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Research article

Geometric invariants and focal surfaces of spacelike curves in de Sitter space from a caustic viewpoint

Haiming Liu and Jiajing Miao*

School of Mathematics, Mudanjiang Normal University, Mudanjiang, 157011, China

* **Correspondence:** Email: jiajing0407@126.com.

Abstract: The focal surface of a generic space curve in Euclidean 3-space is a classical subject which is a two dimensional caustic and has Lagrangian singularities. In this paper, we define the first de Sitter focal surface and the second de Sitter focal surface of de Sitter spacelike curve and consider their singular points as an application of the theory of caustics and Legendrian dualities. The main results state that de Sitter focal surfaces can be seen as two dimensional caustics which have Lagrangian singularities. To characterize these singularities, a useful new geometric invariant $\rho(s)$ is discovered and two dual relationships between focal surfaces and spacelike curve are given. Three examples are used to demonstrate the main results.

Keywords: invariant; caustic; focal surface; de Sitter space; singularity theory
Mathematics Subject Classification: 53A25, 58K35

1. Introduction

The geometry of submainfolds immersed in de Sitter space is an important subject that has fascinated many mathematicians and physicists [1–7]. Up to now, different types of surfaces and curves in de Sitter space such as spacelike curves, timelike curves and null curves have been studied. For instance, in [1], Wang and Pei considered null Cartan curves in de Sitter 3-space and classified the singularities of ruled null surfaces generated by these curves. In [2], the authors investigate the singularities of normal hypersurfaces of de Sitter timelike curves. In [3, 4], the authors classified Weingarten rotation surfaces and Hyperbolic rotation surfaces in de Sitter 3-space. There are also some important works on hypersurfaces immersed in hyperbolic space and lightcone [8–10]. Because there are two kinds of spacelike curves in S^3_1, one is the spacelike curve with spacelike normal vector $\mathbf{n}$ and the other is the spacelike curve with timelike normal vector $\mathbf{n}$, the case of spacelike curve that is immersed in a three-dimensional de Sitter space is more sophisticated and interesting than timelike curves in de Sitter space [5].

The focal surface of a space curve in Euclidean space is the analogue of the evolute of a plane curve which is well defined and is a smooth curve away from the inflection points of the plane curve. It is local bifurcation set of the family of distance squared functions on plane curve and is the critical value of a Lagrangian map, i.e. it is a caustic. As a consequence, it has only Lagrangian singularities. We can conclude that the evolute of a plane curve has an ordinary cusp singularity at points corresponding to ordinary vertices of the plane curve. However, it is very hard to know is there focal surface for spacelike curve in de Sitter space and what does the focal surface look like from the classical differential geometry view point. The classical method has several limitations. For instance, it does not define focal surface in a natural way, explain which singularities could appear in focal surface and how these bifurcate as the original curve is deformed. It is also misses to capture the deep concepts such as caustic which involved. But, it is very powerful to use singularity theory to find new focal surface and describe their singularities more finer. For a spacelike curve $\gamma : I \to S^3_1 \subset \mathbb{R}^4_1$ with spacelike normal vector $\mathbf{n}$ and nowhere vanishing curvature, we find its associated first de Sitter focal surface and for a spacelike curve $\gamma : I \to S^3_1 \subset \mathbb{R}^4_1$ with timelike normal vector $\mathbf{n}$ and nowhere vanishing curvature, we find its associated second de Sitter focal surface. It is shown that de Sitter focal surfaces are two dimensional caustics which have Lagrangian singularities. In order to characterize the types of singularities of de Sitter focal surfaces via differential calculations, we find a de Sitter invariant of γ which is defined to be

$$\rho(s) = \kappa_g^2(s)\tau_g^3(s) + \kappa_g(s)\kappa_g''(s)\tau_g(s) - 2(\kappa_g'(s))^2\tau_g(s) - \kappa_g(s)\kappa_g'(s)\tau_g'(s).$$

For detail, the caustic is a regular surface at an A_2-singularity of $h^S_v(s)$. It is a cuspidal edge at an A_3-singularity of $h^S_v(s)$ and has swallowtail singularity at an A_4-singularity of $h^S_v(s)$. Moreover, we use Legendrian duality to investigate the spacelike curve and de Sitter focal surfaces related by duality. We find that the spacelike curve $\gamma(s)$ and de Sitter focal surfaces are Δ_5-dual to each other (cf., Proposition 5.1). Meanwhile, we summarize the results in Table 1.

Table 1. Geometric conditions for the singularities of focal surfaces.

Curve	Focal surface	Duality	Cuspidal edge	Swallowtail	Contact order
$\gamma(s)$	FDF_γ	Δ_5	$\rho(s_0) \neq 0$	$\rho(s_0) = 0$ and $\rho'(s_0) \neq 0$	4 or 5
$\gamma(s)$	SDF_γ	Δ_5	$\rho(s_0) \neq 0$	$\rho(s_0) = 0$ and $\rho'(s_0) \neq 0$	4 or 5

In Section 2, two classes of focal surfaces of spacelike curves in de Sitter 3-space, where the ones are generated by the spacelike curve with spacelike normal vector $\mathbf{n}$ and the other ones are related to the spacelike curve with timelike normal vector $\mathbf{n}$, and the the main theorems (cf., Theorem 2.1 and Theorem 2.2) are presented. In Section 3, we present de Sitter height function on spacelike curve γ and establish the equivalent relations between A_k-singularities of the height function and geometric invariant $\rho(s)$ (cf., Proposition 3.1 and Proposition 3.2). Moreover, we discuss in detail the geometric meanings of the invariant $\rho(s)$ (cf., Corollary 3.3 and Proposition 3.4). In Section 4, by applying some general results of the singularity theory to de Sitter focal surfaces, we give the proof of the main theorems (cf., Theorem 2.1 and Theorem 2.2) so as to complete the classifications of singularities of the de Sitter focal surfaces. In Section 5, we investigate the relationships between the de Sitter focal surfaces and the spacelike curves by Legendrian dualities [11]. To better illustrate our results, we give three examples in Section 6.

All maps considered here are of class C^∞ unless otherwise stated.

2. Preliminaries

In this section, we give the basic notions on Minkowski 4-space and de Sitter 3-space. Let $\mathbb{R}^4$ be a 4-dimensional vector space. For any two vectors $\mathbf{x} = (x_0, x_1, \cdots, x_3)$, $\mathbf{y} = (y_0, y_1, \cdots, y_3)$ in $\mathbb{R}^4$, their pseudo scalar product is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$. $(\mathbb{R}^4, \langle, \rangle)$ is called 4-dimensional Minkowski space. We denote it as $\mathbb{R}^4_1$. Pseudo vector product of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ is defined by

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \begin{vmatrix} -\mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{vmatrix},$$

where $\mathbf{x} = (x_0, x_1, x_2, x_3)$, $\mathbf{y} = (y_0, y_1, y_2, y_3)$, $\mathbf{z} = (z_0, z_1, z_2, z_3)$ are in $\mathbb{R}^4_1$, $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical basis of $\mathbb{R}^4_1$. We remark that $\langle \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}, \mathbf{w} \rangle = det(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$. A non-zero vector $\mathbf{x}$ in $\mathbb{R}^4_1$ is called spacelike, lightlike or timelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, respectively. The norm of a nonzero vector $\mathbf{x} \in \mathbb{R}^4_1$ is defined by $\| \mathbf{x} \| = \sqrt{| \langle \mathbf{x}, \mathbf{x} \rangle |}$. We define *de Sitter three-space* by

$$S^3_1 = \{\mathbf{x} \in \mathbb{R}^4_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}.$$

Let $\gamma : I \to S^3_1 \subset \mathbb{R}^4_1$ be a smooth spacelike curve parameterized by arc-length parameter s, so we have the unit spacelike tangent vector $\mathbf{t}(s) = \gamma'(s)$. Under the assumption that $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq 1$, one can construct a unit vector $\mathbf{n}(s) = \frac{\mathbf{t}'(s)+\gamma(s)}{\|\mathbf{t}'(s)+\gamma(s)\|}$. Moreover, define $\mathbf{e}(s) = \gamma(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}(s)$, then we can define a pseudo orthonormal frame $\{\gamma(s), \mathbf{t}(s), \mathbf{n}(s), \mathbf{e}(s)\}$ of $\mathbb{R}^4_1$ along $\gamma(s)$. We have the following Frenet-Serret type formula:

$$\begin{pmatrix} \gamma'(s) \\ \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{e}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa_g(s) & 0 \\ 0 & \kappa_g(s)\delta(s) & 0 & \tau_g(s) \\ 0 & 0 & \tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{e}(s) \end{pmatrix},$$

where $\delta(s) = -\langle \mathbf{n}(s), \mathbf{n}(s) \rangle$, $\kappa_g(s) = \| \mathbf{t}'(s) + \gamma(s) \|$ and $\tau_g(s) = \frac{1}{\kappa_g^2(s)} det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))$.

For a vector $\mathbf{v} \in \mathbb{R}^4_1$ and a real number c, we define the hyperplane with pseudo-normal vector $\mathbf{v}$ by $HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}^4_1 \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}$. The $HP(\mathbf{v}, c)$ is called a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\mathbf{v}$ is timelike, spacelike or lightlike respectively. Typical surfaces in de Sitter 3-space are given by the intersection of S^3_1 with a hyperplane in $\mathbb{R}^4_1$. A surface $S^3_1 \cap HP(\mathbf{v}, c)$ is elliptic, hyperbolic or parabolic if $\mathbf{v}$ is timelike, spacelike or lightlike respectively. For any $r \in \mathbb{R}$ and $\mathbf{v}_0 \in S^3_1$, we denote $HPS^1(\mathbf{v}_0, r) = \{\mathbf{v} \in S^3_1 \mid \langle \mathbf{v}, \mathbf{v}_0 \rangle = r\}$. We call $HPS^1(\mathbf{v}_0, r)$ a hyperbolic pseudo-sphere in S^3_1 with the center $\mathbf{v}_0$. If $\delta(s) = -1$, we define *the first de Sitter focal surface* of spacelike curve by

$$FDF_\gamma : I \times \mathbb{R} \to S^3_1;\ FDF_\gamma(s, \theta) = \frac{cosh\theta}{\sqrt{\kappa_g^2(s) + 1}} \Big(\kappa_g(s)\gamma(s) + \mathbf{n}(s)\Big) + sinh\theta \mathbf{e}(s).$$

If $\delta(s) = 1$, we define a map as follow:

$$SDF_\gamma : I \times J \to S^3_1;\ SDF_\gamma(s, \theta) = \frac{cos\theta}{\sqrt{\kappa_g^2(s) - 1}} \Big(-\kappa_g(s)\gamma(s) + \mathbf{n}(s)\Big) + sin\theta \mathbf{e}(s),$$

where $\kappa_g(s) > 1$ and $J \in [0, 2\pi)$, and

$$SDF_\gamma : I \times J \to S^3_1;\ SDF_\gamma(s,\theta) = \frac{sinh\theta}{\sqrt{1-\kappa_g^2(s)}}\Big(-\kappa_g(s)\gamma(s) + \mathbf{n}(s)\Big) + cosh\theta\mathbf{e}(s),$$

where $0 < \kappa_g(s) < 1$ and J is an open interval in $\mathbb{R}$. We call $SDF_\gamma(s,\theta)$ *the second de Sitter focal surface* of γ. By some calculations, we can get that

$$\frac{\partial FDF_\gamma}{\partial s}(s,\theta) = \frac{\kappa_g' cosh\theta}{(1+\kappa_g^2)^{\frac{3}{2}}}\gamma(s) + (-(1+\kappa_g^2)^{-\frac{3}{2}}\kappa_g\kappa_g' cosh\theta + \tau_g sinh\theta)\mathbf{n}(s) + \frac{\tau_g cosh\theta}{(1+\kappa_g^2)^{\frac{1}{2}}}\mathbf{e}(s),$$

$$\frac{\partial FDF_\gamma}{\partial \theta}(s,\theta) = \frac{\kappa_g sinh\theta}{(1+\kappa_g^2)^{\frac{1}{2}}}\gamma(s) + \frac{sinh\theta}{(1+\kappa_g^2)^{\frac{1}{2}}}\mathbf{n}(s) + cosh\theta\mathbf{e}(s).$$

$\frac{\partial FDF_\gamma}{\partial s}(s,\theta)$ and $\frac{\partial FDF_\gamma}{\partial \theta}(s,\theta)$ are linearly dependent if and only if there exists an nonzero $\lambda \in \mathbb{R}$, such that $\frac{\partial FDF_\gamma}{\partial s}(s,\theta) = \lambda\frac{\partial FDF_\gamma}{\partial \theta}(s,\theta)$, i.e.

$$\begin{cases} \lambda\dfrac{\kappa_g sinh\theta}{(1+\kappa_g^2)^{\frac{1}{2}}} = \dfrac{\kappa_g' cosh\theta}{(1+\kappa_g^2)^{\frac{3}{2}}}, \\ \lambda\dfrac{sinh\theta}{(1+\kappa_g^2)^{\frac{1}{2}}} = -(1+\kappa_g^2)^{-\frac{3}{2}}\kappa_g\kappa_g' cosh\theta + \tau_g sinh\theta, \\ \lambda cosh\theta = \dfrac{\tau_g cosh\theta}{(1+\kappa_g^2)^{\frac{1}{2}}}. \end{cases} \tag{2.1}$$

This is equivalent to $\lambda = \frac{\tau_g}{(1+\kappa_g^2)^{\frac{1}{2}}}$ and $\kappa_g(s)\tau_g \sinh\theta - \frac{\kappa_g' cosh\theta}{(1+\kappa_g^2)^{\frac{1}{2}}} = 0$. This means that the first de Sitter focal surface $FDF_\gamma(s,\theta)$ is singular at a point (s_0,θ_0) if and only if

$$tanh\theta_0 = \frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{1+\kappa_g^2(s_0)}}.$$

By similar calculations, when $\delta(s) = 1$ and $\kappa_g(s) > 1$, we can get that

$$\frac{\partial SDF_\gamma}{\partial s}(s,\theta) = \frac{\kappa_g' cos\theta}{(\kappa_g^2-1)^{\frac{3}{2}}}\gamma(s) + (-(\kappa_g^2-1)^{-\frac{3}{2}}\kappa_g\kappa_g' cos\theta + \tau_g sin\theta)\mathbf{n}(s) + \frac{\tau_g cos\theta}{(\kappa_g^2-1)^{\frac{1}{2}}}\mathbf{e}(s),$$

$$\frac{\partial SDF_\gamma}{\partial \theta}(s,\theta) = \frac{\kappa_g sin\theta}{(\kappa_g^2-1)^{\frac{1}{2}}}\gamma(s) - \frac{sin\theta}{(\kappa_g^2-1)^{\frac{1}{2}}}\mathbf{n}(s) + cos\theta\mathbf{e}(s).$$

$\frac{\partial SDF_\gamma}{\partial s}(s,\theta)$ and $\frac{\partial SDF_\gamma}{\partial \theta}(s,\theta)$ are linearly dependent if and only if there exists an nonzero $\lambda \in \mathbb{R}$, such that $\frac{\partial SDF_\gamma}{\partial s}(s,\theta) = \lambda\frac{\partial SDF_\gamma}{\partial \theta}(s,\theta)$, i.e.

$$\begin{cases} \lambda\dfrac{\kappa_g sin\theta}{(\kappa_g^2-1)^{\frac{1}{2}}} = \dfrac{\kappa_g' cos\theta}{(\kappa_g^2-1)^{\frac{3}{2}}}, \\ -\lambda\dfrac{sin\theta}{(\kappa_g^2-1)^{\frac{1}{2}}} = -(\kappa_g^2-1)^{-\frac{3}{2}}\kappa_g\kappa_g' cos\theta + \tau_g sin\theta, \\ \lambda cos\theta = \dfrac{\tau_g cos\theta}{(\kappa_g^2-1)^{\frac{1}{2}}}. \end{cases} \tag{2.2}$$

We get $\lambda = \frac{\tau_g}{(\kappa_g^2-1)^{\frac{1}{2}}}$ and $\kappa_g(s)\tau_g \sin\theta - \frac{\kappa_g' cos\theta}{(\kappa_g^2-1)^{\frac{1}{2}}} = 0$. This is equivalent that

$$\kappa_g(s)\tau_g(s)\sqrt{\kappa_g^2(s)-1}sin\theta - \kappa_g^{'}(s)cos\theta = 0.$$

When $0 < \kappa_g(s) < 1$, we get

$$\kappa_g(s)\tau_g(s)\sqrt{1-\kappa_g^2(s)}cosh\theta - \kappa_g^{'}(s)sinh\theta = 0.$$

This means that the second de Sitter focal surface $SDF_\gamma(s,\theta)$ is singular at a point (s_0,θ_0) if and only if $\kappa_g(s_0)\tau_g(s_0)\sqrt{\kappa_g^2(s_0)-1}sin\theta_0 - \kappa_g^{'}(s_0)cos\theta_0 = 0$, or $\kappa_g(s_0)\tau_g(s_0)\sqrt{1-\kappa_g^2(s_0)}cosh\theta_0 - \kappa_g^{'}(s_0)sinh\theta_0 = 0$. Thus, we obtain geometric information about focal surfaces of de Sitter spacelike curves. This method has several limitations. For instance, it does not explain which singularities could appear in the focal surfaces and how these bifurcate as the original curve is deformed. It also misses to capture the deep concepts involved. To describe the generic singularities of focal surfaces, we should suppose that $cos\theta \neq 0$ or $sinh\theta \neq 0$ for the case $\delta(s) = 1$ unless otherwise stated. Continuing with notation, we give a brief review on Lagrangian singularity theory mainly due to Arnold [12]. The main tool of Lagrangian singularities theory is the notion of generating families and caustic. Let $G : (\mathbb{R}\times\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that G is a Morse family if the mapping

$$\Delta^* G = \left(\frac{\partial G}{\partial s}\right) : (\mathbb{R}\times\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$$

is non-singular, where $(s, \mathbf{v}) = (s, v_1, \ldots, v_n) \in (\mathbb{R}\times\mathbb{R}^n, \mathbf{0})$. In this case we have a smooth n-dimensional submanifold,

$$C_G = \left\{(s,\mathbf{v}) \in (\mathbb{R}\times\mathbb{R}^n, \mathbf{0}) \mid \frac{\partial G}{\partial s}(s,\mathbf{v}) = 0\right\}$$

and the map germ $\Phi_G : (C_G, \mathbf{0}) \longrightarrow T^*\mathbb{R}^n$ defined by

$$\Phi_G(s,\mathbf{v}) = \left(\mathbf{v},\ \frac{\partial G}{\partial v_1}(s,\mathbf{v}), \ldots, \frac{\partial G}{\partial v_n}(s,\mathbf{v})\right)$$

is a Lagrangian immersion germ. We call G a generating family of $\Phi_G(C_G)$. Let $\pi_2 : (\mathbb{R}^k\times\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ denote the canonical projection and consider the map-germ π_{C_G} which is given by the restriction of the projection π_2 to $(C_G, \mathbf{0})$. Thus $\pi_{C_G} : (C_G, \mathbf{0}) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ with $\pi_{C_G}(s,\mathbf{v}) = \mathbf{v}$ for any $(s,\mathbf{v}) \in (C_G, \mathbf{0})$. The map π_{C_G} is the catastrophe map of G and it is a Lagrangian map. A caustic is the set of critical values of a Lagrangian map. Therefore the corresponding caustic is

$$C(\Phi_G) = \left\{\mathbf{v} \in \mathbb{R}^n \mid \exists\, s \in \mathbb{R}\,\text{such that}\, \frac{\partial G}{\partial s}(s,\mathbf{v}) = \frac{\partial^2 G}{\partial s^2}(s,\mathbf{v}) = 0\right\}.$$

We sometimes denote $\mathcal{B}_G = C(\Phi_G)$ and call it the *bifurcation set* of G. Now, we can apply the above arguments to our situation, we find that de Sitter focal surfaces are two dimensional caustics which have Lagrangian singularities. In order to characterize the types of singularities of de Sitter focal surfaces via differential calculations, we find a de Sitter invariant of γ which is defined to be

$$\rho(s) = \kappa_g^2(s)\tau_g^3(s) + \kappa_g(s)\kappa_g^{''}(s)\tau_g(s) - 2(\kappa_g^{'}(s))^2\tau_g(s) - \kappa_g(s)\kappa_g^{'}(s)\tau_g^{'}(s).$$

To state the main results, we respectively call $C\times\mathbb{R}=\{(x_1,x_2)\mid x_1^2=x_2^3\}\times\mathbb{R}$ a cuspidal edge,

$$SW=\{(x_1,x_2,x_3)\mid x_1=3u^4+u^2v,x_2=4u^3+2uv,x_3=v\}$$

a swallowtail, $C(2,3,4)=\{(t^2,t^3,t^4)\in\mathbb{R}^3\mid t\in\mathbb{R}\}$ a $(2,3,4)$-cusp (cf., Figure 1). The main results in this paper are given as follows:

Theorem 2.1. *Let $\gamma: I\to S_1^3$ be a unit speed spacelike curve with $\kappa_g(s)\tau_g(s)\neq 0$, then de Sitter focal surfaces are two dimensional caustics which have Lagrangian singularities. For details, if $\delta(s)=-1$, then we have the following:*

(1) *The first de Sitter focal surface $FDF_\gamma(s,\theta)$ is singular at a point (s_0,θ_0) if and only if*

$$tanh\theta_0=\frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{1+\kappa_g^2(s_0)}}.$$

(2) *The first de Sitter focal surface $FDF_\gamma(s,\theta)$ is locally diffeomorphic to cuspidal edge $C\times\mathbb{R}$ at (s_0,θ_0) if*

$$tanh\theta_0=\frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{1+\kappa_g^2(s_0)}},$$

$\rho(s_0)\neq 0$. *Under this condition, the osculating hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0,r_0)$ and $\gamma(s)$ have contact of order 4 for s_0 and $FDF_\gamma(s,\theta(s))$ is locally diffeomorphic to the line.*

(3) *The first de Sitter focal surface $FDF_\gamma(s,\theta)$ of spacelike curve γ is locally diffeomorphic to the SW at (s_0,θ_0) if*

$$tanh\theta_0=\frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{1+\kappa_g^2(s_0)}},$$

$\rho(s_0)=0$ *and* $\rho'(s_0)\neq 0$. *Under this condition, the osculating hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0,r_0)$ and $\gamma(s)$ have contact of order 5 for s_0 and $FDF_\gamma(s,\theta(s))$ is locally diffeomorphic to the $(2,3,4)$-cusp.*

Theorem 2.2. *Let $\gamma: I\to S_1^3$ be a unit speed spacelike curve with $\kappa_g(s)\tau_g(s)\neq 0$, then de Sitter focal surfaces are two dimensional caustics which have Lagrangian singularities. For details:*

(A) *Suppose that $\delta(s)=1$ and $\kappa_g(s)>1$, then we have the following:*

(1) *The second de Sitter focal surface $SDF_\gamma(s,\theta)$ of spacelike curve γ is singular at a point (s_0,θ_0) if and only if $\kappa_g(s_0)\tau_g(s_0)\sqrt{\kappa_g^2(s_0)-1}sin\theta_0-\kappa_g'(s_0)cos\theta_0=0$.*

(2) *The second de Sitter focal surface $SDF_\gamma(s,\theta)$ of spacelike curve γ is locally diffeomorphic to cuspidal edge $C\times\mathbb{R}$ at (s_0,θ_0) if*

$$tan\theta_0=\frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{\kappa_g^2(s_0)-1}},$$

$\rho(s_0)\neq 0$. *Under this condition, the osculating hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0,r_0)$ and $\gamma(s)$ have contact of order 4 for s_0 and $SDF_\gamma(s,\theta(s))$ is locally diffeomorphic to the line.*

(3) *The second de Sitter focal surface* $SDF_\gamma(s,\theta)$ *of spacelike curve* γ *is locally diffeomorphic to the* SW *at* (s_0,θ_0) *if*

$$tan\theta_0 = \frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{\kappa_g^2(s_0)-1}},$$

$\rho(s_0) = 0$ *and* $\rho'(s_0) \neq 0$. *Under this condition, the osculating hyperbolic pseudo-sphere* $HPS^1(\mathbf{v}_0, r_0)$ *and* $\gamma(s)$ *have contact of order* 5 *for* s_0 *and* $SDF_\gamma(s,\theta(s))$ *is locally diffeomorphic to the* $(2,3,4)$-*cusp.*

(B) *Suppose that* $\delta(s) = 1$ *and* $0 < \kappa_g(s) < 1$, *then we have the following:*

(1) *The second de Sitter focal surface* $SDF_\gamma(s,\theta)$ *of spacelike curve* γ *is singular at a point* (s_0,θ_0) *if and only if* $\kappa_g(s_0)\tau_g(s_0)\sqrt{1-\kappa_g^2(s_0)}cosh\theta_0 - \kappa_g'(s_0)sinh\theta_0 = 0$.

(2) *The second de Sitter focal surface* $SDF_\gamma(s,\theta)$ *of spacelike curve* γ *is locally diffeomorphic to cuspidal edge* $C \times \mathbb{R}$ *at* (s_0,θ_0) *if*

$$coth\theta_0 = \frac{k_g'(s_0)}{k_g(s_0)\tau_g(s_0)\sqrt{1-k_g^2(s_0)}},$$

and $\rho(s_0) \neq 0$. *Under this condition, the osculating hyperbolic pseudo-sphere* $HPS^1(\mathbf{v}_0, r_0)$ *and* $\gamma(s)$ *have contact of order* 4 *for* s_0 *and* $SDF_\gamma(s,\theta(s))$ *is locally diffeomorphic to the line.*

(3) *The second de Sitter focal surface* $SDF_\gamma(s,\theta)$ *of spacelike curve* γ *is locally diffeomorphic to the* SW *at* (s_0,θ_0) *if*

$$coth\theta_0 = \frac{k_g'(s_0)}{k_g(s_0)\tau_g(s_0)\sqrt{1-k_g^2(s_0)}},$$

$\rho(s_0) = 0$ *and* $\rho'(s_0) \neq 0$. *Under this condition, the osculating hyperbolic pseudo-sphere* $HPS^1(\mathbf{v}_0, r_0)$ *and* $\gamma(s)$ *have contact of order* 5 *for* s_0 *and* $SDF_\gamma(s,\theta(s))$ *is locally diffeomorphic to the* $(2,3,4)$-*cusp.*

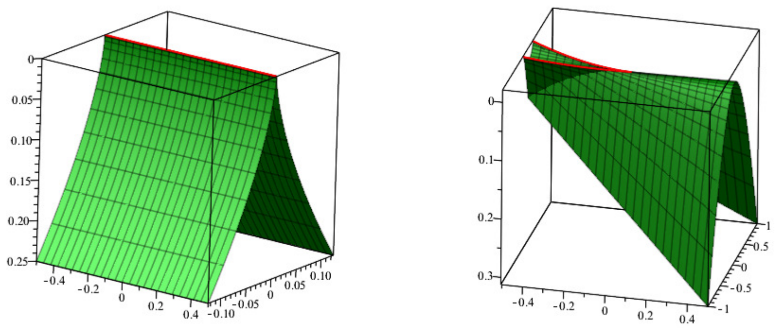

Figure 1. Cuspidal edge and Swallowtail.

3. De Sitter height functions and geometric invariant

In order to study the singularities of the first de Sitter focal surface and the second de Sitter focal surface of spacelike curve in S_1^3, we introduce a very useful family of function on spacelike curve in de Sitter 3-space. Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve, we now define a function $H^S : I \times S^3_1 \longrightarrow \mathbb{R}$ by $H^S(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle$. For any fixed $\mathbf{v} \in S^3_1$, we denote $h^S_v(s) = H^S(s, \mathbf{v})$. We call H^S the *de Sitter height function* on the curve γ. We will consider the contact of a spacelike curve $\gamma : I \to S^3_1$ with hyperbolic pseudo-sphere. By definition, a hyperbolic pseudo-sphere in S^3_1 with the centre $\mathbf{v}_0$ and radius r is the level set $h^S_v(s) = r$. We stress that the contact of γ with the level sets of h^S_v can be measured by the vanishing of successive derivatives of the function $g(s) = h^S_v(\gamma(s)) = \langle \gamma(s), \mathbf{v} \rangle$. In particular, a point $\gamma(s_0)$ is on a hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ of centre $\mathbf{v}_0$ and radius r if and only if $g(s_0) = r$. Furthermore, the curve γ and the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ have an ordinary tangency at $\gamma(s_0)$ if and only if $g(s_0) = r, g'(s_0) = 0$ and $g''(s_0) \neq 0$. Higher orders of tangency between γ and the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ are captured by the vanishing of successive derivatives of the function g at s_0. We say that γ and the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ have $(k+1)$-point contact at s_0 if $g^{'}(s_0) = g^{''}(s_0) = \cdots = g^{(k)}(s_0) = 0$ but $g^{(k+1)}(s_0) \neq 0$. Then s_0 is said to be a singularity of g of type A_k. We say that γ and the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ have at least k-point contact at $s = s_0$ if $g^{'}(s_0) = g^{''}(s_0) = \cdots = g^{(k)}(s_0) = 0$ and call s_0 a singularity of g of type $A_{\geq k}$. If the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ and a de Sitter spacelike curve have contact of at least order 3 for a points s_0, we call $HPS^1(\mathbf{v}_0, r)$ the osculating hyperbolic pseudo-sphere of $\gamma(s)$ at s_0. Then we have the following proposition which contains a geometric invariant $\rho(s)$.

Proposition 3.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve with $\kappa_g(s)\tau_g(s) \neq 0$ and $\delta(s) = -1$, $HPS^1(\mathbf{v}, r)$ be a hyperbolic pseudo-sphere of centre $\mathbf{v}_0$ and radius r. Suppose that $g(s_0) = r$ for some s_0. Then g has a singularity of type A_1, A_2, A_3 or A_4 at s_0 if and only if the geometric conditions in Table 2 are satisfied.*

Table 2. Geometric conditions for the singularities of g.

g	Conditions	The centre of $HPS^1(\mathbf{v}, r)$	Contact order
A_1	$\mathbf{v} = \lambda\gamma(s_0) + \mu\mathbf{n}(s_0) + \nu\mathbf{e}(s_0)$, $\lambda^2 + \mu^2 - \nu^2 = 1, \lambda \neq \mu\kappa_g$.	lies on the normal space to γ at s_0 but is not on the focal surface of γ.	2
A_2	$\mathbf{v} = FDF_\gamma(s_0, \theta)$, $tanh\theta \neq \frac{\kappa_g^{'}(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{1+\kappa_g^2(s_0)}}$.	lies on the regular part of the focal surface of γ at s_0 .	3
A_3	$\mathbf{v} = FDF_\gamma(s_0, \theta)$, $tanh\theta = \frac{\kappa_g^{'}(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{1+\kappa_g^2(s_0)}}$, $\rho(s_0) \neq 0$.	lies on the non-degenerate singular part of the focal surface of γ at s_0 .	4
A_4	$\mathbf{v} = FDF_\gamma(s_0, \theta)$, $tanh\theta = \frac{\kappa_g^{'}(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{1+\kappa_g^2(s_0)}}$, $\rho(s_0) = 0, \rho'(s_0) \neq 0$.	lies on the degenerate singular part of focal surface of γ at s_0 .	5

Proof. By definition and the Frenet-Serret type formulae, we have

(a) $(h^S_\mathbf{v})'(s) = \langle \mathbf{t}(s), \mathbf{v} \rangle$

(b) $(h^S_\mathbf{v})''(s) = \langle -\gamma(s) + \kappa_g(s)\mathbf{n}(s), \mathbf{v} \rangle$.

(c) $(h^S_\mathbf{v})^{(3)}(s) = \langle (\delta(s)\kappa_g^2(s) - 1)\mathbf{t}(s) + \kappa_g'(s)\mathbf{n}(s) + \kappa_g(s)\tau_g(s)\mathbf{e}(s), \mathbf{v} \rangle$,

(d) $(h^S_{\mathbf{v}})^{(4)}(s) = \langle 3\delta(s)\kappa_g(s)\kappa_g'(s)\mathbf{t}(s) + (\delta(s)\kappa_g^3(s) - \kappa_g(s) + \kappa_g''(s) + \kappa_g(s)\tau_g^2(s))\mathbf{n}(s) + (2\kappa_g'(s)\tau_g(s) + \kappa_g(s)\tau_g'(s))\mathbf{e}(s) - (\delta(s)\kappa_g^2(s) - 1)\gamma(s), \mathbf{v}\rangle$,

(e) $(h^S_{\mathbf{v}})^{(5)}(s) = \langle -5\delta(s)\kappa_g(s)\kappa_g'(s)\gamma(s) + (3\delta(s)(\kappa_g'(s))^2 + 4\delta(s)\kappa_g(s)\kappa_g''(s) + \kappa_g^4(s) - 2\delta(s)\kappa_g^2(s) + \delta(s)\kappa_g^2(s)\tau_g^2(s) + 1)\mathbf{t}(s) + (6\delta(s)\kappa_g^2(s)\kappa_g'(s) - \kappa_g'(s) + \kappa_g'''(s) + 3\kappa_g'(s)\tau_g^2(s) + 3\kappa_g(s)\tau_g(s)\tau_g'(s))\mathbf{n}(s) + (\delta(s)\kappa_g^3(s)\tau_g(s) - \kappa_g(s)\tau_g(s) + 3\kappa_g''(s)\tau_g(s) + \kappa_g(s)\tau_g^3(s) + 3\kappa_g'(s)\tau_g'(s) + \kappa_g(s)\tau_g''(s))\mathbf{e}(s), \mathbf{v}\rangle$.

By the conditions that$(h^S_{\mathbf{v}})'(s) = 0$, $\mathbf{v} \in S^3_1$ and $\delta(s) = -1$, we have that there are real numbers λ, μ, ν such that $\mathbf{v} = \lambda\gamma(s) + \mu\mathbf{n}(s) + \nu\mathbf{e}(s)$ and $\lambda^2 + \mu^2 - \nu^2 = 1$. The converse direction also holds. This is equivalent that g has a singularity of type A_1, which is also equivalent that the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ and the de Sitter spacelike curve have contact of at least order 2. Meanwhile, it is equivalent that the centre of the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ lies on the normal space to γ at s_0 but is not on the focal surface of γ. By the above formula (b), we have $(h^S_{\mathbf{v}})'(s) = (h^S_{\mathbf{v}})''(s) = 0$ if and only if $\lambda = \mu\kappa_g(s)$. By the fact that $\lambda^2 + \mu^2 - \nu^2 = 1$, we have $\mu^2(\kappa_g^2(s) + 1) - \nu^2 = 1$. Let $\mu = \frac{cosh\theta}{\sqrt{1+\kappa_g^2(s)}}$, $\nu = sinh\theta$. Thus, we have

$$\mathbf{v} = \frac{cosh\theta}{\sqrt{\kappa_g^2(s) + 1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + sinh\theta\mathbf{e}(s).$$

This is equivalent that g has a singularity of type A_2, which is also equivalent that the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ and the de Sitter spacelike curve have contact of at least order 3. And it is also equivalent that the centre of the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ lies on the regular part of the focal surface of γ at s_0. By the similar arguments to above and formula (c), we have $(h^S_{\mathbf{v}})'(s) = (h^S_{\mathbf{v}})''(s) = (h^S_{\mathbf{v}})^{(3)}(s) = 0$ if and only if $\frac{cosh\theta}{\sqrt{\kappa_g^2(s)+1}}(\kappa_g'(s)) - \kappa_g(s)\tau_g(s)sinh\theta = 0$. This is equivalent to the condition $tanh\theta = \frac{\kappa_g'(s)}{\kappa_g(s)\tau_g(s)\sqrt{1+\kappa_g^2(s)}}$. Combining with the condition that $\mathbf{v} = \frac{cosh\theta}{\sqrt{\kappa_g^2(s)+1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + sinh\theta\mathbf{e}(s)$. Therefore, we have

$$\mathbf{v} = \frac{cosh\theta}{\sqrt{\kappa_g^2(s) + 1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + sinh\theta\mathbf{e}(s)$$

and

$$tanh\theta = \frac{\kappa_g'(s)}{\kappa_g(s)\tau_g(s)\sqrt{1 + \kappa_g^2(s)}}.$$

In the meanwhile, this is equivalent that g has a singularity of type A_3, which is also equivalent that the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ and the de Sitter spacelike curve have contact of at least order 4. In addition, it is equivalent that the centre of the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ lies on the non-degenerate singular part of the focal surface of γ at s_0. Subsequently, by the above formula (d), $(h^S_{\mathbf{v}})'(s) = (h^S_{\mathbf{v}})''(s) = (h^S_{\mathbf{v}})^{(3)}(s) = (h^S_{\mathbf{v}})^{(4)}(s) = 0$ if and only if

$$\frac{cosh\theta}{\sqrt{1 + \kappa_g^2(s)}}(\kappa_g''(s) + \kappa_g(s)\tau_g^2(s)) - sinh\theta(2\kappa_g'(s)\tau_g(s) + \kappa_g(s)\tau_g'(s)) = 0.$$

Substituting

$$tanh\theta = \frac{\kappa_g'(s)}{\kappa_g(s)\tau_g(s)\sqrt{1 + \kappa_g^2(s)}}$$

in the above condition, we get

$$\kappa_g^2(s)\tau_g^3(s) + \kappa_g(s)\kappa_g^{''}(s)\tau_g(s) - 2(\kappa_g^{'}(s))^2\tau_g(s) - \kappa_g(s)\kappa_g^{'}(s)\tau_g^{'}(s) = 0.$$

Therefore, we have the conditions

$$tanh\theta = \frac{\kappa_g^{'}(s)}{\kappa_g(s)\tau_g(s)\sqrt{1+\kappa_g^2(s)}},$$

$$\mathbf{v} = \frac{cosh\theta}{\sqrt{\kappa_g^2(s)+1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + sinh\theta\mathbf{e}(s)$$

and $\rho(s) = 0$ which indicate g has a singularity of type A_4. Under these conditions, the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ and the de Sitter spacelike curve have contact of at least order 5 and the centre of the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r)$ lies on the degenerate singular part of the focal surface of γ at s_0. Finally, by the similar arguments to the above cases we can show that $(h^S_{\mathbf{v}})'(s) = (h^S_{\mathbf{v}})''(s) = (h^S_{\mathbf{v}})^{(3)}(s) = (h^S_{\mathbf{v}})^{(4)}(s) = (h^S_{\mathbf{v}})^{(5)}(s) = 0$ if and only if $\frac{cosh\theta}{\sqrt{1+\kappa_g^2(s)}}(-\kappa_g^2(s)\kappa_g^{'}(s) - \kappa_g^{'}(s) + \kappa_g^{'''}(s) + 3\kappa_g^{'}(s)\tau_g^2(s) + 3\kappa_g(s)\tau_g(s)\tau_g^{'}(s)) - sinh\theta(-\kappa_g^3(s)\tau_g(s) - \kappa_g(s)\tau_g(s) + 3\kappa_g^{''}(s)\tau_g(s) + \kappa_g(s)\tau_g^3(s) + 3\kappa_g^{'}(s)\tau_g^{'}(s) + \kappa_g(s)\tau_g^{''}(s)) = 0$. This is equivalent that $\kappa_g(s)\kappa_g^{'''}(s)\tau_g(s) + 2\kappa_g(s)\kappa_g^{'}(s)(\tau_g(s))^3 + 3\kappa_g^2(s)\tau_g^2(s)\tau_g^{'}(s) - 3\kappa_g^{'}(s)\kappa_g^{''}(s)\tau_g(s) - 3(\kappa_g^{'}(s))^2\tau_g^{'}(s) - \kappa_g(s)\kappa_g^{'}(s)\tau_g^{''}(s) = 0$. In particular, we get

$$tanh\theta = \frac{\kappa_g^{'}(s)}{\kappa_g(s)\tau_g(s)\sqrt{1+\kappa_g^2(s)}},$$

$$\mathbf{v} = \frac{cosh\theta}{\sqrt{\kappa_g^2(s)+1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + sinh\theta\mathbf{e}(s)$$

and $\rho(s) = \rho^{'}(s) = 0$. This completes the proof. □

By the similar arguments to the above cases we can show that the following proposition holds.

Proposition 3.2. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve with $\kappa_g(s)\tau_g(s) \neq 0$, $\kappa_g(s) \neq 1$ and $\delta(s) = 1$. $HPS^1(\mathbf{v}, r)$ be a hyperbolic pseudo-sphere of centre $\mathbf{v}_0$ and radius r. Suppose that $g(s_0) = r$ for some s_0. Then g has a singularity of type A_1, A_2, A_3 or A_4 at s_0 if and only if the geometric conditions in Table 3 are satisfied.*

Proof. (1) From $(h^S_{\mathbf{v}})'(s) = \langle \mathbf{t}(s), \mathbf{v}\rangle = 0$, $\mathbf{v} \in S^3_1$ and $\delta(s) = 1$, we get that there are real numbers λ, μ, ν such that $\mathbf{v} = \lambda\gamma(s) + \mu\mathbf{n}(s) + \nu\mathbf{e}(s)$ and $\lambda^2 - \mu^2 + \nu^2 = 1$. Furthermore, we calculate that $(h^S_{\mathbf{v}})''(s) = \langle -\gamma(s) + k_g(s)\mathbf{n}(s), \mathbf{v}\rangle = 0$, which indicates that $\lambda = -\mu k_g(s)$, hence $\mu^2(k_g^2(s) - 1) + \nu^2 = 1$. When $k_g(s) > 1$, let $\mu = \frac{cos\theta}{\sqrt{k_g^2(s)-1}}$, $\nu = sin\theta$. Thus, we have

$$\mathbf{v} = \frac{cos\theta}{\sqrt{k_g^2(s)-1}}(-k_g(s)\gamma(s) + \mathbf{n}(s)) + sin\theta\mathbf{e}(s).$$

When $0 < k_g(s) < 1$, let $\mu = \frac{\sinh\theta}{\sqrt{1-k_g^2(s)}}$, $\nu = \cosh\theta$. It follows that

$$\mathbf{v} = \frac{\sinh\theta}{\sqrt{1-k_g^2(s)}}(-k_g(s)\gamma(s) + \mathbf{n}(s)) + \cosh\theta\mathbf{e}(s).$$

Subsequently, it can be examined that $(h^S_{\mathbf{v}})^{(3)}(s) = \langle(\delta(s)k_g^2(s)-1)\mathbf{t}(s)+k_g'(s)\mathbf{n}(s)+k_g(s)\tau_g(s)\mathbf{e}(s), \mathbf{v}\rangle = 0$, which implies $\frac{cos\theta}{\sqrt{k_g^2(s)-1}}(-k_g'(s)) + k_g(s)\tau_g(s)sin\theta = 0$ or $\frac{\sinh\theta}{\sqrt{1-k_g^2(s)}}(-k_g'(s)) + k_g(s)\tau_g(s)\cosh\theta = 0$. This is equivalent to the condition $tan\theta = \frac{k_g'(s)}{k_g(s)\tau_g(s)\sqrt{k_g^2(s)-1}}$ or $coth\theta = \frac{k_g'(s)}{k_g(s)\tau_g(s)\sqrt{1-k_g^2(s)}}$. Moreover, calculating the 4th derivative of $h^S_{\mathbf{v}}$, $(h^S_{\mathbf{v}})^{(4)}(s) = \langle 3\delta(s)k_g(s)k_g'(s)\mathbf{t}(s) + (\delta(s)k_g^3(s) - k_g(s) + k_g''(s) + k_g(s)\tau_g^2(s))\mathbf{n}(s) + (2k_g^{'}(s)\tau_g(s)+k_g(s)\tau_g^{'}(s))\mathbf{e}(s)-(\delta(s)k_g^2(s)-1)\gamma(s), \mathbf{v}\rangle$, we can state that $(h^S_{\mathbf{v}})'(s) = (h^S_{\mathbf{v}})''(s) = (h^S_{\mathbf{v}})^{(3)}(s) = (h^S_{\mathbf{v}})^{(4)}(s) = 0$ if and only if there exists θ such that $tan\theta = \frac{k_g'(s)}{k_g(s)\tau_g(s)\sqrt{k_g^2(s)-1}}$ and $\rho(s) = 0$ or $coth\theta = \frac{k_g'(s)}{k_g(s)\tau_g(s)\sqrt{1-k_g^2(s)}}$ and $\rho(s) = 0$. Finally, we calculate $(h^S_{\mathbf{v}})^{(5)}(s) = \langle -5\delta(s)k_g(s)k_g^{'}(s)\gamma(s) + (3\delta(s)(k_g^{'}(s))^2 + 4\delta(s)k_g(s)k_g^{''}(s) + k_g^4(s) - 2\delta(s)k_g^2(s) + \delta(s)k_g^2(s)\tau_g^2(s) + 1)\mathbf{t}(s) + (6\delta(s)k_g^2(s)k_g^{'}(s) - k_g^{'}(s) + k_g^{'''}(s) + 3k_g^{'}(s)\tau_g^2(s) + 3k_g(s)\tau_g(s)\tau_g^{'}(s))\mathbf{n}(s) + (\delta(s)k_g^3(s)\tau_g(s) - k_g(s)\tau_g(s) + 3k_g^{''}(s)\tau_g(s) + k_g(s)\tau_g^3(s) + 3k_g^{'}(s)\tau_g^{'}(s) + k_g(s)\tau_g^{''}(s))\mathbf{e}(s), \mathbf{v}\rangle$. From $(h^S_{\mathbf{v}})'(s) = (h^S_{\mathbf{v}})''(s) = (h^S_{\mathbf{v}})^{(3)}(s) = (h^S_{\mathbf{v}})^{(4)}(s) = (h^S_{\mathbf{v}})^{(5)}(s) = 0$, when $k_g(s) > 1$, we have -$\frac{cos(\theta)}{\sqrt{1+k_g^2(s)}}(-k_g^2(s)k_g^{'}(s) - k_g^{'}(s) + k_g^{'''}(s) + 3k_g^{'}(s)\tau_g^2(s) + 3k_g(s)\tau_g(s)\tau_g^{'}(s)) + sin(\theta)(-k_g^3(s)\tau_g(s) - k_g(s)\tau_g(s) + 3k_g^{''}(s)\tau_g(s) + k_g(s)\tau_g^3(s) + 3k_g^{'}(s)\tau_g^{'}(s) + k_g(s)\tau_g^{''}(s)) = 0$. Combining with the condition that

$$tan\theta = \frac{k_g^{'}(s)}{k_g(s)\tau_g(s)\sqrt{k_g^2(s)-1}}.$$

This is equivalent that $k_g(s)k_g^{'''}(s)\tau_g(s) + 2k_g(s)k_g^{'}(s)(\tau_g(s))^3 + 3k_g^2(s)\tau_g^2(s)\tau_g^{'}(s) - 3k_g^{'}(s)k_g^{''}(s)\tau_g(s) - 3(k_g^{'}(s))^2\tau_g^{'}(s) - k_g(s)k_g^{'}(s)\tau_g^{''}(s) = 0$, which means $\rho^{'}(s) = 0$. Therefore, we get the condition

$$tan\theta = \frac{k_g^{'}(s)}{k_g(s)\tau_g(s)\sqrt{k_g^2(s)-1}},$$

$$\mathbf{v} = \frac{cos(\theta)}{k_g(s)\tau_g(s)\sqrt{k_g^2(s)-1}}(-k_g^2(s)\tau_g(s)\gamma(s) + k_g(s)\tau_g(s)\mathbf{n}(s) + k_g^{'}(s)\mathbf{e}(s))$$

and $\rho(s) = \rho^{'}(s) = 0$. When $0 < k_g(s) < 1$, by the similar arguments to above, we can get

$$coth\theta = \frac{k_g^{'}(s)}{k_g(s)\tau_g(s)\sqrt{1-k_g^2(s)}},$$

$$\mathbf{v} = \frac{sinh(\theta)}{k_g(s)\tau_g(s)\sqrt{1-k_g^2(s)}}(-k_g^2(s)\tau_g(s)\gamma(s) + k_g(s)\tau_g(s)\mathbf{n}(s) + k_g^{'}(s)\mathbf{e}(s))$$

and $\rho(s) = \rho^{'}(s) = 0$. By the similar arguments to above proposition, these assertions hold. □

Table 3. Geometric conditions for the singularities of g.

g	Conditions	The centre of $HPS^1(\mathbf{v}, r)$	Contact order
A_1	$\mathbf{v} = \lambda\gamma(s_0) + \mu\mathbf{n}(s_0) + \nu\mathbf{e}(s_0)$, $\lambda^2 - \mu^2 + \nu^2 = 1$.	lies on the normal space to γ at s_0 but is not on the focal surface of γ.	2
A_2	$\mathbf{v} = SDF_\gamma(s_0, \theta)$, $tan\theta \neq \frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{\kappa_g^2(s_0)-1}}$ or $coth\theta \neq \frac{k_g'(s_0)}{k_g(s_0)\tau_g(s_0)\sqrt{1-k_g^2(s_0)}}$.	lies on the regular part of the focal surface of γ at s_0 .	3
A_3	$\mathbf{v} = SDF_\gamma(s_0, \theta)$, $tan\theta = \frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{\kappa_g^2(s_0)-1}}$, $\rho(s_0) \neq 0$ or $coth\theta = \frac{k_g'(s_0)}{k_g(s_0)\tau_g(s_0)\sqrt{1-k_g^2(s_0)}}$, $\rho(s_0) \neq 0$.	lies on the non-degenerate singular part of the focal surface of γ at s_0 .	4
A_4	$\mathbf{v} = SDF_\gamma(s_0, \theta)$ $tan\theta = \frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{\kappa_g^2(s_0)-1}}$, $\rho(s_0) = 0, \rho'(s_0) \neq 0$ or $coth\theta = \frac{k_g'(s_0)}{k_g(s_0)\tau_g(s_0)\sqrt{1-k_g^2(s_0)}}$ $\rho(s_0) = 0, \rho'(s_0) \neq 0$.	lies on the degenerate singular part of focal surface of γ at s_0 .	5

Corollary 3.3. *Let $\gamma : I \to S^3_1$ be a unit speed space curve with $\kappa_g(s)\tau_g(s) \neq 0$.*

(A) *Suppose that $\delta(s) = -1$, then we have the following:*

(1) *There exists an osculating hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r_0)$ of $\gamma(s)$ at a point s_0.*

(2) *Under the above notations, $\gamma(s)$ and the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r_0)$ have at least a 4-point (respectively, 5-point) contact at $\gamma(s_0)$ if and only if $\rho(s_0) \neq 0$ (respectively, $\rho(s_0) = 0$ and $\rho'(s_0) \neq 0$).*

(B) *Suppose that $\delta(s) = 1$, then we have the following:*

(1) *There exists an osculating hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r_0)$ of $\gamma(s)$ at a point s_0 if and only if $\kappa_g(s_0) \neq 1$.*

(2) *Suppose that $\kappa_g(s_0) \neq 1$. Then $\gamma(s)$ and the hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r_0)$ have at least a 4-point (respectively, 5-point) contact at $\gamma(s_0)$ if and only if $\rho(s_0) \neq 0$ (respectively, $\rho(s_0) = 0$ and $\rho'(s_0) \neq 0$).*

Proof. (A) Under the condition that $\delta(s) = -1$. Let $\mathcal{H}^S : S^3_1 \times S^3_1 \to \mathbb{R}$ be a function defined by $\mathcal{H}^S(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$. We claim that $\mathcal{H}^S_{\mathbf{v}}$ is a submersion and $(\mathfrak{h}^S)^{-1}_{v_0}(c)$ is a hyperbolic pseudo-sphere for any $\mathbf{v} \in S^3_1$, where $\mathfrak{h}^S_v(\mathbf{u}) = \mathcal{H}^S(\mathbf{u}, \mathbf{v})$. We have that $\mathfrak{h}^S_v \circ \gamma(s) = h^S_{v_0}(s)$, here $h^S_{v_0}(s) = H^S(s, \mathbf{v}_0)$. Therefore we have $(h^S_{\mathbf{v_0}})^{-1}(r_0)$ is an osculating hyperbolic pseudo-sphere if and only if $h^S_{v_0}(s) = r_0$ and $(h^S_{\mathbf{v}})'(s) = (h^S_{\mathbf{v}})''(s) = 0$. By Proposition 3.1, this condition is equivalent to the condition that

$$\mathbf{v} = \frac{cosh\theta}{\sqrt{\kappa_g^2(s) + 1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + \kappa_g'(s)\mathbf{e}(s),$$

where

$$tanh\theta = \frac{\kappa_g^{'}(s)}{\kappa_g(s)\tau_g(s)\sqrt{1+\kappa_g^2(s)}}$$

and $r_0 = H^S(s_0, \mathbf{v}_0)$. It is clearly that there exist such $\mathbf{v}_0$ and $\theta(s)$ for any $\kappa_g(s)$. The assertion (2) follows from the rest assertions of Proposition 3.1.

(B) Under the condition that $\delta(s) = 1$. The assertion follows from exactly the same arguments as those of the previous case. □

By some calculations, we can get the following proposition which contains the geometric meanings of the important invariant $\rho(s)$.

Proposition 3.4. *Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve with $\kappa_g(s)\tau_g(s) \neq 0$.*

(A) *Suppose that $\delta(s) = -1$. Let $\theta(s)$ be a function defined by $tanh(\theta(s)) = \frac{\kappa_g^{'}(s)}{\kappa_g(s)\tau_g(s)\sqrt{1+\kappa_g^2(s)}}$. Then the following conditions are equivalent:*

(1)*$FDF_\gamma(s, \theta(s))$ is a constant vector,*

(2) *$\rho(s) = 0$,*

(3) *$Im(\gamma(s)) \subset HPS^1(\mathbf{v}_0, r)$.*

(B) *Suppose that $\delta(s) = 1$ and $\kappa_g(s) \neq 1$. Let $\theta(s)$ be a function defined by $tan(\theta(s)) = \frac{\kappa_g^{'}(s)}{\kappa_g(s)\tau_g(s)\sqrt{\kappa_g^2(s)-1}}$ or $coth(\theta(s)) = \frac{k_g'(s)}{k_g(s)\tau_g(s)\sqrt{1-k_g^2(s)}}$. Then the following conditions are equivalent:*

(1) *$SDF_\gamma(s, \theta(s))$ is a constant vector,*

(2) *$\rho(s) = 0$,*

(3) *$Im(\gamma(s)) \subset HPS^1(\mathbf{v}_0, r)$.*

Proof. First, we consider the assertion (A). Let $\theta = \theta(s)$, by some calculations, we have

$$\begin{aligned} FDF_\gamma'(s, \theta(s)) &= \frac{\kappa_g' cosh(\theta(s))}{(1+\kappa_g^2)^{\frac{3}{2}}}\gamma(s) + \frac{-\kappa_g\kappa_g' cosh(\theta(s)) + (1+\kappa_g^2)^{\frac{3}{2}}\tau_g(s) sinh(\theta(s))}{(1+\kappa_g^2)^{\frac{3}{2}}}\mathbf{n}(s) + \frac{\tau_g cosh(\theta(s))}{(1+\kappa_g^2)^{\frac{1}{2}}}\mathbf{e}(s) \\ &\quad + \frac{\kappa_g sinh(\theta(s))\theta'(s)}{(1+\kappa_g^2)^{\frac{1}{2}}}\gamma(s) + \frac{sinh(\theta(s))\theta'(s)}{(1+\kappa_g^2)^{\frac{1}{2}}}\mathbf{n}(s) + cosh(\theta(s))\theta'(s)\mathbf{e}(s) \\ &= cosh(\theta(s))\frac{\kappa_g' + (1+\kappa_g^2)\kappa_g tanh(\theta(s))\theta'(s)}{(1+\kappa_g^2)^{\frac{3}{2}}}\gamma(s) + cosh(\theta(s))\frac{\tau_g + \sqrt{1+\kappa_g^2}\theta'(s)}{(1+\kappa_g^2)^{\frac{1}{2}}}\mathbf{e}(s) \\ &\quad + cosh(\theta(s))\frac{-\kappa_g\kappa_g' + (1+\kappa_g^2)^{\frac{3}{2}}\tau_g(s) tanh(\theta(s)) + (1+\kappa_g^2) tanh(\theta(s))\theta'(s)}{(1+\kappa_g^2)^{\frac{3}{2}}}\mathbf{n}(s). \end{aligned} \tag{3.1}$$

Differentiating both sides of the equation

$$\kappa_g' cosh(\theta(s)) - \sqrt{1+\kappa_g^2}\kappa_g\tau_g sinh(\theta(s)) = 0$$

on the variable s, we get

$$\theta'(s) cosh(\theta(s))(\kappa_g' tanh(\theta(s)) - \kappa_g\tau_g\sqrt{1+\kappa_g^2})$$

$$= cosh(\theta(s))[-\kappa_g'' + (\frac{\kappa_g^2\kappa_g'\tau_g}{\sqrt{1+\kappa_g^2}} + \kappa_g'\tau_g\sqrt{1+\kappa_g^2} + \kappa_g\tau_g'\sqrt{1+\kappa_g^2})tanh(\theta(s))]).$$

This is equivalent to

$$\theta'(s)\frac{(\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2)}{\kappa_g\tau_g\sqrt{1+\kappa_g^2}} = \frac{-\kappa_g''\kappa_g\tau_g(1+\kappa_g^2) + \kappa_g^2(\kappa_g')^2\tau_g + (\kappa_g')^2\tau_g(1+\kappa_g^2) + \kappa_g\kappa_g'\tau_g'(1+\kappa_g^2)}{\kappa_g\tau_g(1+\kappa_g^2)}.$$

Therefore,

$$\theta'(s) = \frac{(1+\kappa_g^2)\kappa_g\kappa_g'\tau_g' + \tau_g((1+2\kappa_g^2)(\kappa_g')^2 - \kappa_g(1+\kappa_g^2)\kappa_g'')}{\sqrt{1+\kappa_g^2}((\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2))}.$$

Substituting $tanh(\theta(s))$ and $\theta'(s)$ into the left of the following equations, we get

$$\begin{aligned}
&\kappa_g' + (1+\kappa_g^2)\kappa_g tanh(\theta(s))\theta'(s)\\
&= \kappa_g' + \frac{(1+\kappa_g^2)\kappa_g\kappa_g'}{\kappa_g\tau_g\sqrt{1+\kappa_g^2}}\frac{(1+\kappa_g^2)\kappa_g\kappa_g'\tau_g' + \tau_g((1+2\kappa_g^2)(\kappa_g')^2 - \kappa_g(1+\kappa_g^2)\kappa_g'')}{\sqrt{1+\kappa_g^2}((\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2))}\\
&= \frac{(1+\kappa_g^2)\kappa_g'(-\kappa_g^2\tau_g^3 + \kappa_g\kappa_g'\tau_g' - \kappa_g\kappa_g''\tau_g) + 2(\kappa_g')^3\tau_g(1+\kappa_g^2)}{\tau_g[(\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2)]}\\
&= \frac{(1+\kappa_g^2)\kappa_g'(-\kappa_g^2\tau_g^3 + \kappa_g\kappa_g'\tau_g' - \kappa_g\kappa_g''\tau_g + 2(\kappa_g')^2\tau_g)}{\tau_g[(\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2)]}\\
&= \frac{(1+\kappa_g^2)\kappa_g'(-\rho(s))}{\tau_g[(\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2)]},
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\tau_g + \sqrt{1+\kappa_g^2}\theta'(s)\\
&= \tau_g + \sqrt{1+\kappa_g^2}\frac{(1+\kappa_g^2)\kappa_g\kappa_g'\tau_g' + \tau_g((1+2\kappa_g^2)(\kappa_g')^2 - \kappa_g(1+\kappa_g^2)\kappa_g'')}{\sqrt{1+\kappa_g^2}((\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2))}\\
&= \frac{(1+\kappa_g^2)(-\kappa_g^2\tau_g^3 + \kappa_g\kappa_g'\tau_g' - \kappa_g\kappa_g''\tau_g) + 2(\kappa_g')^2\tau_g(1+\kappa_g^2)}{(\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2)}\\
&= \frac{(1+\kappa_g^2)(-\kappa_g^2\tau_g^3 + \kappa_g\kappa_g'\tau_g' - \kappa_g\kappa_g''\tau_g + 2(\kappa_g')^2\tau_g)}{(\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2)}\\
&= \frac{(1+\kappa_g^2)(-\rho(s))}{(\kappa_g')^2 - \kappa_g^2\tau_g^2(1+\kappa_g^2)},
\end{aligned} \tag{3.3}$$

and

$$-\kappa_g\kappa_g' + (1+\kappa_g^2)^{\frac{3}{2}}\tau_g(s)tanh(\theta(s)) + (1+\kappa_g^2)tanh(\theta(s))\theta'(s)$$

$$
\begin{aligned}
&= \frac{(1+\kappa_g^2)\kappa_g'(-\kappa_g^2\tau_g^3+\kappa_g\kappa_g'\tau_g'-\kappa_g\kappa_g''\tau_g+2(\kappa_g')^2\tau_g)}{\kappa_g\tau_g[(\kappa_g')^2-\kappa_g^2\tau_g^2(1+\kappa_g^2)]}\\
&= \frac{(1+\kappa_g^2)\kappa_g'(-\rho(s))}{\kappa_g\tau_g[(\kappa_g')^2-\kappa_g^2\tau_g^2(1+\kappa_g^2)]}. \qquad (3.4)
\end{aligned}
$$

Furthermore, substituting Eqs (3.2),(3.3) and (3.4) into (3.1), we get

$$
\begin{aligned}
FDF_\gamma'(s,\theta(s)) \;&=\; \rho(s)\Big(\frac{-\kappa_g' cosh(\theta(s))}{(1+\kappa_g^2)^{\frac{1}{2}}\tau_g[(\kappa_g')^2-\kappa_g^2\tau_g^2(1+\kappa_g^2)]}\gamma(s)+\frac{\sqrt{1+\kappa_g^2}cosh(\theta(s))}{(\kappa_g')^2-\kappa_g^2\tau_g^2(1+\kappa_g^2)}\mathbf{e}(s)\\
&\quad+\frac{-\kappa_g' cosh(\theta(s))}{\sqrt{1+\kappa_g^2}\kappa_g\tau_g[(\kappa_g')^2-\kappa_g^2\tau_g^2(1+\kappa_g^2)]}\mathbf{n}(s)\Big).
\end{aligned}
$$

Therefore, $FDF_\gamma'(s,\theta(s))\equiv 0$ if and only if $\rho(s)\equiv 0$. This means that the conditions (1) and (2) are equivalent. Assume that the condition (1) holds. Then we have $\langle\gamma(s), FDF_\gamma(s,\theta(s))\rangle = \frac{\kappa_g cosh(\theta(s))}{\sqrt{1+\kappa_g^2}}$, which is constant. Under this condition, we put $r=\frac{\kappa_g cosh(\theta(s))}{\sqrt{1+\kappa_g^2}}$ and $\mathbf{v}_0=\frac{cosh(\theta(s))}{\kappa_g(s)\tau_g(s)\sqrt{\kappa_g^2(s)+1}}(\kappa_g^2(s)\tau_g(s)\gamma(s)+\kappa_g(s)\tau_g(s)\mathbf{n}(s)+\kappa_g'(s)\mathbf{e}(s))$. Then it is easy to show that $\gamma(s)$ is a part of the pseudo-sphere $HPS^1(\mathbf{v}_0,r)$. The condition (3) holds. For the converse, we assume that $\langle\gamma(s),\mathbf{v}\rangle=c$ for a constant vector $\mathbf{v}$ and a real number c. Since $h_v^S(s)=c$, we have $\mathbf{v}=\frac{cosh(\theta(s))}{\kappa_g(s)\tau_g(s)\sqrt{\kappa_g^2(s)+1}}(\kappa_g^2(s)\tau_g(s)\gamma(s)+\kappa_g(s)\tau_g(s)\mathbf{n}(s)+\kappa_g'(s)\mathbf{e}(s))$, by Proposition 3.1, so that the condition (1) holds.

For the proof of the assertion (B), when $\kappa_g>1$, since $tan\theta(s)=\frac{\kappa_g'}{\kappa_g\tau_g\sqrt{\kappa_g^2-1}}$, locally, we get $\theta(s)=arctan\frac{\kappa_g'}{\kappa_g\tau_g\sqrt{\kappa_g^2-1}}$. A direct computation shows that

$$
\begin{aligned}
\theta'(s) \;&=\; \frac{1}{1+\frac{(\kappa_g')^2}{\kappa_g^2\tau_g^2(\kappa_g^2-1)}}\times\left(\frac{\kappa_g'}{\kappa_g\tau_g\sqrt{\kappa_g^2-1}}\right)'(s)\\
&=\; \frac{(\kappa_g^2-1)(\kappa_g\kappa_g''\tau_g-(\kappa_g')^2\tau_g-\kappa_g\kappa_g'\tau_g')-\kappa_g^2(\kappa_g')^2\tau_g}{\sqrt{\kappa_g^2-1}(\kappa_g^2\tau_g^2(\kappa_g^2-1)+(\kappa_g')^2)}(s).
\end{aligned}
$$

and

$$
\begin{aligned}
SDF_\gamma'(s,\theta(s)) \;&=\; \frac{\kappa_g' cos(\theta(s))}{(\kappa_g^2-1)^{\frac{3}{2}}}\gamma(s)+\frac{-\kappa_g\kappa_g' cos(\theta(s))+(\kappa_g^2-1)^{\frac{3}{2}}\tau_g(s)sin(\theta(s))}{(\kappa_g^2-1)^{\frac{3}{2}}}\mathbf{n}(s)+\frac{\tau_g cos(\theta(s))}{(\kappa_g^2-1)^{\frac{1}{2}}}\mathbf{e}(s)\\
&\quad+\frac{\kappa_g sin(\theta(s))\theta'(s)}{(\kappa_g^2-1)^{\frac{1}{2}}}\gamma(s)-\frac{sin(\theta(s))\theta'(s)}{(\kappa_g^2-1)^{\frac{1}{2}}}\mathbf{n}(s)+cos(\theta(s))\theta'(s)\mathbf{e}(s)\\
&= cos(\theta(s))\frac{\kappa_g'+(\kappa_g^2-1)\kappa_g tan(\theta(s))\theta'(s)}{(\kappa_g^2-1)^{\frac{3}{2}}}\gamma(s)+cos(\theta(s))\frac{\tau_g+\sqrt{\kappa_g^2-1}\theta'(s)}{(\kappa_g^2-1)^{\frac{1}{2}}}\mathbf{e}(s)
\end{aligned}
$$

$$+cos(\theta(s))\frac{-\kappa_g\kappa_g' + (\kappa_g^2-1)^{\frac{3}{2}}\tau_g(s)tan(\theta(s)) - (\kappa_g^2-1)tan(\theta(s))\theta'(s)}{(\kappa_g^2-1)^{\frac{3}{2}}}\mathbf{n}(s)$$

$$= \rho(s)\Big(\frac{\kappa_g' cos(\theta(s))}{(\kappa_g^2-1)^{\frac{1}{2}}\tau_g[(\kappa_g')^2+\kappa_g^2\tau_g^2(\kappa_g^2-1)]}\gamma(s) + \frac{\sqrt{\kappa_g^2-1}cos(\theta(s))}{(\kappa_g')^2+\kappa_g^2\tau_g^2(\kappa_g^2-1)}\mathbf{e}(s)$$
$$-\frac{\kappa_g' cos(\theta(s))}{\sqrt{\kappa_g^2-1}\kappa_g\tau_g[(\kappa_g')^2+\kappa_g^2\tau_g^2(\kappa_g^2-1)]}\mathbf{n}(s)\Big).$$

Therefore, $SDF'_\gamma(s,\theta(s)) \equiv 0$ if and only if $\rho(s) \equiv 0$. This means that the conditions (1) and (2) are equivalent. Assume that the condition (1) holds. Then we have $\langle\gamma(s), SDF_\gamma(s,\theta(s))\rangle = -\frac{\kappa_g cos(\theta(s))}{\sqrt{\kappa_g^2-1}}$, which is constant. Under this condition, we put $r = -\frac{\kappa_g cos(\theta(s))}{\sqrt{\kappa_g^2-1}}$ and $\mathbf{v}_0 = \frac{cos(\theta(s))}{\kappa_g(s)\tau_g(s)\sqrt{\kappa_g^2(s)-1}}(-\kappa_g^2(s)\tau_g(s)\gamma(s) + \kappa_g(s)\tau_g(s)\mathbf{n}(s) + \kappa_g'(s)\mathbf{e}(s))$. Then it is easy to show that $\gamma(s)$ is a part of the pseudo-sphere $HPS^1(\mathbf{v}_0, r)$. The condition (3) holds. For the converse, we assume that $\langle\gamma(s),\mathbf{v}\rangle = c$ for a constant vector $\mathbf{v}$ and a real number c. Since $h_v^S(s) = c$, we have $\mathbf{v} = \frac{cos(\theta(s))}{\kappa_g(s)\tau_g(s)\sqrt{\kappa_g^2(s)-1}}(-\kappa_g^2(s)\tau_g(s)\gamma(s) + \kappa_g(s)\tau_g(s)\mathbf{n}(s) + \kappa_g'(s)\mathbf{e}(s))$, by Proposition 3.2, so that the condition (1) holds. When $0 < \kappa_g(s) < 1$, the proof is similar to those of (A). This completes the proof. □

4. Proof of the main results

Let function germ $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has A_k-singularity ($k \geq 1$) at s_0, where $f(s) = F(s, x_0)$. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by

$$j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s, x_0)\right)(s_0) = \sum_{j=1}^{k-1} a_{ji}(s-s_0)^j,\ i = 1,\cdots,r.$$

Then F is called a $\mathcal{R}^+$-versal unfolding if the $(k-1)\times r$ matrix of coefficients (a_{ji}) has rank $k-1$, $(k-1 \leq r)$. The *bifurcation set of order l* concerning the unfolding F is defined by

$$\mathfrak{B}_F^l = \left\{\mathbf{x}\in\mathbb{R}^r \mid \exists s\in\mathbb{R}, \frac{\partial F}{\partial s}(s,\mathbf{x}) = \frac{\partial^2 F}{\partial s^2}(s,\mathbf{x}) = \cdots = \frac{\partial^{l+1} F}{\partial s^{l+1}}(s,\mathbf{x}) = 0\right\}.$$

Then $\mathfrak{B}_F^1 = \mathfrak{B}_F$ is the bifurcation set and $\mathfrak{B}_F^2$ is the set of singular points of $\mathfrak{B}_F$. We have the following classification result (cf., [13]).

Theorem 4.1. *Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, \mathbf{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has the A_k singularity at s_0. Suppose that $F(s,\mathbf{x})$ is an $\mathcal{R}^+$-versal unfolding, then we have the following claims.*

(a) *If $k = 2$, then $\mathfrak{B}_F$ is locally diffeomorphic to $\{0\}\times\mathbb{R}^{r-1}$ and $\mathfrak{B}_F^2 = \emptyset$.*

(b)*If $k = 3$, then $\mathfrak{B}_F$ is locally diffeomorphic to $C(2,3)\times\mathbb{R}^{r-2}$, $\mathfrak{B}_F^2$ is diffeomorphic to $\{0\}\times\mathbb{R}^{r-2}$ and $\mathfrak{B}_F^3 = \emptyset$.*

(c) *If $k = 4$, then $\mathfrak{B}_F$ is locally diffeomorphic to $SW\times\mathbb{R}^{r-3}$, $\mathfrak{B}_F^2$ is diffeomorphic to $C(2,3,4)\times\mathbb{R}^{r-3}$, $\mathfrak{B}_F^3$ is diffeomorphic to $\{0\}\times\mathbb{R}^{r-3}$ and $\mathfrak{B}_F^4 = \emptyset$.*

We consider that $H^S(s,\mathbf{v})$ is an unfolding of $h_{\mathbf{v}_0}^S(s)$. Then we have the following proposition:

Proposition 4.2. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve with $\kappa_g(s)\tau_g(s) \neq 0$, $H^S : I \times S^3_1 \longrightarrow \mathbb{R}$ be the de Sitter height function on $\gamma(s)$. If $h^S_{\mathbf{v}_0}(s)$ has an A_k-singularity $(k = 3, 4)$ at s_0, then $H^S(s, \mathbf{v})$ is a $\mathcal{R}^+$-versal unfolding of $h^S_{\mathbf{v}_0}(s)$.*

Proof. Suppose that $\delta(s) = -1$. We consider the pseudo orthonormal basis

$$\mathbf{e}_0 = \gamma(s), \mathbf{e}_1 = \mathbf{t}(s), \mathbf{e}_2 = \mathbf{n}(s), \mathbf{e}_3 = \mathbf{e}(s)$$

instead of the canonical basis of $\mathbb{R}^4_1$. Let $\gamma(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ and $\mathbf{v} = (v_1, \cdots, v_4) \in S^3_1$. Because $\mathbf{v} \in S^3_1$ is a nonzero vector, we have $-v_1^2 + v_2^2 + \cdots + v_4^2 = 1$. Without loss of the generality, we might assume that $v_1 > 0$, hence $v_1 = \sqrt{-1 + v_2^2 + \cdots + v_4^2}$. By a straightforward calculation,

$$H^S(s, \mathbf{v}) = -x_1(s)v_1 + \sum_{i=2}^{4} x_i(s)v_i,$$

$$\frac{\partial H^S}{\partial v_i}(s, \mathbf{v}) = x_i(s) - \frac{v_i}{v_1}x_1(s),\ \frac{\partial}{\partial s}\frac{\partial H^S}{\partial v_i}(s, \mathbf{v}) = x_i'(s) - \frac{v_i}{v_1}x_1'(s),$$
$$\frac{\partial^2}{\partial s^2}\frac{\partial H^S}{\partial v_i}(s, \mathbf{v}) = x_i''(s) - \frac{v_i}{v_1}x_1''(s),\ \frac{\partial^3}{\partial s^3}\frac{\partial H^S}{\partial v_i}(s, \mathbf{v}) = x_i'''(s) - \frac{v_i}{v_1}x_1'''(s), i = 2, 3, 4.$$

Therefore the 3-jet of $\frac{\partial H^S}{\partial v_i}(s, \mathbf{v})(i = 2, 3, 4)$ at s_0 is given by

$$\begin{aligned} j^3\left(\frac{\partial H^S}{\partial v_i}(s, \mathbf{v}_0)\right)(s_0) &= \frac{\partial}{\partial s}\frac{\partial H^S}{\partial v_i}(s - s_0) + \frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial H^S}{\partial v_i}(s - s_0)^2 + \frac{1}{6}\frac{\partial^3}{\partial s^3}\frac{\partial H^S}{\partial v_i}(s - s_0)^3 \\ &= a_{1i}(s - s_0) + \frac{1}{2}a_{2i}(s - s_0)^2 + \frac{1}{6}a_{3i}(s - s_0)^3. \end{aligned}$$

When $h^S_{v_0}$ has A_3-singularity at s_0, we require the 2×3 matrix

$$\begin{pmatrix} x_2'(s) - \frac{v_2}{v_1}x_1'(s) & x_3'(s) - \frac{v_3}{v_1}x_1'(s) & x_4'(s) - \frac{v_4}{v_1}x_1'(s) \\ x_2''(s) - \frac{v_2}{v_1}x_1''(s) & x_3''(s) - \frac{v_3}{v_1}x_1''(s) & x_4''(s) - \frac{v_4}{v_1}x_1''(s) \end{pmatrix}$$

to be nonsingular at s_0. It is enough to show that the rank of the matrix A at s_0 is 3, where

$$A = \begin{pmatrix} x_2'(s) - \frac{v_2}{v_1}x_1'(s) & x_3'(s) - \frac{v_3}{v_1}x_1'(s) & x_4'(s) - \frac{v_4}{v_1}x_1'(s) \\ x_2''(s) - \frac{v_2}{v_1}x_1''(s) & x_3''(s) - \frac{v_3}{v_1}x_1''(s) & x_4''(s) - \frac{v_4}{v_1}x_1''(s) \\ x_2'''(s) - \frac{v_2}{v_1}x_1'''(s) & x_3'''(s) - \frac{v_3}{v_1}x_1'''(s) & x_4'''(s) - \frac{v_4}{v_1}x_1'''(s) \end{pmatrix}.$$

When $h^S_{v_0}$ has A_4-singularity at s_0, we require the rank of the 3×3 matrix A at s_0 is 3. Denote that

$$\mathbf{b}_i = \begin{pmatrix} x_i'(s) \\ x_i''(s) \\ x_i'''(s) \end{pmatrix}, i = 1, 2, 3, 4,$$

then we obtain

$$detA = -\frac{v_2 det(\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_4)}{v_1} - \frac{v_3 det(\mathbf{b}_2, \mathbf{b}_1, \mathbf{b}_4)}{v_1} - \frac{v_4 det(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_1)}{v_1} + \frac{v_1}{v_1}det(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$$

$$= \frac{v_1}{v_1}det(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4) - \frac{v_2}{v_1}det(\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_4) + \frac{v_3}{v_1}det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4) - \frac{v_4}{v_1}det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3).$$

Because

$$\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s) = (-det(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4), -det(\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_4), det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4), -det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3))$$

and

$$\mathbf{v} = \frac{cosh(\theta)}{\kappa_g(s)\tau_g(s)\sqrt{\kappa_g^2(s)+1}}\left(\kappa_g^2(s)\tau_g(s)\gamma(s) + \kappa_g(s)\tau_g(s)\mathbf{n}(s)) + \kappa_g'(s)\mathbf{e}(s)\right),$$

therefore

$$\begin{aligned}
detA &= \langle(\frac{v_1}{v_1}, \frac{v_2}{v_1}, \frac{v_3}{v_1}, \frac{v_4}{v_1}), \gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s)\rangle \\
&= \frac{1}{v_1}\langle \mathbf{v}, \gamma(s) \wedge \mathbf{t}(s) \wedge \kappa_g'(s)\mathbf{n}(s) + \gamma(s) \wedge \mathbf{t}(s) \wedge \kappa_g(s)\tau_g(s)\mathbf{e}(s) + \kappa_g^2(s)\tau_g(s)\mathbf{t}(s) \wedge \mathbf{n}(s) \wedge \mathbf{e}(s)\rangle \\
&= \frac{1}{v_1}\langle \mathbf{v}, \kappa_g'(s)\mathbf{e}(s) + \kappa_g(s)\tau_g(s)\mathbf{n}(s) + \kappa_g^2(s)\tau_g(s)\gamma(s)\rangle \\
&= \frac{cosh\theta}{v_1\kappa_g(s)\tau_g(s)\sqrt{1+\kappa_g^2(s)}}\left(-(\kappa_g'(s))^2 + \kappa_g^2(s)\tau_g^2(s)(1+(\kappa_g(s))^2)\right) \\
&= \frac{cosh\theta}{v_1\kappa_g(s)\tau_g(s)\sqrt{1+\kappa_g^2(s)}}\left(\kappa_g^2(s)\tau_g^2(s)(1+(\kappa_g(s))^2)sech^2\theta\right) \\
&= \frac{sech\theta\kappa_g(s)\tau_g(s)\sqrt{1+\kappa_g^2(s)}}{v_1}.
\end{aligned}$$

Because $\kappa_g(s)\tau_g(s) \neq 0, sech\theta(s) \neq 0$ for any s, therefore, $\kappa_g(s_0)\tau_g(s_0) \neq 0, sech\theta(s_0) \neq 0$. Hence, $detA \neq 0$ at s_0, the rank of the matrix A is three. In conclution, $H^S(s, \mathbf{v})$ is a $\mathcal{R}^+$-versal unfolding of $h^S_{\mathbf{v}_0}(s)$.

For the case of $\delta(s) = 1$, when $\kappa_g(s) > 1$,

$$\begin{aligned}
detA &= \frac{1}{v_1}\langle \mathbf{v}, \kappa_g'(s)\mathbf{e}(s) + \kappa_g(s)\tau_g(s)\mathbf{n}(s) - \kappa_g^2(s)\tau_g(s)\gamma(s)\rangle \\
&= \frac{sinh\theta}{v_1\kappa_g(s)\tau_g(s)\sqrt{1-\kappa_g^2(s)}}\left((\kappa_g'(s))^2 + \kappa_g^2(s)\tau_g^2(s)((1-\kappa_g(s))^2)\right) \\
&= \frac{sinh\theta}{v_1\kappa_g(s)\tau_g(s)\sqrt{1-\kappa_g^2(s)}}\left(\kappa_g^2(s)\tau_g^2(s)((\kappa_g(s))^2-1)sec^2\theta\right) \\
&= \frac{csch\theta\kappa_g(s)\tau_g(s)\sqrt{1-\kappa_g^2(s)}}{v_1}.
\end{aligned}$$

Under the assumption that $\kappa_g(s)\tau_g(s) \neq 0, cos\theta(s) \neq 0$ for any s, therefore, $\kappa_g(s_0)\tau_g(s_0) \neq 0, sec\theta(s_0) \neq 0$. Hence, $detA \neq 0$ at s_0, the rank of the matrix A is also three. When $0 < \kappa_g(s) < 1$, we get

$$detA = \frac{1}{v_1}\langle \mathbf{v}, \kappa_g'(s)\mathbf{e}(s) + \kappa_g(s)\tau_g(s)\mathbf{n}(s) - \kappa_g^2(s)\tau_g(s)\gamma(s)\rangle$$

$$\begin{aligned}
&= \frac{sinh\theta}{v_1\kappa_g(s)\tau_g(s)\sqrt{1-\kappa_g^2(s)}}\Big((\kappa_g'(s))^2-\kappa_g^2(s)\tau_g^2(s)((\kappa_g(s))^2-1)\Big)\\
&= \frac{sinh\theta}{v_1\kappa_g(s)\tau_g(s)\sqrt{1-\kappa_g^2(s)}}\Big(\kappa_g^2(s)\tau_g^2(s)((\kappa_g(s))^2-1)csch^2\theta\Big)\\
&= \frac{csch\theta\kappa_g(s)\tau_g(s)\sqrt{1-\kappa_g^2(s)}}{v_1}\\
&\neq 0.
\end{aligned}$$

In conclusion, if $h^S_{\mathbf{v}_0}(s)$ has an A_k-singularity ($k = 3, 4$) at s_0, then $H^S(s, \mathbf{v})$ is a $\mathcal{R}^+$-versal unfolding of $h^S_{\mathbf{v}_0}(s)$. This completes the proof. □

By Proposition 4.2 and the definition of caustic, we can prove the following Proposition 4.3.

Proposition 4.3. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve with $\kappa_g(s)\tau_g(s) \neq 0$, then de Sitter focal surfaces are two dimensional caustics which have Lagrangian singularities. For detail, the caustic is a regular surface at an A_2-singularity of $h^S_v(s)$. It is a cuspidal edge at an A_3-singularity of $h^S_v(s)$ and has swallowtail singularity at an A_4-singularity of $h^S_v(s)$.*

Proof. Taking

$$H^S(s,v) = -x_1(s)\sqrt{-1+v_2^2+\cdots+v_4^2}+\sum_{i=2}^{4}x_i(s)v_i$$

in Proposition 4.2 for example. We get the Jacobi matrix of $\Delta^*H^S = (\frac{\partial H^S}{\partial s})$ is

$$J\Delta^*H^S = \Big(\ \langle -\gamma(s)+\kappa_g(s)\mathbf{n}(s),\mathbf{v}\rangle \quad x_2'(s)-\tfrac{v_2}{v_1}x_1'(s) \quad x_3'(s)-\tfrac{v_3}{v_1}x_1'(s) \quad x_4'(s)-\tfrac{v_4}{v_1}x_1'(s)\ \Big).$$

Since the rank of the matrix A in the proof of the Proposition 4.2 is 3, the rank of the matrix $J\Delta^*H^S$ is 1. This means that $H^S(s, \mathbf{v})$ is a Morse family. Suppose that $\delta(s) = -1$, we have a smooth 3-dimensional submanifold,

$$\begin{aligned}
C_{H^S} &= \{(s,\mathbf{v})\in\mathbb{R}\times S^3_1 \mid \frac{\partial H^S}{\partial s}(s,\mathbf{v})=0\}\\
&= \{(s,\mathbf{v})\in\mathbb{R}\times S^3_1 \mid \mathbf{v}=\lambda\gamma(s)+\mu\mathbf{n}(s)+\nu\mathbf{e}(s),\lambda^2+\mu^2-\nu^2=1,\lambda\neq\mu\kappa_g, s\in I\}.
\end{aligned}$$

We denote $\pi_{C_{H^S}}$ and the map germ $\Phi^S_H : C_{H^S} \longrightarrow T^*S^3_1$ defined by

$$\Phi^S_H(s,\mathbf{v}) = \left(\mathbf{v},\ \frac{\partial H^S}{\partial v_2}(s,\mathbf{v}),\ldots,\frac{\partial H^S}{\partial v_4}(s,\mathbf{v})\right)$$

is a Lagrangian immersion germ. Therefore H^S is a generating family of $\Phi^S_H(C_{H^S})$. Let $\pi_2 : \mathbb{R}\times S^3_1 \longrightarrow S^3_1$ denote the canonical projection and consider the map-germ $\pi_{C_{H^S}}$ which is given by the the restriction of the projection π_2 to C_{H^S}. Thus $\pi_{C_{H^S}} : C_{H^S} \longrightarrow S^3_1$ with $\pi_{C_{H^S}}(s, \mathbf{v}) = \mathbf{v}$ for any $(s, \mathbf{v}) \in C_{H^S}$. The map $\pi_{C_{H^S}}$ is the catastrophe map of H^S and it is a Lagrangian map. Therefore, the corresponding caustic is

$$C(\Phi^S_H) = \{\mathbf{v}\in S^3_1 \mid \exists\, s\in\mathbb{R}\ \text{such that}\ \frac{\partial G}{\partial s}(s,\mathbf{v}) = \frac{\partial^2 G}{\partial s^2}(s,\mathbf{v}) = 0\}$$

$$= \{\mathbf{v} \in S^3_1 \mid \mathbf{v} = \frac{cosh\theta}{\sqrt{\kappa_g^2(s)+1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + sinh\theta\mathbf{e}(s) \mid s \in I, \theta \in \mathbb{R}\},$$

and is precisely the bifurcation set of H^S, i.e. $\mathcal{B}^S_H = C(\Phi^S_H)$. It follows that for a generic curve, the caustic $C(\Phi^S_H)$ of $\gamma(s)$ is locally either a regular surface, or has cuspidal edge singularity, or swallowtail. The local models of the caustic at $\mathbf{v}$ corresponding to $s \in I$ depend on the $\mathcal{R}^+$-singularity type of $h^S_v(s)$ at s. For a generic $\gamma(s)$, $h^S_v(s)$ has local singularities of types A_1, A_2, A_3 or A_4. The caustic is a regular surface at an A_2-singularity of $h^S_v(s)$. It is a cuspidal edge at an A_3-singularity of $h^S_v(s)$ and has swallowtail singularity at an A_4-singularity of $h^S_v(s)$. For the case of $\delta(s) = 1$, we have the similar arguments to the above proof, so that we omit it. □

Proof of Theorem 2.1. Proposition 4.3 states that de Sitter focal surfaces are two dimensional caustics which have Lagrangian singularities and the bifurcation set of $H^S(s, \mathbf{v})$ is

$$\mathfrak{B}_{H^S} = \{\mathbf{v} = \frac{cosh(\theta)}{\sqrt{\kappa_g^2(s)+1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + sinh(\theta)\mathbf{e}(s) \mid s \in I, \theta \in \mathbb{R}\}.$$

This means that the bifurcation set of the de Sitter height function is the image of the first de Sitter focal surface of $\gamma(s)$. It follows from Proposition 3.1 that $h^S_{v_0}$ has the A_3 -type singularity (respectively, the A_4 -type singularity) at s_0 if and only if

$$tanh\theta_0 = \frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{1+\kappa_g^2(s_0)}},$$

$$\mathbf{v} = \frac{cosh\theta_0}{\sqrt{\kappa_g^2(s_0)+1}}(\kappa_g(s_0)\gamma(s_0) + \mathbf{n}(s_0)) + sinh\theta\mathbf{e}(s_0),$$

$\rho(s_0) \neq 0$. (respectively,

$$tanh(\theta_0) = \frac{\kappa_g'(s_0)}{\kappa_g(s_0)\tau_g(s_0)\sqrt{1+\kappa_g^2(s_0)}},$$

$$\mathbf{v} = \frac{cosh\theta_0}{\sqrt{\kappa_g^2(s_0)+1}}(\kappa_g(s_0)\gamma(s_0) + \mathbf{n}(s_0)) + sinh\theta\mathbf{e}(s_0),$$

$\rho(s_0) = 0, \rho'(s_0) \neq 0$) By Theorem 4.1 and Proposition 4.2, we have the assertions (2) (respectively, (3)). By Corollary 3.3, this means that the osculating hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r_0)$ and $\gamma(s)$ have contact of order 4 for $s = s_0$. (respectively, the osculating hyperbolic pseudo-sphere $HPS^1(\mathbf{v}_0, r_0)$ and $\gamma(s)$ have contact of order 5 for $s = s_0$.) Since the locus of the singularities of CE is locally diffeomorphic to the line, $FDF_\gamma(s, \theta(s))$ is locally diffeomorphic to the line holds (respectively, since the locus of singularities of SW is $C(2, 3, 4)$, $FDF_\gamma(s, \theta(s))$ is locally diffeomorphic to $C(2, 3, 4)$ holds). □

For the proof of the Theorem 2.2, we apply Proposition 3.2, Theorem 4.1, Proposition 4.2 similar to the Theorem 2.1. This completes the proof.

5. Legendrian dualities between focal surfaces and spacelike curves

In this section, we investigate the relationships between de Sitter focal surfaces and spacelike curve by Legendrian dualities [11]. Firstly, we introduce the Legendrian dualities between pseudo-spheres in Minkowski space-time which have been proved to be a basic tool for the study of surfaces in pseudo-spheres in Minkowski space. One-forms on $\mathbb{R}^4_1 \times \mathbb{R}^4_1$ are defined by $\langle d\mathbf{v}, \mathbf{w}\rangle = -w_0 dv_0 + \sum_{i=1}^{3} w_i dv_i$ and $\langle \mathbf{v}, d\mathbf{w}\rangle = -v_0 dw_0 + \sum_{i=1}^{3} v_i dw_i$. We consider the following three double fibrations.

(1) (a) $S^3_1 \times S^3_1 \supset \Delta_5 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w}\rangle = 0\}$,
(b) $\pi_{11} : \Delta_5 \longrightarrow S^3_1, \pi_{12} : \Delta_5 \longrightarrow S^3_1$,
(c) $\theta_{11} = \langle d\mathbf{v}, \mathbf{w}\rangle|_{\Delta_5}, \theta_{12} = \langle \mathbf{v}, d\mathbf{w}\rangle|_{\Delta_5}$.

Here $\pi_{11}(\mathbf{v}, \mathbf{w}) = \mathbf{v}, \pi_{12}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$. We remark that $\theta_{11}^{-1}(0)$ and $\theta_{12}^{-1}(0)$ define the same tangent hyperplane field over Δ_5 which is denoted by K_5. The basic duality theorem is that (Δ_5, K_5) is a contact manifold and both of $\pi_{1j} (j = 1, 2)$ are Legendrian fibrations. If there exists an isotropic mapping $i : L \longrightarrow \Delta_5$, which means that $i^*\theta_{11} = 0$, we say that $\pi_{11}(i(L))$ and $\pi_{12}(i(L))$ are Δ_5-dual to each other. It is easy to see that the condition $i^*\theta_{11} = 0$ is equivalent to $i^*\theta_{12} = 0$. Then we have the following proposition on the relationships among the first de Sitter focal surface, the second de Sitter focal surface, and the spacelike curve with the help of the above Legendrian dualities.

Proposition 5.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve with $\kappa_g \tau_g(s) \neq 0$, then we have the following claims.*

(1) *For the case that $\delta(s) = -1$, $\gamma(s)$ and $FDF_\gamma(s, \theta)$ are Δ_5-dual to each other, $\mathbf{t}(s)$ and $FDF_\gamma(s, \theta)$ are Δ_5-dual to each other.*

(2) *For the case that $\delta(s) = 1$ and $\kappa_g(s) \neq 1$, $\gamma(s)$ and $SDF_\gamma(s, \theta)$ are Δ_5-dual to each other, $\mathbf{t}(s)$ and $SDF_\gamma(s, \theta)$ are Δ_5-dual to each other.*

Proof. (1) Consider the mapping $\mathfrak{L}_{11}(s, \theta) = (FDF_\gamma(s, \theta), \gamma(s))$ and $\mathfrak{L}_{12}(s, \theta) = (FDF_\gamma(s, \theta), \mathbf{t}(s))$. Then we have $\langle FDF_\gamma(s, \theta), \gamma(s)\rangle = 0$, $\langle FDF_\gamma(s, \theta), \mathbf{t}(s)\rangle = 0$ and

$$\begin{aligned}
\mathfrak{L}_{11}^*\theta_{12}(s, \theta) &= \langle FDF_\gamma(s, \theta), \gamma'(s)\rangle \\
&= \langle \frac{cosh(\theta)}{\sqrt{\kappa_g^2(s) + 1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + sinh(\theta)\mathbf{e}(s), \mathbf{t}(s)\rangle \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{L}_{12}^*\theta_{12}(s, \theta) &= \langle FDF_\gamma(s, \theta), \mathbf{t}'(s)\rangle \\
&= \langle \frac{cosh(\theta)}{\sqrt{\kappa_g^2(s) + 1}}(\kappa_g(s)\gamma(s) + \mathbf{n}(s)) + sinh(\theta)\mathbf{e}(s), -\gamma(s) + \kappa_g(s)\mathbf{n}(s)\rangle \\
&= -\frac{\kappa_g(s)cosh(\theta)}{\sqrt{\kappa_g^2(s) + 1}} + \frac{\kappa_g(s)cosh(\theta)}{\sqrt{\kappa_g^2(s) + 1}} \\
&= 0.
\end{aligned}$$

The assertion (1) holds.

(2) Using the same computation as the proof of (A), we consider the mapping $\mathfrak{L}_{21}(s,\theta) = (SDF_\gamma(s,\theta), \gamma(s))$ and $\mathfrak{L}_{22}(s,\theta) = (SDF_\gamma(s,\theta), \mathbf{t}(s))$. Then we have $\langle SDF_\gamma(s,\theta), \gamma(s)\rangle = 0$, $\langle SDF_\gamma(s,\theta), \mathbf{t}(s)\rangle = 0$. When $\kappa_g(s) > 1$,

$$\begin{aligned}\mathfrak{L}_{21}^*\theta_{12}(s,\theta) &= \langle SDF_\gamma(s,\theta), \gamma'(s)\rangle\\ &= \langle \frac{cos(\theta)}{\sqrt{\kappa_g^2(s)-1}}(-\kappa_g(s)\gamma(s)+\mathbf{n}(s)) + sin(\theta)\mathbf{e}(s), \mathbf{t}(s)\rangle\\ &= 0,\end{aligned}$$

$$\begin{aligned}\mathfrak{L}_{22}^*\theta_{12}(s,\theta) &= \langle SDF_\gamma(s,\theta), \mathbf{t}'(s)\rangle\\ &= \langle \frac{cos(\theta)}{\sqrt{\kappa_g^2(s)-1}}(-\kappa_g(s)\gamma(s)+\mathbf{n}(s)) + sin(\theta)\mathbf{e}(s), -\gamma(s)+\kappa_g(s)\mathbf{n}(s)\rangle\\ &= \frac{\kappa_g(s)cos(\theta)}{\sqrt{\kappa_g^2(s)-1}} - \frac{\kappa_g(s)cos(\theta)}{\sqrt{\kappa_g^2(s)-1}}\\ &= 0.\end{aligned}$$

When $0 < \kappa_g(s) < 1$,

$$\begin{aligned}\mathfrak{L}_{21}^*\theta_{12}(s,\theta) &= \langle SDF_\gamma(s,\theta), \gamma'(s)\rangle\\ &= \langle \frac{sinh(\theta)}{\sqrt{1-\kappa_g^2(s)}}(-\kappa_g(s)\gamma(s)+\mathbf{n}(s)) + cosh(\theta)\mathbf{e}(s), \mathbf{t}(s)\rangle\\ &= 0,\end{aligned}$$

$$\begin{aligned}\mathfrak{L}_{22}^*\theta_{12}(s,\theta) &= \langle SDF_\gamma(s,\theta), \mathbf{t}'(s)\rangle\\ &= \langle \frac{sinh(\theta)}{\sqrt{1-\kappa_g^2(s)}}(-\kappa_g(s)\gamma(s)+\mathbf{n}(s)) + cosh(\theta)\mathbf{e}(s), -\gamma(s)+\kappa_g(s)\mathbf{n}(s)\rangle\\ &= \frac{\kappa_g(s)sinh(\theta)}{\sqrt{1-\kappa_g^2(s)}} - \frac{\kappa_g(s)sinh(\theta)}{\sqrt{1-\kappa_g^2(s)}}\\ &= 0.\end{aligned}$$

In conclusion, the assertion (2) holds. □

6. Examples

In order to better illustrate the main results, we give three examples that consist of de Sitter focal surfaces.

Example 6.1. Let $\gamma(s)$ be a unit speed spacelike curve on S^3_1 defined by

$$\gamma(s)=\left(\cos(\frac{\sqrt{3}}{3}s)\sinh(\frac{\sqrt{6}}{3}s),\ \cos(\frac{\sqrt{3}}{3}s)\cosh(\frac{\sqrt{6}}{3}s),\ \sin(\frac{\sqrt{3}}{3}s)\cos(-\frac{\sqrt{6}}{3}s),\ \sin(\frac{\sqrt{3}}{3}s)\sin(-\frac{\sqrt{6}}{3}s)\right)$$

with respect to an arclength parameter s, where $s \in (0.5, 1.4)$.

We get that $\langle \mathbf{t}'(s), \mathbf{t}'(s)\rangle = \frac{8}{9}(\cos(\frac{\sqrt{3}}{3}s))^2 + \frac{1}{9} \neq 1$, $\kappa_g(s) = \frac{2}{3}\sqrt{8(\cos(\frac{\sqrt{3}}{3}s))^2 - 2}$,

$$\tau_g(s) = -\frac{1}{3}\frac{4(\cos(\frac{\sqrt{3}}{3}s))^4 - 12(\cos(\frac{\sqrt{3}}{3}s))^2 + 5}{(4\cos(\frac{\sqrt{3}}{3}s))^2 - 1}$$

and $\mathbf{n}(s) = (n_1(s), n_2(s), n_3(s), n_4(s))$, where

$$n_1(s) = \frac{2\cos(\frac{\sqrt{3}}{3}s)\sinh(\frac{\sqrt{6}}{3}s) - \sqrt{2}\sin(\frac{\sqrt{3}}{3}s)\cos(\frac{\sqrt{6}}{3}s)}{\sqrt{8(\cos(\frac{\sqrt{3}}{3}s))^2 - 2}},$$

$$n_2(s) = \frac{2\cos(\frac{\sqrt{3}}{3}s)\cosh(\frac{\sqrt{6}}{3}s) - \sqrt{2}\sin(\frac{\sqrt{3}}{3}s)\sinh(\frac{\sqrt{6}}{3}s)}{\sqrt{8(\cos(\frac{\sqrt{3}}{3}s))^2 - 2}},$$

$$n_3(s) = \frac{-\sqrt{2}\cos(\frac{\sqrt{3}}{3}s)\sin(\frac{\sqrt{6}}{3}s)}{\sqrt{8(\cos(\frac{\sqrt{3}}{3}s))^2 - 2}}, n_4(s) = \frac{-\sqrt{2}\cos(\frac{\sqrt{3}}{3}s)\cos(\frac{\sqrt{6}}{3}s)}{\sqrt{8(\cos(\frac{\sqrt{3}}{3}s))^2 - 2}}.$$

Thus, we can get $\delta(s) = -1$, the first de Sitter focal surface $FDF_\gamma(s,\theta)$ and the singular locus of the first de Sitter focal surface $SFDF_\gamma(s)$. We see that $\rho(s) = 0$ for $s = 1.360349523$, $\rho'(s) = 0$ for $s = 0.5113872103$. Hence, we have that the first de Sitter focal surface $FDF_\gamma(s,\theta)$ is locally diffeomorphic to cuspidal edge at its singular points and the singular locus of the first de Sitter focal surface $SFDF_\gamma(s)$ is locally diffeomorphic to a line for $s \neq 1.360349523$. The first de Sitter focal surface of spacelike curve $FDF_\gamma(s,\theta)$ is locally diffeomorphic to the SW at its singular points and the singular locus of the first de Sitter focal surface $SFDF_\gamma(s)$ is locally diffeomorphic to the $(2,3,4)-$cusp for $s \neq 0.5113872103$.

Example 6.2. Let $\gamma(s)$ be a unit speed spacelike curve on S^3_1 defined by

$$\gamma(s) = \left(\frac{\sqrt{2}}{2}\sinh(\sqrt{2}s),\ \frac{\sqrt{2}}{2}\cosh(\sqrt{2}s),\ \frac{\sqrt{2}}{2}\sin(2s),\ \frac{\sqrt{2}}{2}\cos(2s)\right)$$

with respect to an arclength parameter s. We draw the pictures of the spacelike curve γ , by projecting them into three-dimensional spaces, see Figure 2. Then we get that $\langle \mathbf{t}'(s), \mathbf{t}'(s)\rangle = 10 \neq 1$, $\kappa_g(s) = 3$ and $\tau_g(s) = 2\sqrt{2}$. Thus, we can get $\rho(s) = 144\sqrt{2}$. We obtain one of normal vector $\mathbf{n}(s)$ which is given by

$$\mathbf{n}(s) = \left(\frac{\sqrt{2}}{2}\sinh(\sqrt{2}s),\ \frac{\sqrt{2}}{2}\cosh(\sqrt{2}s), -\frac{\sqrt{2}}{2}\sin(2s),\ \frac{\sqrt{2}}{2}\cos(2s)\right).$$

It is easy to get $\delta(s) = -1$. Let $\sinh(\theta) = u,\ \cosh(\theta) = \sqrt{1+u^2}$. Thus, the first de Sitter focal surface is given by

$$FDF_\gamma(u,s) = (x_1(u,s), x_2(u,s), x_3(u,s), x_4(u,s))$$

and we obtain the vector parametric equations of the singular locus of the first de Sitter focal surface as follow:

$$SFDF_\gamma(s) = \left(\frac{2\sqrt{5}}{5}\sinh(\sqrt{2}s), \frac{2\sqrt{5}}{5}\cosh(\sqrt{2}s), \frac{\sqrt{5}}{5}\sin(2s), \frac{\sqrt{5}}{5}\cos(2s)\right),$$

where

$$\begin{cases} x_1(u,s) = \dfrac{2\sqrt{5}}{5}\sqrt{1+u^2}\sinh(\sqrt{2}s) + \sqrt{2}u\cosh(\sqrt{2}s), \\ x_2(u,s) = \dfrac{2\sqrt{5}}{5}\sqrt{1+u^2}\cosh(\sqrt{2}s) + \sqrt{2}u\sinh(\sqrt{2}s), \\ x_3(u,s) = \dfrac{\sqrt{5}}{5}\sqrt{1+u^2}\sin(2s) + u\cos(2s), \\ x_4(u,s) = \dfrac{\sqrt{5}}{5}\sqrt{1+u^2}\cos(2s) - u\sin(2s). \end{cases}$$

We see that $\rho(s) = 144\sqrt{2} \neq 0$ for arbitrary real numbers $s > 0$. Hence, we have that the first de Sitter focal surface $FDF_\gamma(u,s)$ is locally diffeomorphic to cuspidal edge at its singular points and the singular locus of the first de Sitter focal surface $SFDF_\gamma(s)$ is locally difeomorphic to a line. The structure of the spacelike curve γ and the focal surface is not easily imagined but it is possible to project them into three-dimensional spaces. We draw the pictures of the spacelike curve γ , the focal surface and its singular locus by projecting them into three-dimensional spaces, see Figure 2 and Figure 3.

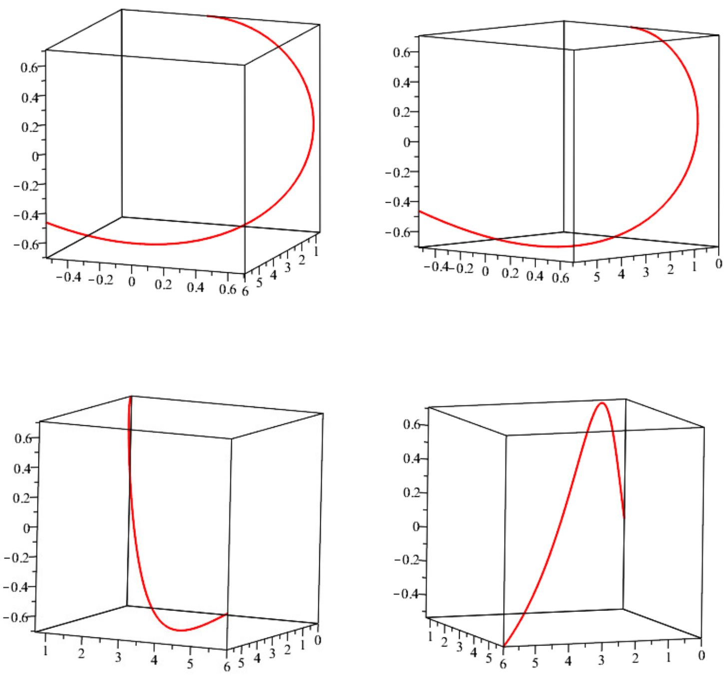

Figure 2. Projection of γ respectively on $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$.

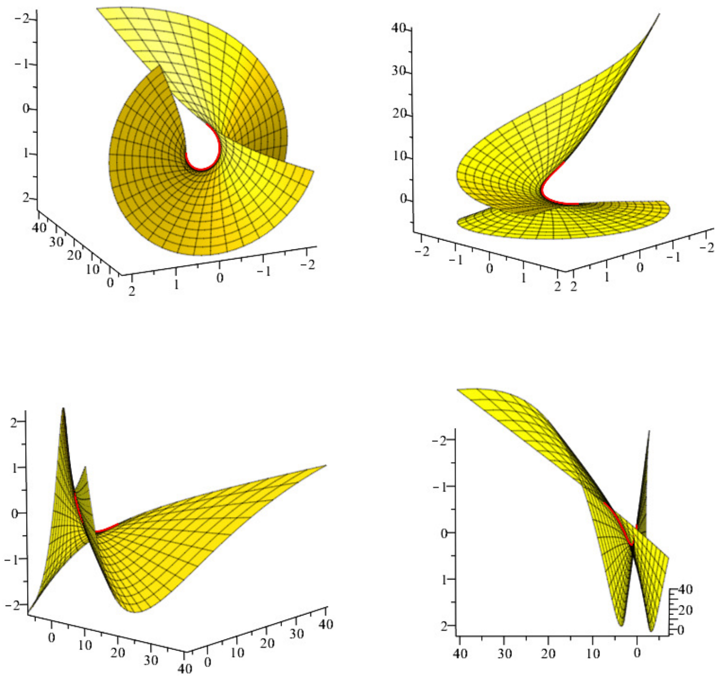

Figure 3. Projection of the first de Sitter focal surface respectively on $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$.

Example 6.3. Let $\gamma(s)$ be a unit speed spacelike curve on S^3_1 defined by

$$\gamma(s) = \left(\cosh(\frac{\sqrt{2}}{2}s),\ \sinh(\frac{\sqrt{2}}{2}s),\ \sqrt{2}\sin(\frac{1}{2}s),\ \sqrt{2}\cos(\frac{1}{2}s) \right),$$

where the arclength parameter $s \in (0, 2)$. Furthermore, the tangent vector

$$\mathbf{t}(s) = (\ \frac{\sqrt{2}}{2}\sinh(\frac{\sqrt{2}}{2}s),\ \frac{\sqrt{2}}{2}\cosh(\frac{\sqrt{2}}{2}s),\ \frac{\sqrt{2}}{2}\cos(\frac{1}{2}s),\ -\frac{\sqrt{2}}{2}\sin(\frac{1}{2}s)\ ).$$

Then we get that $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle = -\frac{1}{8} \neq 1$, $\kappa_g(s) = \frac{3}{4}\sqrt{2}$ and $\tau_g(s) = -\frac{\sqrt{2}}{4}$. Subsequently, it can be examined that $\rho(s) = -\frac{9\sqrt{2}}{256}$. It is calculated that the timelike normal vector $\mathbf{n}(s)$ which is given by

$$\mathbf{n}(s) = \left(\sqrt{2}\cosh(\frac{\sqrt{2}}{2}s),\ \sqrt{2}\sinh(\frac{\sqrt{2}}{2}s),\ \sin(\frac{1}{2}s),\ \cos(\frac{1}{2}s) \right).$$

So, we can get $\delta(s) = 1$, we calculate the other normal vector

$$\mathbf{e}(s) = (\ \frac{\sqrt{2}}{2}\sinh(\frac{\sqrt{2}}{2}s),\ \frac{\sqrt{2}}{2}\cosh(\frac{\sqrt{2}}{2}s),\ -\frac{\sqrt{2}}{2}\cos(\frac{1}{2}s),\ \frac{\sqrt{2}}{2}\sin(\frac{1}{2}s)\ ).$$

Moreover, the second de Sitter focal surface is formulated as

$$SDF_\gamma(s, \theta) = (x_1(s, \theta), x_2(s, \theta), x_3(s, \theta), x_4(s, \theta)),$$

where

$$\begin{cases} x_1(s,\theta) = \cos(\theta)\cosh(\frac{\sqrt{2}}{2}s) + \frac{\sqrt{2}}{2}\sin(\theta)\sinh(\frac{\sqrt{2}}{2}s), \\ x_2(s,\theta) = \cos(\theta)\sinh(\frac{\sqrt{2}}{2}s) + \frac{\sqrt{2}}{2}\sin(\theta)\cosh(\frac{\sqrt{2}}{2}s), \\ x_3(s,\theta) = -\sqrt{2}\cos(\theta)\sin(\frac{1}{2}s) - \frac{\sqrt{2}}{2}\sin(\theta)\cos(\frac{1}{2}s), \\ x_4(s,\theta) = -\sqrt{2}\cos(\theta)\cos(\frac{1}{2}s) + \frac{\sqrt{2}}{2}\sin(\theta)\sin(\frac{1}{2}s). \end{cases}$$

In addition, we obtain the vector parametric equations of the singular locus of the second de Sitter focal surface as follow:

$$SSDF_\gamma(s) = \Big(-\cosh(\frac{\sqrt{2}}{2}s), -\sinh(\frac{\sqrt{2}}{2}s),\ \sqrt{2}\sin(\frac{1}{2}s),\ \sqrt{2}\cos(\frac{1}{2}s)\Big).$$

Noticing that $\rho(s) = -\frac{9\sqrt{2}}{256} \neq 0$ for arbitrary real numbers $s > 0$. Hence, we have that the second de Sitter focal surface $SDF_\gamma(s,\theta)$ is locally difeomorphic to cuspidal edge at its singular points and the singular locus of the second de Sitter focal surface $SSDF_\gamma(s)$ is locally difeomorphic to a line. We draw the projection of the image of the second de Sitter focal surface $SDF_\gamma(s,\theta)$ (in orange) and its critical value set $SSDF_\gamma(s)$ (in red) to three-dimensional spaces (see Figure 4).

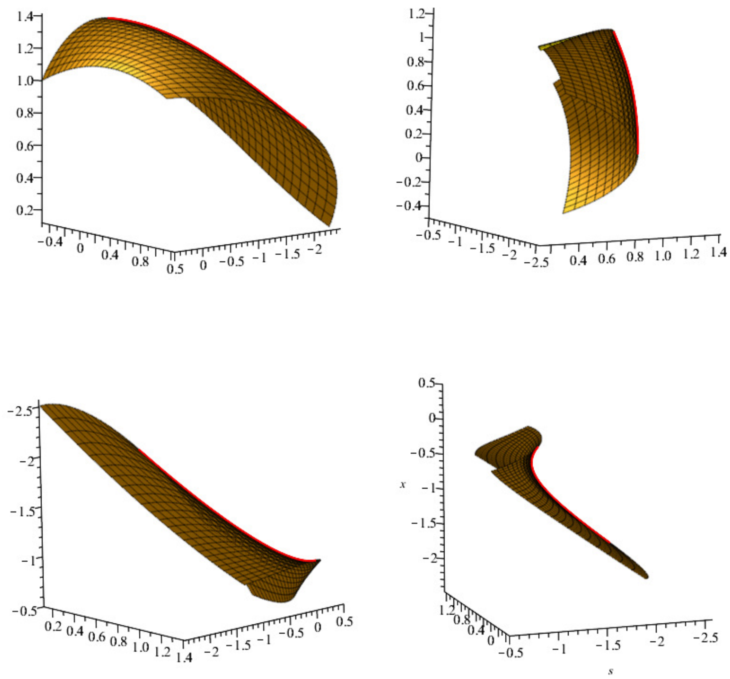

Figure 4. Projection of the second de Sitter focal surface respectively on $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$.

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Conflict of interest

The authors declare that there is no conflicts of interests in this work.

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