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Research article

Strongly essential set of vector Ky Fan's points problem and its applications

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Abstract: In this paper, several existence results of strongly essential set of the solution set for Ky Fan's section problems and vector Ky Fan's point problems are obtained. Firstly, two kinds of strongly essential sets of Ky Fan's section problems are defined, and some further results on existence of the strongly essential component of solutions set of Ky Fan's section problems are proved, which generalize the conclusion in [22], and further generalize the conclusions in [21, 28]. Secondly, based on the above results, two classes of stronger perturbations of vector-valued inequality functions are proposed respectively, and several existence results of the strongly essential component of set of vector Ky Fan's points are obtained. By comparing several metrics, we give some strong and weak relationships among the various metrics involved in the text. The main results of this paper actually generalize the relevant conclusions in the current literature. Finally, as an application, we obtain an existence result of the strongly essential component of weakly Pareto-Nash equilibrium for multiobjective games.

Keywords: strong essential set; strong essential component; Ky Fan's section problems; vector Ky Fan's points; Hausdorff upper semimetric; weakly Pareto-Nash equilibrium; multiobjective games
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1. Introduction

Let X be a nonempty compact convex set of Hausdorff linear topological space and $f : X \times X \to \mathbb{R}^k$ be a vector-valued function. In this paper, we consider the following vector Ky Fan's point problem (in short, **(VKF)**, called vector-valued Ky Fan's inequality problem by some authors) (see, [27]):

(VKF) *Find a point* $y^* \in X$ *such that* $f(x, y^*) \notin \mathrm{int}\mathbb{R}^k_+$ *for any* $x \in X$.

The above function $f : X \times X \to \mathbb{R}^k$ is called a vector-valued inequality function, and $y^* \in X$ is called a vector Ky Fan's point of f. In particular, when $k = 1$, the above vector-valued Ky Fan's inequality problem is just the usual Ky Fan's inequality problem, and the corresponding vector Ky Fan's point becomes the usual Ky Fan's point (see, [12, 20]).

Vector-valued Ky Fan's inequalities are natural generalizations of the Ky Fan's inequality to vector-valued functions, vector-valued Ky Fan's inequality theorem plays a very important role in the research of nonlinear and convex analysis. As the applications of vector-valued Ky Fan's inequality in many mathematical problems, such as vector optimization [9], vector variational inequality [6, 8, 13, 23], vector complementarity and multi-objective games [16, 27], etc., the study on the existence, stability and related applications of solutions of vector-valued Ky Fan's inequality have made rapid developments in the last twenty years [10, 22, 25, 27], and the references therein. Many researchers have achieved a lot of research results and extended it to various generalized forms, such as vector quasi-equilibrium problems [4], bilevel vector equilibrium problems [2, 15], vector quasi-variational inequality [7, 14, 19], generalized quasi-variational inclusion [3], set-valued Ky Fan's inequality [17] and stochastic Ky Fan's inequality etc. [22, 25, 27], and the references therein.

Many researchers focused on the stability of Ky Fan's points and vector Ky Fan's points. Tan et al. [20] and Yu et al. [24, 26] proposed the generic stability and essential components of Ky Fan's point with respect to the perturbation of inequality functions based on sup-norm metric, respectively. Similarly, Yang and Yu [27] obtained the existence of essential components of vector Ky Fan's point with respect to the perturbation of vector-valued inequality functions based on sup-norm metric. As we know, Ky Fan's section theorem is an equivalent form of (vector-valued) Ky Fan's inequality (see, [5, 11]), but there no longer has any function form. To discuss the stability of Ky Fan's section theorem, Zhou et al. [28] introduced a maximum Hausdorff metric of section mappings and obtained the existence of essential component of the set of solutions of Ky Fan's section theorem, which set up an alternative way to study the stable set of Ky Fan's point defined by the perturbation of section mappings.

In both of these cases, two perturbations were proposed by the sup-norm of inequality functions and the maximum Hausdorff metric of section mappings, respectively. Nevertheless, an example (see, Example 1) shows that there is no direct relationship between these two kinds of perturbations. Besides, both the sup-norm metric and the maximum Hausdorff metric must be defined on the total set, so it is very difficult to discuss the stability with respect to set perturbation generated from the uncertainty of cognition and choice. Therefore, there are two questions that deserves attention: (1) Is it possible to establish a perturbation which can include these two perturbations defined by the sup-norm and the maximum Hausdorff metric respectively? (2) Is it possible to define a class of essential sets which have stronger stability and provide a method to deal with the perturbations of strategic sets? Around these questions, Xiang et al. [21, 22] established the strongly stability of Ky Fan's points by introducing a class of stronger perturbations of section mapping and graph defined by the Hausdorff semi-metric on suitable set, respectively, both can be include two perturbations mentioned in question (1), and Xiang et al. [22] further provided a strongly stability analysis method for perturbation of set. However, Xiang et al. [22] only considered the special case in which the initial set is always the total space X, and the perturbation of set variation can only be inward reduced to $X' \in CK(X)$. So there are other two questions worth studying: (3) Is it possible to discuss more general case where the initial set X_0 is arbitrary compact convex subset of X and its perturbation way of set is also arbitrary? (4) In the case of vector value function, is the above method still applicable?

Inspired by the above research works, in this paper, we further investigate the strongly stability of the solution set for Ky Fan's section problems and vector Ky Fan's point problems, to solve the above four questions. Firstly, we shall introduce two kinds of stronger perturbations defined by the Hausdorff semi-metric on $X \times X$ and the Hausdorff metric on X, both can be include two perturbations mentioned in question (1). In some special case, we also compare the relationships among various metrics to obtain the strong and weak relations among these perturbations discussed. Next, based on these perturbations, some concepts of strongly essential set of solutions of Ky Fan's section problems are introduced, respectively. And in the general case where the initial set X_0 is arbitrary compact convex subset of X and its perturbation way of set is also arbitrary, we study and generalize the existence of strongly essential component of solution set for Ky Fan's section problems. Furthermore, we define two kinds of strongly essential sets of solutions of problem **(VKF)**, and based on the above existence results, we investigate the stability of problem **(VKF)**, and obtain some existence results of strongly essential component of solution set for problem **(VKF)**. Finally, as an application, we deduce an existence result of the strongly essential component of weakly Pareto-Nash equilibrium for multiobjective games by means of the above results, which provide a method to discuss the stability of set of weakly Pareto-Nash equilibrium for multiobjective games with respect to general perturbation of strategic set.

2. Preliminaries

Unless otherwise mentioned, we will restrict our discussion domain to a normed linear space $(E, \|\cdot\|)$. Let X be a nonempty convex compact subset of $(E, \|\cdot\|)$, denote $K(X)$ and $CK(X)$ the set of nonempty compact subsets of X and the set of all nonempty compact convex subsets of X respectively. Define a metric d on $X \times X$ as

$$d(x, y) = \|x_1 - y_1\| + \|x_2 - y_2\|$$

for $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $X \times X$. And denote H_d the Hausdorff metric on $K(X)$ or $K(X) \times K(X)$. Let $\mathbb{R}^k_+ = \{x = (x_1, \cdots, x_k) \in \mathbb{R}^k : x_i \geq 0, i = 1, \cdots, k\}$ and $\text{int}\mathbb{R}^k_+ = \{x = (x_1, \cdots, x_k) \in \mathbb{R}^k : x_i > 0, i = 1, \cdots, k\}$. It is clear that $\mathbb{R}^k_+$ is a nonempty closed convex and pointed cone in $\mathbb{R}^k$ with $\text{int}\mathbb{R}^k_+ \neq \emptyset$, and $\text{int}\mathbb{R}^k_+ + \mathbb{R}^k_+ = \text{int}\mathbb{R}^k_+$ (Ref. [27]).

Now, we first recall some basic concepts which will be used in the follows (Ref. [9, 18]).

Definition 1. *A mapping $T : Z \to 2^Y$ is said to be upper semicontinuous at z, if for any $\epsilon > 0$, there exists some $\delta > 0$ such that $T(z') \subset [T(z) + B_\epsilon(0)]$ for any $z' \in Z$ with $d(z', z) < \delta$; And T is said to be upper semicontinuous, if T is upper semicontinuous at any $z \in Z$; And $T : Z \to 2^Y$ is said to be a usco mapping, if T is upper semicontinuous and compact-valued on Z.*

Definition 2. *A vector-valued function $f : X \to \mathbb{R}^k$ is said to be $\mathbb{R}^k_+$-lower semicontinuous at $x \in X$, if for any open neighbourhood V of original point $\mathbf{0}$ in $\mathbb{R}^k$, there exists some open neighbourhood $O(x)$ of x such that $f(x') \in f(x) + V + \mathbb{R}^k_+$ for all $x' \in O(x)$; f is said to be $\mathbb{R}^k_+$-lower semicontinuous on X, if it is $\mathbb{R}^k_+$-lower semicontinuous at any $x \in X$; f is said to be $\mathbb{R}^k_+$-upper semicontinuous on X, if $-f$ is $\mathbb{R}^k_+$-lower semicontinuous on X; And f is said to be $\mathbb{R}^k_+$-continuous on X, if it is both $\mathbb{R}^k_+$-lower semicontinuous and $\mathbb{R}^k_+$-upper semicontinuous on X.*

Definition 3. *A vector-valued function $f : X \to \mathbb{R}^k$ is said to be $\mathbb{R}^k_+$-quasiconcave, if for any $x_1, x_2 \in X$ and any $\lambda \in (0, 1)$, such that $f(\lambda x_1 + (1 - \lambda)x_2) \in y + \mathbb{R}^k_+$ whenever $f(x_1) \in y + \mathbb{R}^k_+$ and $f(x_2) \in y + \mathbb{R}^k_+$ for any $y \in \mathbb{R}^k$.*

For convenience, we recall the Ky Fan's section theorem and the vector-valued Ky Fan's inequality theorem as Theorem **A** and Theorem **B** , respectively, (see [27, 28]).

Theorem A *Let X be a nonempty convex compact subset of space E, $A \subset X \times X$ satisfies:*
(1) *for each $x \in X$, $\{y \in X : (x, y) \in A\}$ is closed;*
(2) *for each $y \in X$, $\{x \in X : (x, y) \notin A\}$ is a convex or empty set;*
(3) *for each $x \in X$, $(x, x) \in A$.*
Then there exists $y_0 \in X$, such that $X \times \{y_0\} \subset A$.

Theorem B *Let X be a nonempty convex compact subset of space E, $\varphi : X \times X \to \mathbb{R}^k$ satisfies:*
(1) *for every $x \in X$, $y \to \varphi(x, y)$ is $\mathbb{R}^k_+$-lower semi-continuous;*
(2) *for every $y \in X$, $x \to \varphi(x, y)$ is $\mathbb{R}^k_+$-quasi-concave;*
(3) *for every $x \in X$, $\varphi(x, x) \notin \text{int}\mathbb{R}^k_+$.*
Then there exists $y^* \in X$, such that $\varphi(x, y^*) \notin \text{int}\mathbb{R}^k_+$ for every $x \in X$.

Note that y_0 in Theorem **A** is called a solution of section problem A, and $y^* \in X$ in Theorem **B** is called a vector Ky Fan's point of φ. In particular, if $k = 1$, then Theorem **B** is just the usual existence theorem of solutions of Ky Fan's inequality (see [12]).

In order to investigate the stability of solutions to Theorem **A** and Theorem **B**, as in [22, 28], denote

$$\mathcal{A} = \{A | A \subset X \times X \ is\ closed\ and\ satisfies\ (1) - (3)\ of\ Theorem\ \mathbf{A}\};$$
$$\mathcal{F} = \{\varphi | \varphi : X \times X \to \mathbb{R}^k\ is\ \mathbb{R}^k_+ - lower\ semicontinuous\ and\ satisfies\ (2), (3)\ of\ Theorem\ \mathbf{B}\}.$$

For each $\varphi \in \mathcal{F}$, denote
$$A_\varphi = \left\{(x, y) \in X \times X : \varphi(x, y) \notin \text{int}\mathbb{R}^k_+\right\}.$$

It can be to verify that $A_\varphi \in \mathcal{A}$. For each $A \in \mathcal{A}$ and $\varphi \in \mathcal{F}$, define the section mappings $E_A : X \to 2^X$ and $E_\varphi : X \to 2^X$ as
$$E_A(x) = \{y \in X : (x, y) \in A\},\ \forall x \in X;$$
$$E_\varphi(x) = \left\{y \in X : \varphi(x, y) \notin \text{int}\mathbb{R}^k_+\right\},\ \forall x \in X.$$

Then $E_\varphi = E_{A_\varphi}$ for any $\varphi \in \mathcal{F}$.

For each $A \in \mathcal{A}$, denote the solution set of problem A by $F_s(A) = \bigcap_{x \in X} E_A(x)$. And for each $\varphi \in \mathcal{F}$, denote the set of all vector Ky Fan's points of φ by $F_K(\varphi) = \bigcap_{x \in X} E_\varphi(x)$. It is obvious that $F_K(\varphi) = F_s(A_\varphi)$ for any $\varphi \in \mathcal{F}$. By Theorem **A** and **B**, the solution mappings $F_s : \mathcal{A} \to K(X)$ and $F_K : \mathcal{F} \to K(X)$ are both well-defined.

The sup-norm metric on $\mathcal{F}$ is introduced in general, that is,

$$\rho_m(\varphi_1, \varphi_2) = \sup_{(x,y) \in X \times X} \|\varphi_1(x, y) - \varphi_2(x, y)\|,\ \varphi_1, \varphi_2 \in \mathcal{F}.$$

Moreover, two metrics on $\mathcal{A}$ and $\mathcal{F}$ are defined by the maximum Hausdorff metric H_d as follows (Ref. [22, 28]):
$$\rho_s(A_1, A_2) = \sup_{x \in X} H_d(E_{A_1}(x), E_{A_2}(x)),\ A_1, A_2 \in \mathcal{A};$$
$$\rho_1(\varphi_1, \varphi_2) = \sup_{x \in X} H_d(E_{\varphi_1}(x), E_{\varphi_2}(x)),\ \varphi_1, \varphi_2 \in \mathcal{F}.$$

Now, some concepts of essentiality about vector Ky Fan's points with respect to ρ_m and ρ_1 are recalled (Ref. [27]).

Definition 4. *Let $\varphi \in \mathcal{F}$. A nonempty closed subset $e(\varphi) \subset F_K(\varphi)$ is said to be an essential set of $F_K(\varphi)$ with respect to ρ_m (or ρ_1), if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $F_K(\varphi') \bigcap [e(\varphi) + B_\varepsilon(0)] \neq \emptyset$ for any $\varphi' \in \mathcal{F}$ with $\rho_m(\varphi', \varphi) < \delta$ (or $\rho_1(\varphi', \varphi) < \delta$).*

Remark 1. *(1) A component C of $F_K(\varphi)$ is said to be an essential component of $F_K(\varphi)$ with respect to ρ_m (or ρ_1), if the component C of $F_K(\varphi)$ is essential with respect to ρ_m (or ρ_1). (2) Similar as Definition 4 and (1) of Remark 1, we may define the essential set and essential component of $F_s(A)$ with respect to ρ_s for $A \in \mathcal{A}$.*

Note that Yang and Yu [27], Yu and Peng [25] prove the existence of essential component for vector Ky Fan's points with respect to ρ_m and ρ_1, respectively. However, the following example shows that the essentiality of set of vector Ky Fan's points based on ρ_m is not necessarily related to the essentiality based on the sup-norm metric ρ_1.

Example 1. *Let $X = [0, 1]$. For $n = 1, 2, \cdots$, define $\varphi, \varphi^n, \phi^n : X \times X \mapsto \mathbb{R}^k$ as*

$$\begin{aligned}
&\varphi(x, y) = (0, \cdots, 0);\\
&\varphi^n(x, y) = (-1, \cdots, -1);\\
&\phi^n(x, y) = \left(\tfrac{1}{n}(y - x), \cdots, \tfrac{1}{n}(y - x)\right), \forall (x, y) \in X \times X.
\end{aligned}$$

Then $\varphi, \varphi^n, \phi^n \in \mathcal{F}$. It is easy to see that $E_\varphi(x) = E_{\varphi^n}(x) = [0, 1]$ and $E_{\phi^n}(x) = [0, x]$ for each $x \in X$. Then $\rho_1(\varphi^n, \varphi) = 0$, while $\rho_m(\varphi^n, \varphi) = \sqrt{k}$ does not converges to 0. On the other hand, it is clear that $\rho_m(\phi^n, \varphi) \to 0$ while $\rho_1(\phi^n, \varphi) = 1$ does not converges to 0. Therefore, the essentiality of the set of vector Ky Fan's points is not necessarily associated with these two kinds of perturbations defined by the metric ρ_1 and ρ_m respectively. This shows that the perturbation of the vector-valued inequality function, even defined by the strong sup-norm metric, when it is sufficiently small, can not guarantee that the perturbation of their section mappings is also sufficiently small.

According to Example 1, a question yields: How can a kind of perturbation be defined such that it includes perturbations defined by ρ_1 and ρ_m? To discuss this question, similar as in [21], we introduce two types of semi-metrics ρ_u^s and ρ_u^k on $\mathcal{A}$ and $\mathcal{F}$, respectively:

$$\rho_u^s(A_2, A_1) = \sup\nolimits_{x \in X} H_u(E_{A_2}(x), E_{A_1}(x)),\ \forall A_2, A_1 \in \mathcal{A};$$

$$\rho_u^k(\varphi_2, \varphi_1) = \sup\nolimits_{x \in X} H_u(E_{\varphi_2}(x), E_{\varphi_1}(x)),\ \forall \varphi_1, \varphi_1 \in \mathcal{F}.$$

where $H_u(A, B) = \sup_{z \in A} d(z, B)$ is the Hausdorff upper semi-metric on $K(X)$. It is clear that $\rho_u^k(\varphi_2, \varphi_1) = \rho_u^s(A_{\varphi_2}, A_{\varphi_1})$. The relations among the metrics $\rho_u^s, \rho_u^k, \rho_s, \rho_1$ and ρ_m are revealed below.

Proposition 1. (1) $\rho_u^s(A_2, A_1) \leq \rho_s(A_2, A_1)$, $\forall A_2, A_1 \in \mathcal{A}$;
(2) $\rho_u^k(\varphi_2, \varphi_1) \leq \rho_1(\varphi_2, \varphi_1)$, $\forall \varphi_2, \varphi_1 \in \mathcal{F}$;
(3) *Let $\varphi, \varphi_n \in \mathcal{F}$, if $\rho_m(\varphi_n, \varphi) \to 0$ $(n \to \infty)$, then $\rho_u^k(\varphi_n, \varphi) \to 0$.*

Proof. The conclusions (1) and (2) follow immediately from the definitions of ρ_s, ρ_1 and ρ_u^s, ρ_u^k.

(3) If the conclusion is not true, then there exist $\epsilon_0 > 0$, $\delta_n > 0$ with $\delta_n \to 0$, and a corresponding sequence of functions φ_n with $\rho_m(\varphi_n, \varphi) < \delta_n$, such that $\rho_u^k(\varphi_n, \varphi) \geq \epsilon_0$, that is,

$$\rho_u^k(\varphi_n, \varphi) = \sup_{x \in X} H_u\left(E_{\varphi_n}(x), E_\varphi(x)\right) = \sup_{x \in X}\left(\sup_{z \in E_{\varphi_n}(x)} d\left(z, E_\varphi(x)\right)\right) \geq \epsilon_0.$$

Then there exist $x_0 \in X$ and $y_n \in E_{\varphi_n}(x_0) \subset X$ such that $d(y_n, E_\varphi(x_0)) \geq \frac{1}{2}\epsilon_0$. From $E_{\varphi_n}(x_0) = \{y \in X : \varphi_n(x_0, y) \notin \text{int}\mathbb{R}^k_+\}$, we have $\varphi_n(x_0, y_n) \notin \text{int}\mathbb{R}^k_+$. Since X is compact, there exists a convergent subsequence of $\{y_n\}$, without loss of generality, we may assume that $y_n \to y_0 \in X$. As $\rho_m(\varphi_n, \varphi) < \delta_n \to 0$, i.e. $\varphi_n \to \varphi$ (with respect to ρ_m), there must be $\varphi(x_0, y_0) \notin \text{int}\mathbb{R}^k_+$. In fact, if it were not true, that is $\varphi(x_0, y_0) \in \text{int}\mathbb{R}^k_+$, then there exists an open neighbourhood V of original point $\mathbf{0}$ in $\mathbb{R}^k$, such that $\varphi(x_0, y_0) + V \in \text{int}\mathbb{R}^k_+$. And since $\rho_m(\varphi_n, \varphi) < \delta_n \to 0$ and φ is $\mathbb{R}^k_+$-lower semicontinuous, then $\varphi_n(x_0, y_n) \in \varphi(x_0, y_n) + \frac{1}{2}V$ and $\varphi(x_0, y_n) \in \varphi(x_0, y_0) + \frac{1}{2}V + \text{int}\mathbb{R}^k_+$ for sufficiently large n. Consequently,

$$\varphi_n(x_0, y_n) \in \varphi(x_0, y_n) + \frac{1}{2}V \subset \varphi(x_0, y_0) + V + \text{int}\mathbb{R}^k_+ \subset \text{int}\mathbb{R}^k_+ + \mathbb{R}^k_+ = \text{int}\mathbb{R}^k_+,$$

which contradicts with $\varphi_n(x_0, y_n) \notin \text{int}\mathbb{R}^k_+$. Thus, $\varphi(x_0, y_0) \notin \text{int}\mathbb{R}^k_+$. From $E_\varphi(x_0) = \{y \in X : \varphi(x_0, y) \notin \text{int}\mathbb{R}^k_+\}$, we have $y_0 \in E_\varphi(x_0)$, which is a contradiction with $d(y_n, E_\varphi(x_0)) \geq \frac{1}{2}\epsilon_0$ and $y_n \to y_0$. The proof is complete. □

Remark 2. *(1) The conclusions of Proposition 1 extend the conclusions of Proposition 2.1 in [21]. In fact, in the special case of $n = 1$, the conclusions of Proposition 1 is just the corresponding results of Proposition 2.1 in [21].*

(2) From (2) and (3) in Proposition 1, we know that the perturbation defined by ρ^k_u includes two perturbations defined by ρ_m and ρ_1. That is to say, the perturbation under ρ^k_u is sufficiently small whenever the perturbation under ρ_m or ρ_1 is small enough.

Moreover, in order to further study the case of perturbation generated from variation of sets, we introduce some notations as follows (Ref. [22]).

For each $A \in \mathcal{A}$ and $X' \in CK(X)$, denote

$$A|X' = \{(x, y) \in X' \times X' : (x, y) \in A\},$$

and

$$\mathcal{A}_X = \{A|X' : A \in \mathcal{A}, X' \in CK(X)\}.$$

It is obvious that $A|X' \subset X' \times X'$ is closed, and so it is compact by the compactness of $X' \times X'$.

For each $\varphi \in \mathcal{F}$ and $X' \in CK(X)$, denote $\varphi|X'$ the restriction of φ on the set X', i.e. $\varphi : X' \times X' \to \mathbb{R}^k$. Denote

$$\mathcal{F}_X = \{\varphi|X' : \varphi \in \mathcal{F}, X' \in CK(X)\}.$$

And for each $\varphi|X' \in \mathcal{F}_X$, let

$$A_{\varphi|X'} = \{(x, y) \in X' \times X' : \varphi(x, y) \notin \text{int}\mathbb{R}^k_+\};$$

$$E_{\varphi|X'}(x) = \{y \in X' : (x, y) \in A_{\varphi|X'}\} = \{y \in X' : \varphi(x, y) \notin \text{int}\mathbb{R}^k_+\}, \quad \forall x \in X'.$$

Then $A_{\varphi|X'} = A_\varphi|X'$, and $E_{\varphi|X'} : X' \to 2^{X'}$ is a section mapping of $\varphi|X'$. In particular, when $X' = X$, we have $A|X' = A$, $\varphi|X' = \varphi$, $A_{\varphi|X'} = A_\varphi$ and $E_{\varphi|X'} = E_\varphi$.

Now, similar as in [22], we introduce two kinds of semi-metrics ρ^s_H, ρ^k_H on $\mathcal{A}_X, \mathcal{F}_X$, respectively:

$$\rho^s_H(A_2|X_2, A_1|X_1) = H_u(A_2|X_2, A_1|X_1) + H_d(X_2, X_1), \ \forall A_2|X_2, A_1|X_1 \in \mathcal{A}_X;$$

$$\rho^k_H(\varphi_2|X_2, \varphi_1|X_1) = H_u(A_{\varphi_2|X_2}, A_{\varphi_1|X_1}) + H_d(X_2, X_1), \ \forall \varphi_2|X_2, \varphi_1|X_1 \in \mathcal{F}_X.$$

where $H_u(A, B) = \sup_{z\in A} d(z, B)$ is the Hausdorff upper semi-metric on $K(X) \times K(X)$ (see [18]). It is easy to see that $\rho^k_H(\varphi_2|X_2, \varphi_1|X_1) = \rho^s_H(A_{\varphi_2|X_2}, A_{\varphi_1|X_1}) = \rho^s_H(A_{\varphi_2}|X_2, A_{\varphi_1}|X_1)$.

In particular, if the perturbation of sets need not be considered, that is, $\mathcal{A}_X = \mathcal{A}, \mathcal{F}_X = \mathcal{F}$, then for $A_1, A_2 \in \mathcal{A}$ and $\varphi_1, \varphi_2 \in \mathcal{F}$, we have $A_1|X = A_1$, $A_2|X = A_2$ and $\varphi_1|X = \varphi_1, \varphi_2|X = \varphi_2$. Then $\rho^s_H(A_2|X, A_1|X) = \rho^s_H(A_2, A_1)$, $\rho^k_H(\varphi_2|X, \varphi_1|X) = \rho^k_H(\varphi_2, \varphi_1)$.

In the special case in which $\mathcal{A}_X = \mathcal{A}$ and $\mathcal{F}_X = \mathcal{F}$, the relations among the metrics ρ^s_u, ρ^k_u and ρ^s_H, ρ^k_H are revealed below.

Proposition 2. *(1) $\rho^s_H(A_2, A_1) \le \rho^s_u(A_2, A_1), \forall A_2, A_1 \in \mathcal{A}$;*
(2) $\rho^k_H(\varphi_2, \varphi_1) \le \rho^k_u(\varphi_2, \varphi_1), \forall \varphi_2, \varphi_1 \in \mathcal{F}$.

Proof. (1) It follows immediately from the proof of (1) of Proposition 2.1 in [22].

(2) For any $\varphi_2, \varphi_1 \in \mathcal{F}$, assume that $\rho^k_u(\varphi_2, \varphi_1) = r$, we need only to show that $\rho^k_H(\varphi_2, \varphi_1) \le r$ holds. Since $\rho^k_u(\varphi_2, \varphi_1) = \sup_{x\in X} H_u(E_{\varphi_2}(x), E_{\varphi_1}(x)) = r$, we have $E_{\varphi_2}(x) \subset E_{\varphi_1}(x) + B_r(0)$ for $x \in X$. And

$$\begin{aligned} A_{\varphi_1} &= \{(x, y) \in X \times X | \varphi_1(x, y) \notin \text{int}\mathbb{R}^k_+\} = \{(x, y) \in X \times X | y \in E_{\varphi_1}(x)\}; \\ A_{\varphi_2} &= \{(x, y) \in X \times X | \varphi_2(x, y) \notin \text{int}\mathbb{R}^k_+\} = \{(x, y) \in X \times X | y \in E_{\varphi_2}(x)\} \\ &\subset \{(x, y) \in X \times X | y \in E_{\varphi_1}(x) + B_r(0)\} \subset A_{\varphi_1} + B_r(0). \end{aligned}$$

Hence, $\rho^k_H(\varphi_2, \varphi_1) = H_u(A_{\varphi_2}, A_{\varphi_1}) = \sup_{z\in A_{\varphi_2}} d(z, A_{\varphi_1}) \le r$. The proof is complete. □

Remark 3. *Proposition 2 illustrates that the perturbation defined by ρ^s_H includes the perturbation defined by ρ^s_u, and the perturbation defined by ρ^k_H includes the perturbation defined by ρ^k_u.*

From the Proposition 1 and 2, in the special case in which $\mathcal{A}_X = \mathcal{A}$ and $\mathcal{F}_X = \mathcal{F}$, we immediately obtain the relations among the metrics $\rho^s_u, \rho^k_u, \rho^s_H, \rho^k_H, \rho_s, \rho_1$ and ρ_m as below.

Corollary 1. *(1) $\rho^s_H(A_2, A_1) \le \rho^s_u(A_2, A_1) \le \rho_s(A_2, A_1)$, $\forall A_2, A_1 \in \mathcal{A}$;*
(2) $\rho^k_H(\varphi_2, \varphi_1) \le \rho^k_u(\varphi_2, \varphi_1) \le \rho_1(\varphi_2, \varphi_1)$, $\forall \varphi_2, \varphi_1 \in \mathcal{F}$;
(3) Let $\varphi, \varphi_n \in \mathcal{F}$, if $\rho_m(\varphi_n, \varphi) \to 0$ $(n \to \infty)$, then $\rho^k_u(\varphi_n, \varphi) \to 0$, which further implies $\rho^k_H(\varphi_n, \varphi) \to 0$.

Remark 4. *(1) It is clear that Corollary 1 contains the conclusions of Proposition 2.1 in [22] as a special case in which $n = 1$.*

(2) Corollary 1 shows that the perturbation defined by ρ^k_H includes the perturbation defined by ρ^k_u, and further includes the perturbations defined by ρ_m and ρ_1. That is to say, when the perturbation under ρ_m or ρ_1 is sufficiently small, the perturbation under ρ^k_u is sufficiently small, and furthermore, the perturbation under ρ^k_H is also sufficiently small.

Looking back on Example 1, note that $E_\varphi(x) = E_{\varphi^n}(x) = [0, 1]$ and $E_{\phi^n}(x) = [0, x]$ for each $x \in X$, we have $E_{\varphi^n}(x) = E_\varphi(x)$ and $E_{\phi^n}(x) \subset E_\varphi(x)$, then $\rho^k_u(\varphi^n, \varphi) \to 0$ and $\rho^k_u(\phi^n, \varphi) \to 0$. Furthermore, we have $A_\varphi = [0, 1] \times [0, 1]$, $A_{\varphi^n} = [0, 1] \times [0, 1]$, and $A_{\phi^n} = \{(x, y) \in [0, 1] \times [0, 1] : y \le x\}$, consequently, $A_{\varphi^n} = A_\varphi, A_{\phi^n} \subset A_\varphi$, then $\rho^k_H(\varphi^n, \varphi) \to 0, \rho^k_H(\phi^n, \varphi) \to 0$.

3. Existence of strongly essential set of vector Ky Fan's points

Let us introduce the concepts of strongly essential set of solution for Ky Fan's section theorem and vector-valued Ky Fan's inequality with respect to ρ^s_u, ρ^s_H and ρ^k_u, ρ^k_H.

Definition 5. *(1) (Ref. [22]) Let $A \in \mathcal{A}$. A nonempty closed subset $e(A) \subset F_s(A)$ is said to be a strongly essential set of $F_s(A)$ with respect to ρ_u^s, if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $F_s(A') \bigcap [e(A) + B_\varepsilon(0)] \neq \emptyset$ for any $A' \in \mathcal{A}$ with $\rho_u^s(A', A) < \delta$.*

(2) Let $A|X_0 \in \mathcal{A}_X$. A nonempty closed subset $e(A|X_0) \subset F_s(A|X_0)$ is said to be a strongly essential set of $F_s(A|X_0)$ with respect to ρ_H^s, if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $F_s(A'|X') \bigcap [e(A|X_0) + B_\varepsilon(0)] \neq \emptyset$ for any $A'|X' \in \mathcal{A}_X$ with $\rho_H^s(A'|X', A|X_0) < \delta$.

Definition 6. *(1) (Ref. [21]) Let $\varphi \in \mathcal{F}$. A nonempty closed subset $e(\varphi) \subset F_K(\varphi)$ is said to be a strongly essential set of $F_K(\varphi)$ with respect to ρ_u^k, if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $F_K(\varphi') \bigcap [e(\varphi) + B_\varepsilon(0)] \neq \emptyset$ for any $\varphi' \in \mathcal{F}$ with $\rho_u^k(\varphi', \varphi) < \delta$.*

(2) Let $\varphi|X_0 \in \mathcal{F}_X$. A nonempty closed subset $e(\varphi|X_0) \subset F_K(\varphi|X_0)$ is said to be a strongly essential set of $F_K(\varphi|X_0)$ with respect to ρ_H^k, if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $F_K(\varphi'|X') \bigcap [e(\varphi|X_0) + B_\varepsilon(0)] \neq \emptyset$ for any $\varphi'|X' \in \mathcal{F}_X$ with $\rho_H^k(\varphi'|X', \varphi|X_0) < \delta$.

Remark 5. *(1) Similar to Remark 1 (1), we may define the strongly essential component in Definition 5 and 6.*

(2) Let e_1 and e_2 are two nonempty closed subsets of $F_s(A|X_0)$ with $e_1 \subset e_2$. If e_1 is a strongly essential set of $F_s(A|X_0)$ with respect to ρ_H^s, then so is e_2. Similarly, there are same results in three other cases.

(3) If S is a minimal element of the family $\mathcal{S}$ of all strongly essential sets with partial order defined by the inclusion relation, then S is said to be a strongly minimal essential set. A connected strongly minimal essential set S is called a strongly stable set.

(4) Obviously, the stability defined by strongly essential set includes the perturbation of sets. Besides, in the special case in which the perturbation is not focused on the sets, that is, $X' = X_0 \equiv X$, by Corollary 1, it follows that the perturbations defined by ρ_H^s and ρ_H^k include those defined by ρ_s, ρ_m and ρ_1. Thus the strongly essential set has stronger stability than those defined in Definition 1 and Remark 1.

(5) Let $\mathcal{F}' \subset \mathcal{F}_X$ and $\varphi|X_0 \in \mathcal{F}'$. Then we may define the strongly essential set on subspace $\mathcal{F}'$: A nonempty closed subset $e(\varphi|X_0) \subset F_K(\varphi|X_0)$ is said to be a strongly essential set of $F_K(\varphi|X_0)$ on $\mathcal{F}'$, if $\forall \epsilon > 0$, there exists $\delta > 0$, such that $F_K(\varphi'|X') \bigcap [e(\varphi|X_0) + B_\varepsilon(0)] \neq \emptyset$ for any $\varphi'|X' \in \mathcal{F}'$ with $\rho_H^k(\varphi'|X', \varphi|X_0) < \delta$. It is easy to see that a strongly essential set of $F_K(\varphi|X_0)$ is also a strongly essential set of $F_K(\varphi|X_0)$ on $\mathcal{F}'$.

Now, we recall some basic results on solution mapping $F_s : (\mathcal{A}_X, \rho_H^s) \to K(X)$ for Ky Fan's section problems, which refer to the Lemma 3.1, Lemma 3.3 and Theoren 3.1 in [22].

Lemma 1. (1) *$F_s : (\mathcal{A}_X, \rho_H^s) \to K(X)$ is an usco mapping;*
(2) *For each $A|X_0 \in \mathcal{A}_X$, $F_s(A|X_0)$ has at least one strongly minimal essential set with respect to ρ_H^s;*
(3) *For each $A = A|X \in \mathcal{A}_X$, $F_s(A)$ has at least one strongly essential component with respect to ρ_H^s.*

Proof. The conclusion (1) and the conclusion (3) see the Lemma 3.1 and Theoren 3.1 in [22], and the conclusion (2) follow similar to the proof of Lemma 3.3 in [22]. □

Noting $\rho_H^k(\varphi_2|X_2, \varphi_1|X_1) = \rho_H^s(A_{\varphi_2}|X_2, A_{\varphi_1}|X_1)$, by means of (1) of Lemma 1, we obtain a similar result for problems **(VKF)**.

Lemma 2. *$F_K : (\mathcal{F}_X, \rho_H^k) \to K(X)$ is an usco mapping.*

Proof. It is suffices to show that $F_K : (\mathcal{F}_X, \rho_H^k) \to K(X)$ is upper semicontinuous for each $\varphi|X_0 \in \mathcal{F}_X$. Note that $\varphi|X_0 \in \mathcal{F}_X$ implies $A_\varphi|X_0 \in \mathcal{A}_X$, it follows from Lemma 1 that F_s is upper semicontinuous at $A_\varphi|X_0$, then for every $\epsilon > 0$, there exist $\delta > 0$ such that

$$A_{\varphi'}|X' \in \mathcal{A}_X \text{ and } \rho_H^s(A_{\varphi'}|X', A_\varphi|X_0) = \rho_H^k(\varphi'|X', \varphi|X_0) < \delta$$

for any $\varphi'|X' \in \mathcal{F}_X$ with $\rho_H^k(\varphi'|X', \varphi|X_0) < \delta$, and then $F_s(A_{\varphi'}|X') \subset [F_s(A_\varphi|X_0) + B_\epsilon(0)]$. Consequently,

$$F_K(\varphi'|X') = F_s(A_{\varphi'}|X') \subset [F_s(A_\varphi|X_0) + B_\epsilon(0)] = [F_K(\varphi|X_0) + B_\epsilon(0)].$$

That is, F_K is upper semicontinuous at $\varphi|X_0 \in \mathcal{F}_X$. The proof is complete. □

In order to prove our main results, we need the following lemmas, which will be useful below (Ref. [1]).

Lemma 3. *([1]) Let X be a nonempty compact convex subset of linear topological space E. If $A_1, A_2, \cdots, A_n \in CK(X)$, then $co\left(\bigcup_{i=1}^n A_i\right) = \{\sum_{i=1}^n \lambda_i x_i : x_i \in A_i, \lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1\}$ and $co\left(\bigcup_{i=1}^n A_i\right) \in CK(X)$, where $co(A)$ denotes the convex hull of the set A.*

Lemma 4. *Let X be a nonempty compact convex subset of linear topological space E. For any $A_1, A_2, B_1, B_2 \in CK(X)$, we have*

$$H_d\left(co(A_1 \cup A_2), co(B_1 \cup B_2)\right) \leq \max\{H_d(A_1, B_1), H_d(A_2, B_2)\}.$$

Proof. Form Lemma 3, $\forall x \in co(A_1 \cup A_2)$, there exist $x_1 \in A_1, x_2 \in A_2$ and $t \in [0, 1]$, such that $x = tx_1 + (1-t)x_2$, and then

$$\begin{aligned} d\left(x, co(B_1 \cup B_2)\right) &= d\left(tx_1 + (1-t)x_2, tco(B_1 \cup B_2) + (1-t)co(B_1 \cup B_2)\right) \\ &\leq d\left(tx_1, tco(B_1 \cup B_2)\right) + d\left((1-t)x_2, (1-t)co(B_1 \cup B_2)\right) \\ &= td\left(x_1, co(B_1 \cup B_2)\right) + (1-t)d\left(x_2, co(B_1 \cup B_2)\right) \\ &\leq td(x_1, B_1) + (1-t)d(x_2, B_2) \leq tH_u(A_1, B_1) + (1-t)H_u(A_2, B_2) \\ &\leq \max\left\{H_u(A_1, B_1), H_u(A_2, B_2)\right\} \leq \max\left\{H_d(A_1, B_1), H_d(A_2, B_2)\right\}, \end{aligned}$$

which implies that

$$H_u\left(co(A_1 \cup A_2), co(B_1 \cup B_2)\right) = \max_{x \in co(A_1 \cup A_2)} d\left(x, co(B_1 \cup B_2)\right) \leq \max\{H_d(A_1, B_1), H_d(A_2, B_2)\}.$$

Similarly, we can verify that $H_l\left(co(A_1 \cup A_2), co(B_1 \cup B_2)\right) \leq \max\{H_d(A_1, B_1), H_d(A_2, B_2)\}$, where $H_l(A, B) = H_u(B, A)$ denotes the Hausdorff lower semi-metric of the sets A and B on $K(X)$.

Therefore, we have

$$\begin{aligned} &H_d\left(co(A_1 \cup A_2), co(B_1 \cup B_2)\right) \\ &\quad = \max\left\{H_u\left(co(A_1 \cup A_2), co(B_1 \cup B_2)\right), H_l\left(co(A_1 \cup A_2), co(B_1 \cup B_2)\right)\right\} \\ &\quad \leq \max\{H_d(A_1, B_1), H_d(A_2, B_2)\}. \end{aligned}$$

The proof is completed. □

In what follows, based on the general case for Ky Fan's section problem $A|X_0 \in \mathcal{A}_X$, which the initial set $X_0 \in CK(X)$ is arbitrary and the perturbation of the set variation is also arbitrary, we will generalize and prove the further existence results of strongly essential component of solutions set for Ky Fan's section problems.

Theorem 1. *For each $A|X_0 \in \mathcal{A}_X$, $F_s(A|X_0)$ has at least one strongly essential component with respect to ρ_H^s.*

Proof. By (2) of Lemma 1, for each $A|X_0 \in \mathcal{A}_X$, there exists $m(A|X_0)$ which is the strongly minimal essential set of $F_s(A|X_0)$. In what follows, we will prove that $m(A|X_0)$ is connected.

If it does not hols, then there exist closed subsets $C_1, C_2 \subset F_s(A|X_0)$ with $C_1 \neq \emptyset, C_2 \neq \emptyset$, such that $C_1 \cap C_2 = \emptyset$ and $m(A|X_0) = C_1 \cup C_2$. Since C_1, C_2 are closed, and so are compact, there exists an $\epsilon > 0$ and two open sets V_1, V_2 on X, such that $V_1 \supset C_1 + B_\epsilon(0), V_2 \supset C_2 + B_\epsilon(0)$ and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$. Then $(C_1 \cup C_2) + B_\epsilon(0) \subset (V_1 \cup V_2)$.

From the minimality of $m(A|X_0)$, we have C_1 and C_2 are not essential, then there exist a sequence $\{\delta_n > 0\}$ with $\delta_n \to 0$ and $A_n^1|X_n^1$, $A_n^2|X_n^2 \in \mathcal{A}_X$ corresponding to δ_n, such that $\rho_H^s(A_n^1|X_n^1, A|X_0) < \frac{\delta_n}{5}$, $\rho_H^s(A_n^2|X_n^2, A|X_0) < \frac{\delta_n}{5}$, and $F_s(A_n^1|X_n^1) \bigcap V_1 = \emptyset$, $F_s(A_n^2|X_n^2) \bigcap V_2 = \emptyset$.

Let $X_n = co(X_n^1 \bigcup X_n^2)$, it follows from Lemma 3 that $X_n \in CK(X)$.

Note that $\rho_H^s(A_n^1|X_n^1, A|X_0) = H_u(A_n^1|X_n^1, A|X_0) + H_d(X_n^1, X_0) < \frac{\delta_n}{5}$ implies $H_d(X_n^1, X_0) < \frac{\delta_n}{5}$. Similarly, one can obtain $H_d(X_n^2, X_0) < \frac{\delta_n}{5}$. By Lemma 4, we have

$$H_d(X_n, X_0) = H_d\left(co(X_n^1 \cup X_n^2), co(X_0 \cup X_0)\right) \leq \max\left\{H_d(X_n^1, X_0), H_d(X_n^2, X_0)\right\} < \frac{\delta_n}{5} \to 0,$$

which means that $X_n \to X_0 \in CK(X)$.

Let

$$E_n^1(x) = \left\{y \in X : (x, y) \in A_n^1\right\}, E_n^2(x) = \left\{y \in X : (x, y) \in A_n^2\right\},\ \forall x \in X.$$

Define $E_n : X \to K(X)$ as

$$E_n(x) = \left[E_n^1(x) \setminus V_2\right] \bigcup \left[E_n^2(x) \setminus V_1\right], \forall x \in X,$$
$$A_n = \{(x, y) \in X \times X : y \in E_n(x), x \in X\}.$$

Firstly, similar as in the proof of Theorem 3.1 of [22], we can easy to verify that $A_n \in \mathcal{A}$. Therefore, $A_n|X_n \in \mathcal{A}_X$.

Secondly, we check that $F_s(A_n|X_n) \bigcap (V_1 \bigcup V_2) = \emptyset$. Suppose by contradiction that there exists a $y_0 \in F_s(A_n|X_n) \bigcap (V_1 \bigcup V_2)$. Then $y_0 \in (V_1 \bigcup V_2)$. Without loss of generality, we may assume $y_0 \in V_1$. Since $F_s(A_n^1|X_n^1) \bigcap V_1 = \emptyset$, then $y_0 \notin F_s(A_n^1|X_n^1)$, and then there is some $x_0 \in X_n^1 \subset X$, such that $y_0 \notin E_n^1(x_0)$. Moreover, since $y_0 \in F_s(A_n|X_n)$, we have $y_0 \in E_n(x_0)$, then $y_0 \in E_n^2(x_0) \setminus V_1$, which contradicts with the assumption $y_0 \in V_1$.

Finally, we prove that $\rho_H^s(A_n|X_n, A|X_0) < \delta_n \to 0$. For any $(x_n, y_n) \in A_n|X_n \subset A_n$, we have $(x_n, y_n) \in X_n \times X_n$ and $y_n \in E_n(x_n)$. Note that $y_n \in E_n(x_n)$ implies $y_n \in E_n^1(x_n)$ or $y_n \in E_n^2(x_n)$, that is $(x_n, y_n) \in A_n^1$ or $(x_n, y_n) \in A_n^2$. Thus, we have $(x_n, y_n) \in A_n^1|X_n$ or $(x_n, y_n) \in A_n^2|X_n$. If $(x_n, y_n) \in A_n^1|X_n$ holds, for

$\rho_H^s(A_n^1|X_n^1, A|X_0) < \frac{\delta_n}{5}$, we have $H_d(X_n, X_0) < \frac{\delta_n}{5}$. By the triangle inequality, we get

$$\begin{aligned} d((x_n, y_n), A|X_0) &\le d((x_n, y_n), A_n^1|X_n^1) + H_u(A_n^1|X_n^1, A|X_0) \\ &\le H_u(A_n^1|X_n, A_n^1|X_n^1) + H_u(A_n^1|X_n^1, A|X_0) \\ &\le H_u(X_n \times X_n, X_n^1 \times X_n^1) + H_u(A_n^1|X_n^1, A|X_0) \\ &\le 2H_d(X_n, X_n^1) + H_u(A_n^1|X_n^1, A|X_0) \\ &\le 2(H_d(X_n, X_0) + H_d(X_0, X_n^1)) + H_u(A_n^1|X_n^1, A|X_0) \\ &\le 2H_d(X_n, X_0) + H_d(X_0, X_n^1) + \rho_H^s(A_n^1|X_n^1, A|X_0) \\ &< 4 \times \frac{\delta_n}{5} = \frac{4}{5}\delta_n. \end{aligned}$$

In a similar way, if $(x_n, y_n) \in A_n^2|X_n$ holds, we also obtain $d((x_n, y_n), A|X_0) < \frac{4}{5}\delta_n$ and $H_d(X_n, X_0) < \frac{\delta_n}{5}$. Therefore, from the arbitrariness of $(x_n, y_n) \in A_n|X_n$, one has

$$H_u(A_n|X_n, A|X_0) = \sup_{(x_n, y_n) \in A_n|X_n} d((x_n, y_n), A|X_0) \le \frac{4}{5}\delta_n.$$

And so $\rho_H^s(A_n|X_n, A|X_0) = H_u(A_n|X_n, A|X_0) + H_d(X_n, X_0) < \frac{4}{5}\delta_n + \frac{\delta_n}{5} = \delta_n \to 0$.

Note that $m(A|X_0)$ is the essential set of $F_s(A|X_0)$ with respect to ρ_H^s and $\rho_H^s(A_n|X_n, A|X_0) \to 0$ imply $F_s(A_n|X_n) \cap [m(A|X_0) + B_\epsilon(0)] \ne \emptyset$ for sufficiently large positive integer n. On the other hand, we also have

$$F_s(A_n|X_n) \cap [m(A|X_0) + B_\epsilon(0)] \subset F_s(A_n|X_n) \cap [C_1 \cup C_2 + B_\epsilon(0)] \subset F_s(A_n|X_n) \cap [V_1 \cup V_2] = \emptyset,$$

for each positive integer n, which leads to a contradiction. So the strongly minimal essential set $m(A|X_0)$ is connected.

Hence there exists a component C_α of $F_s(A|X_0)$ such that $m(A|X_0) \subset C_\alpha$. By (2) of Remark 5, it deduces that C_α is a strongly essential set of $F_s(A|X_0)$, and so is a strongly essential component of $F_s(A|X_0)$ with respect to ρ_H^s. The proof is complete. □

Remark 6. *By the proof of Theorem 1, we know that the strongly minimal essential set $m(A|X_0)$ is connected, and so is a strongly stable set. In Theorem 1, we study the more general case where the initial set $X_0 \in CK(X)$ is arbitrary for problem $A|X_0 \in \mathcal{A}_X$, and the perturbation of the set variation is also arbitrary. But the Theorem 3.1 in [22] only considers the special case in which the initial set is the total space X, and the perturbation of the set variation can only be reduced to $X' \in CK(X)$. Therefore, Theorem 1 actually generalizes the conclusion of Theorem 3.1 in [22] (see, (3) in Lemma 1). In fact, Theorem 3.1 in [22] is just the special case of Theorem 1 in this paper when $X_0 \equiv X$. Besides, by Corollary 1, we immediately draw the following conclusions, which takes the Theorem 3.1 in [21] and the Theorem 3.3 in [28] as its special cases.*

Corollary 2. (1) *For each $A \in \mathcal{A}_X$, $F_s(A)$ has at least one strongly essential component with respect to ρ_H^s;*
(2) *For each $A \in \mathcal{A}$, $F_s(A)$ has at least one strongly essential component with respect to ρ_u^s;*
(3) *For each $A \in \mathcal{A}$, $F_s(A)$ has at least one essential component with respect to ρ_s*

By Theorem 1, we can deduce the existence of strongly essential components of the set of vector Ky Fan's points with respect to ρ_H^k.

Theorem 2. *For each $\varphi|X_0 \in \mathcal{F}_X$, $F_K(\varphi|X_0)$ has at least one strongly essential component with respect to ρ_H^k.*

Proof. For each $\varphi|X_0 \in \mathcal{F}_X$, it means that $A_\varphi|X_0 = A_{\varphi|X_0} \in \mathcal{A}_X$. By Theorem 1, $F_s(A_\varphi|X_0)$ has at least one strongly essential component with respect to ρ_H^s, denote it by C_α. Then $\forall\epsilon > 0$, there exists $\delta > 0$, such that $F_s(A'|X') \bigcap [C_\alpha + B_\epsilon(0)] \neq \emptyset$ for any $A'|X' \in \mathcal{A}_X$ with $\rho_H^s(A'|X', A_\varphi|X_0) < \delta$.

Note that $\rho_H^k(\varphi'|X', \varphi|X_0) = \rho_H^s(A_{\varphi'|X'}, A_{\varphi|X_0}) = \rho_H^s(A_{\varphi'}|X', A_\varphi|X_0)$ and $F_K(\varphi|X_0) = F_s(A_\varphi|X_0)$, $F_K(\varphi'|X') = F_s(A_{\varphi'}|X')$. Clearly, if $\varphi'|X' \in \mathcal{F}_X$ and $\rho_H^k(\varphi'|X', \varphi|X_0) < \delta$, which imply $A_{\varphi'}|X' \in \mathcal{A}_X$ and $\rho_H^s(A_{\varphi'}|X', A_\varphi|X_0) < \delta$, then $F_s(A_{\varphi'}|X') \bigcap [C_\alpha + B_\epsilon(0)] \neq \emptyset$, namely, $F_K(\varphi'|X') \bigcap [C_\alpha + B_\epsilon(0)] \neq \emptyset$. By Definition 6 and (1) of Remark 5, C_α is an essential component of $F_K(\varphi|X_0)$. The proof is complete. □

By (5) of Remark 5 and Theorem 2, we can easy to deduce immediately the existence result of strongly essential component for the problems **(VKF)** on the subspace $\mathcal{F}$ of $\mathcal{F}_X$.

Corollary 3. *Let $\mathcal{F}' \subset \mathcal{F}_X$ and $\varphi|X_0 \in \mathcal{F}'$. Then $F_K(\varphi|X_0)$ has at least one strongly essential component with respect to ρ_H^k on $\mathcal{F}'$.*

Proof. By Theorem 2, for each $\varphi|X_0 \in \mathcal{F}' \subset \mathcal{F}_X$, there exists a strongly essential component with respect to ρ_H^k in $\mathcal{F}_X$ denoted by C. By Remark 5 (5), it is clear that C is also strongly essential component with respect to ρ_H^k on $\mathcal{F}'$. □

By Corollary 1, the strongly essential component of $F_K(\varphi)$ has stronger stability than those based on the metric ρ_m or ρ_1. Therefore, by Corollary 1 and Theorem 2, we can obtain the existence of essential component of solution set of the problems **(VKF)** with respect to ρ_u^k, ρ_1 or ρ_m.

Corollary 4. *Let $\varphi \in \mathcal{F}$. Then*
(1) $F_K(\varphi)$ has at least one strongly essential component with respect to ρ_u^k;
(2) $F_K(\varphi)$ has at least one essential component with respect to ρ_1;
(3) $F_K(\varphi)$ has at least one essential component with respect to ρ_m.

Proof. (1) For each $\varphi \in \mathcal{F} \subset \mathcal{F}_X$, by Theorem 2, $F_K(\varphi)$ has at least one strongly essential component with respect to ρ_H^k, which is denoted by C_α. Then $\forall\epsilon > 0$, there exists $\delta > 0$, such that $F_K(\varphi'|X') \bigcap [C_\alpha + B_\epsilon(0)] \neq \emptyset$ for any $\varphi'|X' \in \mathcal{F}_X$ with $\rho_H^k(\varphi'|X', \varphi) < \delta$. In particular, when $\rho_H^k(\varphi', \varphi) < \delta$, one has $F_K(\varphi') \bigcap [C_\alpha + B_\epsilon(0)] \neq \emptyset$. By (2) of Corollary 1, when $\rho_u^k(\varphi', \varphi) < \delta$, we have $\rho_H^k(\varphi', \varphi) \leq \rho_u^k(\varphi', \varphi) < \delta$, then $F_K(\varphi') \bigcap [C_\alpha + B_\epsilon(0)] \neq \emptyset$. By Definition 6 and Remark 5, we have that C_α is an essential component of $F_K(\varphi)$ with respect to ρ_u^k.

The proofs of conclusions (2) and (3) similar to (1), by (2) and (3) of Corollary 1, we may complete the proofs. □

It is easy to see that the strongly essential component of $F_K(\varphi|X_0)$ has stronger stability, so it may provide a more applicable and convenient approach for eliminating the solutions with relatively weak stability.

4. Application: Strong stability of weakly Pareto-Nash equilibrium of multi-objective game

Let $N = \{1, 2, \cdots, n\}$ be the set of players, and denote by $\Gamma(X, f)$ a multi-objective game by a $2n$-tuple $(X_1, \cdots, X_n; f_1, \cdots, f_n)$, where X_i is the strategy set of i-th player, $X = \prod_{i=1}^n X_i$ is the strategy profile set of the multi-objective game Γ, and $f_i = \{f_i^1, \cdots, f_i^k\} : X = \prod_{i=1}^n X_i \to \mathbb{R}^k$ is the vector-valued payoff function of i-th player, respectively, where $f_i^j : X = \prod_{i=1}^n X_i \to \mathbb{R}$ for each $j = 1, \cdots, k$. For each $i \in N$, denote $X_{-i} = (X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$, and for each $x = (x_1, \cdots, x_n) \in X$, denote $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in X_{-i}$.

Definition 7. $x^* = (x_i^*, x_{-i}^*) \in X$ *is called a weakly Pareto-Nash equilibrium of multi-objective game* $\Gamma(X, f)$*, if*

$$f_i(y_i, x_{-i}^*) - f_i(x_i^*, x_{-i}^*) \notin \operatorname{int}\mathbb{R}_+^k,\ \forall y \in X_i,\ \forall i \in N.$$

Assume that a multi-objective game $\Gamma(X, f)$ satisfies the condition **C** : (1) $\forall i \in N$, f_i is $\mathbb{R}_+^k$-continuous on X; and (2) $\forall y_{-i} \in X_{-i}$, $x_i \to \sum_{i\in N} f_i(x_i, y_{-i})$ is $\mathbb{R}_+^k$-quasi-concave on X_i.

Denote

$$\mathcal{G} = \{\Gamma(X, f) : \Gamma(X, f) \ satisfies\ the\ condition\ \mathbf{C}\}.$$

It is easy to verify that for $\Gamma(X, f) \in \mathcal{G}$, the corresponding function $\psi_f(x, y) = \sum_{i\in N} f_i(x_i, y_{-i}) - \sum_{i\in N} f_i(y_i, y_{-i})$ satisfies the conditions of vector-valued Ky Fan's inequality theorem and $\psi_f \in \mathcal{F}$.

To study the stability of weakly Pareto-Nash equilibrium based on perturbations including of the variations of the strategy set, we introduce some notations and definitions.

Let multi-objective game $\Gamma(X, f) \in \mathcal{G}$. For each $X' \in CK(X)$, denote by $\Gamma(X', f)$ a multi-objective game with the strategy profile set $X' = \prod_{i=1}^n X_i'$ and the vector-valued payoff function $f = (f_1, \cdots, f_n)$. In particular, if $X' = X$, we have $\Gamma(X', f) = \Gamma(X, f)$. Denote

$$\mathcal{G}_X = \{\Gamma(X', f) \in \mathcal{G} : X' \in CK(X)\}.$$

For each $X' \in CK(X)$ and $\Gamma(X', f) \in \mathcal{G}_X$, denoted by $N(\Gamma(X', f))$ the set of all weakly Pareto-Nash equilibria of multi-objective game $\Gamma(X', f)$. Let

$$U(x, y) = \sum_{i\in N} f_i(x_i, y_{-i}), \forall x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in X.$$

It is easy to see that

$$\psi_f(x, y) = U(x, y) - U(y, y) = \sum_{i\in N} f_i(x_i, y_{-i}) - \sum_{i\in N} f_i(y_i, y_{-i}), \forall x, y \in X.$$

Denote $\psi_f|X' : X' \times X' \to \mathbb{R}^k$ as follows:

$$(\psi_f|X')(x, y) = \psi_f(x, y), \forall x, y \in X'.$$

We define the collective-better-reply correspondence (Ref. [28]) $CBR_{\Gamma(X', f)} : X' \to 2^{X'}$ as follows:

$$CBR_{\Gamma(X', f)}(x) = \left\{y \in X' : U(x, y) - U(y, y) \notin \operatorname{int}\mathbb{R}_+^k\right\}, \forall x \in X'.$$

Then

$$\begin{aligned} CBR_{\Gamma(X', f)}(x) &= E_{\psi_f|X'}(x) = \left\{y \in X : (\psi_f|X')(x, y) \notin \operatorname{int}\mathbb{R}_+^k\right\} \\ &= \left\{y \in X' : \psi_f(x, y) \notin \operatorname{int}\mathbb{R}_+^k\right\}, \forall x \in X'. \end{aligned}$$

Moreover, by Definition 7, we have $N(\Gamma(X', f')) = F_K(\psi_{f'}|X')$. From Theorem **B** , one can see that for each $\Gamma(X', f') \in \mathcal{G}_X$, we have $F_K(\psi_{f'}|X') \neq \emptyset$, then $N(\Gamma(X', f')) \neq \emptyset$. Thus $N : \mathcal{G}_X \to K(X)$ is a set-value mapping with nonempty value.

We now introduce two metrics on $\mathcal{G}_X$ as follows: for any $\Gamma(X, f), \Gamma(X, g) \in \mathcal{G}_X$,

$$\rho_m(\Gamma(X, f), \Gamma(X, g)) = \sup_{x\in X} \|f(x) - g(x)\|,$$
$$\rho_2(\Gamma(X, f), \Gamma(X, g)) = \sup_{x\in X} H_d(CBR_{\Gamma(X,f)}(x), CBR_{\Gamma(X,g)}(x)).$$

By the definitions of metrics, $\rho_2(\Gamma(X, f), \Gamma(X, g)) = \rho_1(\psi_f, \psi_g)$ and $\rho_m(\Gamma(X, f), \Gamma(X, g)) = \rho_m(\psi_f, \psi_g)$.

Next, we further define a semi-metric on $\mathcal{G}_X$ as follows: for any $\Gamma(X^2, f^2), \Gamma(X^1, f^1) \in \mathcal{G}_X$,

$$\rho^g_H(\Gamma(X_2, f_2), \Gamma(X_1, f_1)) = H_u(A_{\psi_{f_2}}|X_2, A_{\psi_{f_1}}|X_1) + H_d(X_2, X_1).$$

Then $\rho^g_H(\Gamma(X_2, f_2), \Gamma(X_1, f_1)) = \rho^k_H(\psi_{f_2}|X_2, \psi_{f_1}|X_1)$.

Definition 8. *Given $\Gamma(X_0, f) \in \mathcal{G}_X$. A nonempty closed subset $e \subset N(\Gamma(X_0, f))$ is said to be a strongly essential set of $N(\Gamma(X_0, f))$ with respect to ρ^g_H, if $\forall\epsilon > 0$, there exists $\delta > 0$ such that $N(\Gamma(X', f'))\bigcap[e + B_\varepsilon(0)] \neq \emptyset$ for any $\Gamma(X', f') \in \mathcal{G}_X$ with $\rho^g_H(\Gamma(X', f'), \Gamma(X_0, f)) < \delta$. And if a component C of $N(\Gamma(X_0, f))$ is a strongly essential set of $N(\Gamma(X, f))$, then C is called a strongly essential component of $N(\Gamma(X_0, f))$.*

By Theorem 2 and Definition 8, we can deduce the existence of strongly essential component of weakly Pareto-Nash equilibrium for multi-objective games.

Theorem 3. *For each $\Gamma(X_0, f) \in \mathcal{G}_X$, $N(\Gamma(X_0, f))$ has at least one strongly essential component with respect to ρ^g_H.*

Proof. For each $\Gamma(X_0, f) \in \mathcal{G}_X$, then $\psi_f|X_0 \in \mathcal{F}_X$. By Theorem 2, $F_K(\psi_f|X_0)$ has at least one strongly essential component denoted by C_α. Then $\forall\epsilon > 0$, there exists $\delta > 0$, such that $F_K(\psi_{f'}|X')\bigcap[C_\alpha + B_\epsilon(0)] \neq \emptyset$ for any $\psi_{f'}|X' \in \mathcal{F}_X$ with $\rho^k_H(\psi_{f'}|X', \psi_f|X_0) < \delta$.

Moreover, for any $\Gamma(X', f') \in \mathcal{G}_X$ with $\rho^g_H(\Gamma(X', f'), \Gamma(X_0, f)) < \delta$, we have $\psi_{f'}|X' \in \mathcal{F}_X$ and $\rho^k_H(\psi_{f'}|X', \psi_f|X_0) = \rho^g_H(\Gamma(X', f'), \Gamma(X_0, f)) < \delta$, and then $F_K(\psi_{f'}|X')\bigcap[C_\alpha + B_\epsilon(0)] \neq \emptyset$, which means that $N(\Gamma(X', f'))\bigcap[C_\alpha + B_\varepsilon(0)] \neq \emptyset$. By Definition 8, we have that C_α is a strongly essential component of $N(\Gamma(X_0, f))$ with respect to ρ^g_H. □

Remark 7. *(1) For multi-objective games, the perturbation based on ρ^g_H includes not only the perturbations of vector-valued payoff functions but also the perturbation of strategy sets. Here, the strategy set of i-th player shifts from X_i to X'_i and the vector-valued payoff functions from f_i to f'_i generated from the uncertainty in strategy choices.*

(2) Also, if the perturbation of strategy need not be considered in multi-objective game, that is, $X' = X_0 = X$, then

$$\rho^g_H(\Gamma(X, f'), \Gamma(X, f)) = \rho^k_H(\psi_{f'}, \psi_f)$$

By (2) of Corollary 1, we have

$$\rho^g_H(\Gamma(X, f'), \Gamma(X, f)) = \rho^k_H(\psi_{f'}, \psi_f) \leq \rho_1(\psi_{f'}, \psi_f) = \rho_2(\Gamma(X, f'), \Gamma(X, f)).$$

Then a strongly essential component C of $N(\Gamma(X, f))$ with respect to ρ^g_H must be a essential component with respect to ρ_2, and hence C has stronger stability. Moreover, by (3) of Corollary 1 and $\rho_m(\Gamma(X, f), \Gamma(X, g)) = \rho_m(\psi_f, \psi_g)$, the result also holds for $\rho_m(\Gamma(X, f'), \Gamma(X, f))$.

5. Conclusions

In this paper, the existence of strongly essential components of the solution set for Ky Fan's section problems and vector Ky Fan's point problems are studied. Firstly, we propose two kinds of stronger perturbations for Ky Fan's section problems and the problems **(VKF)** defined by the Hausdorff semi-metric of graphic and section mapping respectively. By comparing the relationships among various metrics to obtain some strong and weak relations among these perturbations (see, Proposition 1, 2 and Corollary 1), and some further results on existence of the strongly essential component of solutions set of Ky Fan's section problems are obtained (see, Theorem 1, Corollary 2). In Theorem 1, we investigate the more general case where the initial set $X_0 \in CK(X)$ is arbitrary for problem $A|X_0 \in \mathcal{A}_X$, and the perturbation of the set variation is also arbitrary. But the Theorem 3.1 in [22] only considers the special case in which the initial set is always the total space X, and the perturbation of the set variation can only be reduced inward to $X' \in CK(X)$. Therefore, Theorem 1 actually generalizes the conclusion of Theorem 3.1 in [22]. In fact, Theorem 3.1 in [22] is just the special case of Theorem 1 when $X_0 \equiv X$. Besides, by Corollary 1, we immediately take the Theorem 3.1 in [21] and the Theorem 3.3 in [28] as its special cases. Secondly, based on the above results, two kinds of stronger perturbations of vector-valued inequality functions is proposed by means of the Hausdorff upper semi-metric of graphic and section mapping of problems **(VKF)** respectively, and several existence results of the strongly essential component of set of vector Ky Fan's points are obtained (see, Theorem 2, Corollary 3 and 4). Finally, as an application, we use the equivalence of weakly Pareto-Nash equilibrium with vector Ky Fan's points to obtain the existence of the strongly essential component of weakly Pareto-Nash equilibrium for multiobjective games, which provide a method to investigate the stability of set of weakly Pareto-Nash equilibrium for multiobjective games with respect to general perturbation of strategic set.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

1. C. D. Aliprantis, K. C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3 Eds., Berlin: Springer-Verlag, 2006.
2. L. Q. Anh, N. V. Hung, The existence and stability of solutions for symmetric generalized quasivariational inclusion problems, *Filomat,* **29** (2015), 2147–2165.

3. L. Q. Anh, N. V. Hung, Stability of solution mappings for parametric bilevel vector equilibrium problems, *Comput. Appl. Math.*, **37** (2018), 1537–1549.

4. Q. H. Ansari, J. C. Yao, Systems of vector quasi-equilibrium problems and their applications, in *Variational Analysis and Generalized Differentiation in Optimization and Control*, New York: Springer, **47** (2010), 1–42.

5. H. Brézis, L. Nirenberg, G. Stampacchia, A remark on Ky Fan's minimax principle, *Boll. Un. Mat. Ital.*, **6** (1972), 293–300.

6. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63** (1994), 123–145.

7. S. S. Chang, Salahuddin, G. Wang, On the existence theorems of solutions for generalized vector variational inequalities, *J. Inequal. Appl.*, **2015** (2015), 365.

8. G. Y. Chen, Existence of solutions for a vector variational inequalities: An extension of Hartman-Stampacchia theorems, *J. Optim. Theory Appl.*, **74** (1992), 445–456.

9. G. Y. Chen, X. X. Huang, X. Q. Yang, *Vector Optimization: Set-Valued and Variational Analysis*, Berlin: Springer-Verlag, 2005.

10. X. P. Ding, KKM type theorems and minimax inequalities and saddle point theorems in generalized convex spaces, *Acta Math. Sin.*, **47** (2004), 711–722.

11. K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Annl.*, **142** (1961), 305–310.

12. K. Fan, A minimax inequality and applications, *Inequalities*, **3** (1972), 103–113.

13. F. Giannessi, *Vector Variational Inequalities and Vector Equilibria*, Springer Science & Business Media, 2013.

14. N. J. Huang, On the generalized implicit quasivariational inequalities, *J. Math. Anal. Appl.*, **216** (1997), 197–210.

15. N. V. Hung, D. O'Regan, Bilevel equilibrium problems with lower and upper bounds in locally convex Hausdorff topological vector spaces, *Topol. Appl.*, **2020** (2020), 106939.

16. N. V. Hung, V. M. Tam, D. O'Regan, Y. J. Cho, A new class of generalized multiobjective games in bounded rationality with fuzzy mappings: Structural (λ, ϵ)-stability and (λ, ϵ)-robustness to ϵ-equilibria, *J. Comput. Appl. Math.*, **372** (2020), 112735.

17. W. S. Jia, S. W. Xiang, J. H. He, Y. L. Yang, The existence and stability for weakly Ky Fan's points of set-valued mappings, *J. Inequal. Appl.*, **2012** (2012), 199.

18. E. Klein, A. C. Thompson, *Theory of Correspondences*, New York: A Wiley-Inter Science Publication, 1984.

19. U. Mosco, Implicit variational problems and quasivariational inequalities, in *Nonlinear Operators and the Calculus of Variations*, Berlin: Springer-Verlag, (1976), 83–156.

20. K. K. Tan, J. Yu, X. Z. Yuan, The stability of Ky Fan's points, *Proc. Amer. Math. Soc.*, **123** (1995), 1511–1519.

21. S. W. Xiang, J. H. He, C. W. Liu, W. S. Jia, Y. L. Yang, Some further results on the stability of Ky Fan's points, *J. Inequal. Appl.*, **2017** (2017), 1–12.

22. S. W. Xiang, D. J. Zhang, R. Li, Y. L. Yang, Strongly essential set of Ky Fan's points and the stability of Nash equilibrium, *J. Math. Anal. Appl.*, **459** (2018), 839–851.

23. Z. Yang, On existence and essential stability of solutions of symmetric variational relation problems, *J. Inequalities Appl.*, **2014** (2014), 1–13.

24. J. Yu, Q. Luo, On essential components of the solution set of generalized games, *J. Math. Anal. Appl.*, **230** (1999), 303–310.

25. J. Yu, D. T. Peng, Solvability of vector Ky Fan inequalities with applications, *J. Syst. Sci. Complexity*, **26** (2016), 978–990.

26. J. Yu, S. W. Xiang, On essential components of the set of Nash equilibrium points, *Nonlinear Anal.: Theory Methods Appl.*, **38** (1999), 259–264.

27. J. Yu, H. Yang, Essential components of the set of weakly Pareto-Nash equilibrium points, *Appl. Math. Lett.*, **15** (2002), 553–560.

28. Y. H. Zhou, S. W. Xiang, H. Yang, Stability of solutions for Ky Fan's section theorem with some applications, *Nonlinear Anal.: Theory Methods Appl.*, **62** (2005), 1127–1136.