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Research article

On discrete-time laser model with fuzzy environment

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Abstract: In this work, dynamical behaviors of discrete time laser model with fuzzy parameters are studied. It provides a flexible model to fit fuzzy uncertainty data. For four different fuzzy parameters and fuzzy initial conditions, using a generalization of division (g-division) of fuzzy number, it can represent dynamical behaviors including boundedness and global asymptotical stability of positive solution. Finally, two examples are given to demonstrate the effectiveness of the results obtained.

Keywords: discrete-time laser model; boundedness; stability; g-division
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1. Introduction

Mathematical models of physics, chemistry, ecology, physiology, psychology, engineering and social sciences have been governed by differential equation and difference equation. With the development of computers, compared with continuous-time model, discrete-time models described by difference equations are better formulated and analyzed in the past decades. Recently laser model has vast application in medical sciences, industries, highly security areas in army [1–7]. Laser, whose basic principal lies on the Einstein theory of light proposed in 1916, is a device that produces intense beam of monochromatic and coherent light. Since then it is developed by Gordon Gould in 1957. In 1960, the first working ruby laser was invented by Theodore Maiman. Laser light is coherent, highly directional and monochromatic which makes it different from ordinary light. The working principle of laser is based on the spontaneous absorption, spontaneous emission, stimulated emission, and population inversion are essential for the laser formation. The readers can refer to [8–10]. For instance, Hakin [11] proposed a simple continuous-time laser model in 1983. Khan and Sharif [12] proposed a discrete-time laser model and studied extensively dynamical properties about fixed points, the existence of prime period and periodic points, and transcritical bifurcation of a one-dimensional discrete-time laser model in R^+.

In fact, the identification of the parameters of the model is usually based on statistical method, starting from data experimentally obtained and on the choice of some method adapted to the identification. These models, even the classic deterministic approach, are subjected to inaccuracies (fuzzy uncertainty) that can be caused by the nature of the state variables, by parameters as coefficients of the model and by initial conditions. In fact, fuzzy difference equation is generation of difference equation whose parameters or the initial values are fuzzy numbers, and its solutions are sequences of fuzzy numbers. It has been used to model a dynamical systems under possibility uncertainty. Due to the applicability of fuzzy difference equation for the analysis of phenomena where imprecision is inherent, this class of difference equation is a very important topic from theoretical point of view and also its applications. Recently there has been an increasing interest in the study of fuzzy difference equations (see [13–25]).

In this paper, by virtue of the theory of fuzzy sets, we consider the following discrete-time laser model with fuzzy uncertainty parameters and initial conditions.

$$x_{n+1} = Ax_n + \frac{Bx_n}{Cx_n + H}, n = 0, 1, \cdots, \tag{1.1}$$

where x_n is the number of laser photon at the nth time, A, B, C, H and the initial value x_0 are positive fuzzy numbers.

The main aim of this work is to study the existence of positive solutions of discrete-time laser model (1.1). Furthermore, according to a generation of division (g-division) of fuzzy numbers, we derive some conditions so that every positive solution of discrete-time laser model (1.1) is bounded. Finally, under some conditions we prove that discrete-time laser model (1.1) has a fixed point 0 which is asymptotically stable, and a unique positive fixed point x.

2. Preliminaries and definitions

Firstly, we give the following definitions and lemma needed in the sequel.

Definition 2.1. [26] $u : R \to [0, 1]$ is said to be a fuzzy number if it satisfies conditions (i)–(iv) written below:

(i) u is normal, i. e., there exists an $x \in R$ such that $u(x) = 1$;

(ii) u is fuzzy convex, i. e., for all $t \in [0, 1]$ and $x_1, x_2 \in R$ such that

$$u(tx_1 + (1 - t)x_2) \geq \min\{u(x_1), u(x_2)\};$$

(iii) u is upper semicontinuous;

(iv) The support of u,supp$u = \overline{\bigcup_{\alpha\in(0,1]}[u]^\alpha} = \overline{\{x : u(x) > 0\}}$ is compact.

For $\alpha \in (0, 1]$, the $\alpha-$cuts of fuzzy number u is $[u]^\alpha = \{x \in R : u(x) \geq \alpha\}$ and for $\alpha = 0$, the support of u is defined as supp$u = [u]^0 = \overline{\{x \in R | u(x) > 0\}}$.

A fuzzy number can also be described by a parametric form.

Definition 2.2. [26] A fuzzy number u in a parametric form is a pair (u_l, u_r) of functions $u_l, u_r, 0 \leq \alpha \leq 1$, which satisfies the following requirements:

(i) $u_l(\alpha)$ is a bounded monotonic increasing left continuous function,
(ii) $u_r(\alpha)$ is a bounded monotonic decreasing left continuous function,
(iii) $u_l(\alpha) \leq u_r(\alpha), 0 \leq \alpha \leq 1$.

A crisp (real) number x is simply represented by $(u_l(\alpha), u_r(\alpha)) = (x, x), 0 \leq \alpha \leq 1$. The fuzzy number space $\{(u_l(\alpha), u_r(\alpha))\}$ becomes a convex cone E^1 which could be embedded isomorphically and isometrically into a Banach space (see [26]).

Definition 2.3. [26] The distance between two arbitrary fuzzy numbers u and v is defined as follows:

$$D(u, v) = \sup_{\alpha \in [0,1]} \max\{|u_{l,\alpha} - v_{l,\alpha}|, |u_{r,\alpha} - v_{r,\alpha}|\}. \tag{2.1}$$

It is clear that (E^1, D) is a complete metric space.

Definition 2.4. [26] Let $u = (u_l(\alpha), u_r(\alpha)), v = (v_l(\alpha), v_r(\alpha)) \in E^1, 0 \leq \alpha \leq 1$, and $k \in R$. Then
(i) $u = v$ iff $u_l(\alpha) = v_l(\alpha), u_r(\alpha) = v_r(\alpha)$,
(ii) $u + v = (u_l(\alpha) + v_l(\alpha), u_r(\alpha) + v_r(\alpha))$,
(iii) $u - v = (u_l(\alpha) - v_r(\alpha), u_r(\alpha) - v_l(\alpha))$,
(iv) $ku = \begin{cases} (ku_l(\alpha), ku_r(\alpha)), & k \geq 0; \\ (ku_r(\alpha), ku_l(\alpha)), & k < 0, \end{cases}$
(v) $uv = (\min\{u_l(\alpha)v_l(\alpha), u_l(\alpha)v_r(\alpha), u_r(\alpha)v_l(\alpha), u_r(\alpha)v_r(\alpha)\}, \max\{u_l(\alpha)v_l(\alpha), u_l(\alpha)v_r(\alpha), u_r(\alpha)v_l(\alpha), u_r(\alpha)v_r(\alpha)\})$.

Definition 2.5. [27] Suppose that $u, v \in E^1$ have α-cuts $[u]^\alpha = [u_{l,\alpha}, u_{r,\alpha}], [v]^\alpha = [v_{l,\alpha}, v_{r,\alpha}]$, with $0 \notin [v]^\alpha, \forall \alpha \in [0, 1]$. The generation of division (g-division) $\div_g$ is the operation that calculates the fuzzy number $s = u \div_g v$ having level cuts $[s]^\alpha = [s_{l,\alpha}, s_{r,\alpha}]$(here $[u]^{\alpha-1} = [1/u_{r,\alpha}, 1/u_{l,\alpha}]$) defined by

$$[s]^\alpha = [u]^\alpha \div_g [v]^\alpha \Longleftrightarrow \begin{cases} (i) \quad [u]^\alpha = [v]^\alpha [s]^\alpha, \\ \text{or} \\ (ii) \quad [v]^\alpha = [u]^\alpha [s]^{\alpha-1} \end{cases} \tag{2.2}$$

provided that s is a proper fuzzy number $s_{l,\alpha}$ is nondecreasing,$s_{r,\alpha}$ is nonincreasing, $s_{l,1} \leq s_{r,1}$.

Remark 2.1. According to [27], in this paper the fuzzy number is positive, if $u \div_g v = s \in E^1$ exists, then the following two cases are possible

Case I. if $u_{l,\alpha}v_{r,\alpha} \leq u_{r,\alpha}v_{l,\alpha}, \forall \alpha \in [0, 1]$, then $s_{l,\alpha} = \frac{u_{l,\alpha}}{v_{l,\alpha}}, s_{r,\alpha} = \frac{u_{r,\alpha}}{v_{r,\alpha}}$,

Case II. if $u_{l,\alpha}v_{r,\alpha} \geq u_{r,\alpha}v_{l,\alpha}, \forall \alpha \in [0, 1]$, then $s_{l,\alpha} = \frac{u_{r,\alpha}}{v_{r,\alpha}}, s_{r,\alpha} = \frac{u_{l,\alpha}}{v_{l,\alpha}}$.

Definition 2.6. [26] A triangular fuzzy number (TFN) denoted by A is defined as (a, b, c) where the membership function

$$A(x) = \begin{cases} 0, & x \leq a; \\ \frac{x-a}{b-a}, & a \leq x \leq b; \\ 1, & x = b; \\ \frac{c-x}{c-b}, & b \leq x \leq c; \\ 0, & x \geq c. \end{cases}$$

The $\alpha-$cuts of $A = (a, b, c)$ are described by $[A]^\alpha = \{x \in R : A(x) \geq \alpha\} = [a + \alpha(b - a), c - \alpha(c - b)] = [A_{l,\alpha}, A_{r,\alpha}]$, $\alpha \in [0, 1]$, it is clear that $[A]^\alpha$ are closed interval. A fuzzy number A is positive if supp$A \subset (0, \infty)$.

The following proposition is fundamental since it characterizes a fuzzy set through the α-levels.

Proposition 2.1. [26] If $\{A^\alpha : \alpha \in [0,1]\}$ is a compact, convex and not empty subset family of R^n such that
(i) $\overline{\bigcup A^\alpha} \subset A^0$.
(ii) $A^{\alpha_2} \subset A^{\alpha_1}$ if $\alpha_1 \leq \alpha_2$.
(iii) $A^\alpha = \bigcap_{k\geq 1} A^{\alpha_k}$ if $\alpha_k \uparrow \alpha > 0$.
Then there is $u \in E^n(E^n$ denotes n dimensional fuzzy number space) such that $[u]^\alpha = A^\alpha$ for all $\alpha \in (0,1]$ and $[u]^0 = \overline{\bigcup_{0<\alpha\leq 1} A^\alpha} \subset A^0$.

The fuzzy analog of the boundedness and persistence (see [15, 16]) is as follows:

Definition 2.7. A sequence of positive fuzzy numbers (x_n) is persistence (resp. bounded) if there exists a positive real number M (resp. N) such that

$$\text{supp } x_n \subset [M,\infty)(\text{resp. supp } x_n \subset (0,N]), n = 1,2,\cdots,$$

A sequence of positive fuzzy numbers (x_n) is bounded and persistence if there exist positive real numbers $M, N > 0$ such that

$$\text{supp } x_n \subset [M,N], n = 1,2,\cdots.$$

A sequence of positive fuzzy numbers $(x_n), n = 1,2,\cdots$, is an unbounded if the norm $\|x_n\|, n = 1,2,\cdots$, is an unbounded sequence.

Definition 2.8. x_n is a positive solution of (1.1) if (x_n) is a sequence of positive fuzzy numbers which satisfies (1.1). A positive fuzzy number x is called a positive equilibrium of (1.1) if

$$x = Ax + \frac{Bx}{Cx+H}.$$

Let (x_n) be a sequence of positive fuzzy numbers and x is a positive fuzzy number, $x_n \to x$ as $n \to \infty$ if $\lim_{n\to\infty} D(x_n, x) = 0$.

Lemma 2.1. [26] Let $f : R^+ \times R^+ \times R^+ \times R^+ \to R^+$ be continuous, A, B, C, D are fuzzy numbers. Then

$$[f(A,B,C,D)]^\alpha = f([A]^\alpha, [B]^\alpha, [C]^\alpha, [D]^\alpha), \quad \alpha \in (0,1]. \tag{2.3}$$

3. Main results

3.1. Existence of positive solution of discrete-time laser model

Firstly we give the existence of positive solutions of discrete-time laser model with fuzzy environment.

Theorem 3.1. Let parameters A, B, C, H and initial value x_0 of discrete-time laser model be fuzzy numbers. Then, for any positive fuzzy number x_0, there exists a unique positive solution x_n of discrete-time laser model with initial conditions x_0.

Proof. The proof is similar to those of Proposition 2.1 [25]. So we omit the proof of Theorem 3.1.

Noting Remark 2.1, taking α-cuts, one of the following two cases holds
Case I

$$[x_{n+1}]^\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha}L_{n,\alpha} + \frac{B_{l,\alpha}L_{n,\alpha}}{C_{l,\alpha}L_{n,\alpha} + H_{l,\alpha}}, A_{r,\alpha}R_{n,\alpha} + \frac{B_{r,\alpha}R_{n,\alpha}}{C_{r,\alpha}R_{n,\alpha} + H_{r,\alpha}}\right] \tag{3.1}$$

Case II

$$[x_{n+1}]^{\alpha} = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha}L_{n,\alpha} + \frac{B_{r,\alpha}R_{n,\alpha}}{C_{r,\alpha}R_{n,\alpha} + H_{r,\alpha}}, A_{r,\alpha}R_{n,\alpha} + \frac{B_{l,\alpha}L_{n,\alpha}}{C_{l,\alpha}L_{n,\alpha} + H_{l,\alpha}}\right] \tag{3.2}$$

3.2. Dynamics of discrete-time laser model (1.1)

To study the dynamical behavior of the positive solutions of discrete-time laser model (1.1), according to Definition 2.5, we consider two cases.

First, if Case I holds true, we need the following lemma.

Lemma 3.1. Consider the following difference equation

$$y_{n+1} = ay_n + \frac{by_n}{cy_n + h}, \quad n = 0, 1, \cdots, \tag{3.3}$$

where $a \in (0, 1), b, c, h \in (0, +\infty), y_0 \in (0, +\infty)$, then the following statements are true:
(i) Every positive solution of (3.3) satisfies

$$0 < y_n \leq \frac{b}{c(1-a)} + y_0. \tag{3.4}$$

(ii) The equation has a fixed point $y^* = 0$ if $b \leq (1-a)h$.
(iii) The equation has two fixed points $y^* = 0, y^* = \frac{b}{c(1-a)} - \frac{h}{c}$ if $b > (1-a)h$.

Proof. (i) Let y_n be a positive solution of (3.3). It follows from (3.3) that, for $n \geq 0$,

$$0 < y_{n+1} = ay_n + \frac{by_n}{cy_n + h} \leq ay_n + \frac{b}{c}.$$

From which we have

$$0 < y_n \leq \frac{b}{c(1-a)} + \left(y_0 - \frac{b}{c(1-a)}\right)a^{n+1} \leq \frac{b}{c(1-a)} + y_0.$$

This completes the proof of (i).

If y^* is fixed point of (3.3), i.e., $y_n = y^*$. So from (3.3), we have

$$y^* = ay^* + \frac{by^*}{cy^* + h}. \tag{3.5}$$

After some manipulation, from (3.5), we can get

$$y^* = 0, \quad y^* = \frac{b}{c(1-a)} - \frac{h}{c}. \tag{3.6}$$

From (3.3), we can summarized the existence of fixed points as follows
(ii) If $b < h(1-a)$, then $y^* = \frac{b}{c(1-a)} - \frac{h}{c}$ is not a positive number. And hence if $b \leq h(1-a)$ then (3.3) has a fixed point $y^* = 0$.
(iii) If $b > h(1-a)$, then $y^* = \frac{b}{c(1-a)} - \frac{h}{c}$ is a positive number. And hence if $b > h(1-a)$ then (3.3) has two fixed points $y^* = 0, y^* = \frac{b}{c(1-a)} - \frac{h}{c}$.

Proposition 3.1. The following statements hold true
(i) The fixed point $y^* = 0$ of (3.3) is stable if $b < (1-a)h$.
(ii) The fixed point $y^* = 0$ of (3.3) is unstable if $b > (1-a)h$.
(iii) The fixed point $y^* = 0$ of (3.3) is non-hyperbolic if $b = (1-a)h$.

Proof. From (3.3), let

$$f(y) := ay + \frac{by}{cy+h} \tag{3.7}$$

From (3.7), it follows that

$$f'(y) = a + \frac{bh}{(cy+h)^2} \tag{3.8}$$

From (3.8), it can get

$$\left|f'(y)|_{y^*=0}\right| = \left|a + \frac{b}{h}\right| \tag{3.9}$$

Therefore from (3.9), it can conclude that $y^* = 0$ is stable if $b < (1-a)h$, unstable if $b > (1-a)h$, non-hyperbolic if $b = (1-a)h$.

Proposition 3.2. The fixed point $y^* = \frac{b}{c(1-a)} - \frac{h}{c}$ of (3.3) is globally asymptotically stable if $b > (1-a)h$.

Proof. From (3.8), it can get

$$\left|f'(y)|_{y^*=\frac{b}{c(1-a)}-\frac{h}{c}}\right| = \left|a + \frac{h(1-a)^2}{b}\right|. \tag{3.10}$$

Therefore from (3.10), it can conclude that, if $a + \frac{h(1-a)^2}{b} < 1$, i.e., $b > (1-a)h$ then the fixed point $y^* = \frac{b}{c(1-a)} - \frac{h}{c}$ is stable.

On the other hand, it follows from (3.4) that (y_n) is bounded. And from (3.8), we have $f'(y) > 0$. Namely (y_n) is monotone increasing. So we have

$$\lim_{n\to\infty} y_n = y^* = \frac{b}{c(1-a)} - \frac{h}{c}. \tag{3.11}$$

Therefore the fixed point $y^* = \frac{b}{c(1-a)} - \frac{h}{c}$ is globally asymptotically stable.

Proposition 3.3. The fixed point $y^* = 0$ of (3.3) is globally asymptotically stable if $b < (1-a)h$.

Proof. From (3.3), we can get that

$$y_{n+1} \le (a + \frac{b}{h})y_n \tag{3.12}$$

From (3.12), it follows that

$$\begin{aligned} y_1 &\le (a + \tfrac{b}{h})y_0 \\ y_2 &\le (a + \tfrac{b}{h})^2 y_0 \\ &\vdots \\ y_n &\le (a + \tfrac{b}{h})^n y_0 \end{aligned} \tag{3.13}$$

Since $b < (1-a)h$, then $\lim_{n\to\infty} y_n = 0$. Therefore the fixed point $y^* = 0$ of (3.3) is globally asymptotically stable.

Theorem 3.2. Consider discrete-time laser model (1.1), where A, B, C, H and initial value x_0 are positive fuzzy numbers. There exists positive number N_A, $\forall\alpha \in (0, 1]$, $A_{r,\alpha} < N_A < 1$. If

$$\frac{B_{l,\alpha}L_{n,\alpha}}{B_{r,\alpha}R_{n,\alpha}} \leq \frac{C_{l,\alpha}L_{n,\alpha} + H_{l,\alpha}}{C_{r,\alpha}R_{n,\alpha} + H_{r,\alpha}}, \forall\alpha \in (0, 1]. \tag{3.14}$$

and

$$B_{l,\alpha} < H_{l,\alpha}(1 - A_{l,\alpha}), \quad B_{r,\alpha} < H_{r,\alpha}(1 - A_{r,\alpha}), \quad \forall\alpha \in (0, 1]. \tag{3.15}$$

Then (1.1) has a fixed point $x^* = 0$ which is globally asymptotically stable.

Proof. Since A, B, C, H are positive fuzzy numbers and (3.14) holds true, taking α-cuts of model (1.1) on both sides, we can have the following difference equation system with parameters

$$L_{n+1,\alpha} = A_{l,\alpha}L_{n,\alpha} + \frac{B_{l,\alpha}L_{n,\alpha}}{C_{l,\alpha}L_{n,\alpha} + H_{l,\alpha}}, \;\; R_{n+1,\alpha} = A_{r,\alpha}R_{n,\alpha} + \frac{B_{r,\alpha}R_{n,\alpha}}{C_{r,\alpha}R_{n,\alpha} + H_{r,\alpha}}. \tag{3.16}$$

Since (3.15) holds true, using Proposition 3.2, we can get

$$\lim_{n\to\infty} L_{n,\alpha} = 0, \;\; \lim_{n\to\infty} R_{n,\alpha} = 0. \tag{3.17}$$

On the other hand, let $x_n = x^*$, where $[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}] = [L_\alpha, R_\alpha] = [x^*]^\alpha, \alpha \in (0, 1]$, be the fixed point of (1.1). From (3.16), one can get

$$L_\alpha = A_{l,\alpha}L_\alpha + \frac{B_{l,\alpha}L_\alpha}{C_{l,\alpha}L_\alpha + H_{l,\alpha}}, \;\; R_\alpha = A_{r,\alpha}R_\alpha + \frac{B_{r,\alpha}R_\alpha}{C_{r,\alpha}R_\alpha + H_{r,\alpha}}. \tag{3.18}$$

Since (3.15) is satisfied, from (3.16), it follows that

$$L_\alpha = 0, \;\; R_\alpha = 0, \; \lim_{n\to\infty} D(x_n, x^*) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \{\max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\}\} = 0. \tag{3.19}$$

Therefore, by virtue of Proposition 3.3, the fixed point $x^* = 0$ is globally asymptotically stable.

Theorem 3.3. Consider discrete-time laser model (1.1), where A, B, C, H and initial value x_0 are positive fuzzy numbers, there exists positive number N_A, $\forall\alpha \in (0, 1]$, $A_{r,\alpha} < N_A < 1$. If (3.14) holds true, and

$$B_{l,\alpha} > H_{l,\alpha}(1 - A_{l,\alpha}), \quad B_{r,\alpha} > H_{r,\alpha}(1 - A_{r,\alpha}), \quad \forall\alpha \in (0, 1], \tag{3.20}$$

then the following statements are true.
(i) Every positive solution of (1.1) is bounded.
(ii) Equation (1.1) has a unique positive fixed point x^* which is asymptotically stable.

Proof. (i) Since A, B, C, H and x_0 are positive fuzzy numbers, there exist positive constants $M_A, N_A, M_B, N_B, M_C, N_C, M_H, N_H, M_0, N_0$ such that

$$\begin{cases} [A]^\alpha = [A_{l,\alpha}, A_{r,\alpha}] \subset \overline{\bigcup_{\alpha\in(0,1]}[A_{l,\alpha}, A_{r,\alpha}]} \subset [M_A, N_A] \\ [B]^\alpha = [B_{l,\alpha}, B_{r,\alpha}] \subset \overline{\bigcup_{\alpha\in(0,1]}[B_{l,\alpha}, B_{r,\alpha}]} \subset [M_B, N_B] \\ [C]^\alpha = [C_{l,\alpha}, C_{r,\alpha}] \subset \overline{\bigcup_{\alpha\in(0,1]}[C_{l,\alpha}, C_{r,\alpha}]} \subset [M_C, N_C] \\ [H]^\alpha = [H_{l,\alpha}, H_{r,\alpha}] \subset \overline{\bigcup_{\alpha\in(0,1]}[H_{l,\alpha}, H_{r,\alpha}]} \subset [M_H, N_H] \\ [x_0]^\alpha = [L_{0,\alpha}, R_{0,\alpha}] \subset \overline{\bigcup_{\alpha\in(0,1]}[L_{0,\alpha}, R_{0,\alpha}]} \subset [M_0, N_0] \end{cases} \tag{3.21}$$

Using (i) of Lemma 3.1, we can get that

$$0 < L_n \leq \frac{B_{l,\alpha}}{C_{l,\alpha}(1 - A_{l,\alpha})} + L_{0,\alpha}, \quad 0 < R_n \leq \frac{B_{r,\alpha}}{C_{r,\alpha}(1 - A_{r,\alpha})} + R_{0,\alpha}. \tag{3.22}$$

From (3.21) and (3.22), we have that for $\alpha \in (0, 1]$

$$[L_{n,\alpha}, R_{n,\alpha}] \subset [0, N], \ n \geq 1. \tag{3.23}$$

where $N = \frac{N_B}{M_C(1-N_A)} + N_0$. From (3.22), we have for $n \geq 1, \bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset (0, N]$, so $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subset (0, N]$. Thus the proof of (i) is completed.

(ii) We consider system (3.18), then the positive solution of (3.18) is given by

$$L_\alpha = \frac{B_{l,\alpha}}{C_{l,\alpha}(1 - A_{l,\alpha})} - \frac{H_{l,\alpha}}{C_{l,\alpha}}, \ R_\alpha = \frac{B_{r,\alpha}}{C_{r,\alpha}(1 - A_{r,\alpha})} - \frac{H_{r,\alpha}}{C_{r,\alpha}}, \alpha \in (0, 1]. \tag{3.24}$$

Let x_n be a positive solution of (1.1) such that $[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0, 1], n = 0, 1, 2, \cdots$. Then applying Proposition 3.2 to system (3.16), we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha \tag{3.25}$$

From (3.23) and (3.25), we have, for $0 < \alpha_1 < \alpha_2 < 1$,

$$0 < L_{\alpha_1} \leq L_{\alpha_2} \leq R_{\alpha_2} \leq R_{\alpha_1}. \tag{3.26}$$

Since $A_{l,\alpha}, A_{r,\alpha}, B_{l,\alpha}, B_{r,\alpha}, C_{l,\alpha}, C_{r,\alpha}, H_{l,\alpha}, H_{r,\alpha}$ are left continuous. It follows from (3.24) that L_α, R_α are also left continuous.

From (3.21) and (3.24), it follows that

$$c = \frac{M_B}{N_C(1 - M_A)} - \frac{N_H}{M_C} \leq L_\alpha \leq R_\alpha \leq \frac{N_B}{M_A(1 - N_A)} - \frac{M_H}{N_C} = d. \tag{3.27}$$

Therefore (3.27) implies that $[L_\alpha, R_\alpha] \subset [c, d]$, and so $\bigcup_{\alpha\in(0,1]}[L_\alpha, R_\alpha] \subset [c, d]$. It is clear that

$$\overline{\bigcup_{\alpha\in(0,1]} [L_\alpha, R_\alpha]} \text{ is compact and } \overline{\bigcup_{\alpha\in(0,1]} [L_\alpha, R_\alpha]} \subset (0, \infty). \tag{3.28}$$

So from Definition 2.2, (3.24), (3.26), (3.28) and since $L_\alpha, R_\alpha, \alpha \in (0, 1]$ determine a fuzzy number x^* such that

$$x^* = Ax^* + \frac{Bx^*}{Cx^* + H}, \quad [x^*]^\alpha = [L_\alpha, R_\alpha], \ \alpha \in (0, 1]. \tag{3.29}$$

Suppose that there exists another positive fixed point $\bar{x}$ of (1.1), then there exist functions $\overline{L}_\alpha, \overline{R}_\alpha$: $(0, 1) \to (0, \infty)$ such that

$$\bar{x} = A\bar{x} + \frac{B\bar{x}}{C\bar{x} + H}, \quad [x]^\alpha = [\overline{L}_\alpha, \overline{R}_\alpha], \ \alpha \in (0, 1].$$

From which we have

$$\overline{L}_\alpha = A_{l,\alpha}\overline{L}_\alpha + \frac{B_{l,\alpha}\overline{L}_\alpha}{C_{l,\alpha}\overline{L}_\alpha + H_{l,\alpha}}, \quad \overline{R}_\alpha = A_{r,\alpha}\overline{R}_\alpha + \frac{B_{r,\alpha}\overline{R}_\alpha}{C_{r,\alpha}\overline{R}_\alpha + H_{r,\alpha}}.$$

So $\overline{L}_\alpha = L_\alpha, \overline{R}_\alpha = R_\alpha, \alpha \in (0,1]$. Hence $\bar{x} = x^*$, namely x^* is a unique positive fixed point of (1.1).

From (3.25), we have

$$\lim_{n\to\infty} D(x_n, x^*) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\} = 0. \tag{3.30}$$

Namely, every positive solution x_n of (1.1) converges the unique fixed point x^* with respect to D as $n \to \infty$. Applying Proposition 3.2, it can obtain that the positive fixed point x^* is globally asymptotically stable.

Secondly, if Case II holds true, it follows that for $n \in \{0, 1, 2, \cdots\}, \alpha \in (0, 1]$

$$L_{n+1,\alpha} = A_{l,\alpha}L_{n,\alpha} + \frac{B_{r,\alpha}R_{n,\alpha}}{C_{r,\alpha}R_{n,\alpha} + H_{r,\alpha}},\ \ R_{n+1,\alpha} == A_{r,\alpha}R_{n,\alpha} + \frac{B_{l,\alpha}L_{n,\alpha}}{C_{l,\alpha}L_{n,\alpha} + H_{l,\alpha}} \tag{3.31}$$

We need the following lemma.

Lemma 3.2. Consider the system of difference equations

$$y_{n+1} = a_1y_n + \frac{b_2z_n}{c_2z_n + h_2},\ \ z_{n+1} = a_2z_n + \frac{b_1y_n}{c_1y_n + h_1},\ \ n = 0, 1, \cdots, \tag{3.32}$$

where $a_i \in (0, 1), b_i, c_i, h_i \in (0, +\infty)(i = 1, 2), y_0, z_0 \in (0, +\infty)$. If

$$a_1 + a_2 < 1. \tag{3.33}$$

and

$$b_1b_2 > h_1h_2(1 - a_1)(1 - a_2). \tag{3.34}$$

Then the following statements are true.
(i) Every positive solution (y_n, z_n) of (3.32) satisfy

$$0 < y_n \le \frac{b_2}{(1 - a_1)c_2} + y_0,\quad 0 < z_n \le \frac{b_1}{(1 - a_2)c_1} + z_0. \tag{3.35}$$

(ii) System (3.32) has fixed point $(y, z) = (0, 0)$ which is asymptotically stable.
(iii) System (3.32) has a unique fixed point

$$y = \frac{(1 - a_2)K}{b_1c_2 + h_2c_1(1 - a_2)}, z = \frac{(1 - a_1)K}{b_2c_1 + h_1c_2(1 - a_1)}, \tag{3.36}$$

where $K = \frac{b_1b_2 - h_1h_2(1-a_1)(1-a_2)}{(1-a_1)(1-a_2)}$.

Proof. (i) Let (y_n, z_n) be a positive solution of (3.32). It follows from (3.32) that, for $n \ge 0$,

$$0 < y_{n+1} = a_1y_n + \frac{b_2z_n}{c_2z_n + h_2} \le a_1y_n + \frac{b_2}{c_2},\ \ 0 < z_{n+1} = a_2z_n + \frac{b_1y_n}{c_1y_n + h_1} \le a_2z_n + \frac{b_1}{c_1}.$$

From which, we have

$$\begin{cases} 0 < y_n \le \frac{b_2}{c_2(1-a_1)} + \left(y_0 - \frac{b_2}{c_2(1-a_1)}\right)a_1^n \le \frac{b_2}{c_2(1-a_1)} + y_0 \\ 0 < z_n \le \frac{b_1}{c_1(1-a_2)} + \left(z_0 - \frac{b_1}{c_1(1-a_2)}\right)a_2^n \le \frac{b_1}{c_1(1-a_2)} + z_0. \end{cases}$$

This completes the proof of (i).

(ii) It is clear that $(0,0)$ is a fixed point of (3.32). We can obtain that the linearized system of (3.32) about the fixed point $(0,0)$ is

$$X_{n+1} = D_1 X_n, \tag{3.37}$$

where $X_n = (x_n, y_n)^T$ and

$$D_1 = \begin{pmatrix} a_1 & \frac{b_2}{c_2} \\ \frac{b_1}{h_1} & a_2 \end{pmatrix}.$$

Thus the characteristic equation of (3.37) is

$$\lambda^2 - (a_1 + a_2)\lambda + a_1 a_2 - \frac{b_1 b_2}{h_1 h_2} = 0. \tag{3.38}$$

Since (3.33) and (3.34) hold true, we have

$$a_1 + a_2 + a_1 a_2 - \frac{b_1 b_2}{h_1 h_2} < a_1 + q_2 + a_1 a_2 - (1 - a_1)(1 - a_2) < 1 \tag{3.39}$$

By virtue of Theorem 1.3.7 [28], we obtain that the fixed point $(0,0)$ is asymptotically stable.

(iii) Let $(y_n, z_n) = (y, z)$ be fixed point of (3.32). We consider the following system

$$y = a_1 y + \frac{b_1 z}{c_1 z + h_1}, \quad z = a_2 z + \frac{b_2 y}{c_2 y + h_2}. \tag{3.40}$$

It is clear that the positive fixed point (y, z) can be written by (3.36).

Theorem 3.4. Consider the difference Eq (1.1), where A, B, C, H are positive fuzzy numbers. There exists positive number $N_A, \forall \alpha \in (0,1], A_{r,\alpha} < N_A < 1$. If

$$\frac{B_{l,\alpha} L_{n,\alpha}}{B_{r,\alpha} R_{n,\alpha}} \geq \frac{C_{l,\alpha} L_{n,\alpha} + H_{l,\alpha}}{C_{r,\alpha} R_{n,\alpha} + H_{r,\alpha}}, \forall \alpha \in (0,1], \tag{3.41}$$

$$A_{l,\alpha} + A_{r,\alpha} < 1, \forall \alpha \in (0,1], \tag{3.42}$$

and

$$B_{l,\alpha} B_{r,\alpha} > H_{l,\alpha} H_{r,\alpha} (1 - A_{l,\alpha})(1 - A_{r,\alpha}), \forall \alpha \in (0,1]. \tag{3.43}$$

Then the following statements are true

(i) Every positive solution of (1.1) is bounded.

(ii) The Eq (1.1) has a fixed point 0 which is globally asymptotically stable.

(iii) The Eq (1.1) has a unique positive fixed point x such that

$$[x]^\alpha = [L_\alpha, R_\alpha],\ L_\alpha = \frac{(1 - A_{r,\alpha}) K_\alpha}{B_{l,\alpha} C_{r,\alpha} + H_{r,\alpha} C_{l,\alpha} (1 - A_{r,\alpha})}, R_\alpha = \frac{(1 - A_{l,\alpha}) K_\alpha}{B_{r,\alpha} C_{l,\alpha} + H_{l,\alpha} C_{r,\alpha} (1 - A_{l,\alpha})},$$

where

$$K_\alpha = \frac{B_{l,\alpha} B_{r,\alpha} - H_{l,\alpha} H_{r,\alpha} (1 - A_{l,\alpha})(1 - A_{r,\alpha})}{(1 - A_{l,\alpha})(1 - A_{r,\alpha})}. \tag{3.44}$$

Proof. (i) Let x_n be a positive solution of (1.1). Applying (i) of Lemma 3.2, we have

$$0 < L_n \le \frac{B_{r,\alpha}}{(1-A_{l,\alpha})C_{r,\alpha}} + L_{0,\alpha},\ \ 0 < R_n \le \frac{B_{l,\alpha}}{(1-A_{r,\alpha})C_{l,\alpha}} + R_{0,\alpha}. \tag{3.45}$$

Next, the proof is similar to (i) of Theorem 3.3. So we omit it.

(ii) The proof is similar to those of Theorem 3.2. We omit it.

(iii) Let $x_n = x$ be a fixed point of (1.1), then

$$x = Ax + \frac{Bx}{Cx+H}. \tag{3.46}$$

Taking α-cuts on both sides of (3.46), since (3.41) holds true, one gets the following system

$$L_\alpha = A_{l,\alpha}L_\alpha + \frac{B_{r,\alpha}R_\alpha}{C_{r,\alpha}R_\alpha + H_{r,\alpha}},\ \ R_\alpha = A_{r,\alpha}R_\alpha + \frac{B_{l,\alpha}L_\alpha}{C_{l,\alpha}L_\alpha + H_{l,\alpha}},\ \ \alpha \in (0,1]. \tag{3.47}$$

From which we obtain that

$$L_\alpha = \frac{(1-A_{r,\alpha})K_\alpha}{B_{l,\alpha}C_{r,\alpha} + H_{r,\alpha}C_{l,\alpha}(1-A_{r,\alpha})}, R_\alpha = \frac{(1-A_{l,\alpha})K_\alpha}{B_{r,\alpha}C_{l,\alpha} + H_{l,\alpha}C_{r,\alpha}(1-A_{l,\alpha})}, \tag{3.48}$$

Next, we can show that L_α, R_α constitute a positive fuzzy number x such that $[x]^\alpha = [L_\alpha, R_\alpha], \alpha \in (0,1]$. The proof is similar to (ii) of Theorem 3.3. We omit it.

Remark 3.1. In dynamical system model, the parameters of model derived from statistic data with vagueness or uncertainty. It corresponds to reality to use fuzzy parameters in dynamical system model. Compared with classic discrete time laser model, the solution of discrete time fuzzy laser model is within a range of value (approximate value), which are taken into account fuzzy uncertainties. Furthermore the global asymptotic behaviour of discrete time laser model are obtained in fuzzy context. The results obtained is generation of discrete time Beverton-Holt population model with fuzzy environment [25].

4. Numerical simulations

In this section, we give two numerical examples to verify the effectiveness of theoretic results obtained.

Example 4.1. Consider the following fuzzy discrete time laser model

$$x_{n+1} = Ax_n + \frac{Bx_n}{Cx_n + H},\ n = 0, 1, \cdots, \tag{4.1}$$

we take A, B, C, H and the initial values x_0 such that

$$A(x) = \begin{cases} 10x-4, & 0.4 \le x \le 0.5 \\ -10x+6, & 0.5 \le x \le 0.6 \end{cases},\ \ B(x) = \begin{cases} 5x-4, & 0.8 \le x \le 1 \\ -5x+6, & 1 \le x \le 1.2 \end{cases} \tag{4.2}$$

$$C(x)=\begin{cases}2x-2, & 1\le x\le 1.5\\ -2x+4, & 1.5\le x\le 2\end{cases},\quad H(x)=\begin{cases}2x-6, & 3\le x\le 3.5\\ -2x+8, & 3.5\le x\le 4\end{cases} \tag{4.3}$$

$$x_0(x)=\begin{cases}x-6, & 6\le x\le 7\\ -x+8, & 7\le x\le 8\end{cases} \tag{4.4}$$

From (4.2), we get

$$[A]^\alpha=\left[0.4+\frac{1}{10}\alpha, 0.6-\frac{1}{10}\alpha\right],\ [B]^\alpha=\left[0.8+\frac{1}{5}\alpha, 1.2-\frac{1}{5}\alpha\right],\ \alpha\in(0,1]. \tag{4.5}$$

From (4.3) and (4.4), we get

$$[C]^\alpha=\left[1+\frac{1}{2}\alpha, 2-\frac{1}{2}\alpha\right],\ \ [H]^\alpha=\left[3+\frac{1}{2}\alpha, 4-\frac{1}{2}\alpha\right], [x_0]^\alpha=[6+\alpha, 8-\alpha],\ \alpha\in(0,1]. \tag{4.6}$$

Therefore, it follows that

$$\overline{\bigcup_{\alpha\in(0,1]}[A]^\alpha}=[0.4,0.6],\ \overline{\bigcup_{\alpha\in(0,1]}[B]^\alpha}=[0.8,1.2],\ \overline{\bigcup_{\alpha\in(0,1]}[C]^\alpha}=[1,2],\ \overline{\bigcup_{\alpha\in(0,1]}[H]^\alpha}=[3,4].\ \overline{\bigcup_{\alpha\in(0,1]}[x_0]^\alpha}=[6,8]. \tag{4.7}$$

From (4.1), it results in a coupled system of difference equations with parameter α,

$$L_{n+1,\alpha}=A_{l,\alpha}L_{n,\alpha}+\frac{B_{l,\alpha}L_{n,\alpha}}{C_{l,\alpha}L_{n,\alpha}+H_{l,\alpha}},\ \ R_{n+1,\alpha}=A_{r,\alpha}R_{n,\alpha}+\frac{B_{r,\alpha}R_{n,\alpha}}{C_{r,\alpha}R_{n,\alpha}+H_{r,\alpha}},\ \alpha\in(0,1]. \tag{4.8}$$

Therefore, it is clear that $A_{r,\alpha}<1, \forall\alpha\in(0,1]$, (3.14) and (3.15) hold true. so from Theorem 3.2, we have that every positive solution x_n of Eq (4.1) is bounded In addition, from Theorem 3.2, Eq (4.1) has a fixed point 0. Moreover every positive solution x_n of Eq (4.1) converges the fixed point 0 with respect to D as $n\to\infty$. (see Figures 1–3).

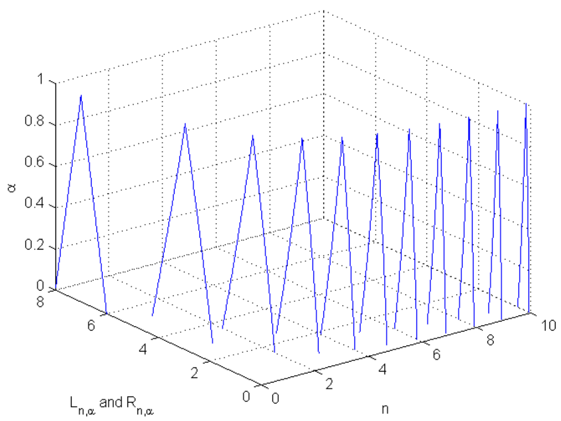

Figure 1. The Dynamics of system (4.8).

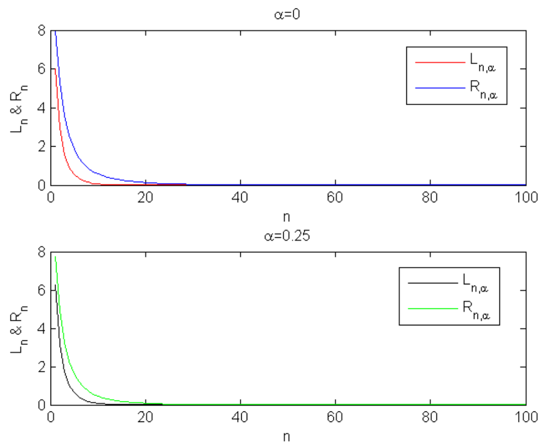

Figure 2. The solution of system (4.8) at $\alpha = 0$ and $\alpha = 0.25$.

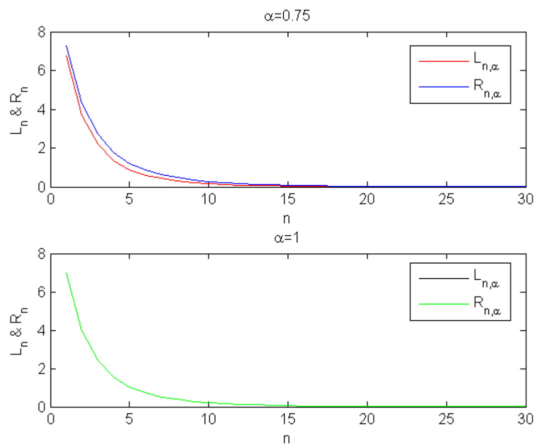

Figure 3. The solution of system (4.8) at $\alpha = 0.75$ and $\alpha = 1$.

Example 4.2. Consider the following fuzzy discrete time laser model (4.1). where A, C, H and the initial values x_0 are same as Example 4.1.

$$B(x) = \begin{cases} x - 2, & 2 \leq x \leq 3 \\ \\ -x + 4, & 3 \leq x \leq 4 \end{cases} \tag{4.9}$$

From (4.9), we get

$$[B]^\alpha = [2 + \alpha, 4 - \alpha], \tag{4.10}$$

Therefore, it follows that

$$\overline{\bigcup_{\alpha \in (0,1]}} [B]^\alpha = [2, 4], \alpha \in (0, 1]. \tag{4.11}$$

It is clear that (3.20) is satisfied, so from Theorem 3.3, Eq (4.1) has a unique positive equilibrium $\overline{x} = (0.341, 1.667, 3)$. Moreover every positive solution x_n of Eq (4.1) converges the unique equilibrium $\overline{x}$ with respect to D as $n \to \infty$. (see Figures 4–6)

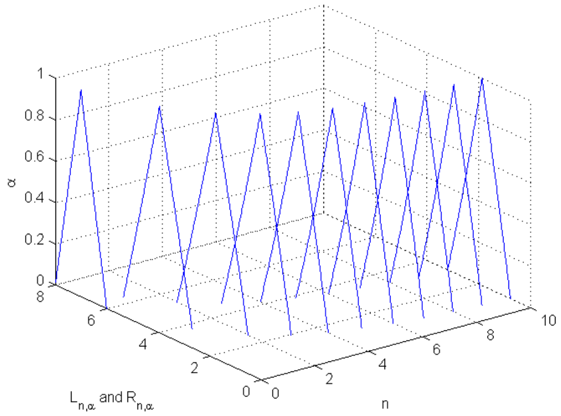

Figure 4. The Dynamics of system (4.8).

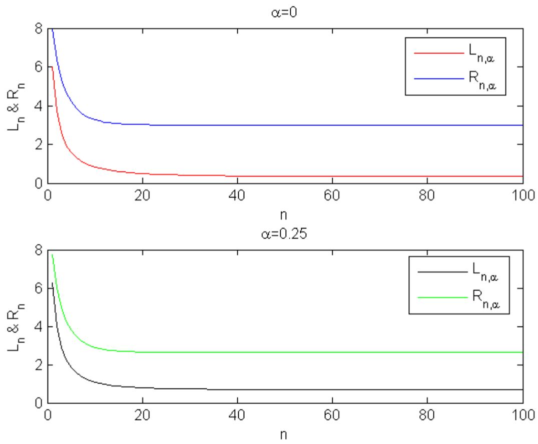

Figure 5. The solution of system (4.8) at $\alpha = 0$ and $\alpha = 0.25$.

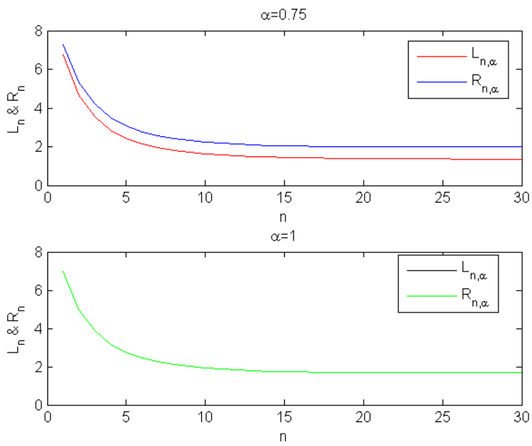

Figure 6. The solution of system (4.8) at $\alpha = 0.75$ and $\alpha = 1$.

5. Conclusions

In this work, according to a generalization of division (g-division) of fuzzy number, we study the fuzzy discrete time laser model $x_{n+1} = Ax_n + \frac{Bx_n}{Cx_n+H}$. The existence of positive solution and qualitative behavior to (1.1) are investigated. The main results are as follows

(i) Under Case I, the positive solution is bounded if $B_{l,\alpha} < H_{l,\alpha}(1 - A_{l,\alpha}), B_{r,\alpha} < H_{r,\alpha}(1 - A_{r,\alpha}), \alpha \in (0, 1]$. Moreover system (1.1) has a fixed point 0 which is globally asymptotically stable. Otherwise, if $B_{l,\alpha} \geq H_{l,\alpha}(1 - A_{l,\alpha}), B_{r,\alpha} \geq H_{r,\alpha}(1 - A_{r,\alpha}), \alpha \in (0, 1]$. Then system (1.1) has a unique positive fixed point x^* which is asymptotically stable.

(ii) Under Case II, if $A_{l,\alpha} + A_{r,\alpha} < 1$ and $B_{l,\alpha}B_{r,\alpha} > H_{l,\alpha}H_{r,\alpha}(1 - A_{l,\alpha})(1 - A_{r,\alpha}), \alpha \in (0, 1]$, then the positive solution is bounded. Moreover system (1.1) has a unique positive fixed point x and fixed point 0 which is global asymptotically stable.

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Conflict of interest

The authors declare that they have no competing interests.

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