Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(4): 3037–3052.
DOI:10.3934/math.2021185
Received: 23 October 2020
Accepted: 08 January 2021
Published: 12 January 2021

Research article

On q-analogue of meromorphic multivalent functions in lemniscate of Bernoulli domain

Bakhtiar Ahmad[1], Muhammad Ghaffar Khan[2], Basem Aref Frasin[3], Mohamed Kamal Aouf[4], Thabet Abdeljawad[5,6,7,*], Wali Khan Mashwani[2] and Muhammad Arif[1]

[1] Abdul Wali Khan University Mardan, 23200 Mardan, Pakistan
[2] Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat, Pakistan
[3] Faculty of Science, Department of Mathematics, AL AL-Bayt University Mafraq, Jordon
[4] Department of Mathematics, Faculty of Science, Mansoura 35516, Egypt
[5] Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia
[6] Department of Medical Research, China Medical University Taichung 40402, Taiwan
[7] Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan

* **Correspondence:** Email: tabdeljawad@psu.edu.sa.

Abstract: Utilizing the concepts from q-calculus in the field of geometric function theory, we introduce a subclass of p-valent meromorphic functions relating to the domain of lemniscate of Bernoulli. The well known problem of Fekete-Szegö for this class is evaluated. Also some geometric results related to subordinations are evaluated for this class in connection with Janowski functions.

Keywords: meromorphic functions; q-calculus; lemniscate of Bernoulli; Janowski functions
Mathematics Subject Classification: 30C45, 30D30

1. Introduction

1.1. Background of the research

The q-calculus (Quantum Calculus) is a branch of mathematics related to calculus in which the concept of limit is replaced by the parameter q. This field of study has motivated the researchers in the recent past with its numerous applications in applied sciences like Physics and Mathematics, e.g., optimal control problems, the field of ordinary fractional calculus, q-transform analysis, q-difference and q-integral equations. The applications of q-generalization in special functions and quantum physics are of high value which makes the study pertinent and interesting in these fields. While the q-difference operator has a vital importance in the theory of special functions and quantum theory, number theory, statistical mechanics, etc. The q-generalization of the concepts of differentiation and integrations were introduced and studied by Jackson [1]. Similarly, Aral and Gupta [2, 3] used some what similar concept by introducing the q-analogue of operator of Baskakov Durrmeyer by using q-beta function. Later, Aral and Anastassiu et. al. in [4, 5] generalized some complex operators, q-Gauss-Weierstrass singular integral and q-Picard operators. For more details on the topic one can see, for example [6–17]. Some of latest inovations in the field can be seen in the work of Arif et al. [18] in which they investigated the q-generalization of Harmonic starlike functions. While Srivastava with his co-authors in [19, 20] investigated some general families in q-analogue related to Janowski functions and obatained some interesting results. Later, Shafiq et al. [21] extended this idea of generalization to close to convex functions. Recently, more research seem to have diversified this field with the introduction of operator theory. Some of the details of such work can be seen in the work of Shi and co-authors [22]. Also some new domains have been explored such as Sine domain in the recent work [23]. Motivated from the discussion above we utilize the concepts of q-calculus and introduce a subclass of p-valent meromorphic functions and investigate some of their nice geometric properties.

1.2. Preliminaries

Before going into our main results we give some basic concepts relating to our work.
Let $\mathcal{M}_p$ represents the class of meromorphic multivalent functions which are analytic in $\mathfrak{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ with the representation

$$f(z) = \frac{1}{z^p} + \sum_{k=p+1}^{\infty} a_k z^k, \ \ (z \in \mathfrak{D}^*). \tag{1.1}$$

Let $f(z)$ and $g(z)$ be analytic in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then the function $f(z)$ is subordinated to $g(z)$ in $\mathfrak{D}$, written as $f(z) \prec g(z)$, $z \in \mathfrak{D}$, if there exist a Schwarz function $\omega(z)$ such that $f(z) = g(\omega(z))$, where $\omega(z)$ is analytic in $\mathfrak{D}$, with $w(0) = 0$ and $w(z) < 1$, $z \in \mathfrak{D}$.

Let $\mathcal{P}$ denote the class of analytic function $l(z)$ normalized by

$$l(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.2}$$

such that $Re(z) > 0$.
We now consider a class of functions in the domain of lemniscate of Bernoulli. All functions $l(z)$ will belong to such a class if it satisfy;

$$h(z) \prec \sqrt{1+z}. \tag{1.3}$$

These functions lie in the right-half of the lemniscate of Bernoulli and with this geometrical representation is the reason behind this name.

With simple calculations the above can be written as

$$\left|(h(z))^2 - 1\right| < 1.$$

Similarly $\mathcal{SL}^*$, in parallel comparison to starlike functions, for analytic functions is

$$\mathcal{SL}^* = \left\{ f(z) \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z} \right\}, \tag{1.4}$$

where $\mathcal{A}$ represents the class of analytic functions and $z \in \mathfrak{D}$. Alternatively

$$\mathcal{SL}^* = \left\{ f(z) \in \mathcal{A} : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \right\},$$

Sokol and Stankiewicz [24] introduced this alongwith some properties. Further study on this was made by different authors in [25–27]. Upper bounds for the coefficients of this class are evaluated in [28].

An important problem in the field of analytic functions is to study a functional $\left|a_3 - va_2^2\right|$ called the Fekete-Szegö functional. Where a_2 and a_3 the coefficients of the original function with a parameter v over which the extremal value of the functional is evaluated. The problem of obtaining the upper bound of this functional for subclasses of normalized functions is called the Fekete-Szegö problem or inequality. M. Fekete and G. Szegö [29], were the first to estimate this classical functional for the class S. While Pfluger [30] utilized Jenkin's method to prove that this result holds for complex μ such that $Re\frac{\mu}{1-\mu} \geq 0$. For other related material on the topic reader is reffered to [31–33].

Similarly the class of Janowski functions is defined for the function $J(z)$ with $-1 \leq B < A \leq 1$

$$J(z) \prec \frac{1 + Az}{1 + Bz}$$

equivalently the functions of this class satisfies

$$\left| \frac{J(z) - 1}{A - BJ(z)} \right| < 1$$

more details on Janowski functions can be seen in [34].

The q-derivative, also known as the q-difference operator, for a function is

$$D_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \tag{1.5}$$

with $z \neq 0$ and $0 < q < 1$. With simple calculations for $n \in \mathbb{N}$ and $z \in \mathfrak{D}^*$, one can see that

$$D_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}, \tag{1.6}$$

with

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^{n-1} q^l \ \text{ and } \ [0]_q = 0.$$

Now we define our new class and we discuss the problem of Fekete-Szegö for this class. Some geometric properties of this class related to subordinations are discussed in connection with Janowski functions.

1.3. The class $\mathcal{MSL}_{p,q}^*$

We introduce $\mathcal{MSL}_{p,q}^*$, a family of meromorphic multivalent functions associated with the domain of lemniscate of Bernoulli in q-analogue as:

If $f(z) \in \mathcal{M}_p$, then it will be in the class $\mathcal{MSL}^*_{p,q}$ if the following holds

$$-\frac{q^p z D_q f(z)}{[p]_q f(z)} \prec \sqrt{1+z}, \tag{1.7}$$

we note that $\lim_{q\to 1^-} \mathcal{MSL}^*_{p,q} = \mathcal{MSL}^*_p$,where

$$\mathcal{MSL}^*_p = \left\{ f(z) \in \mathcal{M}_p : -\frac{zf'(z)}{pf(z)} \prec \sqrt{1+z}, z \in \mathfrak{D}^* \right\}.$$

In this research article we investigate some properties of meromorphic multivalent functions in association with lemniscate of Bernoulli in q-analogue. The important inequality of Fekete-Szegö is evaluated in the beginning of main results. Then we evaluate some bounds of ξ which associate $1 + \xi\frac{z^{p+1}D_q f(z)}{[p]_q}$, $1+\xi\frac{zD_q f(z)}{[p]_q f(z)}$, $1+\xi\frac{z^{1-p}D_q f(z)}{[p]_q (f(z))^2}$ and $1+\xi\frac{z^{1-2p}D_q f(z)}{[p]_q (f(z))^3}$ with Janowski functions and $z^p f(z) \prec \sqrt{1+z}$. Utilizing these theorems along with some conditions we prove that a function may be a member of $\mathcal{MSL}^*_{p,q}$.

2. Sets of Lemma

The following Lemmas are important as they help in our main results.

2.1. Lemma

[35]. If $l(z)$ is in $\mathcal{P}$ given by (1.2), then

$$\left|p_2 - \lambda p_1^2\right| \leq 2 \max\{1; |2\lambda - 1|\}, \nu \in \mathbb{C}.$$

2.2. Lemma

[35]. If $l(z)$ is in $\mathcal{P}$ given by (1.2), then

$$\left|p_2 - \nu p_1^2\right| \leq \begin{cases} -4\nu + 2 & (\nu \leq 0), \\ 2 & (0 \leq \nu \leq 1) \\ 4\nu - 2 & (\nu \geq 1). \end{cases}$$

2.3. Lemma

[36]. (q-Jack's lemma) For an analytic function $\omega(z)$ in $U = \{z \in \mathbb{C} : |z| < 1\}$ with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 = re^{i\theta}$, for $\theta \in [-\pi, \pi]$, we can write that for $0 < q < 1$

$$z_0 D_q \omega(z_0) = m\omega(z_0),$$

with m is real and $m \geq 1$.

3. Main results

In this section we start with Fekete-Szegö problem in the first two theorems. Then some important results relating to subordination are proved using q-Jack's Lemma and with the help of these results the functions are shown to be in the class of $\mathcal{MSL}^*_{p,q}$ in the form of some corollaries.

3.1. Theorem

Let $f \in \mathcal{MSL}^{*}_{p,q}$ and are of the form (1.1), then

$$\left|a_{p+2}-\lambda a_{p+1}^{2}\right| \leq \frac{[p]_q}{2\left([p+2]_q-[p]_q\right)} \max\{1,|\mu|\},$$

where

$$\mu=\frac{\left(q^p\left([p+1]_q\right)^2+3q^p\left([p]_q\right)^2-2\lambda\left([p]_q\right)^2+2\lambda[p+2]_q[p]_q-4q^p[p+1]_q[p]_q\right)}{4q^p\left([p+2]_q-[p]_q\right)}.$$

Proof. Let $f \in \mathcal{MSL}^{*}_{p,q}$, then we have

$$-\frac{q^p zD_q f(z)}{[p]_q f(z)}=\sqrt{1+\omega(z)}, \tag{3.1}$$

where $|\omega(z)| \leq 1$ and $\omega(0)=0$.Let

$$\Phi(\omega(z))=\sqrt{1+\omega(z)}.$$

Thus for

$$l(z)=1+p_1 z+p_2 z^2+\cdots=\frac{1+\omega(z)}{1-\omega(z)}, \tag{3.2}$$

we have $l(z)$ is in $\mathcal{P}$ and

$$\omega(z)=\frac{p_1 z+p_2 z^2+p_3 z^3+\cdots}{2+p_1 z+p_2 z^2+p_3 z^3+\cdots}=\frac{l(z)-1}{l(z)+1}.$$

Now as

$$\sqrt{\frac{2l(z)}{l(z)+1}}=1+\frac{1}{4}p_1 z+\left(\frac{1}{4}p_2-\frac{5}{32}p_1^2\right)z^2+\cdots.$$

So from (3.1), we get

$$-q^p zD_q f(z)=[p]_q\sqrt{\frac{2l(z)}{l(z)+1}}\,f(z),$$

thus

$$\frac{[p]_q}{z^p}-q^p\sum_{k=p+1}^{\infty}[k]_q a_k z^k=$$

$$[p]_q\left(1+\frac{1}{4}p_1 z+\left(\frac{1}{4}p_2-\frac{5}{32}p_1^2\right)z^2+\cdots\right)\left(\frac{[p]_q}{z^p}+\sum_{k=p+1}^{\infty}a_k z^k\right)$$

By comparing of coefficients of z^{k+p}, we get

$$a_{p+1} = -\frac{[p]_q}{4q^p\left([p+1]_q-[p]_q\right)}p_1, \tag{3.3}$$

$$a_{p+2} = -\frac{[p]_q}{q^p\left([p+2]_q-[p]_q\right)}\left(\frac{1}{4}p_2-\frac{5[p+1]_q-7[p]_q}{32\left([p+1]_q-[p]_q\right)}p_1^2\right). \tag{3.4}$$

Form (3.3) and (3.4)

$$\left|a_{p+2}-\lambda a_{p+1}^2\right| =$$

$$\frac{[p]_q}{4q^p\left([p+2]_q-[p]_q\right)}\left|\begin{array}{c} p_2- \\ \frac{5q^p\left([p+1]_q\right)^2-12q^p[p+1]_q[p]_q+7q^p\left([p]_q\right)^2-2\lambda\left([p]_q\right)^2+2[p+2]_q[p]_q\lambda}{8q^p\left([p+1]_q-[p]_q\right)^2}p_1^2\end{array}\right|,$$

Using Lemma 2.1

$$\left|a_{p+2}-\lambda a_{p+1}^2\right| \leq \frac{[p]_q}{2\left([p+2]_q-[p]_q\right)}\max\{1,|\mu|\},$$

with μ is defined as above. □

3.2. Theorem

If $f\in\mathcal{MSL}^*_{p,q}$ and of the form (1.1), then

$$\left|a_{p+2}-\lambda a_{p+1}^2\right| \leq$$

$$\begin{cases} \frac{\gamma}{4q^p(\alpha-\gamma)}\frac{-q^p\beta^2+4q^p\beta\gamma-3q^p\gamma^2}{2q^p(\beta-\gamma)^2}+\frac{\gamma}{4q^p(\alpha-\gamma)}\frac{2\alpha\gamma-2\gamma^2}{2q^p(\beta-\gamma)^2}\lambda, & \frac{5q^p\beta^2-12q^p\beta\gamma+7q^p\gamma^2-2\lambda\gamma^2+2\alpha\gamma\lambda}{8q^p(\beta-\gamma)^2}\leq 0 \\ \frac{\gamma}{2q^p(\alpha-\gamma)}, & 0\leq\frac{5q^p\beta^2-12q^p\beta\gamma+7q^p\gamma^2-2\lambda\gamma^2+2\alpha\gamma\lambda}{8q^p(\beta-\gamma)^2}\leq 1 \\ \frac{\gamma}{4q^p(\alpha-\gamma)}\frac{q^p\beta^2-4q^p\beta\gamma+3q^p\gamma^2}{2q^p(\beta-\gamma)^2}-\frac{\gamma}{4q^p(\alpha-\gamma)}\frac{2\alpha\gamma-2\gamma^2}{2q^p(\beta-\gamma)^2}\lambda, & \frac{5q^p\beta^2-12q^p\beta\gamma+7q^p\gamma^2-2\lambda\gamma^2+2\alpha\gamma\lambda}{8q^p(\beta-\gamma)^2}\geq 1, \end{cases}$$

where $\lambda\in\mathbb{R}$, $\alpha=[p+2]_q$, $\beta=[p+1]_q$ and $\gamma=[p]_q$.

Proof. From (3.3) and (3.4) it follows that

$$a_{p+2}-\lambda a_{p+1}^2 = \frac{[p]_q}{4q^p\left([p+2]_q-[p]_q\right)}\Bigg(p_2-$$

$$\left(\frac{5q^p\left([p+1]_q\right)^2-12q^p[p+1]_q[p]_q+7q^p\left([p]_q\right)^2-2\lambda\left([p]_q\right)^2+2[p+2]_q[p]_q\lambda}{8q^p\left([p+1]_q-[p]_q\right)^2}\right)p_1^2\Bigg),$$

using above notations, we get

$$a_{p+2}-\lambda a_{p+1}^2 = \frac{\gamma}{4q^p(\alpha-\gamma)}\left(p_2-\frac{5q^p\beta^2-12q^p\beta\gamma+7q^p\gamma^2-2\lambda\gamma^2+2\alpha\gamma\lambda}{8q^p(\beta-\gamma)^2}p_1^2\right).$$

Let $v=\frac{5q^p\beta^2-12q^p\beta\gamma+7q^p\gamma^2-2\lambda\gamma^2+2\alpha\gamma\lambda}{8q^p(\beta-\gamma)^2}\leq 0$, using Lemma 2.2, we have

$$\left|a_{p+2}-\lambda a_{p+1}^2\right| \leq \frac{\gamma}{4q^p(\alpha-\gamma)}\left[-4\left(\frac{5q^p\beta^2-12q^p\beta\gamma+7q^p\gamma^2-2\lambda\gamma^2+2\alpha\gamma\lambda}{8q^p(\beta-\gamma)^2}\right)+2\right]$$

$$\leq \quad \frac{\gamma}{4q^p(\alpha-\gamma)}\frac{-q^p\beta^2+4q^p\beta\gamma-3q^p\gamma^2}{2q^p(\beta-\gamma)^2}+\frac{\gamma}{4(\alpha-\gamma)}\frac{2\alpha\gamma-2\gamma^2}{2q^p(\beta-\gamma)^2}\lambda.$$

Let $v=\frac{5q^p\beta^2-12q^p\beta\gamma+7q^p\gamma^2-2\lambda\gamma^2+2\alpha\gamma\lambda}{8q^p(\beta-\gamma)^2}$, where $v\in[0,1]$ using Lemma 2.2, we get the second inequality. Now for $v=\frac{5q^p\beta^2-12q^p\beta\gamma+7q^p\gamma^2-2\lambda\gamma^2+2\alpha\gamma\lambda}{8q^p(\beta-\gamma)^2}\geq 1$, using Lemma 2.2, we have

$$\left|a_{p+2}-\lambda a_{p+1}^2\right|\leq\frac{\gamma}{4q^p(\alpha-\gamma)}\left[4\left(\frac{5\beta^2-12\beta\gamma+7\gamma^2-\lambda\gamma^2}{8(\beta-\gamma)^2}\right)-2\right]$$

$$\leq\frac{\gamma}{4q^p(\alpha-\gamma)}\frac{q^p\beta^2-4q^p\beta\gamma+3q^p\gamma^2}{2q^p(\beta-\gamma)^2}-\frac{\gamma}{4q^p(\alpha-\gamma)}\frac{2\alpha\gamma-2\gamma^2}{2q^p(\beta-\gamma)^2}\lambda,$$

and hence the proof. □

3.3. Theorem

If $f(z)\in\mathcal{M}_p$, then for $-1\leq B<A\leq 1$ with

$$|\xi|\geq\frac{2^{\frac{3}{2}}(A-B)[p]_q}{1-|B|-4p(1+|B|)},\tag{3.5}$$

and if

$$1+\xi\frac{z^{p+1}D_qf(z)}{[p]_q}\prec\frac{1+Az}{1+Bz},\tag{3.6}$$

holds, then

$$z^pf(z)\prec\sqrt{1+z}.$$

Proof. Suppose that

$$J(z)=1+\xi\frac{z^{p+1}D_qf(z)}{[p]_q}\tag{3.7}$$

and consider

$$z^pf(z)=\sqrt{1+\omega(z)}.\tag{3.8}$$

Now to prove the required result it will be enough if we prove that $|\omega(z)|<1$.

Using (3.7) and (3.8)

$$J(z)=1+\frac{\xi}{[p]_q}\left(\frac{zD_q\omega(z)}{2\sqrt{1+\omega(z)}}-p\sqrt{1+\omega(z)}\right)$$

and so

$$\left|\frac{J(z)-1}{A-BJ(z)}\right|=\left|\frac{\frac{\xi}{[p]_q}\left(\frac{zD_q\omega(z)}{2\sqrt{1+\omega(z)}}-p\sqrt{1+\omega(z)}\right)}{A-B\left(1+\frac{\xi}{[p]_q}\left(\frac{zD_q\omega(z)}{2\sqrt{1+\omega(z)}}-p\sqrt{1+\omega(z)}\right)\right)}\right|$$

$$=\left|\frac{\xi zD_q\omega(z)-2p\xi(1+\omega(z))}{2[p]_q(A-B)\sqrt{1+\omega(z)}-B\left(\xi zD_q\omega(z)-2p\xi(1+\omega(z))\right)}\right|$$

Now if $\omega(z)$ attains its maximum value at some $z = z_0$, which is $|\omega(z_0)| = 1$. Then by Lemma 2.3, with $m \geq 1$ we have, $\omega(z_0) = e^{i\theta}$ and $z_0 D_q \omega(z_0) = m\omega(z_0)$, with $\theta \in [-\pi, \pi]$ so

$$\begin{aligned}
&\left|\frac{J(z_0)-1}{A-BJ(z_0)}\right| \\
= &\left|\frac{\xi(m\omega(z_0)-2p(1+\omega(z_0)))}{2(A-B)[p]_q\sqrt{1+\omega(z_0)}-B(\xi(m\omega(z_0)-2p(1+\omega(z_0))))}\right| \\
\geq &\frac{|\xi|\left(m-2p\left(\left|1+e^{i\theta}\right|\right)\right)}{2(A-B)[p]_q\sqrt{|1+e^{i\theta}|}+|B|(|\xi|(m-2p(|1+e^{i\theta}|)))} \\
= &\frac{|\xi|\left(m-2p\sqrt{2+2\cos\theta}\right)}{2(A-B)[p]_q(2+2\cos\theta)^{\frac{1}{4}}+|B|\left(|\xi|\left(m-2p\sqrt{2+2\cos\theta}\right)\right)} \\
\geq &\frac{|\xi|(m-4p)}{|B||\xi|(m+4p)+2^{\frac{3}{2}}(A-B)[p]_q}.
\end{aligned}$$

Consider

$$\begin{aligned}
\phi(m) &= \frac{|\xi|(m-4p)}{|B||\xi|(m+4p)+2^{\frac{3}{2}}(A-B)[p]_q} \\
&\Rightarrow \phi'(m) = \frac{8p|\xi|^2|B|+2^{\frac{3}{2}}|\xi|(A-B)[p]_q}{\left(|B||\xi|(m+4p)+2^{\frac{3}{2}}(A-B)[p]_q\right)^2} > 0,
\end{aligned}$$

showing the increasing behavior of $\phi(m)$ so minimum of $\phi(m)$ will be at $m = 1$ with

$$\left|\frac{J(z_0)-1}{A-BJ(z_0)}\right| \geq \frac{|\xi|(1-4p)}{2^{\frac{3}{2}}(A-B)[p]_q+|B||\xi|(1+4p)},$$

so from(3.5)

$$\left|\frac{J(z_0)-1}{A-BJ(z_0)}\right| \geq 1$$

contradicting (3.6), thus $|\omega(z)| < 1$ and so we get the desired result. □

3.4. Corollary

Let $-1 \leq B < A \leq 1$ and $f(z) \in \mathcal{M}_p$. If

$$|\xi| \geq \frac{2^{\frac{3}{2}}[p]_q(A-B)}{1-|B|-4(1+|B|)p},$$

and

$$1-\left(1-p+\frac{zD_q^2 f(z)}{D_q f(z)}-\frac{zD_q f(z)}{f(z)}\right)\frac{\xi z D_q f(z)}{[p]_q^2 f(z)} \prec \frac{1+Az}{1+Bz}, \tag{3.9}$$

then $f(z) \in \mathcal{MSL}^*_{p,q}$.

Proof. Suppose that

$$l(z) = \frac{q^p z^{1-p} D_q f(z)}{[p]_q f(z)}. \tag{3.10}$$

From (3.10) it follows that

$$z^{p+1} D_q l(z) = \left(1 - p + \frac{z D_q^2 f(z)}{D_q f(z)} - \frac{z D_q f(z)}{f(z)}\right) \frac{z D_q f(z)}{[p]_q^2 f(z)},$$

Using the condition (3.9) ,we have

$$1 - \xi z^{p+1} D_q l(z) \prec \frac{1 + Az}{1 + Bz}.$$

Now using Theorem 3.3, we get

$$-z^p l(z) = -\frac{q^p z D_q f(z)}{[p]_q f(z)} \prec \sqrt{1 + z},$$

thus $f(z) \in \mathcal{MSL}_p^*$. □

3.5. Theorem

Let $-1 \leq B < A \leq 1$ and $f(z) \in \mathcal{M}_p$. If

$$|\xi| \geq \frac{4 [p]_q (A - B)}{1 - |B| - 4(1 + |B|) p} \tag{3.11}$$

and

$$1 + \xi \frac{z D_q f(z)}{[p]_q f(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{3.12}$$

then

$$z^p f(z) \prec \sqrt{1 + z}.$$

Proof. We define a function

$$J(z) = 1 + \xi \frac{z D_q f(z)}{[p]_q f(z)}. \tag{3.13}$$

Now as

$$z^p f(z) = \sqrt{1 + \omega(z)} \tag{3.14}$$

Using Logarithmic differentiation on (3.14) , from (3.13) we obtain that

$$J(z) = 1 + \frac{\xi}{[p]_q} \left(\frac{z D_q \omega(z)}{2(1 + \omega(z))} - p\right)$$

and so

$$\left|\frac{J(z) - 1}{A - BJ(z)}\right| = \left|\frac{\frac{\xi}{[p]_q}\left(\frac{z D_q \omega(z)}{2(1+\omega(z))} - p\right)}{A - B\left(1 + \frac{\xi}{[p]_q}\left(\frac{z D_q \omega(z)}{2(1+\omega(z))} - p\right)\right)}\right|$$

$$= \left| \frac{\xi \left(zD_q\omega(z) - 2p(1+\omega(z)) \right)}{2(A-B)[p]_q(1+\omega(z)) - B\left(\xi zD_q\omega(z) - 2p\xi(1+\omega(z)) \right)} \right|.$$

If at some $z = z_0$, $\omega(z)$ attains its maximum value i.e. $|\omega(z_0)| = 1$. Then using Lemma 2.3, we have

$$\begin{aligned} & \left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right| \\ = & \left| \frac{\xi(m\omega(z_0) - 2p(1+\omega(z_0)))}{2(A-B)[p]_q(1+\omega(z_0)) - B(\xi m\omega(z_0) - 2p\xi(1+\omega(z_0)))} \right| \\ \geq & \frac{|\xi|\left(m - 2p\left|1+e^{i\theta}\right|\right)}{2(A-B)[p]_q|1+e^{i\theta}| + |\xi||B|(m+2p|1+e^{i\theta}|)} \\ = & \frac{|\xi|m - 2p|\xi|\sqrt{2+2\cos\theta}}{2\left((A-B)[p]_q + p|B||\xi|\right)\sqrt{2+2\cos\theta} + |\xi||B|m} \\ \geq & \frac{|\xi|(m-4p)}{4\left((A-B)[p]_q + p|B||\xi|\right) + |\xi||B|m}. \end{aligned}$$

Now let

$$\begin{aligned} \phi(m) &= \frac{|\xi|(m-4p)}{4\left((A-B)[p]_q + p|\xi||B|\right) + |B||\xi|m} \\ &\Rightarrow \phi'(m) = \frac{|\xi|\left(8p|B| + 4(A-B)[p]_q\right)}{\left(4\left((A-B)[p]_q + p|B||\xi|\right) + |B||\xi|m\right)^2} > 0, \end{aligned}$$

which shows that the increasing nature of $\phi(m)$ and so its minimum value will be at $m = 1$ thus

$$\left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right| \geq \frac{(1-4p)|\xi|}{4\left(p|\xi||B| + (A-B)[p]_q\right) + |B||\xi|},$$

hence by(3.11)

$$\left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right| \geq 1,$$

which contradicts (3.12) , therefore $|\omega(z)| < 1$ and so the desired result. □

3.6. Corollary

Let $-1 \leq B < A \leq 1$ and $f(z) \in \mathcal{M}_p$. If

$$|\xi| \geq \frac{4[p]_q(A-B)}{1 - |B| - 4(1+|B|p)},$$

and

$$1 - \left(1 - p + \frac{zD_q^2 f(z)}{D_q f(z)} - \frac{zD_q f(z)}{f(z)} \right) \frac{\xi}{[p]_q} \prec \frac{1+Az}{1+Bz},$$

then $f(z) \in \mathcal{MSL}^*_{p,q}$.

3.7. Theorem

Let $-1 \leq B < A \leq 1$ and $f(z) \in \mathcal{M}_p$. If

$$|\xi| \geq \frac{2^{\frac{5}{2}} [p]_q (A-B)}{1-|B|-4(1+|B|)p} \tag{3.15}$$

and

$$1+\xi \frac{z^{1-p} D_q f(z)}{[p]_q (f(z))^2} \prec \frac{1+Az}{1+Bz},$$

then $z^p f(z) \prec \sqrt{1+z}$.

Proof. Here we define a function

$$J(z) = 1+\xi \frac{z^{1-p} D_q f(z)}{[p]_q f^2(z)}.$$

So if

$$z^p f(z) = \sqrt{1+\omega(z)},$$

using some simplification we obtain that

$$J(z) = 1+\frac{\xi}{[p]_q}\left(\frac{zD_q\omega(z)}{2(1+\omega(z))^{\frac{3}{2}}} - \frac{p}{\sqrt{1+\omega(z)}}\right),$$

and so

$$\left|\frac{J(z)-1}{A-BJ(z)}\right| = \left|\frac{\frac{\xi}{[p]_q}\left(\frac{zD_q\omega(z)}{2(1+\omega(z))^{\frac{3}{2}}} - \frac{p}{\sqrt{1+\omega(z)}}\right)}{A-B\left(1+\frac{\xi}{[p]_q}\left(\frac{zD_q\omega(z)}{2(1+\omega(z))^{\frac{3}{2}}} - \frac{p}{\sqrt{1+\omega(z)}}\right)\right)}\right|$$

$$= \left|\frac{\xi\left(zD_q\omega(z) - 2p(1+\omega(z))\right)}{2(A-B)[p]_q(1+\omega(z))^{\frac{3}{2}} + 2p\xi B(1+\omega(z)) - B\xi zD_q\omega(z)}\right|.$$

Now if $\omega(z)$ attains, at some $z = z_0$, its maximum value which is $|\omega(z_0)| = 1$. Then by Lemma 2.3, with $m \geq 1$ we have, $\omega(z_0) = e^{i\theta}$ and $z_0 D_q\omega(z_0) = m\omega(z_0)$, with $\theta \in [-\pi, \pi]$ so

$$\begin{aligned}
&\left|\frac{J(z_0)-1}{A-BJ(z_0)}\right| \\
= &\left|\frac{\xi(m\omega(z_0) - 2p(1+\omega(z_0)))}{2(A-B)[p]_q(1+\omega(z_0))^{\frac{3}{2}} + 2p\xi B(1+\omega(z_0)) - B\xi m\omega(z_0)}\right| \\
\geq &\frac{|\xi| m - 2p|\xi|\left|1+e^{i\theta}\right|}{2(A-B)[p]_q|1+e^{i\theta}|^{\frac{3}{2}} + |B||\xi| m + 2p|\xi||B||1+e^{i\theta}|} \\
= &\frac{|\xi| m - 2p|\xi|\sqrt{2+2\cos\theta}}{2(A-B)[p]_q(2+2\cos\theta)^{\frac{3}{4}} + |B||\xi| m + 2p|\xi||B|\sqrt{2+2\cos\theta}} \\
\geq &\frac{(m-4p)|\xi|}{2^{\frac{5}{2}}(A-B)[p]_q + 4p|\xi||B| + |B||\xi| m}.
\end{aligned}$$

Now let

$$\phi(m) = \frac{|\xi|(m-4p)}{2^{\frac{5}{2}}(A-B)[p]_q + |B||\xi|(m+4p)}$$

$$\Rightarrow \phi'(m) = \frac{|\xi|\left(2^{\frac{5}{2}}(A-B) + 8p|\xi||B|\right)}{\left(2^{\frac{5}{2}}(A-B)[p]_q + 4p|\xi||B| + |B||\xi|m\right)^2} > 0,$$

this shows $\phi(m)$ an increasing function which implies that at $m = 1$ it will have its minimum value and

$$\left|\frac{J(z_0)-1}{A-BJ(z_0)}\right| \geq \frac{(1-4p)|\xi|}{2^{\frac{5}{2}}(A-B)[p]_q + |B||\xi| + 4p|\xi||B|},$$

now by (3.15) we have

$$\left|\frac{J(z_0)-1}{A-BJ(z_0)}\right| \geq 1,$$

this is a contradiction as $J(z) \prec \frac{1+Az}{1+Bz}$, thus $|\omega(z)| < 1$ and so the result. □

3.8. Corollary

Let $-1 \leq B < A \leq 1$ and $f(z) \in \mathcal{M}_p$. If

$$|\xi| \geq \frac{2^{\frac{5}{2}}[p]_q(A-B)}{1-|B|-4(1+|B|)p},$$

and

$$1 - \left(1 - p + \frac{zD_q^2 f(z)}{D_q f(z)} - \frac{zD_q f(z)}{f(z)}\right)\frac{\xi f(z)}{zD_q f(z)} \prec \frac{1+Az}{1+Bz},$$

then $f(z) \in \mathcal{MSL}^*_{p,q}$.

3.9. Theorem

Let $-1 \leq B < A \leq 1$ and $f(z) \in \mathcal{M}_p$. If

$$|\xi| \geq \frac{8[p]_q(A-B)}{1-|B|-4(1+|B|)p} \tag{3.16}$$

and

$$1 + \xi\frac{z^{1-2p}D_q f(z)}{[p]_q (f(z))^3} \prec \frac{1+Az}{1+Bz}, \tag{3.17}$$

then $z^p f(z) \prec \sqrt{1+z}$.

Proof. Suppose that

$$J(z) = 1 + \xi\frac{z^{1-2p}D_q f(z)}{[p]_q (f(z))^3}.$$

Now if

$$z^p f(z) = \sqrt{1+\omega(z)},$$

with simple calculations we can easily obtain

$$J(z) = 1 + \frac{\xi}{[p]_q (1+\omega(z))}\left(\frac{zD_q\omega(z)}{2[p]_q(1+\omega(z))} - p\right),$$

and so

$$\left|\frac{J(z)-1}{A-BJ(z)}\right| = \left|\frac{\frac{\xi}{[p]_q(1+\omega(z))}\left(\frac{zD_q\omega(z)}{2[p]_q(1+\omega(z))}-p\right)}{A-B\left(1+\frac{\xi}{[p]_q(1+\omega(z))}\left(\frac{zD_q\omega(z)}{2[p]_q(1+\omega(z))}-p\right)\right)}\right|$$

$$= \left|\frac{\xi\left(zD_q\omega(z) - 2p(1+\omega(z))\right)}{2(A-B)[p]_q(1+\omega(z))^2 + 2p\xi B(1+\omega(z)) - B\xi zD_q\omega(z)}\right|,$$

if at some $z = z_0$, $\omega(z)$ attains its maximum value i.e. $|\omega(z_0)| = 1$. Then using Lemma 2.3,

$$\begin{aligned}
&\left|\frac{J(z_0)-1}{A-BJ(z_0)}\right| \\
= &\left|\frac{\xi(m\omega(z_0) - 2p(1+\omega(z_0)))}{2(A-B)[p]_q(1+\omega(z_0))^2 - 2p\xi B(1+\omega(z_0)) + B\xi m\omega(z_0)}\right| \\
\geq &\frac{|\xi|\left(m - 2p\left|1+e^{i\theta}\right|\right)}{2(A-B)[p]_q|1+e^{i\theta}|^2 + 2p|\xi||B||1+e^{i\theta}| + |B||\xi|m} \\
= &\frac{|\xi|\left(m-2p\sqrt{2+2\cos\theta}\right)}{2(A-B)[p]_q(2+2\cos\theta) + 2p|\xi||B|\sqrt{2+2\cos\theta} + |B||\xi|m} \\
\geq &\frac{(m-4p)|\xi|}{8(A-B)[p]_q + |B||\xi|(m+4p)}.
\end{aligned}$$

Now let

$$\phi(m) = \frac{(m-4p)|\xi|}{8(A-B)[p]_q + |B||\xi|(m+4p)}$$

$$\Rightarrow \phi'(m) = \frac{8|\xi|(A-B)[p]_q + 8p|\xi|^2|B|}{\left(8(A-B)[p]_q + |B||\xi|m + 4p|\xi||B|\right)^2} > 0$$

which shows that the increasing nature of $\phi(m)$ and so its minimum value will be at $m = 1$ thus

$$\left|\frac{J(z_0)-1}{A-BJ(z_0)}\right| \geq \frac{(1-4p)|\xi|}{8(A-B)[p]_q + |B||\xi|(1+8p)},$$

and hence

$$\left|\frac{J(z_0)-1}{A-BJ(z_0)}\right| \geq 1,$$

thus a contradiction by (3.17), so $|\omega(z)| < 1$ and so we get the desired proof. □

3.10. Corollary

Let $-1 \leq B < A \leq 1$ and $f(z) \in \mathcal{M}_p$. If

$$|\xi| \geq \frac{8(A-B)[p]_q}{1-|B|-4p(1+|B|)}$$

and

$$1-\xi[p]_q\left(1-p+\frac{zD_q^2 f(z)}{D_q f(z)}-\frac{zD_q f(z)}{f(z)}\right)\left(\frac{f(z)}{zD_q f(z)}\right)^2 \prec \frac{1+Az}{1+Bz},$$

then $f(z) \in \mathcal{MSL}^*_{p,q}$.

3.11. Remark

Letting $q \to 1^-$ in our results we obtain results for the class $\mathcal{MSL}^*_p$.

4. Conclusion

The main purpose of this article is to seek some applications of the q-calculus in Geometric Function theory, which is the recent attraction for many researchers these days. The methods and ideas of q-calculus are used in the introduction of a new subclass of p-valent meromorphic functions with the help of subordinations. The domain of lemniscate of Bernoulli is considered in defining this class. Working on the coefficients of these functions we obtained a very important result of Fekete-Szegö for this class. Furthermore the functionals $1+\xi\frac{z^{p+1}D_q f(z)}{[p]_q}$, $1+\xi\frac{zD_q f(z)}{[p]_q f(z)}$, $1+\xi\frac{z^{1-p}D_q f(z)}{[p]_q (f(z))^2}$ and $1+\xi\frac{z^{1-2p}D_q f(z)}{[p]_q (f(z))^3}$ are connected with Janowski functions with the help of some conditions on ξ which ensures that a function to be a member of the class $\mathcal{MSL}^*_{p,q}$.

Acknowledgment

The authors are grateful to the editor and anonymous referees for their comments and remarks to improve this manuscript. The author Thabet Abdeljawad would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Method in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Conflict of interest

The authors declare that they have no competing interests.

References

1. F. H. Jackson, On q-definite integrals, *Q. J. Pure Appl. Math.*, **41** (1910), 193–203.
2. A. Aral, V. Gupta, Generalized q-Baskakov operators, *Math. Slovaca*, **61** (2011), 619–634.
3. A. Aral, V. Gupta, On q-Baskakov type operators, *Demonstr. Math.,* **42** (2009), 109–122.

4. A. Aral, On the generalized Picard and Gauss Weierstrass singular integrals, *J. Compu. Anal. Appl*, **8** (2006), 249–261.
5. G. A. Anastassiu, S. G. Gal, Geometric and approximation properties of generalized singular integrals, *J. Korean Math. Soci.,* **23** (2006), 425–443.
6. S. Kanas, D. Răducanu, Some class of analytic functions related to conic domains, *Math. Slovaca,* **64** (2014), 1183–1196.
7. H. Aldweby, M. Darus, Some subordination results on q-analogue of Ruscheweyh differential operator, *Abstr. Appl. Anal.,* (2014), Article ID 958563.
8. S. Mahmmod, J. Sokół, New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator, *Res. Math.*, **71** (2017), 1345–1357.
9. T. M. Seoudy, M. K. Aouf, Convolution properties for certain classes of analytic functions defined by q-derivative operator, *Abstr. Appl. Anal.*, (2014), Article ID 846719.
10. T. M. Seoudy, M. K. Aouf, Coefficient estimates of new classes of q-starlike and q-convex functions of complex order, *J. Math. Inequal.*, **10** (2016), 135–145.
11. C. Ramachandran, T. Soupramanien, B. A. Frasin, New subclasses of analytic function associated with q-difference operator, *Eur. J. Pure Appl. Math.*, **10** (2017), 348–362.
12. S. Kavitha, N. E. Cho, G. Murugusundaramoorthy, *On (p, q)-Quantum Calculus Involving Quasi-Subordination, Trend in mathematics, Advance in Algebra and Analysis International Conference on Advance in Mathematical Sciences,* Vellore, India, December 2017, Vol. 1, 215–223.
13. B. A. Frasin, G. Murugusundaramoorthy, A subordination results for a class of analytic functions defined by q-differential operator, *Ann. Univ. Paedagog. Crac. Stud. Math.*, **19** (2020), 53–64.
14. B. Khan, Z. G. Liu, H. M. Serivastava, N. Khan, M. Darus, M. Tahir, A study of some families of multivalent q-starlike functions involving higher-order q-derivatives, *Mathematics*, **8** (2020), 1–12.
15. H. M. Srivastava, N. Khan, M. Darus, S. Khan, Q. Z. Ahmad, S. Hussain, Fekete-Szego type problems and their applications for a subclass of q-starlike functions with respect to symmetrical points, *Mathematics*, **8** (2020), 1–18.
16. M. Shafiq, H. M. Srivastava, N. Khan, Q. Z. Ahmad, M. Darus, S. Kiran, An upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with k-Fibonacci numbers, *Symmetry*, **12** (2020), 1–17.
17. M. G. Khan, B. Ahmad, B. A. Frasin, J. Abdel, On Janowski analytic (p; q)-starlike functions in symmetric circular domain, *J. Funct. Spaces,* (2020), Article ID 4257907.
18. M. Arif, O. Barkub, H. M. Srivastava, S. Abdullah, A. Khan, Some Janowski type Harmonic q-starlike functions associated with symmetric points, *Mathematics,* **8** (2020), Article ID 629.
19. H. M. Srivastava, D. Bansal, Close-to-convexity of a certain family of q-Mittag-Leffler functions, *J. Nonlinear Var. Anal.,* **1** (2017), 61–69.
20. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general families of q-starlike functions associated with the Janowski functions, *Filomat.*, **33** (2019), 2613–2626.
21. M. Shafiq, N. Khan, H. M. Srivastava, B. Khan, Q. Z. Ahmad, M. Tahir, Generalization of close to convex functions associated with Janowski functions, *Maejo Int. J. Sci. Technol.,* **14** (2020), 141–155.

22. L. Shi, M. G. Khan, B. Ahmad, Some geometric properties of a family of analytic functions involving a generalised q-operator, *Symmetry,* **12** (2020),1–11.
23. S. Islam, M. G. Khan, B. Ahmad, M. Arif, R. Chinram, Q-Extension of starlike functions subordinated with a trignometric sine function, *Mathematics,* **8** (2020), Article ID 1676.
24. J. Sokól, J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, *Zesz. Nauk. Politech. Rzeszowskiej Mat.,* **19** (1996), 101–105.
25. J. Sokól, Radius problem in the class $\mathcal{SL}^{*}$, *Appl. Math. Comput.,* **214** (2009), 569–573.
26. S. A. Halim, R. Omar, Applications of certain functions associated with lemniscate Bernoulli, *J. Indones. Math. Soc.,* **18** (2012), 93–99.
27. R. M. Ali, N. E. Chu, V. Ravichandran, S. S. Kumar, First order differential subordination for functions associated with the lemniscate of Bernoulli, *Taiwan. J. Math.,* **16** (2012), 1017–1026.
28. J. Sokól, Coefficient estimates in a class of strongly starlike functions, *Kyungpook Math. J.*, **49** (2009), 349–353.
29. M, Fekete, G. Szegö, Eine Bemerkung über ungerade schlichte Funktionen, *J. Lond. Math. Soc.,* **8** (1933), 85–89.
30. A. Pfluger, The Fekete-Szegö inequality for complex parameters, *Complex Var. Theory Appl.*, **7** (1986), 149–160.
31. F. R. Keogh, E. P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Am. Math. Soc.,* **20** (1969), 8–12.
32. W. Ma, D. Minda, An internal geometric characterization of strongly starlike functions, *Ann. Univ. Mariae Curie-Sklodowska, Sect. A.,* **45** (1991), 89–97.
33. W. Ma, D. Minda, Coefficient inequalities for strongly close-to-convex functions, *J. Math. Anal. Appl.,* **205**, 537–553.
34. W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.,* **23** (1970), 159–177.
35. W. Ma, D. Minda, A unified treatment of some special classes of univalent functions, In: Z. Li, F. Ren, L. Yang, S. Zhang (eds.), *Proceeding of the conference on Complex Analysis,* (Tianjin, 1992), Int. Press, Cambridge, 1994, 157–169.
36. K. Ademogullari, Y. Kahramaner, q-harmonic mappings for which analytic part is q-convex functions, *Nonlinear Anal. Di . Eqns.*, **4** (2016), 283–293.