^{m-1}([b, T])$, Then

$$D_{b,t}^{-p} D_{b,t}^p g(t) = g(t) - \sum_{j=0}^{m-1} \frac{g^{(j)}(b)}{j!} (t-b)^j. \quad (2.4)$$

Lemma 2.5. [59] Let $g(s) \in L^2([0, T])$, $h(s) \in L^2([0, T])$, then

$$\left(\int_0^T |g(s)h(s)| ds \right)^2 \leq \left(\int_0^T |g(s)|^2 ds \right) \left(\int_0^T |h(s)|^2 ds \right) \quad (2.5)$$

Assumption 1. [60] We assume that the non-linear functions $g_j(\cdot)$ and $\sigma_{ij}(\cdot, \cdot)$ satisfy the following conditions: There exist positive constants L_j and η_{ij} such that

$$|g_j(x) - g_j(y)| \leq F_j |x - y|, |\sigma_{ij}(x, \bar{x}) - \sigma_{ij}(y, \bar{y})| \leq \eta_{ij} (|x - y|^2 + |\bar{x} - \bar{y}|^2),$$

for any $x, y, \bar{x}, \bar{y} \in \mathcal{R}$, $i, j = 1, \dots, n$. For convenience, we introduce the following notation related to model Eq (3.1).

$$\begin{aligned} \|F\| &= \sum_{k=1}^n F_k^2, \|A\| = \sum_{i=1}^n \max_{\forall k} \{a_{ik}^2\}, \\ \|B\| &= \max_{\forall i} \{b_i^2\}, \|D\| = \sum_{i=1}^n \max_{\forall k} \{d_{ik}^2\}, \\ \|K\| &= 8n \max_{\forall k} \{\eta_{ik}\}, \|V\| = \max_{\forall i} \{v_i^2\}, \\ \|W\| &= \max_{\forall i} \{w_i^2\}. \end{aligned}$$

3. Main results

In this paper, the stochastic fractional-order competitive neural networks with leakage delays is defined by

$$\begin{aligned} D^p z_i(t) &= -b_i z_i(t - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(t)) + \sum_{k=1}^n d_{ik} g_k(z_k(t - \eta)) + c_i \sum_{l=1}^m r_{il}(t) \xi_l \\ D^p r_{il}(t) &= -v_i r_{il}(t - \delta) + p_l w_i g_i(z_i(t)), i = 1, 2, \dots, n, l = 1, 2, \dots, m, \end{aligned} \quad (3.1)$$

where D^p denotes Caputo fractional derivative of order p with $0 < p < 1$, $z_i(t)$ is the corresponding state variable of the number of n neurons at time t , $b_i > 0$ and $v_i > 0$, represent the self feedback connection weight matrices, a_{ik} , d_{ik} are represents the synaptic connection weight matrix and delayed synaptic connection weight matrix, respectively to i th and k th neurons, p_l denotes the constant external stimulus, c_i denotes the external strengths of the stimulus, r_{il} denotes the synaptic efficiency, $\mu > 0$, $\delta > 0$ denote the leakage delays, $g_k(z_k(t))$ and $g_k(z_k(t - \eta))$ are referred the bounded neuron output activation, where the time varying delay η is bounded and differentiable.

After setting $s_i(t) = \sum_{l=1}^m r_{il}(t) \xi_l = r_i^T(t) \xi$, where $r_i(t) = (r_{i1}, \dots, r_{im})^T$, $\xi = (\xi_1, \dots, \xi_m)^T$. Without loss of generality, ξ is assumed to normalized with unit magnitude $|\xi|^2 = 1$, where ξ is the input stimulus. Then the model (3.1) can be simplified the following state-space form such as

$$\begin{aligned} D^p z_i(t) &= -b_i z_i(t - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(t)) + \sum_{k=1}^n d_{ik} g_k(z_k(t - \eta)) + c_i s_i(t) \\ D^p s_i(t) &= -v_i r_{il}(t - \delta) + w_i g_i(z_i(t)), i = 1, 2, \dots, n, \end{aligned} \quad (3.2)$$

Initial conditions of the model(3.1) is described as:

$$z_i = \phi_i(\gamma), s_i = \psi_i(\gamma), \gamma \in [-\tau, 0], i = 1, 2, \dots, n. \quad (3.3)$$

Now we applying stochastic terms in the above equation we get,

$$\begin{aligned} D^p z_i(t) &= -b_i z_i(t - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(t)) + \sum_{k=1}^n d_{ik} g_k(z_k(t - \eta)) + c_i s_i(t) \\ &\quad + \sum_{k=1}^n \sigma_{ik}(z_k(t), z_k(t - \eta)) \frac{dw_k(t)}{dt}, t \in [0, T], \\ D^p s_i(t) &= -v_i s_i(t - \delta) + w_i g_i(z_i(t)); i = 1, 2, \dots, n, \end{aligned} \quad (3.4)$$

$\sigma(\cdot, \cdot) = (\sigma(\cdot, \cdot))_{n \times n}$ is the diffusion coefficient matrix and $\omega(\cdot) = (\omega_1(\cdot), \dots, \omega_n(\cdot))^T$ is an n-dimensional Brownian motion defined on a complete probability space (Ω, F, P) with a natural filtration $\{F_t\}_{t \geq 0}$. $\phi_i(t)$ is the initial function where $\phi_i(t) \in L_{F_0}^2([-\tau, 0], R)$, here $L_{F_0}^2([-\tau, 0], R)$ denotes the family of all C-valued random processes $\gamma(s)$ such that $\gamma(s)$ is F_0 -measurable and $\int_{-\tau}^0 E\|\gamma(s)\|^2 ds < \infty$.

Theorem 3.1. If assume Assumption 1 hold, then the system Eq (3.1) has a unique solution.

Proof. According to the properties of the fractional calculus, one can obtain that system Eq (3.4) is equivalent to the following Volterra fractional integral with memory

$$\begin{aligned} z_i(t) &= \phi_i(0) + I^p D^p z_i(t) \\ &= \phi_i(0) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left[-b_i z_i(s - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(s)) \right. \\ &\quad \left. + \sum_{k=1}^n d_{ik} g_k(z_k(s - \eta)) + c_i s_i(s) + \sum_{k=1}^n \sigma_{ik}(z_k(s), z_k(s - \eta)) \frac{dw_k(s)}{ds} \right] ds, \end{aligned} \quad (3.5)$$

where $t \in [0, T]$. We consider a mapping $\phi : R^n \rightarrow R^n$, defined by:

$$\begin{aligned} \phi_i z_i(t) &= \phi_i(0) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left[-b_i z_i(s - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(s)) \right. \\ &\quad \left. + \sum_{k=1}^n d_{ik} g_k(z_k(s - \eta)) + c_i s_i(s) + \sum_{k=1}^n \sigma_{ik}(z_k(s), z_k(s - \eta)) \frac{dw_k(s)}{ds} \right] ds, \end{aligned} \quad (3.6)$$

For any two different functions $(z_1(t), \dots, z_n(t))^T, (y_1(t), \dots, y_n(t))^T$, we have

$$\begin{aligned} \phi_i y_i(t) - \phi_i z_i(t) &= \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left[-b_i [y_i(s - \mu) - z_i(s - \mu)] \right. \\ &\quad \left. + \sum_{k=1}^n [a_{ik} g_k(y_k(s)) - a_{ik} g_k(z_k(s))] \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^n [d_{ik}g_k(y_k(s-\eta)) - d_{ik}g_k(z_k(s-\eta))] \\
& + \sum_{k=1}^n [\sigma_{ik}(y_k(s), y_k(s-\eta)) \\
& - \sigma_{ik}(z_k(s), z_k(s-\eta))] \frac{dw_k(s)}{ds} ds,
\end{aligned}$$

Then, applying elementary inequality, one sees that

$$\begin{aligned}
|\phi_i y_i(t) - \phi_i z_i(t)|^2 & \leq \frac{4}{\Gamma^2(p)} \left[\left| \int_0^t (t-s)^{p-1} b_i y_i(s-\mu) - z_i(s-\mu) ds \right|^2 \right. \\
& + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} a_{ik} g_k(y_k(s)) - a_{ik} g_k(z_k(s)) ds \right|^2 \\
& + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} d_{ik} g_k(y_k(s-\eta)) - d_{ik} g_k(z_k(s-\eta)) ds \right|^2 \\
& + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} [\sigma_{ik}(y_k(s), y_k(s-\eta)) \right. \\
& \left. - \sigma_{ik}(z_k(s), z_k(s-\eta))] ds \right|^2, \\
e^{-2Rt} |\phi_i y_i(t) - \phi_i z_i(t)|^2 & \leq \frac{4}{\Gamma^2(p)} \left[b_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} y_i(s-\mu) - z_i(s-\mu) ds \right|^2 \right. \\
& + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} e^{-Rt} [a_{ik} g_k(y_k(s)) - a_{ik} g_k(z_k(s))] ds \right|^2 \\
& + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} e^{-Rt} [d_{ik} g_k(y_k(s-\eta)) - d_{ik} g_k(z_k(s-\eta))] ds \right|^2 \\
& + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} e^{-Rt} [\sigma_{ik}(y_k(s), y_k(s-\eta)) \right. \\
& \left. - \sigma_{ik}(z_k(s), z_k(s-\eta))] dw_k(s) \right|^2,
\end{aligned} \tag{3.7}$$

First, we have a tendency to evaluate the primary term of the right hand side of the above inequality by using Cauchy inequality to obtain

$$\begin{aligned}
& b_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} y_i(s-\mu) - z_i(s-\mu) ds \right|^2 \\
& \leq b_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_i(s-\mu) - z_i(s-\mu)|^2 ds \right) \\
& \leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_i(s-\mu) - z_i(s-\mu)|^2 ds \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_{\mu}^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_i(s-\mu) - z_i(s-\mu)|^2 ds \right) \\
&+ \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_0^{\mu} (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |\phi_i(s-\mu) - \phi_i(s-\mu)|^2 ds \right) \\
&\leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_{\mu}^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_i(s-\mu) - z_i(s-\mu)|^2 ds \right) \\
&\leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_0^{t-\mu} (t-\gamma-\mu)^{p-1} e^{-R(t-\gamma-\mu)} e^{-2R\gamma} e^{-2R\mu} |y_i(\gamma) - z_i(\gamma)|^2 d\gamma \right) \\
&\leq \frac{\Gamma(p)}{R^p} b_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} e^{-2R\mu} \int_0^{t-\mu} \xi^{p-1} e^{-R\xi} d\xi \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 b_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\}
\end{aligned} \tag{3.8}$$

Next, we evaluate the second term by using Assumption 1, we have

$$\begin{aligned}
&\sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} e^{-Rt} [a_{ik} g_k(y_k(s)) - a_{ik} g_k(z_k(s))] ds \right|^2 \\
&\leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n |a_{ik}| |g_k(y_k(s)) - g_k(z_k(s))| ds \right)^2 \\
&\leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n |a_{ik}| F_k |y_k(s) - z_k(s)| ds \right)^2 \\
&\leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n |a_{ik}| F_k |y_k(s) - z_k(s)| \right)^2 ds \right) \\
&\leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \left(\sum_{k=1}^n |y_k(s) - z_k(s)|^2 \right) ds \right) \\
&\leq \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} \sum_{k=1}^n e^{-2Rs} |y_k(s) - z_k(s)|^2 ds \right) \\
&\leq \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right)^2 \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\}
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\left| \int_0^t (t-s)^{p-1} e^{-Rt} \left[\sum_{k=1}^n d_{ik} g_k(y_k(s-\eta)) - \sum_{k=1}^n d_{ik} g_k(z_k(s-\eta)) \right] ds \right|^2 \\
&\leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n F_k |d_{ik}| |y_k(s-\eta) - z_k(s-\eta)| ds \right)^2
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n F_k |d_{ik}| |y_k(s-\eta) - z_k(s-\eta)|^2 \right) ds \right) \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n |y_k(s-\eta) - z_k(s-\eta)|^2 \right) ds \right) \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \left[\int_0^\eta (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \sum_{k=1}^n |\phi_k(s-\eta) - \phi_k(s-\eta)|^2 ds \right. \\
&\quad \left. + \int_\eta^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \sum_{k=1}^n |y_k(s-\eta) - z_k(s-\eta)|^2 ds \right] \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \int_0^{t-\eta} (t-\gamma-\eta)^{p-1} e^{-R(t-\gamma-\eta)} e^{-2R\gamma} e^{-2R\eta} \sum_{k=1}^n |y_k(\gamma) - z_k(t)|^2 d\gamma \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{ e^{-2Rt} |y_k(t) - z_k(t)|^2 \} \int_0^{t-\eta} \xi^{p-1} e^{-R\xi} d\xi \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{ e^{-2Rt} |y_k(t) - z_k(t)|^2 \} \tag{3.10}
\end{aligned}$$

However, by using the Burkholder-Davis-Gundy inequality and Assumption 1, we get that

$$\begin{aligned}
&E \left[\sup_{t \in (0, T]} \left| \int_0^t (t-s)^{p-1} e^{-Rt} \sum_{k=1}^n [\sigma_{ik}(y_k(s), y_k(s-\eta)) - \sigma_{ik}(z_k(s), z_k(s-\eta))] dw_k(s) \right|^2 \right] \\
&\leq 4E \int_0^T (T-s)^{(2p-1)-1} e^{-2R(T-s)} e^{-2Rs} n \sum_{k=1}^n |\sigma_{ik}(y_k(s), y_k(s-\eta)) - \sigma_{ik}(z_k(s), z_k(s-\eta))|^2 ds \\
&\leq 4E \int_0^T (T-s)^{(2p-1)-1} e^{-2R(T-s)-1} e^{-2Rs} n \sum_{k=1}^n \eta_{ik} [|y_k(s) - z_k(s)|^2 + |y_k(s-\eta) - z_k(s-\eta)|^2] ds \\
&\leq 4n \max_k \{ \eta_{ik} \} \left\{ E \int_0^T (T-s)^{(2p-1)-1} e^{-2R(T-s)} e^{-2Rs} \sum_{k=1}^n |y_k(s) - z_k(s)|^2 ds \right. \\
&\quad \left. + E \int_0^A (T-s)^{(2p-1)-1} e^{-2R(T-s)} e^{-2Rs} \sum_{k=1}^n |y_k(s-\eta) - z_k(s-\eta)|^2 ds \right\} \\
&\leq 4n \max_k \{ \eta_{ik} \} \left\{ E \int_0^T (T-s)^{(2p-1)-1} e^{-2R(T-s)} e^{-2Rs} \sum_{k=1}^n |y_k(s) - z_k(s)|^2 ds \right. \\
&\quad \left. + E \int_0^\eta (T-s)^{(2p-1)-1} e^{-2R(T-s)} e^{-2Rs} \sum_{k=1}^n |\phi_k(s-\eta) - \phi_k(s-\eta)|^2 ds \right. \\
&\quad \left. + E \int_\eta^T (T-s)^{(2p-1)-1} e^{-2R(T-s)} e^{-2Rs} \sum_{k=1}^n |y_k(s-\eta) - z_k(s-\eta)|^2 ds \right\} \\
&\leq 4n \sum_{k=1}^n \max_k \{ \eta_{ik} \} E \sum_{k=1}^n \sup_{t \in (0, T]} \{ e^{-2Rt} |y_k(t) - z_k(t)|^2 \} \int_0^T (T-s)^{(2p-1)-1} e^{-2R(T-s)} ds
\end{aligned}$$

$$\begin{aligned}
& + E \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} e^{-2R\eta} \int_0^{T-\eta} (T - \gamma - \eta)^{(2p-1)-1} e^{-2R(T-\gamma-\eta)} d\gamma \\
& \leq 8n \sum_{k=1}^n \max_k \{\eta_{ik}\} E \|y(t) - z(t)\|^2 \int_0^t \xi^{(2p-1)} e^{-2R\xi} d\xi \\
& \leq 8n \sum_{k=1}^n \max_k \{\eta_{ik}\} \left[\frac{\Gamma(2p-1)}{R^p} \right] E \|y(t) - z(t)\|^2
\end{aligned} \tag{3.11}$$

Thus, by combining the above inequalities together, one obtains

$$\begin{aligned}
E \|\phi y(t) - \phi z(t)\|^2 & \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 b_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} \\
& + \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \\
& + \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \\
& + 8n \max_k \{\eta_{ik}\} \left[\frac{\Gamma(2p-1)}{R^p} \right] E \|y(t) - z(t)\|^2 \\
& \leq \left[\frac{4\Gamma^2(p)}{R^{2p}} \right] [\|B\| + \|A\| \|F\| + \|D\| \|F\|] E \|y(t) - z(t)\|^2 \\
& + 4\|K\| \left[\frac{\Gamma(2\alpha-1)}{N^\alpha} E \|y(t) - z(t)\|^2 \right] \\
& \leq \left[\frac{4[\|B\| + \|A\| \|F\| + \|D\| \|F\|]}{R^{2p}} + \frac{4\|K\| \Gamma(2p-1)}{\Gamma^2(p) R^p} \right] E \|y(t) - z(t)\|^2
\end{aligned} \tag{3.12}$$

Similarly, by the same procedures as the above inequality, we obtains

$$\begin{aligned}
u_i(t) - s_i(t) & = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} [-v_i[u_i(t-\delta) - s_i(t-\delta)] \\
& + w_i[g_i(y_i(t)) - g_i(z_i(t))]] ds \\
|\phi_i(u_i(t)) - \phi_i(s_i(t))|^2 & \leq \frac{2}{\Gamma^2(p)} \left[\left| \int_0^t (t-s)^{p-1} v_i[u_i(s-\delta) - s_i(s-\delta)] ds \right|^2 \right. \\
& + \left. \left| \int_0^t (t-s)^{p-1} w_i[g_i(y_i(t)) - g_i(x_i(t))] ds \right|^2 \right] \\
e^{-2Rt} |\phi_i u_i(t) - \phi_i s_i(t)|^2 & \leq \frac{2}{\Gamma^2(p)} \left[v_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [u_i(s-\delta) - s_i(s-\delta)] ds \right|^2 \right. \\
& + \left. w_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [g_i(y_i(t)) - g_i(z_i(t))] ds \right|^2 \right]
\end{aligned} \tag{3.13}$$

By using Cauchy inequality

$$v_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [u_i(s-\delta) - s_i(s-\delta)] ds \right|^2$$

$$\begin{aligned}
&\leq v_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |u_i(s-\delta) - s_i(s-\delta)|^2 ds \right) \\
&\leq \frac{\Gamma(p)}{R^p} v_i^2 \left[\int_\delta^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |u_i(s-\delta) - s_i(s-\delta)|^2 ds \right. \\
&\quad \left. + \int_0^\delta (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |\psi_i(s-\delta) - \psi_i(s-\delta)|^2 ds \right] \\
&\leq \frac{\Gamma(p)}{R^p} v_i^2 \left[\int_\delta^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |u_i(s-\delta) - s_i(s-\delta)|^2 ds \right] \\
&\leq \frac{\Gamma(p)}{R^p} v_i^2 \left[\int_0^{t-\delta} (t-\gamma-\delta)^{p-1} e^{-R(t-\gamma-\delta)} e^{-2R\gamma} e^{-2R\delta} |u_i(\gamma) - s_i(\gamma)|^2 d\gamma \right] \\
&\leq \frac{\Gamma(p)}{R^p} v_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |u_i(t) - s_i(t)|^2\} e^{-2Rt} \int_0^{t-\delta} \xi^{p-1} e^{-R\xi} d\xi \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |u_i(t) - s_i(t)|^2\} \tag{3.14}
\end{aligned}$$

$$\begin{aligned}
&w_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [g_i(y_i(s)) - g_i(z_i(s))] ds \right|^2 \\
&\leq w_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} |F_i| |y_i(s) - z_i(s)| ds \right)^2 \\
&\leq w_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right)^2 \\
&\quad \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |F_i| |y_i(s) - z_i(s)|^2 ds \right) \\
&\leq w_i^2 |F_i| \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} \\
&\quad \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right)^2 \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 w_i^2 |F_i| \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} \tag{3.15}
\end{aligned}$$

From (3.14) and (3.15) in (3.13)

$$\begin{aligned}
E\|\phi_i u_i(t) - \phi_i s_i(t)\|^2 &\leq \frac{2}{\Gamma^2(p)} \left[\left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |u_i(t) - s_i(t)|^2\} \right. \\
&\quad \left. + \left[\frac{\Gamma(p)}{R^p} \right]^2 w_i^2 |F_i| \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} \right] \\
E\|\phi_i u_i(t) - \phi_i s_i(t)\|^2 &\leq \frac{2(\|V\|)}{R^{2p}} E\|u(t) - s(t)\|^2 + \frac{2\|W\|\|F\|}{R^{2p}} E\|y(t) - z(t)\|^2 \tag{3.16}
\end{aligned}$$

From (3.12) and (3.16)

$$E\|\phi_i y_i(t) - \phi_i z_i(t)\|^2 \leq \left[\frac{4(\|B\| + \|A\|\|F\| + \|D\|\|F\|)}{R^{2p}} + \frac{4\|K\|\Gamma(2p-1)}{\Gamma^2(p)R^p} \right] E\|y(t) - z(t)\|^2$$

$$E\|\phi_i u_i(t) - \phi_i s_i(t)\|^2 \leq \frac{2(\|V\|)}{R^{2p}} E\|u(t) - s(t)\|^2 + \frac{2\|W\|\|F\|}{R^{2p}} E\|y(t) - z(t)\|^2$$

By combining the both the equations we get

$$\begin{aligned} & E\|\phi_i y_i(t) - \phi_i z_i(t)\|^2 + E\|\phi_i u_i(t) - \phi_i s_i(t)\|^2 \\ & \leq \left[\frac{4[\|B\| + \|A\|\|F\| + \|D\|\|F\|]}{R^{2p}} + \frac{4\|K\|\Gamma(2p-1)}{\Gamma^2(p)R^p} \right] E\|y(t) - z(t)\|^2 \\ & + \frac{2(\|V\|)}{R^{2p}} E\|u(t) - s(t)\|^2 + \frac{2\|W\|\|F\|}{R^{2p}} E\|y(t) - z(t)\|^2 \\ & E\|\phi_i y_i(t) - \phi_i z_i(t)\|^2 + E\|\phi_i u_i(t) - \phi_i s_i(t)\|^2 \\ & \leq \left[\frac{4[\|B\| + \|A\|\|F\| + \|D\|\|F\|]}{R^{2p}} + \frac{4\|K\|\Gamma(2p-1)}{\Gamma^2(p)R^p} + \frac{2\|W\|\|F\|}{R^{2p}} \right] \\ & E\|y(t) - z(t)\|^2 + \frac{2(\|V\|)}{R^{2p}} E\|u(t) - s(t)\|^2 \end{aligned}$$

where

$$K_1 = \left[\frac{4[\|B\| + \|A\|\|F\| + \|D\|\|F\|]}{R^{2p}} + \frac{4\|K\|\Gamma(2p-1)}{\Gamma^2(p)R^p} + \frac{2\|W\|\|F\|}{R^{2p}} \right], \quad (3.17)$$

$$K_2 = \frac{2(\|V\|)}{R^{2p}}. \quad (3.18)$$

$$E\|\phi y(t) - \phi z(t)\|^2 \leq K_1 E\|y(t) - z(t)\|^2, \quad (3.19)$$

$$E\|\phi(u(t)) - \phi(s(t))\|^2 \leq K_2 E\|u(t) - s(t)\|^2. \quad (3.20)$$

$$\begin{aligned} E\|\phi(W) - \phi(V)\|^2 & \leq \text{Max}\{E\|y(t) - z(t)\|^2, E\|u(t) - s(t)\|^2\}, \\ E\|\phi(W) - \phi(V)\|^2 & \leq ME\|W - V\|^2. \end{aligned}$$

Therefore the mapping ϕ is a contraction mapping. As a consequence of the Banach fixed point theorem, the problem Eq (3.5) has a unique fixed point, so that we conclude that system Eq (3.1) has a unique solution, which complete the proof of the theorem. \square

Theorem 3.2. If Assumption 1 hold, the solution of system given by Eq (3.1) satisfying initial condition is uniformly stable in mean square.

Proof. Assume that For any two different functions $(z_1(t), \dots, z_n(t), s_1(t), \dots, s_n(t))^T$, $(y_1(t), \dots, y_n(t), u_1(t), \dots, u_n(t))^T$, solutions of Eq (3.1) with the different initial conditions $z_i = \phi_i(\gamma) \in L^2_{F_0}([-\tau, 0], R)$, $s_i = \psi_i(\gamma) \in L^2_{F_0}([-\tau, 0], R)$, $i \in n$, one has

$$D^p[y_i(t) - z_i(t)] = -b_i[y_i(t - \mu) - z_i(t - \mu)] + \sum_{k=1}^n a_{ik}[g_k(z_k(t)) - g_k(z_k(t))]$$

$$\begin{aligned}
& + \sum_{k=1}^n d_{ik}[g_k(z_k(t-\eta)) - g_k(z_k(t-\eta))] \\
& + \sum_{k=1}^n [\sigma_{ik}(z_k(t), z_k(t-\eta)) - \sigma_{ik}(z_k(t), z_k(t-\eta))] \frac{dw_k(t)}{dt} \\
D^p[u_i(t) - s_i(t)] & = -v_i[u_i(t-\delta) - s_i(t-\delta)] + w_i[g_i(y_i(t)) - g_i(z_i(t))]; i = 1, 2, \dots, n, \quad (3.21)
\end{aligned}$$

Based on Lemma 2.4, the solution of the system Eq (3.21) can be expressed in the following form

$$\begin{aligned}
y_i(t) - z_i(t) & = \psi_i(0) - \phi_i(0) + I^\alpha \left[-b_i[y_i(t-\mu) - z_i(t-\mu)] \right. \\
& + \sum_{k=1}^n a_{ik}[g_k(z_k(t)) - g_k(z_k(t))] \\
& + \sum_{k=1}^n d_{ik}[g_k(z_k(t-\eta)) - g_k(z_k(t-\eta))] \\
& \left. + \sum_{k=1}^n [\sigma_{ik}(z_k(t), z_k(t-\eta)) - \sigma_{ik}(z_k(t), z_k(t-\eta))] \frac{dw_k(t)}{dt} \right] \\
& = \psi_i(0) - \phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[-b_i[y_i(t-\mu) - z_i(t-\mu)] \right. \\
& + \sum_{k=1}^n a_{ik}[g_k(z_k(t)) - g_k(z_k(t))] + \sum_{k=1}^n d_{ik}[g_k(z_k(t-\eta)) - g_k(z_k(t-\eta))] \\
& \left. + \sum_{k=1}^n [\sigma_{ik}(z_k(t), z_k(t-\eta)) - \sigma_{ik}(z_k(t), z_k(t-\eta))] \frac{dw_k(t)}{ds} \right] ds \\
& = \psi_i(0) - \phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[-b_i[y_i(t-\mu) - z_i(t-\mu)] ds \right. \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{k=1}^n a_{ik}[g_k(z_k(t)) - g_k(z_k(t))] ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^n d_{ik}[g_k(z_k(t-\eta)) - g_k(z_k(t-\eta))] ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{k=1}^n [\sigma_{ik}(z_k(t), z_k(t-\eta)) \\
& \left. - \sigma_{ik}(z_k(t), z_k(t-\eta))] dw_k(s) \right] ds \quad (3.22)
\end{aligned}$$

Then we have

$$\begin{aligned}
e^{-2Rt}|y_i(t) - z_i(t)|^2 & \leq 5e^{-2Rt}|\psi_i(0) - \phi_i(0)|^2 \\
& + \frac{5}{\Gamma^2(\alpha)} [b_i^2] \int_0^t (t-s)^{2\alpha-2} e^{-Rt} |y_i(s-\mu) - z_i(s-\mu)|^2 ds \\
& + \left| \int_0^t (t-s)^{\alpha-1} e^{-Rt} \sum_{k=1}^n a_{ik}[g_k(y_k(s)) - g_k(z_k(s))] ds \right|^2
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^t (t-s)^{p-1} e^{-Rt} \sum_{k=1}^n d_{ik} [g_k(z_k(s-\eta)) - g_k(z_k(s-\eta))] ds \right|^2 \\
& + \left| \int_0^t (t-s)^{p-1} e^{-Rt} \sum_{k=1}^n [\sigma_{ik}(y_k, (y_k(s-\eta))) \right. \\
& \left. - \sigma_{ik}(z_k, (z_k(s-\eta)))] dw_k(s) \right|^2
\end{aligned} \tag{3.23}$$

Firstly, we observe that

$$\begin{aligned}
& b_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [y_i(s-\mu) - z_i(s-\mu)] ds \right|^2 \\
& \leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_{-\mu}^0 (t-\gamma-\mu)^{p-1} e^{-R(t-\gamma-\mu)} e^{-2R\gamma} e^{-2R\mu} \sum_{k=1}^n |\psi_k(\gamma) - \phi_k(\gamma)|^2 d\gamma \right. \\
& \left. + \int_0^{t-\mu} (t-\gamma-\mu)^{p-1} e^{-R(t-\gamma-\mu)} e^{-2R\gamma} e^{-2R\mu} \sum_{k=1}^n |y_k(\gamma) - z_k(\gamma)|^2 d\gamma \right) \\
& \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 b_i^2 \sup_{t \in [-\mu, 0]} \{e^{-2Rt} |\psi_k(t) - \phi_k(t)|^2\} \\
& + \left[\frac{\Gamma(p)}{R^p} \right]^2 b_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\}
\end{aligned} \tag{3.24}$$

Secondly, we can get from Assumption 1

$$\begin{aligned}
& \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} e^{-Rt} [a_{ik} g_k(y_k(s)) - a_{ik} g_k(z_k(s))] ds \right|^2 \\
& \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n |a_{ik}| |g_k(y_k(s)) - g_k(z_k(s))| ds \right)^2 \\
& \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n |a_{ik}| F_k |y_k(s) - z_k(s)| ds \right)^2 \\
& \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \sum_{k=1}^n |a_{ik}| F_k |y_k(s) - z_k(s)|^2 ds \right) \\
& \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \left(\sum_{k=1}^n |y_k(s) - z_k(s)|^2 \right) ds \right) \\
& \leq \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_k(s) - z_k(s)|^2 ds \right) \\
& \leq \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right)^2 \\
& \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\}
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
& \left| \int_0^t (t-s)^{p-1} e^{-Rt} \left[\sum_{k=1}^n d_{ik} g_k(y_k(s-\eta)) - \sum_{k=1}^n d_{ik} g_k(z_k(s-\eta)) \right] ds \right|^2 \\
& \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n F_k |d_{ik}| |y_k(s-\eta) - z_k(s-\eta)| ds \right)^2 \\
& \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-2R(t-s)} e^{-2Rs} \sum_{k=1}^n F_k |d_{ik}| |y_k(s-\eta) - z_k(s-\eta)|^2 ds \right) \\
& \leq \left[\frac{\Gamma(p)}{R^p} \right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \left[\int_{-\eta}^0 (t-\gamma-\eta)^{p-1} e^{-R(t-\gamma-\eta)} e^{-2R\gamma} e^{-2R\eta} \sum_{k=1}^n |\psi_k(\gamma) - \phi_k(\gamma)|^2 d\gamma \right. \\
& \quad \left. + \int_0^{t-\eta} (t-\gamma-\eta)^{p-1} e^{-R(t-\gamma-\eta)} e^{-2R\gamma} e^{-2R\eta} \sum_{k=1}^n |y_k(\gamma) - z_k(\gamma)|^2 d\gamma \right] \\
& \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n F_k^2 d_{ik}^2 \right) \sum_{k=1}^n \sup_{t \in [-\eta, 0]} \{ e^{-2Rt} |\psi_k(\gamma) - \phi_k(\gamma)|^2 \} \\
& \quad + \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n F_k^2 d_{ik}^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{ e^{-2Rt} |y_k(\gamma) - z_k(\gamma)|^2 \} \tag{3.26}
\end{aligned}$$

However, by using the Burkholder-Davis-Gundy's inequality and Assumption 1, we get that

$$\begin{aligned}
& E \left[\sup_{t \in (0, T]} \frac{5}{\Gamma^2(p)} \left| \int_0^t (t-s)^{(p-1)} e^{-Rt} \sum_{k=1}^n [\sigma_{ik}(y_k(s), y_k(s-\eta)) - \sigma_{ik}(z_k(s), z_k(s-\eta))] dw_k(s) \right|^2 \right] \\
& \leq \frac{5}{\Gamma^2(p)} E \sum_{k=1}^n \sup_{t \in (0, T]} \left| \int_0^t (t-s)^{(p-1)} e^{-Rt} \sum_{k=1}^n [\sigma_{ik}(y_k(s), y_k(s-\eta)) - \sigma_{ik}(z_k(s), z_k(s-\eta))] dw_k(s) \right|^2
\end{aligned}$$

$$\begin{aligned}
& E \sum_{k=1}^n \sup_{t \in (0, T]} \left| \int_0^t (t-s)^{(p-1)} e^{-Rt} \sum_{k=1}^n [\sigma_{ik}(y_k(s), y_k(s-\eta)) - \sigma_{ik}(z_k(s), z_k(s-\eta))] dw_k(s) \right|^2 \\
& \leq 4E \int_0^A (T-s)^{(2p-1)} e^{-2R(T-s)} e^{-2Rs} n \sum_{k=1}^n \eta_{ik} [|y_k(s) - z_k(s)|^2 + |y_k(s-\eta) - z_k(s-\eta)|^2] dw_k(s) \\
& \leq 4n \max_k \{ \eta_{ik} \} \left\{ E \int_0^T (T-s)^{(2p-1)-1} e^{-2R(T-s)} e^{-2Rs} \sum_{k=1}^n |y_k(s) - z_k(s)|^2 ds \right. \\
& \quad \left. + E \int_{-\eta}^0 (T-\gamma-\eta)^{(2p-1)-1} e^{-2R(T-\gamma-\eta)} \sum_{k=1}^n |\psi_k(\gamma) - \phi_k(\gamma)|^2 d\gamma \right\} \\
& \quad + E \int_0^{t-\eta} (A-\gamma-\eta)^{(2p-1)-1} e^{-2R(T-\gamma-\eta)} e^{-2RT} e^{-2R\eta} \sum_{k=1}^n |y_k(\gamma) - z_k(\gamma)|^2 d\gamma \} \\
& \leq 4n \max_k \{ \eta_{ik} \} \left\{ E \sum_{k=1}^n \sup_{t \in (0, T]} \{ e^{-2Nt} |y_k(s) - z_k(s)|^2 \} \int_0^T (T-s)^{(2p-1)-1} e^{-2R(T-s)} ds \right.
\end{aligned}$$

$$\begin{aligned}
& + E \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Nt} |\psi_k(s) - \phi_k(s)|^2\} e^{-2N\eta} \int_{-\eta}^0 (T - \gamma - \eta)^{(2p-1)} e^{-2R(T-\gamma-\eta)} d\gamma \\
& + E \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Nt} |y_k(s) - z_k(s)|^2\} e^{-2N\eta} \int_0^{t-\eta} (T - \gamma - \eta)^{(2p-1)-1} e^{-2R(T-\gamma-\eta)} d\gamma \} \\
& \leq \frac{\Gamma(2p-1)}{R^p} 4n \max_k \{\eta_{ik}\} \{2E \|y(t) - z(t)\|^2 + E \|\psi(t) - \phi(t)\|^2\}
\end{aligned} \tag{3.27}$$

Consequently, by combining the above inequalities together, we have

$$\begin{aligned}
E \|y_i(t) - z_i(t)\|^2 & \leq 5e^{-2Rt} |\psi_i(0) - \phi_i(0)|^2 \\
& + \frac{5}{\Gamma^2(p)} \left[\frac{\Gamma(p)}{R^p} \right]^2 b_i^2 \sum_{k=1}^n \sup_{t \in [\mu, 0]} \{e^{-2Rt} |\psi_k(\gamma) - \phi_k(\gamma)|^2\} \\
& + \frac{5}{\Gamma^2(p)} \left[\frac{\Gamma(p)}{R^p} \right]^2 b_i^2 \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(\gamma) - z_k(\gamma)|^2\} \\
& + \frac{5}{\Gamma^2(p)} \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \\
& + \frac{5}{\Gamma^2(p)} \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n F_k^2 d_{ik}^2 \right) \sum_{k=1}^n \sup_{t \in (-\eta, 0)} \{e^{-2Rt} |\psi_k(\gamma) - \phi_k(\gamma)|^2\} \\
& + \frac{5}{\Gamma^2(p)} \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n F_k^2 d_{ik}^2 \right) \sum_{k=1}^n \sup_{t \in (-\eta, 0)} \{e^{-2Rt} |y_k(\gamma) - z_k(\gamma)|^2\} \\
& + \frac{5}{\Gamma^2(p)} \frac{\Gamma(2p-1)}{R^p} 4n \max_k \{\eta_{ik}\} \{2E \|y(t) - x(t)\|^2 + E \|\psi(t) - \phi(t)\|^2\} \\
& \leq 5 \|\psi(t) - \phi(t)\| + \frac{5}{R^{2p}} \|B\| \|\psi(t) - \phi(t)\|^2 + \frac{5}{R^{2p}} \|B\| \|y(t) - z(t)\|^2 \\
& + \frac{5}{R^{2p}} \|A\| \|F\| \|y(t) - z(t)\|^2 + \frac{5}{R^{2p}} \|D\| \|F\| \|\psi(t) - \phi(t)\|^2 \\
& + \frac{5}{R^{2p}} \|D\| \|F\| \|y(t) - z(t)\|^2 + \frac{5\Gamma(2p-1)}{R^p \Gamma^2(p)} \|K\| \|y(t) - z(t)\|^2 \\
& + \frac{5\Gamma(2p-1)}{R^p \Gamma^2(p)} \|K\| \|\psi(t) - \phi(t)\|^2 \\
E \|y(t) - z(t)\|^2 & \leq 5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1) \|K\|}{R^p \Gamma^2(p)} \right] E \|\psi(t) - \phi(t)\|^2 \\
& + 5 \left[\frac{\|B\| + \|F\| \|A\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1) \|K\|}{R^p \Gamma^2(p)} \right] E \|y(t) - z(t)\|^2 \\
E \|y(t) - z(t)\|^2 & \leq \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1) \|K\|}{\Gamma^2(p) R^p} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1) \|K\|}{\Gamma^2(p) R^p} \right]} \right] E \|\psi(t) - \phi(t)\|^2
\end{aligned} \tag{3.28}$$

where,

$$L_1 = \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1) \|K\|}{\Gamma^2(p) R^p} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^p} + \frac{\Gamma(2p-1) \|K\|}{\Gamma^2(p) R^p} \right]} \right]$$

Similarly, by the same procedures as the above inequality, we obtain

$$\begin{aligned}
 u_i(t) - s_i(t) &= \psi_i(0) - \phi_i(0) \\
 &+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left[-v_i[u_i(s-\delta) - s_i(s-\delta)] \right. \\
 &\quad \left. + w_i[g_i(y_i(s)) - g_i(z_i(s))] \right] ds \\
 e^{-2Rt}|u_i(t) - s_i(t)|^2 &\leq 5e^{-2Rt}|\psi_i(0) - \phi_i(0)|^2 \\
 &+ \frac{5}{\Gamma^2(p)} \left[v_i^2 \int_0^t (t-s)^{p-1} e^{-Rt} [u_i(s-\delta) - s_i(s-\delta)]^2 \right. \\
 &\quad \left. + w_i^2 \int_0^t (t-s)^{p-1} e^{-Rt} [g_i(y_i(s)) - g_i(z_i(s))]^2 \right] ds
 \end{aligned} \tag{3.29}$$

$$w_i^2 \int_0^t (t-s)^{p-1} e^{-Rt} [g_i(y_i(s)) - g_i(z_i(s))]^2 ds \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 w_i^2 \|F\| \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \tag{3.30}$$

$$\begin{aligned}
 &v_i^2 \int_0^t (t-s)^{p-1} e^{-Rt} [u_i(s-\delta) - s_i(s-\delta)]^2 ds \\
 &\leq \frac{\Gamma(p)}{R^p} v_i^2 \int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \sum_{k=1}^n |u_k(s-\delta) - s_k(s-\delta)|^2 ds \\
 &\leq \frac{\Gamma(p)}{R^p} v_i^2 \int_{-\delta}^0 (t-\gamma-\delta)^{p-1} e^{-R(t-\gamma-\delta)} e^{-2R\gamma} e^{-2R\delta} \sum_{k=1}^n |\psi_k(\gamma) - \phi_k(\gamma)|^2 d\gamma \\
 &+ \int_0^{t-\delta} (t-\gamma-\delta)^{p-1} e^{-R(t-\gamma-\delta)} e^{-2R\gamma} e^{-2R\delta} \sum_{k=1}^n |u_k(\gamma) - s_k(\gamma)|^2 d\gamma \\
 &\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{[-\delta, 0]} \{e^{-2Rt} |\psi_k(t) - \phi_k(t)|^2\} \\
 &+ \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{(0, T]} \{e^{-2Rt} |u_k(t) - s_k(t)|^2\}
 \end{aligned} \tag{3.31}$$

Substituting the above (3.30) and (3.31) in the above equation (3.29) we get

$$\begin{aligned}
 e^{-2Rt}|u_i(t) - s_i(t)|^2 &\leq 5e^{-2Rt}|\psi_i(0) - \phi_i(0)|^2 \\
 &+ \frac{5}{\Gamma^2(p)} \left\{ \left[\frac{\Gamma(p)}{R^p} \right]^2 w_i^2 \|F\| \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \right. \\
 &+ \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{[-\delta, 0]} \{e^{-2Rt} |\psi_k(t) - \phi_k(t)|^2\} \\
 &+ \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{(0, T]} \{e^{-2Rt} |u_k(t) - v_k(t)|^2\} \left. \right\} \\
 &\leq 5\|\psi(t) - \phi(t)\|^2 + \frac{5}{R^{2p}} \|W\| \|F\| \|E\| \|y(t) - z(t)\|^2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{5}{R^{2p}} \|V\| \|E\| \|\psi(t) - \phi(t)\|^2 \\
& + \frac{5}{R^{2p}} \|V\| \|E\| \|u(t) - s(t)\|^2 \\
E\|u(t) - s(t)\|^2 & \leq 5 \left[1 + \frac{\|V\|}{R^{2p}} \right] E\|\psi(t) - \phi(t)\|^2 \\
& + \frac{5}{R^{2p}} \left[\|W\| \|F\| \right] E\|y(t) - z(t)\|^2 + \frac{5}{R^{2p}} \|V\| \|E\| \|u(t) - s(t)\|^2
\end{aligned} \tag{3.32}$$

From Eqs (3.28) and (3.32)

$$\begin{aligned}
E\|y(t) - z(t)\|^2 & \leq \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]} \right] E\|\psi(t) - \phi(t)\|^2 \\
E\|u(t) - s(t)\|^2 & \leq 5 \left[1 + \frac{\|V\|}{R^{2p}} \right] E\|\psi(t) - \phi(t)\|^2 + \frac{5}{R^{2p}} \left[\|W\| \|F\| \right] E\|y(t) - z(t)\|^2 \\
& + \frac{5\|V\|}{R^{2p}} E\|u(t) - s(t)\|^2 \\
\left[1 - \frac{5\|V\|}{R^{2p}} \right] E\|u(t) - s(t)\|^2 & \leq 5 \left[1 + \frac{\|V\|}{R^{2p}} \right] E\|\psi(t) - \phi(t)\|^2 + \frac{5}{R^{2p}} \left[\|W\| \|F\| \right] E\|y(t) - z(t)\|^2 \\
E\|u(t) - s(t)\|^2 & \leq \\
5 \left[1 + \frac{\|V\|}{R^{2p}} \right] E\|\psi(t) - \phi(t)\|^2 + \frac{5}{R^{2p}} \left[\|W\| \|F\| \right] & \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]} \right] E\|\psi(t) - \phi(t)\|^2 \\
& \frac{\left[1 - \frac{5\|V\|}{R^{2p}} \right]}{\left[1 - \frac{5\|V\|}{R^{2p}} \right]} \\
\leq \left\{ \frac{5 \left[1 + \frac{\|V\|}{R^{2p}} \right] + \frac{5}{R^{2p}} \left[\|W\| \|F\| \right] \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]} \right]}{\left[1 - \frac{5\|V\|}{R^{2p}} \right]} \right\} E\|\psi(t) - \phi(t)\|^2
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
L_1 & = \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]} \right] \\
L_2 & = \left\{ \frac{5 \left[1 + \frac{\|V\|}{R^{2p}} \right] + \frac{5}{R^{2p}} \left[\|W\| \|F\| \right] \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1)\|K\|}{\Gamma^2(p)R^p} \right]} \right]}{\left[1 - \frac{5\|V\|}{R^{2p}} \right]} \right\}.
\end{aligned}$$

by the above two equations we get

$$\begin{aligned}
E\|y(t) - z(t)\|^2 & \leq \delta_1 E\|\psi(t) - \phi(t)\|^2 \\
E\|u(t) - s(t)\|^2 & \leq \delta_2 E\|\psi(t) - \phi(t)\|^2
\end{aligned}$$

$$E\|y(t) - z(t)\|^2 \leq \varepsilon_1 \quad (3.34)$$

$$E\|\psi(t) - \phi(t)\|^2 \leq \delta_1 \quad (3.35)$$

$$E\|u(t) - s(t)\|^2 \leq \varepsilon_2 \quad (3.36)$$

$$E\|\psi(t) - \phi(t)\|^2 \leq \delta_2. \quad (3.37)$$

which means that the solution of system Eq (3.1) is uniformly stable in mean square. \square

Remark 1. In the proof of Theorem 3.2, we investigated stochastic fractional-order competitive neural networks with leakage delay without constructing Lyapunov function and by using Cauchy inequality, analysis method and Burkholder DavisGundy inequality.

Remark 2. If there are no stochastic disturbance, then system (3.4) becomes the following fractional-order competitive neural networks with leakage delay:

$$\begin{aligned} D^p z_i(t) &= -b_i z_i(t - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(t)) + \sum_{k=1}^n d_{ik} g_k(z_k(t - \eta)) + c_i s_i(t) \\ D^p s_i(t) &= -v_i s_i(t - \delta) + w_i g_i(z_i(t)); i = 1, 2, \dots, n, \end{aligned} \quad (3.38)$$

Theorem 3.3. If assume Assumption 1 hold, then the system Eq (3.38) has a unique solution.

Proof. According to the properties of the fractional calculus, one can obtain that system Eq (3.38) is equivalent to the following Volterra fractional integral with memory

$$\begin{aligned} z_i(t) &= \phi_i(0) + I^p D^p z_i(t) \\ &= \phi_i(0) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left[-b_i z_i(s - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(s)) \right. \\ &\quad \left. + \sum_{k=1}^n d_{ik} g_k(z_k(s - \eta)) + c_i s_i(s) \right] ds \end{aligned} \quad (3.39)$$

where $t \in [0, T]$. We consider a mapping $\phi : R^n \rightarrow R^n$, defined by:

$$\begin{aligned} \phi_i z_i(t) &= \phi_i(0) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left[-b_i z_i(s - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(s)) \right. \\ &\quad \left. + \sum_{k=1}^n d_{ik} g_k(z_k(s - \eta)) + c_i s_i(s) \right] ds \end{aligned} \quad (3.40)$$

where $\phi(u) = (\phi_1(u), \phi_2(u), \dots, \phi_n(u))^T$. For any two different functions $z(t) = (z_1(t), \dots, z_n(t))^T$, $y(t) = (y_1(t), \dots, y_n(t))^T$, we have

$$\begin{aligned} \phi_i y_i(t) - \phi_i z_i(t) &= \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left[-b_i [y_i(s - \mu) - z_i(s - \mu)] \right. \\ &\quad \left. + \sum_{k=1}^n [a_{ik} g_k(y_k(s)) - a_{ik} g_k(z_k(s))] \right] \end{aligned}$$

$$+ \sum_{k=1}^n [d_{ik}g_k(y_k(s-\eta)) - d_{ik}g_k(z_k(s-\eta))] ds,$$

Then, applying elementary inequality, one sees that

$$\begin{aligned} |\phi_i y_i(t) - \phi_i z_i(t)|^2 &\leq \frac{3}{\Gamma^2(p)} \left[\left| \int_0^t (t-s)^{p-1} b_i y_i(s-\mu) - z_i(s-\mu) ds \right|^2 \right. \\ &\quad + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} a_{ik} g_k(y_k(s)) - a_{ik} g_k(z_k(s)) ds \right|^2 \\ &\quad + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} d_{ik} g_k(y_k(s-\eta)) - d_{ik} g_k(z_k(s-\eta)) ds \right|^2 \\ &\quad \left. , \right. \\ e^{-2Rt} |\phi_i y_i(t) - \phi_i z_i(t)|^2 &\leq \frac{3}{\Gamma^2(p)} \left[b_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} y_i(s-\mu) - z_i(s-\mu) ds \right|^2 \right. \\ &\quad + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} e^{-Rt} [a_{ik} g_k(y_k(s)) - a_{ik} g_k(z_k(s))] ds \right|^2 \\ &\quad + \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} e^{-Rt} [d_{ik} g_k(y_k(s-\eta)) - d_{ik} g_k(z_k(s-\eta))] ds \right|^2 \left. \right], \end{aligned} \tag{3.41}$$

First, we evaluate the first term of the right hand side of the above inequality by using Cauchy's inequality to obtain

$$\begin{aligned} &b_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} y_i(s-\mu) - z_i(s-\mu) ds \right|^2 \\ &\leq b_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_i(s-\mu) - z_i(s-\mu)|^2 ds \right) \\ &\leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_i(s-\mu) - z_i(s-\mu)|^2 ds \right) \\ &\leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_\mu^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_i(s-\mu) - z_i(s-\mu)|^2 ds \right) \\ &\quad + \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_0^\mu (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |\phi_i(s-\mu) - \phi_i(s-\mu)|^2 ds \right) \\ &\leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_\mu^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_i(s-\mu) - z_i(s-\mu)|^2 ds \right) \\ &\leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_0^{t-\mu} (t-\gamma-\mu)^{p-1} e^{-R(t-\gamma-\mu)} e^{-2R\gamma} e^{-2R\mu} |y_i(\gamma) - z_i(\gamma)|^2 d\gamma \right) \\ &\leq \frac{\Gamma(p)}{R^p} b_i^2 \sup_{t \in (0, T]} \{ e^{-2Rt} |y_i(t) - z_i(t)|^2 \} e^{-2R\mu} \int_0^{t-\mu} \xi^{p-1} e^{-R\xi} d\xi \end{aligned}$$

$$\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 b_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} \quad (3.42)$$

Next, we evaluate the second term of Eq (3.41) by using Assumption 1, we have

$$\begin{aligned} & \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} e^{-Rt} [a_{ik} g_k(y_k(s)) - a_{ik} g_k(z_k(s))] ds \right|^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n |a_{ik}| |g_k(y_k(s)) - g_k(z_k(s))| ds \right)^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n |a_{ik}| F_k |y_k(s) - z_k(s)| ds \right)^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n |a_{ik}| F_k |y_k(s) - z_k(s)| ds \right)^2 \right) \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \left(\sum_{k=1}^n |y_k(s) - z_k(s)|^2 ds \right) \right) \\ & \leq \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} \sum_{k=1}^n e^{-2Rs} |y_k(s) - z_k(s)|^2 ds \right) \\ & \leq \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right)^2 \\ & \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \end{aligned} \quad (3.43)$$

$$\begin{aligned} & \left| \int_0^t (t-s)^{p-1} e^{-Rt} \left[\sum_{k=1}^n d_{ik} g_k(y_k(s-\eta)) - \sum_{k=1}^n d_{ik} g_k(z_k(s-\eta)) \right] ds \right|^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n F_k |d_{ik}| |y_k(s-\eta) - z_k(s-\eta)| ds \right)^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n F_k |d_{ik}| |y_k(s-\eta) - z_k(s-\eta)|^2 ds \right) \right) \\ & \leq \left[\frac{\Gamma(p)}{R^p} \right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n |y_k(s-\eta) - z_k(s-\eta)|^2 ds \right) \right) \\ & \leq \left[\frac{\Gamma(p)}{R^p} \right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \left[\int_0^\eta (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \sum_{k=1}^n |\phi_k(s-\eta) - \phi_k(s-\eta)|^2 ds \right. \\ & \quad \left. + \int_\eta^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \sum_{k=1}^n |y_k(s-\eta) - z_k(s-\eta)|^2 ds \right] \\ & \leq \left[\frac{\Gamma(p)}{R^p} \right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \int_0^{t-\eta} (t-\gamma-\eta)^{p-1} e^{-R(t-\gamma-\eta)} e^{-2R\gamma} e^{-2R\eta} \sum_{k=1}^n |y_k(\gamma) - z_k(\gamma)|^2 d\gamma \end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{\Gamma(p)}{R^p}\right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2\right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \int_0^{t-\eta} \xi^{p-1} e^{-R\xi} d\xi \\
&\leq \left[\frac{\Gamma(p)}{R^p}\right]^2 \left(\sum_{k=1}^n d_{ik}^2 F_k^2\right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\}
\end{aligned} \tag{3.44}$$

Thus, by combining the above inequalities together, one obtains

$$\begin{aligned}
E\|\phi y(t) - \phi z(t)\|^2 &\leq \frac{3}{\Gamma^2(p)} \left[\frac{\Gamma(p)}{R^p}\right]^2 b_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} \\
&\quad + \left[\frac{\Gamma(p)}{R^p}\right]^2 \left(\sum_{k=1}^n d_{ik}^2 F_k^2\right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \\
&\quad + \left[\frac{\Gamma(p)}{R^p}\right]^2 \left(\sum_{k=1}^n d_{ik}^2 F_k^2\right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \\
&\leq \left[\frac{3}{R^{2p}}\right] \left[\|B\| + \|A\| \|F\| + \|D\| \|F\|\right] E\|y(t) - z(t)\|^2 \\
&\leq \left[\frac{3[\|B\| + \|A\| \|F\| + \|D\| \|F\|]}{R^{2p}}\right] E\|y(t) - z(t)\|^2
\end{aligned} \tag{3.45}$$

Similarly, by the same procedures as the above inequality, we obtain

$$\begin{aligned}
u_i(t) - s_i(t) &= \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left[-v_i[u_i(t-\delta) - s_i(t-\delta)]\right. \\
&\quad \left.+ w_i[g_i(y_i(t-\delta)) - g_i(z_i(t-\delta))]\right] ds \\
|\phi_i(u_i(t)) - \phi_i s_i(t)|^2 &\leq \frac{2}{\Gamma^2(p)} \left[\left| \int_0^t (t-s)^{p-1} v_i[u_i(s-\delta) - s_i(s-\delta)] ds \right|^2 \right. \\
&\quad \left. + \left| \int_0^t (t-s)^{p-1} w_i[g_i(y_i(t)) - g_i(x_i(t))] ds \right|^2 \right] \\
e^{-2Rt} |\phi_i u_i(t) - \phi_i s_i(t)|^2 &\leq \frac{2}{\Gamma^2(p)} \left[v_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [u_i(s-\delta) - s_i(s-\delta)] ds \right|^2 \right. \\
&\quad \left. + w_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [g_i(y_i(t)) - g_i(z_i(t))] ds \right|^2 \right]
\end{aligned} \tag{3.46}$$

By using Cauchy inequality

$$\begin{aligned}
&v_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [u_i(s-\delta) - s_i(s-\delta)] ds \right|^2 \\
&\leq v_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |u_i(s-\delta) - s_i(s-\delta)|^2 ds \right) \\
&\leq \frac{\Gamma(p)}{R^p} v_i^2 \left[\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |u_i(s-\delta) - s_i(s-\delta)|^2 ds \right. \\
&\quad \left. + \int_0^\delta (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |\psi_i(s-\delta) - \psi_i(s-\delta)|^2 ds \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\Gamma(p)}{R^p} v_i^2 \left[\int_{\delta}^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |u_i(s-\delta) - s_i(s-\delta)|^2 ds \right] \\
&\leq \frac{\Gamma(p)}{R^p} v_i^2 \left[\int_0^{t-\delta} (t-\gamma-\delta)^{p-1} e^{-R(t-\gamma-\delta)} e^{-2R\gamma} e^{-2R\delta} |u_i(\gamma) - s_i(\gamma)|^2 ds \right] \\
&\leq \frac{\Gamma(p)}{R^p} v_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |u_i(t) - s_i(t)|^2\} e^{-2R\delta} \int_0^{t-\delta} \xi^{p-1} e^{-R\xi} d\xi \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |u_i(t) - s_i(t)|^2\} \tag{3.47}
\end{aligned}$$

$$\begin{aligned}
&w_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [g_i y_i(s) - g_i(z_i(s))] ds \right|^2 \\
&\leq w_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} |F_i| |y_i(s) - z_i(s)| ds \right)^2 \\
&\leq w_i^2 \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right)^2 \\
&\quad \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |F_i| |y_i(s) - z_i(s)|^2 ds \right) \\
&\leq w_i^2 |F_i| \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} \\
&\quad \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right)^2 \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 w_i^2 |F_i| \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} \tag{3.48}
\end{aligned}$$

From (3.47) and (3.48) in (3.46)

$$\begin{aligned}
E \|\phi_i u_i(t) - \phi_i s_i(t)\|^2 &\leq \frac{2}{\Gamma^2(p)} \left[\left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |u_i(t) - s_i(t)|^2\} \right. \\
&\quad \left. + \left[\frac{\Gamma(p)}{R^p} \right]^2 w_i^2 |F_i| \sup_{t \in (0, T]} \{e^{-2Rt} |y_i(t) - z_i(t)|^2\} \right] \\
E \|\phi_i u_i(t) - \phi_i s_i(t)\|^2 &\leq \frac{2(\|V\|)}{R^{2p}} E \|u(t) - s(t)\|^2 + \frac{2\|W\|\|F\|}{R^{2p}} E \|y(t) - z(t)\|^2 \tag{3.49}
\end{aligned}$$

From (3.45) and (3.49)

$$\begin{aligned}
E \|\phi_i y_i(t) - \phi_i z_i(t)\|^2 &\leq \left[\frac{3(\|B\| + \|A\|\|F\| + \|D\|\|F\|)}{R^{2p}} \right] E \|y(t) - z(t)\|^2 \\
E \|\phi_i u_i(t) - \phi_i s_i(t)\|^2 &\leq \frac{2(\|V\|)}{R^{2p}} E \|u(t) - s(t)\|^2 + \frac{2\|W\|\|F\|}{R^{2p}} E \|y(t) - z(t)\|^2
\end{aligned}$$

By combining the both the equations we get

$$E \|\phi_i y_i(t) - \phi_i z_i(t)\|^2 + E \|\phi_i u_i(t) - \phi_i s_i(t)\|^2$$

$$\begin{aligned}
&\leq \left[\frac{3(\|B\| + \|A\| \|F\| + \|D\| \|F\|)}{R^{2p}} \right] E\|y(t) - z(t)\|^2 + \frac{2(\|V\|)}{R^{2p}} E\|u(t) - s(t)\|^2 \\
&+ \frac{2\|W\| \|F\|}{R^{2p}} E\|y(t) - z(t)\|^2 \\
&E\|\phi_i y_i(t) - \phi_i z_i(t)\|^2 + E\|\phi_i u_i(t) - \phi_i s_i(t)\|^2 \\
&\leq \left[\frac{3(\|B\| + \|A\| \|F\| + \|D\| \|F\|)}{R^{2p}} + \frac{2\|W\| \|F\|}{R^{2p}} \right] E\|y(t) - z(t)\|^2 \\
&+ \frac{2(\|V\|)}{R^{2p}} E\|u(t) - s(t)\|^2
\end{aligned}$$

where

$$K_1^* = \left[\frac{3(\|B\| + \|A\| \|F\| + \|D\| \|F\|)}{R^{2p}} + \frac{2\|W\| \|F\|}{R^{2p}} \right], \quad (3.50)$$

$$K_2^* = \frac{2(\|V\|)}{R^{2p}}. \quad (3.51)$$

$$E\|\phi y(t) - \phi z(t)\|^2 \leq K_1^* E\|y(t) - z(t)\|^2, \quad (3.52)$$

$$E\|\phi(u(t)) - \phi(s(t))\|^2 \leq K_2^* E\|u(t) - s(t)\|^2. \quad (3.53)$$

Therefore the mapping ϕ is a contraction mapping. As a consequence of the Banach fixed point theorem, the problem Eq (3.40) has a unique fixed point, so that we conclude that system Eq (3.38) has a unique solution, which complete the proof of the theorem. \square

Remark 3. If there are no stochastic disturbance, then system (3.4) becomes the following fractional-order competitive neural networks with leakage delay:

$$\begin{aligned}
D^p z_i(t) &= -b_i z_i(t - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(t)) + \sum_{k=1}^n d_{ik} g_k(z_k(t - \eta)) + c_i s_i(t) \\
D^p s_i(t) &= -v_i s_i(t - \delta) + w_i g_i(z_i(t)); i = 1, 2, \dots, n,
\end{aligned} \quad (3.54)$$

Theorem 3.4. If Assumption 1 hold, the solution of system given by Eq (3.54) satisfying initial condition is uniformly stable in mean square.

Proof. Assume that $z(t) = (z_1(t), \dots, z_n(t), s_1(t), \dots, s_n(t))^T$ and $y(t) = (y_1(t), \dots, y_n(t), u_1(t), \dots, u_n(t))^T$ are solutions of Eq (3.54) with the different initial conditions $z_i = \phi_i(\gamma) \in L^2_{F_0}([- \tau, R], R)$, $s_i = \psi_i(\gamma) \in L^2_{F_0}([- \tau, R], R)$, $i \in n$, one has

$$\begin{aligned}
D^p [y_i(t) - z_i(t)] &= -b_i [y_i(t - \mu) - z_i(t - \mu)] + \sum_{k=1}^n a_{ik} [g_k(z_k(t)) - g_k(z_k(t))] \\
&+ \sum_{i=1}^n d_{ik} [g_k(z_k(t - \eta)) - g_k(z_k(t - \eta))] \\
D^p [u_i(t) - s_i(t)] &= -v_i [u_i(t - \delta) - s_i(t - \delta)] + w_i [g_i(y_i(t)) - g_i(z_i(t))]; i = 1, 2, \dots, n,
\end{aligned} \quad (3.55)$$

Based on Lemma 2.4, the solution of the system Eq (3.55) can be expressed in the following form

$$\begin{aligned}
y_i(t) - z_i(t) &= \psi_i(0) - \phi_i(0) + I^\alpha \left[-b_i[y_i(t - \mu) - z_i(t - \mu)] \right. \\
&\quad + \sum_{k=1}^n a_{ik}[g_k(z_k(t)) - g_k(z_k(t))] \\
&\quad \left. + \sum_{k=1}^n d_{ik}[g_k(z_k(t - \eta)) - g_k(z_k(t - \eta))] \right] \\
&= \psi_i(0) - \phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[-b_i[y_i(t - \mu) - z_i(t - \mu)] \right. \\
&\quad \left. + \sum_{k=1}^n a_{ik}[g_k(z_k(t)) - g_k(z_k(t))] + \sum_{k=1}^n d_{ik}[g_k(z_k(t - \eta)) - g_k(z_k(t - \eta))] \right] ds \\
&= \psi_i(0) - \phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[-b_i y_i(t - \mu) - z_i(t - \mu) \right] ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{k=1}^n a_{ik}[g_k(z_k(t)) - g_k(z_k(t))] ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{k=1}^n d_{ik}[g_k(z_k(t - \eta)) - g_k(z_k(t - \eta))] ds
\end{aligned} \tag{3.56}$$

Then we have

$$\begin{aligned}
e^{-2Rt}|y_i(t) - z_i(t)|^2 &\leq 5e^{-2Rt}|\psi_i(0) - \phi_i(0)|^2 \\
&\quad + \frac{5}{\Gamma^2(\alpha)} \left[b_i^2 \int_0^t (t-s)^{\alpha-1} e^{-Rt} (y_i(s - \mu) - z_i(s - \mu))^2 ds \right. \\
&\quad + \left| \int_0^t (t-s)^{\alpha-1} e^{-Rt} \sum_{k=1}^n a_{ik}[g_k(y_k(s)) - g_k(z_k(s))] ds \right|^2 \\
&\quad \left. + \left| \int_0^t (t-s)^{\alpha-1} e^{-Rt} \sum_{k=1}^n d_{ik}[g_k(z_k(s - \eta)) - g_k(z_k(s - \eta))] ds \right|^2 \right]
\end{aligned} \tag{3.57}$$

Firstly, we observe that

$$\begin{aligned}
&b_i^2 \int_0^t (t-s)^{\alpha-1} e^{-Rt} [y_i(s - \mu) - z_i(s - \mu)]^2 ds \\
&\leq \frac{\Gamma(p)}{R^p} b_i^2 \int_0^t (t-s)^{\alpha-1} e^{-R(t-s)} e^{-2Rs} \sum_{k=1}^n |y_k(s - \mu) - z_k(s - \mu)|^2 ds \\
&\leq \frac{\Gamma(p)}{R^p} b_i^2 \left(\int_{-\mu}^0 (t-\gamma - \mu)^{\alpha-1} e^{-R(t-\gamma-\mu)} e^{-2R\gamma} e^{-2R\mu} \sum_{k=1}^n |\psi_k(\gamma) - \phi_k(\gamma)|^2 d\gamma \right. \\
&\quad \left. + \int_0^{t-\mu} (t-\gamma - \mu)^{\alpha-1} e^{-R(t-\gamma-\mu)} e^{-2R\gamma} e^{-2R\mu} \sum_{k=1}^n |y_k(\gamma) - z_k(\gamma)|^2 d\gamma \right) \\
&\leq \left[\frac{\Gamma(p)}{R^p} \right]^2 b_i^2 \sup_{t \in [-\mu, 0]} \{e^{-2Rt} |\psi_k(t) - \phi_k(t)|^2\}
\end{aligned}$$

$$+ \left[\frac{\Gamma(p)}{R^p} \right]^2 b_i^2 \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \quad (3.58)$$

Secondly, we can get from Assumption 1

$$\begin{aligned} & \sum_{k=1}^n \left| \int_0^t (t-s)^{p-1} e^{-Rt} [a_{ik} g_k y_k(s) - a_{ik} g_k z_k(s)] ds \right|^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n |a_{ik}| |g_k y_k(s) - g_k z_k(s)| ds \right)^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n |a_{ik}| F_k |y_k(s) - z_k(s)| ds \right)^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \sum_{k=1}^n |a_{ik}| F_k |y_k(s) - z_k(s)|^2 ds \right) \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \left(\sum_{k=1}^n |y_k(s) - z_k(s)|^2 ds \right) \right) \\ & \leq \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} |y_k(s) - z_k(s)|^2 ds \right) \\ & \leq \left(\sum_{k=1}^n a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right)^2 \\ & \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(a_{ik}^2 F_k^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \end{aligned} \quad (3.59)$$

$$\begin{aligned} & \left| \int_0^t (t-s)^{p-1} e^{-Rt} \left[\sum_{k=1}^n d_{ik} g_k (y_k(s-\eta)) - \sum_{k=1}^n d_{ik} g_k (z_k(s-\eta)) \right] ds \right|^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-Rs} \sum_{k=1}^n F_k |d_{ik}| |y_k(s-\eta) - z_k(s-\eta)| ds \right)^2 \\ & \leq \left(\int_0^t (t-s)^{p-1} e^{-R(t-s)} ds \right) \left(\int_0^t (t-s)^{p-1} e^{-2R(t-s)} e^{-2Rs} \sum_{k=1}^n F_k |d_{ik}| |y_k(s-\eta) - z_k(s-\eta)|^2 ds \right) \\ & \leq \left[\frac{\Gamma(p)}{R^p} \right] \left(\sum_{k=1}^n d_{ik}^2 F_k^2 \right) \left[\int_{-\eta}^0 (t-\gamma-\eta)^{p-1} e^{-R(t-\gamma-\eta)} e^{-2R\gamma} e^{-2R\eta} \sum_{k=1}^n |\psi_k(\gamma) - \phi_k(\gamma)|^2 d\gamma \right. \\ & \quad \left. + \int_0^{t-\eta} (t-\gamma-\eta)^{p-1} e^{-R(t-\gamma-\eta)} e^{-2R\gamma} e^{-2R\eta} \sum_{k=1}^n |y_k(\gamma) - z_k(\gamma)|^2 d\gamma \right) \\ & \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n F_k^2 d_{ik}^2 \right) \sum_{k=1}^n \sup_{t \in [-\eta, 0]} \{e^{-2Rt} |\psi_k(\gamma) - \phi_k(\gamma)|^2\} \\ & \quad + \left[\frac{\Gamma(p)}{R^p} \right]^2 \left(\sum_{k=1}^n F_k^2 d_{ik}^2 \right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt} |y_k(\gamma) - z_k(\gamma)|^2\} \end{aligned} \quad (3.60)$$

Consequently, by combining the above inequalities together, we have

$$\begin{aligned}
E\|y_i(t) - z_i(t)\|^2 &\leq 5e^{-2Rt}|\psi_i(0) - \phi_i(0)|^2 \\
&+ \frac{5}{\Gamma^2(p)}\left[\frac{\Gamma(p)}{R^p}\right]^2 b_i^2 \sum_{k=1}^n \sup_{t \in [\mu, 0]} \{e^{-2Rt}|\psi_k(\gamma) - \phi_k(\gamma)|^2\} \\
&+ \frac{5}{\Gamma^2(p)}\left[\frac{\Gamma(p)}{R^p}\right]^2 b_i^2 \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt}|y_k(\gamma) - z_k(\gamma)|^2\} \\
&+ \frac{5}{\Gamma^2(p)}\left[\frac{\Gamma(p)}{R^p}\right]^2 \left(\sum_{k=1}^n a_{ik}^2 F_k^2\right) \sum_{k=1}^n \sup_{t \in (0, T]} \{e^{-2Rt}|y_k(t) - z_k(t)|^2\} \\
&+ \frac{5}{\Gamma^2(p)}\left[\frac{\Gamma(p)}{R^p}\right]^2 \left(\sum_{k=1}^n F_k^2 d_{ik}^2\right) \sum_{k=1}^n \sup_{t \in (-\eta, 0)} \{e^{-2Rt}|\psi_k(\gamma) - \phi_k(\gamma)|^2\} \\
&+ \frac{5}{\Gamma^2(p)}\left[\frac{\Gamma(p)}{R^p}\right]^2 \left(\sum_{k=1}^n F_k^2 d_{ik}^2\right) \sum_{k=1}^n \sup_{t \in (-\eta, 0)} \{e^{-2Rt}|y_k(\gamma) - z_k(\gamma)|^2\} \\
&\leq 5\|\psi(t) - \phi(t)\| + \frac{5}{R^{2p}}\|B\|\|\psi(t) - \phi(t)\|^2 + \frac{5}{R^{2p}}\|B\|\|y(t) - z(t)\|^2 \\
&+ \frac{5}{R^{2p}}\|A\|\|F\|\|y(t) - z(t)\|^2 + \frac{5}{R^{2p}}\|D\|\|F\|\|\psi(t) - \phi(t)\|^2 \\
&+ \frac{5}{R^{2p}}\|D\|\|F\|\|y(t) - z(t)\|^2 \\
E\|y(t) - z(t)\|^2 &\leq 5\left[1 + \frac{\|B\| + \|F\|\|D\|}{R^{2p}}\right]E\|\psi(t) - \phi(t)\|^2 \\
&+ 5\left[\frac{\|B\| + \|F\|\|A\| + \|F\|\|D\|}{R^{2p}}\right]E\|y(t) - z(t)\|^2 \\
E\|y(t) - z(t)\|^2 &\leq \left[\frac{5\left[1 + \frac{\|B\| + \|F\|\|D\|}{R^{2p}}\right]}{1 - 5\left[\frac{\|B\| + \|A\|\|F\| + \|F\|\|D\|}{R^{2p}}\right]}\right]E\|\psi(t) - \phi(t)\|^2
\end{aligned} \tag{3.61}$$

where,

$$L_1 = \left[\frac{5\left[1 + \frac{\|B\| + \|F\|\|D\|}{R^{2p}}\right]}{1 - 5\left[\frac{\|B\| + \|A\|\|F\| + \|F\|\|D\|}{R^{2p}}\right]}\right]$$

Similarly, by the same procedures as the above inequality, we obtain

$$\begin{aligned}
u_i(t) - s_i(t) &= \psi_i(0) - \phi_i(0) \\
&+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left[-v_i[u_i(s-\delta) - s_i(s-\delta)] \right. \\
&\quad \left. + w_i[g_i(y_i(s)) - g_i(z_i(s))] \right] ds \\
e^{-2Rt}|u_i(t) - s_i(t)|^2 &\leq 5e^{-2Rt}|\psi_i(0) - \phi_i(0)|^2 \\
&+ \frac{5}{\Gamma^2(p)} \left[v_i^2 \int_0^t (t-s)^{p-1} e^{-Rt} |u_i(s-\delta) - s_i(s-\delta)|^2 \right. \\
&\quad \left. + w_i^2 \int_0^t (t-s)^{p-1} e^{-Rt} |g_i(y_i(s)) - g_i(z_i(s))|^2 \right] ds
\end{aligned} \tag{3.62}$$

$$w_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [g_i(y_i(s)) - g_i(z_i(s))]^2 ds \right| \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 w_i^2 \|F\| \sup_{t \in (0,A]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \quad (3.63)$$

$$\begin{aligned} & v_i^2 \left| \int_0^t (t-s)^{p-1} e^{-Rt} [u_i(s-\delta) - s_i(s-\delta)]^2 ds \right| \\ & \leq \frac{\Gamma(p)}{R^p} v_i^2 \int_0^t (t-s)^{p-1} e^{-R(t-s)} e^{-2Rs} \sum_{k=1}^n |u_k(s-\delta) - s_k(s-\delta)|^2 ds \\ & \leq \frac{\Gamma(p)}{R^p} v_i^2 \int_{-\delta}^0 (t-\gamma-\delta)^{p-1} e^{-R(t-\gamma-\delta)} e^{-2R\gamma} e^{-2R\delta} \sum_{k=1}^n |\psi_k(\gamma) - \phi_k(\gamma)|^2 d\gamma \\ & \quad + \int_0^{t-\delta} (t-\gamma-\delta)^{p-1} e^{-R(t-\gamma-\delta)} e^{-2R\gamma} e^{-2R\delta} \sum_{k=1}^n |u_k(\gamma) - s_k(\gamma)|^2 d\gamma \\ & \leq \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{[-\delta,0]} \{e^{-2Rt} |\psi_k(t) - \phi_k(t)|^2\} \\ & \quad + \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{(0,T]} \{e^{-2Rt} |u_k(t) - s_k(t)|^2\} \end{aligned} \quad (3.64)$$

Substituting the above (3.63) and (3.64) in the above equation (3.62) we get

$$\begin{aligned} e^{-2Rt} |u_i(t) - s_i(t)|^2 & \leq 5e^{-2Rt} |\psi_i(0) - \phi_i(0)|^2 \\ & \quad + \frac{5}{\Gamma^2(p)} \left\{ \left[\frac{\Gamma(p)}{R^p} \right]^2 w_i^2 \|F\| \sup_{t \in (0,T]} \{e^{-2Rt} |y_k(t) - z_k(t)|^2\} \right. \\ & \quad + \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{[-\delta,0]} \{e^{-2Rt} |\psi_k(t) - \phi_k(t)|^2\} \\ & \quad + \left. \left[\frac{\Gamma(p)}{R^p} \right]^2 v_i^2 \sup_{(0,T]} \{e^{-2Rt} |u_k(t) - s_k(t)|^2\} \right\} \\ & \leq 5 \|\psi(t) - \phi(t)\|^2 + \frac{5}{R^{2p}} \|W\| \|F\| \|E\| \|y(t) - z(t)\|^2 \\ & \quad + \frac{5}{R^{2p}} \|V\| \|E\| \|\psi(t) - \phi(t)\|^2 \\ & \quad + \frac{5}{R^{2p}} \|V\| \|E\| \|u(t) - s(t)\|^2 \\ E \|u(t) - s(t)\|^2 & \leq 5 \left[1 + \frac{\|V\|}{R^{2p}} \right] E \|\psi(t) - \phi(t)\|^2 \\ & \quad + \frac{5}{R^{2p}} \left[\|W\| \|F\| \right] E \|y(t) - z(t)\|^2 + \frac{5}{R^{2p}} \|V\| \|E\| \|u(t) - s(t)\|^2 \end{aligned} \quad (3.65)$$

From Eqs (3.61) and (3.65)

$$E \|y(t) - z(t)\|^2 \leq \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^{2p}} \right]} \right] E \|\psi(t) - \phi(t)\|^2$$

$$\begin{aligned}
E\|u(t) - s(t)\|^2 &\leq 5\left[1 + \frac{\|V\|}{R^{2p}}\right]E\|\psi(t) - \phi(t)\|^2 \\
&\quad + \frac{5}{R^{2p}}\left[\|W\|\|F\|\right]E\|y(t) - x(t)\|^2 \\
&\quad + \frac{5\|V\|}{R^{2p}}E\|u(t) - s(t)\|^2 \\
\left[1 - \frac{5\|V\|}{R^{2p}}\right]E\|u(t) - s(t)\|^2 &\leq 5\left[1 + \frac{\|V\|}{R^{2p}}\right]E\|\psi(t) - \phi(t)\|^2 \\
&\quad + \frac{5}{R^{2p}}\left[\|W\|\|F\|\right]E\|y(t) - z(t)\|^2 \\
E\|u(t) - s(t)\|^2 &\leq \frac{5\left[1 + \frac{\|V\|}{R^{2p}}\right]E\|\psi(t) - \phi(t)\|^2 + \frac{5}{R^{2p}}\left[\|W\|\|F\|\right]\left[\frac{5\left[1 + \frac{\|B\| + \|F\|\|D\|}{R^{2p}}\right]}{1 - 5\left[\frac{\|B\| + \|A\|\|D\| + \|F\|\|D\|}{R^{2p}}\right]}\right]E\|\psi(t) - \phi(t)\|^2}{\left[1 - \frac{5\|V\|}{R^{2p}}\right]} \\
&\leq \left\{ \frac{5\left[1 + \frac{\|V\|}{R^{2p}}\right] + \frac{5}{R^{2p}}\left[\|W\|\|F\|\right]\left[\frac{5\left[1 + \frac{\|B\| + \|F\|\|D\|}{R^{2p}}\right]}{1 - 5\left[\frac{\|B\| + \|A\|\|D\| + \|F\|\|D\|}{R^{2p}}\right]}\right]}{\left[1 - \frac{5\|V\|}{R^{2p}}\right]} \right\} E\|\psi(t) - \phi(t)\|^2 \\
L_1^* &= \left[\frac{5\left[1 + \frac{\|B\| + \|F\|\|D\|}{R^{2p}}\right]}{1 - 5\left[\frac{\|B\| + \|A\|\|D\| + \|F\|\|D\|}{R^{2p}}\right]} \right] \\
L_2^* &= \left\{ \frac{5\left[1 + \frac{\|V\|}{R^{2p}}\right] + \frac{5}{R^{2p}}\left[\|W\|\|F\|\right]\left[\frac{5\left[1 + \frac{\|B\| + \|F\|\|D\|}{R^{2p}}\right]}{1 - 5\left[\frac{\|B\| + \|A\|\|D\| + \|F\|\|D\|}{R^{2p}}\right]}\right]}{\left[1 - \frac{5\|V\|}{R^{2p}}\right]} \right\}.
\end{aligned}$$

By the above two equations we get

$$E\|y(t) - z(t)\|^2 \leq \delta_1 E\|\psi(t) - \phi(t)\|^2$$

$$E\|u(t) - s(t)\|^2 \leq \delta_2 E\|\psi(t) - \phi(t)\|^2,$$

$$E\|y(t) - z(t)\|^2 \leq \varepsilon_1 \tag{3.66}$$

$$E\|\psi(t) - \phi(t)\|^2 \leq \delta_1 \tag{3.67}$$

$$E\|u(t) - s(t)\|^2 \leq \varepsilon_2 \tag{3.68}$$

$$E\|\psi(t) - \phi(t)\|^2 \leq \delta_2. \tag{3.69}$$

which means that the solution of system Eq (3.54) is uniformly stable in mean square. \square

Remark 4. The author derived the existence and uniqueness results using Banach contraction fixed point theorem, sufficient conditions for uniform stability of equilibrium point for the networks. But, it is more complicated to study the stochastic fractional-order competitive neural networks with leakage delays. This interesting problem competitive neural networks model.

Remark 5. In this paper we investigated the mean square stability of stochastic fractional-order competitive neural networks with leakage delays with $\frac{1}{2} \leq \alpha < 1$ by using Cauchy Schwarz inequality. Many authors have focused on studying the stability analysis of fractional-order neural networks which depends on the orders α of fractional derivatives. Unlike the previous works, we analyzed the stability of stochastic fractional-order neural networks with delays and leakage terms which are dependent on the orders α and β different fractional derivatives and reflect the close relation between neuron activation functions, time-delay of network parameters, and coefficient terms. This was motivated by the long range delay dependent dynamic process that is part of our current project. However, we propose to investigate the proposed problem for not only $0 < \alpha < 1$, but also more general set of linearly independent multi time-scales.

4. Numerical examples

Example 4.1. Consider the stochastic fractional-order competitive neural networks with leakage delay terms as follows:

$$\begin{aligned} D^p z_i(t) &= -b_i z_i(t - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(t)) + \sum_{i=1}^n d_{ik} g_k(z_k(t - \eta)) + c_i s_i(t) \\ &\quad + \sum_{k=1}^n \sigma_{ik}(z_k(t), z_k(t - \eta)) \frac{dw_k(t)}{dt} \\ D^p s_i(t) &= -v_i s_i(t - \delta) + w_i g_i(z_i(t)); i = 1, 2, \dots, n, \end{aligned} \quad (4.1)$$

where $p = 0.9$, $g_1(z_1(t)) = \frac{1}{2}(|z_1(t) + 1| + |z_1(t) - 1|)$, $g_2(z_2(t)) = z_2(t)$. $\sigma_{ik}(z_k(t), z_k(t - \eta)) = \frac{\sqrt{6}}{40} z_j(t)$, $i, j = 1, 2$. Obviously, $\eta = \frac{1}{2}$. Choose $b_1 = 0.8$, $b_2 = 0.5$, $a_{11} = a_{12} = 0.89$, $a_{21} = \frac{0.6}{7}$, $a_{22} = 0.76$, $d_{11} = 0.98$, $d_{12} = 0.99$, $d_{21} = \frac{0.6}{7}$, $d_{22} = -0.96$. Clearly, the nonlinear functions $g_j(\cdot)$ and $\sigma(\cdot)$ satisfy condition Assumption 1. We can get the following inequality easily $\|B\| = 4.297$, $\|A\| = 6.429$, $\|D\| = 9.267$, $\|F\| = 0.428$, $\|K\| = 8.876$, $\|V\| = 5.245$, $\|W\| = 6.356$, $p = 3.467$, $R = 4.274$.

Substituting in Theorem 3.1, we get

$$\begin{aligned} K_1 &= \left[\frac{4[\|B\| + \|A\|\|F\| + \|D\|\|F\|]}{R^{2p}} + \frac{4\|K\|\Gamma(2p-1)}{\Gamma^2(p)R^p} \right] + \frac{2\|F\|\|W\|}{R^{2p}} = 2.5576, \\ K_2 &= \frac{2\|V\|}{R^{2p}} = 0.0005 \end{aligned}$$

which satisfy $K_1 > K_2$, By Theorem 3.1, the system (4.1) is uniformly stable in mean square. State trajectories of the system are given in Figure 1.

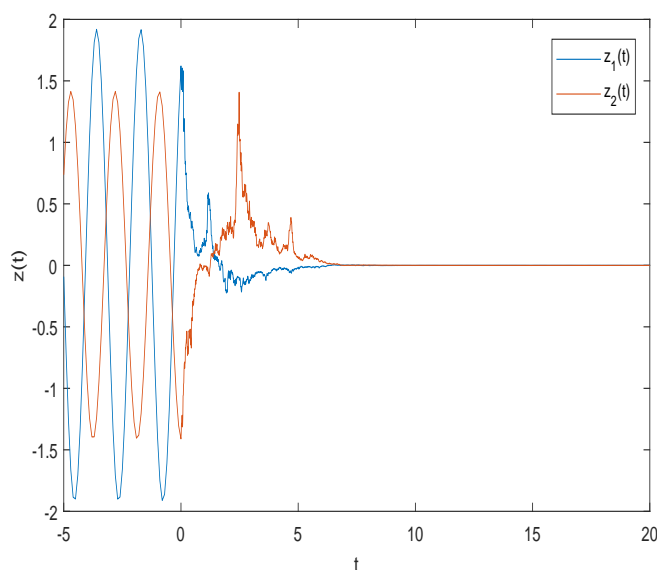


Figure 1. State trajectories of the system in Example 4.1.

Example 4.2. Consider the stochastic fractional-order competitive neural networks with leakage delay terms as follows:

$$\begin{aligned}
 D^p z_i(t) &= -b_i z_i(t - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(t)) + \sum_{i=1}^n d_{ik} g_k(z_k(t - \eta)) + c_i s_i(t) \\
 &\quad + \sum_{k=1}^n \sigma_{ik}(z_k(t), z_k(t - \eta)) \frac{dw_k(t)}{dt} \\
 D^p s_i(t) &= -v_i s_i(t - \delta) + w_i g_i(z_i(t)); i = 1, 2, \dots, n,
 \end{aligned} \tag{4.2}$$

where $p = 0.2$, $g_1(z_1(t)) = \frac{1}{2}(|z_1(t) + 1| + |z_1(t) - 1|)$, $g_2(z_2(t)) = z_2(t)$. $\sigma_{ik}(z_k(t), z_k(t - \eta)) = \frac{\sqrt{3}}{20} z_j(t)$, $i, j = 1, 2$. Obviously, $\eta = \frac{1}{2}$. Choose $b_1 = 0.89$, $b_2 = 0.15$, $a_{11} = a_{12} = 0.19$, $a_{21} = \frac{0.2}{7}$, $a_{22} = 0.16$, $d_{11} = 0.18$, $d_{12} = 0.1$, $d_{21} = \frac{0.2}{7}$, $d_{22} = -2.96$. We can get the following inequality easily $\|B\| = 2.256$, $\|A\| = 4.257$, $\|D\| = 7.749$, $\|F\| = 9.467$, $\|K\| = 6.474$, $\|V\| = 6.667$, $\|W\| = 5.678$, $P = 2.227$, $R = 5.729$.

Substituting in Theorem 3.2, we get

$$\begin{aligned}
 L_1 &= \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1) \|K\|}{\Gamma^2(p) R^p} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1) 2 \|K\|}{\Gamma^2(p) R^p} \right]} \right] = 65.7507. \\
 L_2 &= \left\{ \frac{5 \left[1 + \frac{\|v\|}{R^{2p}} \right] + \frac{5}{R^{2p}} \left[\|w\| \|F\| \right] \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1) \|K\|}{\Gamma^2(p) R^p} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|D\| + \|F\| \|D\|}{R^{2p}} + \frac{\Gamma(2p-1) \|K\|}{\Gamma^2(p) R^p} \right]} \right]}{\left[1 - \frac{5 \|v\|}{R^{2p}} \right]} \right\} \\
 L_2 &= \frac{5.1594}{0.9833} = 5.2470.
 \end{aligned}$$

which satisfy $L_1 > L_2$. By Theorem 3.2, the system (4.2) is uniformly stable in mean square. State trajectories of the system are given in Figure 2.

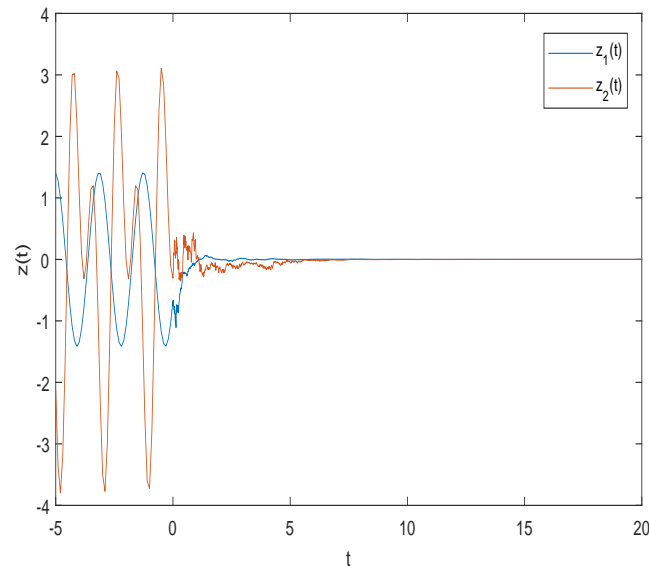


Figure 2. State trajectories of the system in Example 4.2.

Example 4.3. Consider the fractional-order competitive neural networks with leakage delay terms as follows:

$$\begin{aligned} D^p z_i(t) &= -b_i z_i(t - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(t)) + \sum_{i=1}^n d_{ik} g_k(z_k(t - \eta)) + c_i s_i(t) \\ D^p s_i(t) &= -v_i s_i(t - \delta) + w_i g_i(z_i(t)); i = 1, 2, \dots, n, \end{aligned} \quad (4.3)$$

where $p = 0.9$, $g_1(z_1(t)) = \frac{1}{2}(|z_1(t) + 1| + |z_1(t) - 1|)$, $g_2(z_2(t)) = z_2(t)$. $\sigma_{ik}(z_k(t), z_k(t - \eta)) = \frac{\sqrt{7}}{40} z_j(t)$, $i, j = 1, 2$. Obviously, $\eta = \frac{1}{2}$. Choose $b_1 = 0.6$, $b_2 = 0.45$, $a_{11} = a_{12} = 0.39$, $a_{21} = \frac{0.1}{7}$, $a_{22} = 0.64$, $d_{11} = 0.38$, $d_{12} = 0.99$, $d_{21} = \frac{0.2}{7}$, $d_{22} = -2.96$. Clearly, the nonlinear functions $g_j(\cdot)$ and $\eta(\cdot)$ satisfy condition Assumption 1. We can get the following inequality easily, $\|B\| = 0.264$, $\|A\| = 0.467$, $\|D\| = 0.124$, $\|F\| = 1.246$, $\|V\| = 0.276$, $\|W\| = 0.227$, $R = 0.224$. Substituting above values in Theorem 3.3, we get

$$\begin{aligned} K_1^* &= \frac{3[\|B\| + \|A\|\|F\| + \|D\|\|F\|]}{R^{2p}} + \frac{2\|W\|\|F\|}{R^{2p}} = 8.1616, \\ K_2^* &= \frac{2(\|V\|)}{R^{2p}} = 0.9864. \end{aligned}$$

which satisfy $K_1^* > K_2^*$. By Theorem 3.3, the system (4.3) is uniformly stable in mean square. State trajectories of the system are given in Figure 3.

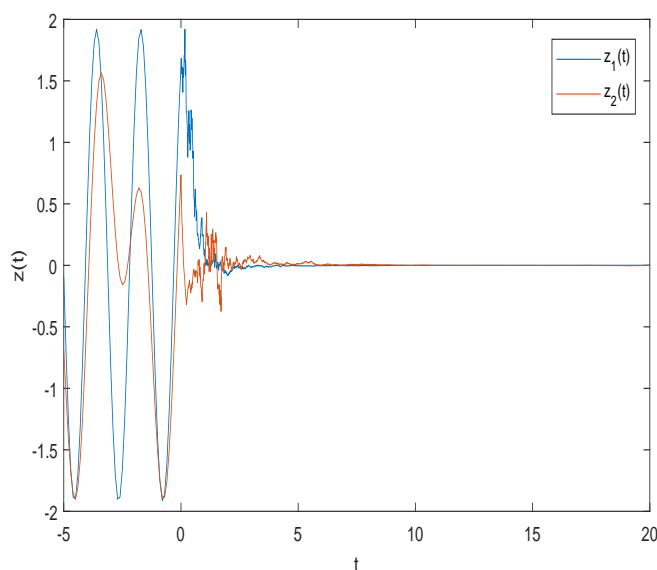


Figure 3. State trajectories of the system in Example 4.3.

Example 4.4. Consider the fractional-order competitive neural networks with leakage delay terms as follows:

$$\begin{aligned} D^p z_i(t) &= -b_i z_i(t - \mu) + \sum_{k=1}^n a_{ik} g_k(z_k(t)) + \sum_{i=1}^n d_{ik} g_k(z_k(t - \eta)) + c_i s_i(t) \\ D^p s_i(t) &= -v_i s_i(t - \delta) + w_i g_i(z_i(t)); i = 1, 2, \dots, n, \end{aligned} \quad (4.4)$$

where $p = 0.9$, $g_1(z_1(t)) = \frac{1}{2}(|z_1(t) + 1| + |z_1(t) - 1|)$, $g_2(z_2(t)) = z_2(t)$. $\sigma_{ik}(z_k(t), z_k(t - \eta)) = \frac{\sqrt{9}}{40} z_j(t)$, $i, j = 1, 2$. Obviously, $\eta = \frac{1}{2}$. Choose $b_1 = 0.2$, $b_2 = 0.1$, $a_{11} = a_{12} = 0.19$, $a_{21} = \frac{0.2}{7}$, $a_{22} = 0.16$, $d_{11} = 0.28$, $d_{12} = 0.19$, $d_{21} = \frac{0.4}{7}$, $d_{22} = -0.46$. We can get the following inequality easily $\|B\| = 2.227$, $\|A\| = 1.246$, $\|D\| = 4.222$, $\|F\| = 3.328$, $\|V\| = 4.242$, $\|W\| = 6.267$, $R = 9.474$. Substituting above values in Theorem 3.4, we get

$$\begin{aligned} L_1^* &= \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|F\| + \|F\| \|D\|}{R^{2p}} \right]} \right] = 3.1883, \\ L_2^* &= \left\{ \frac{5 \left[1 + \frac{\|v\|}{R^{2p}} \right] + \frac{5}{R^{2p}} \left[\|w\| \|F\| \right] \left[\frac{5 \left[1 + \frac{\|B\| + \|F\| \|D\|}{R^{2p}} \right]}{1 - 5 \left[\frac{\|B\| + \|A\| \|F\| + \|F\| \|D\|}{R^{2p}} \right]} \right]}{1 - \frac{5 \|v\|}{R^{2p}}} \right\} \\ &= \frac{6.6008}{-2.8153} = -2.3446 \end{aligned}$$

which satisfy $L_1^* > L_2^*$. By Theorem 3.4, the system (4.4) is uniformly stable in mean square. State trajectories of the system are given in Figure 4.

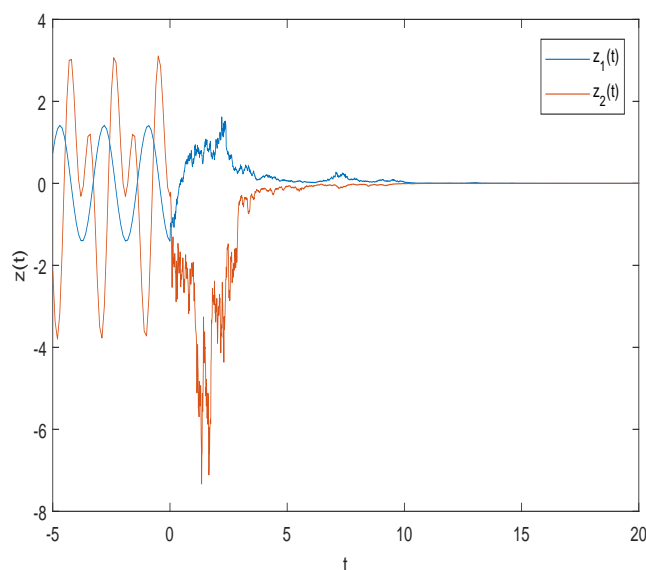


Figure 4. State trajectories of the system in Example 4.4.

5. Conclusion

In this paper, we investigate the stability analysis of stochastic fractional-order competitive neural networks with leakage delay. As is well known, there are many stability results about integer-order neural networks in the past few decades, most of which are obtained by constructing Lyapunov function, but these results and methods could not be extended and applied to fractional-order case. According to the Cauchy-Schwartz inequality, Burkholder-Davis-Gundy inequality, analysis techniques, some sufficient conditions were derived to guarantee the existence and uniqueness and the uniform stability in mean square. Furthermore, the main tools used in this paper are stochastic analysis techniques, fractional calculations and Banach contraction principle. Finally, four numerical examples are given to illustrate the effectiveness of the proposed theories.

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Conflict of interest

All authors declare there is no conflict of interest in this paper.

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