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Research article

Some rigidity theorems on Finsler manifolds

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Abstract: We prove that, for a Finsler manifold with the weighted Ricci curvature bounded below by a positive number, it is a Finsler sphere if and only if the diam attains its maximal value, if and only if the volume attains its maximal value, and if and only if the first closed eigenvalue of the Finsler-Laplacian attains its lower bound. These generalize some rigidity theorems in Riemannian geometry to the Finsler setting.

Keywords: Randers sphere, the maximum diam, the weighted Ricci curvature
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1. Introduction

In Riemannian geometry, Myers's theorem proves that if (M, g) is a complete connected Riemannian n-manifold such that $\mathrm{Ric} \geq (n-1)k$ for some positive number k, then it is compact and $Diam(M) \leq \frac{\pi}{\sqrt{k}}$. Further, Cheng's maximum diam theorem states that if its diameter attains its maximal value, then the manifold is isometric to the Euclidean sphere $\mathbb{S}^n(\frac{1}{\sqrt{k}})$. Under the same curvature condition, Lichnerowicz estimate indicates that the first closed eigenvalue of the Laplacian is not less than nk, while Obata rigidity theorem shows if the first closed eigenvalue attains its lower bound, then it is isometric to $\mathbb{S}^n(\frac{1}{\sqrt{k}})$. By the same token, Bishop-Gromov comparison theorem also demonstrates that if $\mathrm{Vol}(M) = \mathrm{Vol}(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$, then it is isometric to $\mathbb{S}^n(\frac{1}{\sqrt{k}})$. Therefore, on a complete connected Riemannian n-manifold with $\mathrm{Ric} \geq (n-1)k$ for some positive k, the following conditions are equivalent:

- (M, g) is the Euclidean sphere $\mathbb{S}^n(\frac{1}{\sqrt{k}})$;
- $Diam(M) = \frac{\pi}{\sqrt{k}}$;
- the first eigenvalue of Laplacian is $\lambda_1(M) = nk$;
- $\mathrm{Vol}(M) = \mathrm{Vol}(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$.

It is natural to ask:

In the Finsler setting, can we also characterize and determine Finsler spheres by the diam, the first eigenvalue and the volume?

In Finsler geometry, Cheng's maximum diam theorem is studied in [2] for reversible Finsler manifolds and in [10] for the general case. Lichnerowicz estimate and Obata rigidity theorem in Finsler situation are also considered in [8–10]. Along this line, the authors give a positive answer to the problem above. As to Bishop-Gromov comparison theorem, there are several generalized results established in [4, 6, 7, 11], respectively. However, they (see [4, 6, 7]) do not study the rigidity phenomenon when the volume reaches its maximum value. In [11], Zhao-Shen study the rigidity problem and give some characterizations by using constant radial flag curvature and constant radial S curvature. Yet, there is still much to be desired.

Let $\mathfrak{S}^n(\frac{1}{\sqrt{k}})$ denote a Finsler sphere which has Busemann Hausdorff volume form, constant flag curvature k and vanishing S curvature. When the Finsler metric is a Randers metric, the sphere is called a Randers sphere denoted by $\mathcal{S}^n(\frac{1}{\sqrt{k}})$. There are infinitely many Randers spheres, and if the metric is reversible, the sphere is just the Euclidean sphere (see [10]). For more details, we refer to Section 2 below.

In this paper, we characterize Finsler spheres and obtain some rigidity results in the following.

Theorem 1.1. *Let $(M, F, d\mu)$ be a complete connected Finsler n-manifold with Busemann-Hausdorff volume form. If the weighted Ricci curvature* $\mathrm{Ric}_n \geq (n-1)k > 0$*, then the following conditions are equivalent:*

(1) $(M, F, d\mu)$ *is a Finsler sphere* $\mathfrak{S}^n(\frac{1}{\sqrt{k}})$*;*
(2) *Diam*$(M) = \frac{\pi}{\sqrt{k}}$*;*
(3) the first eigenvalue of Finsler-Laplacian is $\lambda_1 = nk$*;*
(4) $\mathrm{vol}_F^{d\mu}(M) = \mathrm{vol}(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$.

Since the classification for Finsler metrics with constant flag curvature is not solved, we can not determine all Finsler sphere metrics. However, for a Randers sphere $\mathcal{S}^n(\frac{1}{\sqrt{k}})$, the metric F can be expressed by navigation data (g, W), where g is the standard sphere metric and W is a Killing vector. Therefore, Randers spheres are of most importance among Finsler spheres. By narrowing the scope in Theorem 1.1, we have

Theorem 1.2. *Let $(M, F, d\mu)$ be a complete connected Randers n-manifold with Busemann-Hausdorff volume form. If the weighted Ricci curvature* $\mathrm{Ric}_n \geq (n-1)k > 0$*, then the following conditions are equivalent:*

(1) $(M, F, d\mu)$ *is a Randers sphere* $\mathcal{S}^n(\frac{1}{\sqrt{k}})$*;*
(2) *Diam*$(M) = \frac{\pi}{\sqrt{k}}$*;*
(3) the first eigenvalue of Finsler-Laplacian is $\lambda_1 = nk$*;*
(4) $\mathrm{vol}_F^{d\mu}(M) = \mathrm{vol}(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$.

Remark 1.3. Theorems 1.1 and 1.2 show that, apart from the Euclid sphere, the maximum diameter, the maximum volume and the lower bound of the first eigenvalue can be attained on countless Finsler (especially Randers) spheres.

The paper is organized as follows. In Section 2, some fundamental concepts and formulas which are necessary for the present paper are given, and some lemmas are contained. The volume comparison theorem and Theorem 1.1 are then proved in Sections 3 and 4, repectively.

2. Preliminaries

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinates on TM with $y = y^i\partial/\partial x^i$. A *Finsler metric* on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) *Regularity*: $F(x, y)$ is smooth in $TM \setminus 0$;

(ii) *Positive homogeneity*: $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$;

(iii) *Strong convexity*: The fundamental quadratic form

$$g := g_{ij}(x, y)dx^i \otimes dx^j, \qquad g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$$

is positively definite.

Let $X = X^i\frac{\partial}{\partial x^i}$ be a differentiable vector field. Then the *covariant derivative* of X by $v \in T_xM$ with reference vector $w \in T_xM\backslash 0$ is defined by

$$D^w_v X(x) := \left\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w)v^jX^k(x) \right\} \frac{\partial}{\partial x^i},$$

where Γ^i_{jk} denote the coefficients of the Chern connection.

Given two linearly independent vectors $V, W \in T_xM\backslash 0$, the flag curvature is defined by

$$K(V, W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2},$$

where R^V is the *Chern curvature*:

$$R^V(X, Y)Z = D^V_X D^V_Y Z - D^V_Y D^V_X Z - D^V_{[X,Y]}Z.$$

Then the Ricci curvature for (M, F) is defined as

$$\mathrm{Ric}(V) = \sum_{\alpha=1}^{n-1} K(V, e_\alpha),$$

where $e_1, \cdots, e_{n-1}, \frac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V.

Let $(M, F, d\mu)$ be a Finsler n-manifold with $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. The distortion is given by

$$\tau(x, V) = \log \frac{\sqrt{\det(g_{ij}(x, V))}}{\sigma(x)}.$$

For the vector $V \in T_xM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = x$, $\dot\gamma(0) = V$. Then the S-curvature measures the rate of changes of the distortion along geodesics

$$S(x, V) := \frac{d}{dt}[\tau(\gamma(t), \dot\gamma(t))]_{t=0}.$$

Define

$$\dot S(x, V) := F^{-2}(V)\frac{d}{dt}[S(\gamma(t), \dot\gamma(t))]_{t=0}.$$

Then the *weighted Ricci curvature* of $(M, F, d\mu)$ is defined by (see [3])

$$\begin{cases} \mathrm{Ric}_n(V) := \begin{cases} \mathrm{Ric}(V) + \dot{S}(V), & \text{for } S(V) = 0, \\ -\infty, & \text{otherwise,} \end{cases} \\ \mathrm{Ric}_N(V) := \mathrm{Ric}(V) + \dot{S}(V) - \frac{S(V)^2}{(N-n)F(V)^2}, \ \forall\ N \in (n, \infty), \\ \mathrm{Ric}_\infty(V) := \mathrm{Ric}(V) + \dot{S}(V), \end{cases}$$

For a smooth function u, the *gradient vector* of u at x is defined by $\nabla u(x) := \mathcal{L}^{-1}(du)$, where $\mathcal{L} : T_xM \to T_x^*M$ is the Legendre transform. Let $V = V^i \frac{\partial}{\partial x^i}$ be a smooth vector field on M. The *divergence* of V with respect to an arbitrary volume form $d\mu$ is defined by

$$\mathrm{div}V := \sum_{i=1}^{n} \left(\frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \log \sigma(x)}{\partial x^i} \right).$$

Then the *Finsler-Laplacian* of u can be defined by

$$\Delta u := \mathrm{div}(\nabla u).$$

The equality is in the weak $W^{1,2}(M)$ sense. Namely, for any $\varphi \in C_0^\infty(M)$, we have

$$\int_M \varphi \Delta u d\mu = -\int_M d\varphi(\nabla u) d\mu.$$

Recall that Bao-Shen [1] found a family Randers sphere metrics on $\mathbb{S}^3$, which shows that the maximum diam can be achieved in non-Riemannian case.

Example 2.1. [1] View $\mathbb{S}^3$ as a compact Lie group. Let $\zeta^1, \zeta^2, \zeta^3$ be the standard right invariant 1-form on $\mathbb{S}^3$ satisfying

$$d\zeta^1 = 2\zeta^2 \wedge \zeta^3, \quad d\zeta^2 = 2\zeta^3 \wedge \zeta^1, \quad d\zeta^3 = 2\zeta^1 \wedge \zeta^2.$$

For $k \geq 1$, define

$$\alpha_k(y) = \sqrt{(k\zeta^1(y))^2 + k(\zeta^2(y))^2 + k(\zeta^3(y))^2}, \quad \beta_k(y) = \sqrt{k^2 - k}\zeta^1(y).$$

Then $F_k = \alpha_k + \beta_k$ is a Randers metric on $\mathbb{S}^3$ satisfying

$$K \equiv 1, \quad S \equiv 0, \quad Diam(\mathbb{S}^3, F_k) = \pi.$$

Inspired by Bao-Shen's example, we give the following definition.

Definition 2.2. A Finsler manifold with Busemann-Hausdorff volume form is said to be a Finsler sphere if it has positively constant flag curvature and vanishing S-curvature. In particular, if the metric is a Randers metric, we call it a Randers sphere.

Generally, we are not able to identify a Finsler sphere metric. However, for a Randers sphere metric, F can be expressed by (see [10])

$$F = \frac{\sqrt{\lambda \mathrm{g}^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \quad \lambda = 1 - \|W\|_{\mathrm{g}}^2,$$

where g is the standard sphere metric, and W is a Killing vector field on $\mathbb{S}^n(\frac{1}{\sqrt{k}})$.

To prove our results, we further introduce the following lemmas.

Lemma 2.3. *[4] Let $(M, F, d\mu)$ be a Finsler n manifold. If the weighted Ricci curvature satisfies* $\mathrm{Ric}_N \geq (N-1)k, N \in [n, \infty)$*, then the Laplacian of the distance function $r(x) = d_F(p, x)$ from any given point $p \in M$ can be estimated as follows:*

$$\Delta r \leq (N-1)\frac{\mathrm{s}_k'(r)}{\mathrm{s}_k(r)},$$

pointwise on $M\backslash(\{p\} \cup Cut(p))$ and in the sense of distributions on $M\backslash\{p\}$, where

$$\boldsymbol{s}_k = \begin{cases} \frac{1}{\sqrt{k}}\sin(\sqrt{k}t), & k > 0; \\ t, & k=0; \\ \frac{1}{\sqrt{-k}}\sinh(\sqrt{-k}t), & k < 0. \end{cases}$$

Lemma 2.4. *[3] Let $(M, F, d\mu)$ be a Finsler n manifold. If the weighted Ricci curvature satisfies* $\mathrm{Ric}_N \geq (N-1)k, N \in [n, \infty)$*, then for any $0 < r < R$ ($\leq \frac{\pi}{\sqrt{k}}$ if $k > 0$), it holds that*

$$\max\left\{\frac{\mathrm{vol}_F^{d\mu} B_x^+(R)}{\mathrm{vol}_F^{d\mu} B_x^+(r)}, \frac{\mathrm{vol}_F^{d\mu} B_x^-(R)}{\mathrm{vol}_F^{d\mu} B_x^-(r)}\right\} \leq \frac{\int_0^R \mathrm{s}_k^{N-1} dt}{\int_0^r \mathrm{s}_k^{N-1} dt}.$$

3. Volume comparison theorems

In [6, 7], the volume comparison theorems are established on Finsler manifolds satisfying $\mathrm{Ric} \geq (n-1)k$ and some S curvature condition. Later, Zhao-Shen [11] generalize them and further obtain the rigidity result. Using the weighted Ricci curvature condition, Ohta [3] obtain another version of the relative volume comparison (see Lemma 2.4 above). However, the problem about the rigidity result remains open. The main obstacle is that, for any volume form, the limit

$$\lim_{r\to 0} \frac{\mathrm{vol}_F^{d\mu}(B_p^+(r))}{\int_0^r (\mathbf{s}_k(t))^{N-1} dt}$$

does not necessarily exists. Therefore, to obtain the rigidity result, it is suitable to give a restriction on the volume form.

Let $(M, F, d\mu)$ be a Finsler n-manifold. Fix a point $p \in M$. Then on T_pM the Finsler metric $F(p, y)$ induces a Riemannian metric $g_p(y) := g_{ij}(p, y)dy^i \otimes dy^j$, which also induces a Riemannian metric $\dot{g}_p$ on $S_pM := \{y | y \in T_pM, F(y) = 1\}$. Let (r, θ) be the polar coordinate around p and write the volume form by $d\mu = \sigma_p(r, \theta)drd\theta$. Then we can give the volume comparison theorem as follows:

Theorem 3.1. *Let $(M, F, d\mu)$ be a forward complete Finsler n-manifold with arbitrary volume form. If the weighted Ricci curvature* $\mathrm{Ric}_n \geq (n-1)k$*, then for some positive number $C_p := \lim_{r\to 0} \int_{S_pM} \frac{\sigma_p(r,\theta)}{r^{n-1}} d\theta$,*

$$\mathrm{vol}_F^{d\mu}(B_p^+(r)) \leq C_p \int_0^r (\boldsymbol{s}_k(t))^{n-1} dt, \quad 0 \leq r \leq \mathbf{i}_p, \tag{3.1}$$

where $B_p^+(r)$ denotes the forward geodesic ball centered at p of radius r, and $\mathbf{i}_p$ is the cut value of p. Moreover, the equality holds for $r_0 > 0$ if and only if for $\forall y \in S_pM$,

$$K(\dot{\gamma}_y(t); \cdot) = k, \qquad 0 \leq t \leq r_0 \leq \mathbf{i}_p,$$

where $\gamma_y(t)$ is the geodesic satisfying $\gamma_y(0) = p, \dot{\gamma}_y(0) = y$. In this case, under the polar coordinate (r, θ) of p, we have

$$g(\nabla r|_{(r,\theta)}) = dr \otimes dr + \mathbf{s}_k^2 \dot{g}_p(\theta),$$

where $\dot{g}_p$ denotes the Riemannian metric on S_pM.

Proof. From Lemma 2.4, we have

$$\frac{\mathrm{vol}_F^{d\mu}(B_p^+(r))}{\int_0^r (\mathbf{s}_k(t))^{n-1} dt}$$

is monotone decreasing on r. By

$$\lim_{r\to 0} \frac{\mathrm{vol}_F^{d\mu}(B_p^+(r))}{\int_0^r (\mathbf{s}_k(t))^{n-1} dt} = \lim_{r\to 0} \frac{\int_0^r \int_{S_pM} \sigma_p(r,\theta) d\theta dr}{\int_0^r (\mathbf{s}_k(t))^{n-1} dt}$$
$$= \lim_{r\to 0} \int_{S_pM} \frac{\sigma_p(r,\theta)}{r^{n-1}} d\theta = C_p,$$

we obtain (3.1).

Let $r(x) = d_F(p, x)$ be the distance function from p. Since $\mathrm{Ric}_n \geq (n-1)k$, by Laplacian comparison theorem (Lemma 2.3), we have

$$\Delta r \leq (n-1) \frac{\mathbf{s}_k'(r)}{\mathbf{s}_k(r)}, \tag{3.2}$$

which yields

$$\Delta r = \frac{\partial}{\partial r} \log \sigma_p \leq \frac{\partial}{\partial r} \log \mathbf{s}_k(r)^{n-1} := \frac{\partial}{\partial r} \log \tilde{\sigma},$$

where $\tilde{\sigma} := \mathbf{s}_k(r)^{n-1}$. Define $f(r) = \frac{\sigma_p(r,\theta)}{\tilde{\sigma}(r)}$. Then

$$f'(r) = \frac{\sigma_p' \tilde{\sigma} - \sigma_p \tilde{\sigma}'}{\tilde{\sigma}^2} = \frac{\sigma_p}{\tilde{\sigma}} \frac{\partial}{\partial r} (\log \sigma_p - \log \tilde{\sigma}) \leq 0.$$

Hence, $f(r)$ is monotone decreasing on r. As a result,

$$\frac{\sigma_p(R,\theta)}{\tilde{\sigma}(R)} \leq \frac{\sigma_p(r,\theta)}{\tilde{\sigma}(r)}, \quad r \leq R.$$

Assume that the equality holds in (3.1). That is,

$$\int_0^r \int_{S_pM} \sigma_p(\rho,\theta) d\theta d\rho = C_p \int_0^r \tilde{\sigma}(\rho) d\rho, \qquad r \leq \mathbf{i}_p.$$

Differentiating it with respect to r on both sides gives

$$\int_{S_pM} \frac{\sigma_p(r,\theta)}{\tilde{\sigma}(r)} d\theta = C_p, \qquad r \leq \mathbf{i}_p.$$

By the monotonicity of $f(r)$, we deduce that

$$C_p = \int_{S_pM} \frac{\sigma_p(R,\theta)}{\tilde{\sigma}(R)} d\theta \leq \int_{S_pM} \frac{\sigma_p(r,\theta)}{\tilde{\sigma}(r)} d\theta = C_p, \quad r \leq R \leq \mathbf{i}_p,$$

which implies

$$\frac{\sigma_p(R,\theta)}{\tilde{\sigma}(R)} = \frac{\sigma_p(r,\theta)}{\tilde{\sigma}(r)}, \quad r \leq R \leq \mathbf{i}_p.$$

Thus the equality holds in (3.2), which gives

$$\frac{\partial}{\partial r}(\Delta r) + \frac{(\Delta r)^2}{n-1} = -(n-1)k. \tag{3.3}$$

Let $S_p(r(x))$ be the forward geodesic sphere of radius $r(x)$ centered at p. Choosing the local $g_{\nabla r}$-orthonormal frame $E_1, \cdots, E_{n-1}$ of $S_p(r(x))$ near x, we get local vector fields $E_1, \cdots, E_{n-1}, E_n = \nabla r$ by parallel transport along geodesic rays. Thus, it follows from [7] that

$$\frac{\partial}{\partial r} \mathrm{tr}_{\nabla r} H(r) = -\mathrm{Ric}(\nabla r) - \sum_{i,j} [H(r)(E_i, E_j)]^2, \tag{3.4}$$

where $H(r)$ is the Hessian of the distance function r. On the other hand, we also have (see [7])

$$\Delta r = \mathrm{tr}_{\nabla r} H(r) - S(\nabla r) = \mathrm{tr}_{\nabla r} H(r). \tag{3.5}$$

Therefore, by (3.4) and (3.5), we obtain

$$\begin{aligned} -(n-1)k &= \frac{\partial}{\partial r}(\Delta r) + \frac{(\Delta r)^2}{n-1} \\ &= \frac{\partial}{\partial r}(\mathrm{tr}_{\nabla r} H(r)) + \frac{1}{n-1}(\mathrm{tr}_{\nabla r} H(r))^2 \\ &\leq \frac{\partial}{\partial r} \mathrm{tr}_{\nabla r} H(r) + \sum_{i,j} [H(r)(E_i, E_j)]^2 \\ &= -\mathrm{Ric}(\nabla r) = -\mathrm{Ric}_n(\nabla r) \\ &\leq -(n-1)k. \end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\sum_{i,j} [H(r)(E_i, E_j)]^2 = \frac{1}{n-1}(\mathrm{tr}_{\nabla r} H(r))^2,$$

which means

$$\begin{aligned} \nabla^2 r(E_i, E_j) :=& H(r)(E_i, E_j) \\ =& \begin{cases} \frac{\mathrm{tr}_{\nabla r} H(r)}{n-1} = \frac{\Delta r}{n-1} = \mathbf{ct}_k(r), & i = j < n, \\ 0, & i \neq j, \end{cases} \end{aligned} \tag{3.7}$$

where $\mathbf{ct}_k(r) := \frac{\mathbf{s}_k'(r)}{\mathbf{s}_k(r)}$. Next we shall compute the flag curvature. From (3.7), we know that $\{E_i\}_{i=1}^{n-1}$ are $(n-1)$ eigenvectors of $\nabla^2 r$. That is,

$$D_{E_i}^{\nabla r}\nabla r = \mathbf{ct}_k(r)E_i, \qquad i = 1, \cdots, n-1.$$

Noticed that ∇r is a geodesic field of (M, F). Therefore, the flag curvature $K(\nabla r; \cdot)$ equals to the sectional curvature of the weighted Riemannian manifold $(M, g_{\nabla r})$. Note that $\{E_i\}_{i=1}^{n-1}$ are $(n-1)$ eigenvectors of $\nabla^2 r$ and parallel along the geodesic ray. By a straightforward computation, we get, for $1 \leq i \leq n-1$,

$$\begin{aligned}
K(\nabla r; E_i) &= R^{\nabla r}(E_i, \nabla r, E_i, \nabla r) = g_{\nabla r}(R^{\nabla r}(E_i, \nabla r)\nabla r, E_i) \\
&= g_{\nabla r}(D_{E_i}^{\nabla r} D_{\nabla r}^{\nabla r}\nabla r - D_{\nabla r}^{\nabla r} D_{E_i}^{\nabla r}\nabla r - D_{[E_i,\nabla r]}^{\nabla r}\nabla r, E_i) \\
&= -g_{\nabla r}(D_{\nabla r}^{\nabla r}(\mathbf{ct}_k(r)E_i) + D_{D_{E_i}^{\nabla r}\nabla r - D_{\nabla r}^{\nabla r}E_i}^{\nabla r}\nabla r, E_i) \\
&= -g_{\nabla r}(\mathbf{ct}_k'(r)E_i + D_{\mathbf{ct}_k(r)E_i}^{\nabla r}\nabla r, E_i) \\
&= -\mathbf{ct}_k'(r) - \mathbf{ct}_k(r) g_{\nabla r}(D_{E_i}^{\nabla r}\nabla r, E_i) \\
&= -\mathbf{ct}_k'(r) - \mathbf{ct}_k(r)^2 \\
&= k.
\end{aligned}$$

We are now to prove that if $K(\dot{\gamma}_y(t); \cdot) = k$, then the equality holds in (3.1). Under the polar coordinate of p, we have $(r, \theta) = (r(q), \theta^1(q), \cdots, \theta^{n-1}(q))$ for $q \in \mathcal{D}_p \backslash \{p\}$, where

$$r(q) = F(y), \quad \theta^\alpha(q) = \bar{\theta}^\alpha\left(\frac{y}{F(y)}\right), \quad y = \exp_p^{-1}(q).$$

Then

$$\frac{\partial}{\partial \theta^\alpha}|_q = (d\exp_p)_y \left(r \frac{\partial}{\partial \bar{\theta}^\alpha} \right).$$

So, for $y \in S_pM$, $\frac{\partial}{\partial \theta^\alpha}$ can be viewed as a Jacobi field on $\gamma_y(t)$, and

$$\lim_{r \to 0} \frac{1}{r} \frac{\partial}{\partial \theta^\alpha}|_q = \frac{\partial}{\partial \bar{\theta}^\alpha}|_y.$$

Since $K(\nabla r; \cdot) = k$, $J_\alpha(t) = \frac{\partial}{\partial \theta^\alpha}|_{(t,y)} = \mathbf{s}_k(t) E_\alpha(t)$ is the Jacobi field satisfying $J(0) = 0$, where $E_\alpha(t)$ is a parallel vector field on $\gamma_y(t)$, and $E_\alpha(0) = \frac{\partial}{\partial \bar{\theta}^\alpha}|_y$. By Gauss lemma, $g_{\nabla r}(\nabla r, \frac{\partial}{\partial \theta^\alpha}) = 0$. Therefore,

$$g(\nabla r|_{(r,\theta)}) = dr \otimes dr + \mathbf{s}_k^2 \dot{g}_p(\theta).$$

Since $S(\dot{\gamma}_y(t)) = 0$, we have

$$\begin{aligned}
0 =& \frac{d}{dt}\tau(t) = \frac{d}{dt}\log\frac{\sqrt{\det g_{ij}(t)}}{\sigma_p(t)} = \frac{d}{dt}\log\frac{\mathbf{s}_k(t)^{n-1}\sqrt{\det(\dot{g}_p(\theta)_{\alpha\beta})}}{\sigma_p(t)} \\
=& \frac{d}{dt}\log\frac{\mathbf{s}_k(t)^{n-1}}{\sigma_p(t)} = \frac{d}{dt}\log\frac{\tilde{\sigma}(t)}{\sigma_p(t)},
\end{aligned}$$

which means that

$$\frac{\sigma_p(R,\theta)}{\tilde{\sigma}(R)} = \frac{\sigma_p(r,\theta)}{\tilde{\sigma}(r)}, \quad 0 \le r \le R \le \mathbf{i}_p.$$

Thus, we obtain

$$\begin{aligned}\int_0^R \int_{S_pM} \sigma_p(R,\theta)d\theta dR &= \int_0^R \int_{S_pM} \tilde{\sigma}(R)\frac{\sigma_p(r,\theta)}{\tilde{\sigma}(r)}d\theta dR \\ &= \int_0^R \tilde{\sigma}(R)dR \int_{S_pM} \frac{\sigma_p(r,\theta)}{\tilde{\sigma}(r)}d\theta \\ &= \int_0^R \tilde{\sigma}(R)dR \lim_{r\to 0}\int_{S_pM} \frac{\sigma_p(r,\theta)}{\tilde{\sigma}(r)}d\theta \\ &= C_p \int_0^R \tilde{\sigma}(R)dR, \quad 0 \le R \le \mathbf{i}_p.\end{aligned}$$

That is,

$$\mathrm{vol}_F^{d\mu}(B^+(r)) = C_p \int_0^r (\mathbf{s}_k(t))^{n-1}dt, \quad 0 \le r \le \mathbf{i}_p.$$

□

From Theorem 3.1, it is easy to obtain the following:

Corollary 3.2. *Let $(M, F, d\mu)$ be a forward complete Finsler n-manifold with arbitrary volume form. If the weighted Ricci curvature* $\mathrm{Ric}_n \ge (n-1)k$*, and there exists some positive number C such that* $\lim_{r\to 0}\int_{S_pM}\frac{\sigma_p(r,\theta)}{r^{n-1}}d\theta = C$ *for* $\forall p \in M$*, then*

$$\mathrm{vol}_F^{d\mu}(B_p^+(r)) = C\int_0^r (s_k(t))^{n-1}dt, \quad 0 \le r \le \mathbf{i}_p, \forall p \in M$$

if and only if $K \equiv k$*.*

4. Rigidity results on Finsler manifolds

Recall that, for a Randers sphere $(\mathcal{S}^n(\frac{1}{\sqrt{k}}))$, the metric F is expressed by (see [10])

$$F = \frac{\sqrt{\lambda \mathrm{g}^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \quad \lambda = 1 - \|W\|_{\mathrm{g}}^2,$$

where g is the standard sphere metric, W is a Killing vector field on $\mathbb{S}^n(\frac{1}{\sqrt{k}})$. Moreover, we have (see [10])

$$\mathrm{vol}_F^{d\mu}(\mathcal{S}^n(\frac{1}{\sqrt{k}})) = \mathrm{vol}_{\mathrm{g}}(\mathbb{S}^n(\frac{1}{\sqrt{k}})); \quad Diam(\mathcal{S}^n(\frac{1}{\sqrt{k}}), F) = \frac{\pi}{\sqrt{k}}.$$

In what follows, we show that the properties above still hold for a general Finsler sphere.

Proposition 4.1. *On a Finsler sphere* $\mathfrak{S}^n(\frac{1}{\sqrt{k}})$*, we have*

(1) $\mathrm{vol}_F^{d\mu}(\mathfrak{S}^n(\frac{1}{\sqrt{k}})) = \mathrm{vol}_g(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$;
(2) $Diam(\mathfrak{S}^n(\frac{1}{\sqrt{k}})) = \frac{\pi}{\sqrt{k}}$.

Proof. Since $d\mu$ is the Busemann-Hausdorff volume form, the constant C_p in (3.1) is

$$C_p = \lim_{r\to 0}\int_{S_pM}\frac{\sigma_p(r,\theta)}{r^{n-1}}d\theta = \lim_{r\to 0}\frac{\mathrm{vol}_F^{d\mu}(B_p^+(r))}{\int_0^r(\mathbf{s}_k(t))^{n-1}dt} = \mathrm{vol}(\mathbb{S}^{n-1}).$$

Thus, from Corollary 3.2, we have

$$\mathrm{vol}_F^{d\mu}(\mathfrak{S}^n(\frac{1}{\sqrt{k}})) = \mathrm{vol}_g(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$$

Now fix $p \in \mathfrak{S}^n(\frac{1}{\sqrt{k}})$. Using $K = k$ and Theorem 3.1 in [5], there exists $q \in \mathfrak{S}^n(\frac{1}{\sqrt{k}})$ such that

$$\exp_p(\frac{\pi}{\sqrt{k}}\xi) = q, \qquad \forall\xi \in S_p(\mathfrak{S}^n(\frac{1}{\sqrt{k}})),$$

where $S_p(\mathfrak{S}^n(\frac{1}{\sqrt{k}})) := \{v|v \in T_p(\mathfrak{S}^n(\frac{1}{\sqrt{k}})), F(v) = 1\}$. From the proof of the volume comparison theorem (Theorem 3.1),

$$\mathrm{vol}_F^{d\mu}(B_p^+(r)) \le \sigma_n(r),$$

where $\sigma_n(r)$ denotes the volume of the metric ball of radius r in $\mathbb{S}^n(\frac{1}{\sqrt{k}})$. The equality holds if and only if $B_p^+(r) \subset \mathcal{D}_p$, i.e., $\mathbf{i}_p \ge r$. By the Bonnet-Myers theorem, $Diam(\mathfrak{S}^n(\frac{1}{\sqrt{k}})) \le \frac{\pi}{\sqrt{k}}$, which means $\overline{B_p^+(\frac{\pi}{\sqrt{k}})} = \mathfrak{S}^n(\frac{1}{\sqrt{k}})$. Therefore,

$$\mathrm{vol}_F^{d\mu}(B_p^+(\frac{\pi}{\sqrt{k}})) = \mathrm{vol}_F^{d\mu}(\mathfrak{S}^n(\frac{1}{\sqrt{k}})) = \mathrm{vol}_g(\mathbb{S}^n(\frac{1}{\sqrt{k}})) = \sigma_n(\frac{\pi}{\sqrt{k}}).$$

We deduce that $\mathbf{i}_p \ge \frac{\pi}{\sqrt{k}}$, which yields $d_F(p,q) = \frac{\pi}{\sqrt{k}}$. □

Theorem 4.2. *Let $(M, F, d\mu)$ be a complete connected Finsler n-manifold with Busemann-Hausdorff volume form. If the weighted Ricci curvature* $\mathrm{Ric}_n \ge (n-1)k > 0$, *and* $\mathrm{vol}_F^{d\mu}(M) = \mathrm{vol}_g(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$*, then (M, F) is isometric to a Finsler sphere.*

Proof. Since $d\mu$ is the Busemann-Hausdorff volume form, the constant C_p in (3.1) is $C_p = \mathrm{vol}(\mathbb{S}^{n-1})$. Then it follows Theorem 3.1 that, for any $p \in M$,

$$\mathrm{vol}_F^{d\mu}(B_p^+(r)) \le \mathrm{vol}(\mathbb{S}^{n-1})\int_0^r(\mathbf{s}_k(t))^{n-1}dt := \sigma_n(r),$$

where $\sigma_n(r)$ denotes the volume of the metric ball of radius r in $\mathbb{S}^n(\frac{1}{\sqrt{k}})$. The equality holds if and only if $B_p^+(r) \subset \mathcal{D}_p$. That is, $\mathbf{i}_p \ge r$. Since

$$\mathrm{vol}_F^{d\mu}(M) = \mathrm{vol}_g(\mathbb{S}^n(\frac{1}{\sqrt{k}})),$$

we deduce that, for $\forall p \in M$ and $\forall r \leq \mathbf{i}_p$,

$$\mathrm{vol}_F^{d\mu}(B_p^+(r)) = \sigma_n(r).$$

Then, from Corollary 3.2, we have

$$K \equiv k.$$

This completes the proof. □

To prove Theorem 1.1, we further need the following theorems.

Theorem 4.3. *[10] Let $(M, F, d\mu)$ be a complete connected Finsler n-manifold with the Busemann-Hausdorff volume form. If the weighted Ricci curvature satisfies $\mathrm{Ric}_n \geq (n-1)k > 0$, and Diam$(M) = \frac{\pi}{\sqrt{k}}$, then (M, F) is isometric to a Finsler sphere.*

Theorem 4.4. *[10] Let $(M, F, d\mu)$ be a complete connected Finsler n-manifold with the Busemann-Hausdorff volume form. If the weighted Ricci curvature satisfies $\mathrm{Ric}_n \geq (n-1)k > 0$, then the first eigenvalue of Finsler-Laplacian $\lambda_1 = nk$ if and only if (M, F) is isometric to a Finsler sphere.*

Proof of Theorem 1.1.

It follows from Proposition 4.1, Theorems 4.2–4.4 directly.

From the proof of Theorem 3.1, we know that the key step is

$$\Delta r = (n-1)\frac{\mathbf{s}_k'(r)}{\mathbf{s}_k(r)}.$$

Therefore, we get another equivalent condition.

Theorem 4.5. *Let $(M, F, d\mu)$ be a complete connected Finsler n-manifold with Busemann-Hausdorff volume form. If the weighted Ricci curvature $\mathrm{Ric}_n \geq (n-1)k > 0$, then the following conditions are equivalent:*

(1) $(M, F, d\mu)$ is a Finsler sphere $\mathfrak{S}^n(\frac{1}{\sqrt{k}})$;
(2) $\Delta r = (n-1)\cot r$ for any distance function $r(x) = d_F(p, x), \forall p \in M$.

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Conflict of interest

The author declares that they have no conflicts of interest.

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