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Research article

Conversion calculation method of multivariate integrals

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Abstract: The new schemes of calculation of double integrals and triple integrals are proposed in this paper. The formulas in which the double integral is converted into a line integral with respect to the arc length, and the triple integral is converted into a surface integral with respect to the area or a line integral with respect to the arc length are given separately. The effectiveness of the proposed methods is verified by several examples. Under certain conditions, these methods become the normal iterated integrals in Cartesian coordinate system or polar coordinate system, and the commonly used triple iterated integrals in Cartesian coordinate system, Cylindrical coordinate system or Spherical coordinate system. The transformation calculation method promoted in this paper points out the intrinsic relationship among double integral, triple integral, line integral and surface integral, which further enriches the theories of multivariate integrals.

Keywords: double integral; line integral with respect to arc length; triple integral; surface integral with respect to area; conversion calculation method
Mathematics Subject Classification: 26B20, 97I50

1. Introduction

Multivariable calculus [1] is a rich and fascinating subject. Multivariate integrals are widely used in various disciplines. For example, in optimization and (multidimensional) optimal control theory, authors [2, 3] studied the optimization of some multiple or curvilinear integral functionals subject to ODEs, PDEs or isoperimetric constraints. In 2018, Mititelu and Treanţă [4] proposed optimality conditions in multiobjective control problems which involve multiple integrals. In the computation method of multivariate integrals, we often convert the double integral into iterated integrals in a Cartesian coordinate system by Fubini's Theorem or a polar coordinate system [5]. The triple integral is converted into iterated integrals in a Cartesian coordinate system, or a cylindrical coordinate system, or a spherical coordinate system. In general, a line integral is converted into a definite integral, and a surface integral is converted into a double integral. In addition, Green's formula, Gauss's formula and Stokes's formula could be applied to perform the calculation [6].

In the last twenty years, more researchers worked on the conversion relationships for several types of multivariate integrals. For example, the surface integral with respect to area is converted into the line integral with respect to the arc length in [7]. In [8], Li and Shi proposed a method of computing the surface integral by curvilinear integral in the special case when the projection of the surface on the coordinate plane was a curve. In the current paper, we propose new transformation algorithms for multivariate integrals, which further enhances the computation methods of multivariate integrals.

2. New methods for calculating double integrals

The main idea is to use the change of variables formula to transform double and triple integrals to iterated integrals which can be done more easily.

Theorem 1. (1) If the bounded closed region D is generated by a family of planar smooth curves

$$L_t : y = y(x,t), a \le t \le b, x_1(t) \le x \le x_2(t), y'_t \ne 0$$

with no double point (i.e., a point traced out twice as a closed curve is traversed), and the function $f(x,y)$ is continuous on the closed region D, then

$$\iint\limits_D f(x,y)\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}t \int_{x_1(t)}^{x_2(t)} f(x, y(x,t)) \left|y'_t\right| \mathrm{d}x \tag{2.1}$$

$$\iint\limits_D f(x,y)\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}t \int\limits_{L_t} f(x,y)\frac{\left|y'_t\right|}{\sqrt{1+y'^2_x}}\,\mathrm{d}s \tag{2.2}$$

where the symbol 'ds' represents the arclength differential.

(2) If the bounded closed region D is generated by a family of planar smooth curves

$$L_t : x = x(y,t), a \le t \le b, y_1(t) \le y \le y_2(t), x'_t \ne 0$$

with no double points, and the function $f(x,y)$ is continuous on the closed region D , then

$$\iint\limits_D f(x,y)\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}t \int_{y_1(t)}^{y_2(t)} f(x(y,t), y) \left|x'_t\right| \mathrm{d}y \tag{2.3}$$

$$\iint\limits_D f(x,y)\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}t \int\limits_{L_t} f(x,y)\frac{\left|x'_t\right|}{\sqrt{1+x'^2_y}}\,\mathrm{d}s \tag{2.4}$$

Proof: Let transformation $y = y(x,t)$, which transforms the D on the xOy plane to

$$D : a \le t \le b, x_1(t) \le x \le x_2(t)$$

on the tOx plane. Notice

$$\frac{\partial(x,y)}{\partial(x,t)} = \begin{vmatrix} 1 & 0 \\ y'_x & y'_t \end{vmatrix} = y'_t \ne 0.$$

According to the integration by substitution, the formula (2.1) holds. On the other hand, since $\mathrm{d}s = \sqrt{1+y_x'^2}\,\mathrm{d}x$ on the curve L_t,

$$\int_{L_t} f(x,y)\frac{\left|y_t'\right|}{\sqrt{1+y_x'^2}}\,\mathrm{d}s = \int_{x_1(t)}^{x_2(t)} f(x,y(x,t))\left|y_t'\right|\mathrm{d}x.$$

The formula (2.2) can be obtained from (2.1). In the same way we can get (2.3) and (2.4).

Example 1. Calculate the double integral $I = \iint\limits_D \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, where the closed region D is bounded by three curves $y = \sqrt{1-x^2}$, $x = \sqrt{y}$ and $y = \sqrt{-x}$.

Method 1. Take a family of concentric circles arcs

$$L_t : y = \sqrt{t^2-x^2}, \frac{1}{2} - \sqrt{t^2+\frac{1}{4}} \le x \le \sqrt{\sqrt{t^2+\frac{1}{4}}-\frac{1}{2}},$$

where $0 \le t \le 1$ as shown in Figure 1. Obviously, the central angle of each arc segment is $\frac{\pi}{2}$, which results in the length of arc L_t is $\frac{\pi t}{2}$. It follows from (2.2) that

$$I = \int_0^1 \mathrm{d}t \int_{L_t} \sqrt{x^2+y^2}\,\mathrm{d}s = \int_0^1 \mathrm{d}t \int_{L_t} t\,\mathrm{d}s = \int_0^1 t\cdot\frac{1}{2}\pi t\,\mathrm{d}t = \frac{\pi}{6}.$$

Method 2. Label regions D_1, D_2 and D_3 as shown in Figure 2. Rotate D_1 clockwise by $\frac{\pi}{2}$ to get D_3. It is not difficult to find that $D = D_1 \cup D_2$ and $D_2 \cup D_3$: $x^2+y^2 \le 1$, $x \ge 0$ $y \ge 0$. Make a rotation transformation $\begin{cases} x = -v, \\ y = u, \end{cases}$, then

$$\iint\limits_{D_1} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{D_3} \sqrt{u^2+v^2}\,\mathrm{d}u\,\mathrm{d}v = \iint\limits_{D_3} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y.$$

So,

$$\begin{aligned} I &= \iint\limits_{D_1} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{D_2} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{D_3} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{D_2} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint\limits_{D_2\cup D_3} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^1 r\cdot r\,\mathrm{d}r = \frac{\pi}{6}. \end{aligned}$$

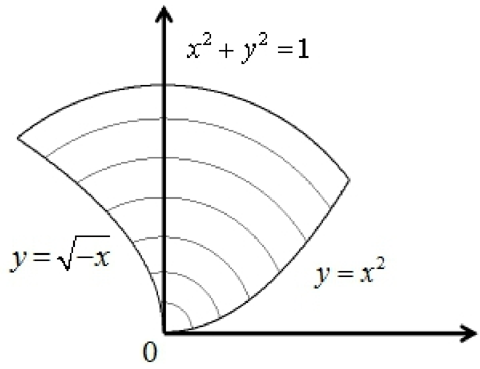

Figure 1. A family of concentric circles arcs.

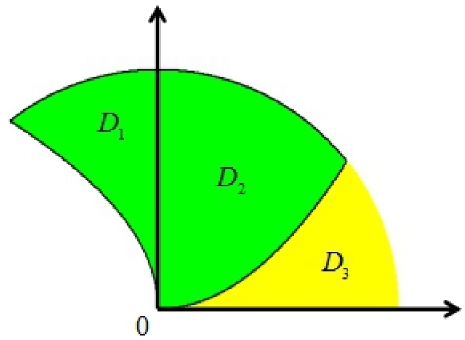

Figure 2. The division of regions.

Example 2. Calculate the double integral $I = \iint\limits_D \frac{1}{x}\,\mathrm{d}x\,\mathrm{d}y$, where the closed region D is in the first quadrant and is bounded by two curves $xy = 2$, $xy = 12$, and two lines $y = x+1$, $y = \frac{1}{4}(x+2)$, as shown in Figure 3.

Since the shape of the integral region D is very complicated (at least it needs to be represented as two sub-regions), as shown in Figure 4, the method of directly converting to the iterated integral in the Cartesian coordinate system or the polar coordinate system is quite cumbersome. Moreover, it is difficult to calculate with the integration by substitution.

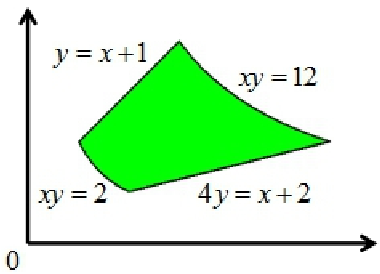

Figure 3. The closed region D.

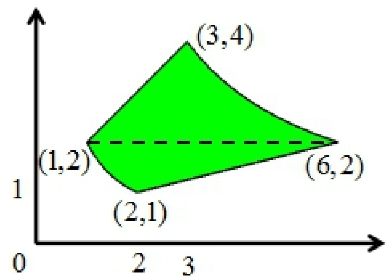

Figure 4. The division of regions.

The methods developed in Theorem 1 work well for this kind of regions.

Solution: Let's take a family of hyperbolics

$$L_t : y = \frac{t}{x}, \frac{\sqrt{4t+1}-1}{2} \leq x \leq \sqrt{4t+1} - 1 (2 \leq t \leq 12)$$

as shown in Figure 5.

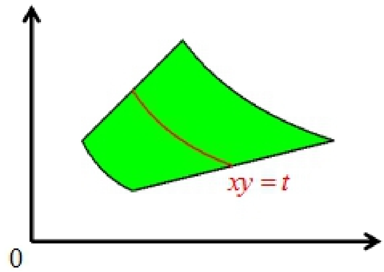

Figure 5. A family of hyperbolics.

It follows from (2.1) that

$$I = \int_2^{12} \mathrm{d}t \int_{\frac{\sqrt{4t+1}-1}{2}}^{\sqrt{4t+1}-1} \frac{1}{x} \cdot \frac{1}{x} \,\mathrm{d}x = \int_2^{12} \frac{1}{\sqrt{4t+1}-1} \,\mathrm{d}t = 2 + \frac{1}{2} \ln 3.$$

Example 3. Calculate the double integral $I = \iint\limits_D \frac{\sqrt{x^2+y^2}}{x} \,\mathrm{d}\sigma$, where the closed region D is bounded by four lines $x = 1, x = 2, y = 0$ and $y = x$, as shown in Figure 6.

If we adopt the method of converting to the iterated integral in the polar coordinate system, the integral will eventually be transformed into the definite integral $\int_0^{\pi/4} \sec^3 x \,\mathrm{d}x$. And the calculation process is complicated.

Solution: Let's take a family of lines

$$L_t : y = tx, 1 \le x \le 2 (0 \le t \le 1)$$

as shown in Figure 7. It follows from (2.1) that

$$I = \int_0^1 \mathrm{d}t \int_1^2 \frac{\sqrt{x^2 + t^2 x^2}}{x} \cdot |x| \mathrm{d}x = \int_0^1 \sqrt{1+t^2} \,\mathrm{d}t \cdot \int_1^2 x \,\mathrm{d}x = \frac{3}{4}(\sqrt{2} + \ln(1+\sqrt{2}).$$

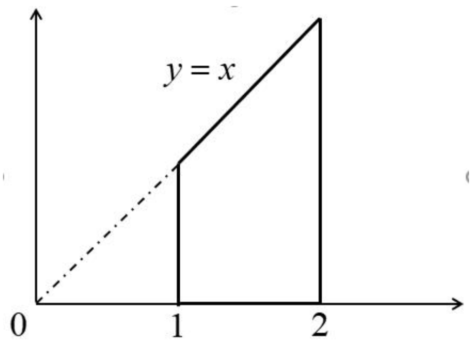

Figure 6. The closed region D.

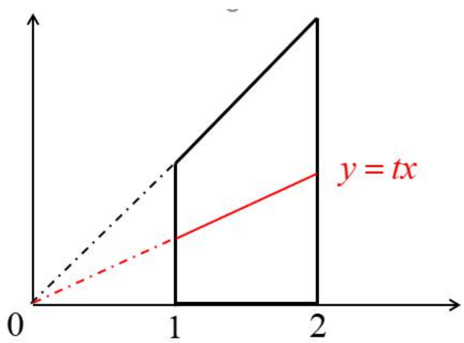

Figure 7. A family of lines.

It can be seen from the above three examples that the calculation method of double integral varies. The calculation method proposed by Theorem 1 is an important component. This method can simplify the calculation process, which has certain theoretical significance for studying the multivariate integrals.

According to the differential method of the implicit function, the following inference is easily obtained by Theorem 1.

Corollary 1. The bounded closed region D is generated by a family of planar smooth curves

$$L_t : F(x, y, t) = 0, a \leq t \leq b \left(F_t' \neq 0, F_x'^2 + F_y'^2 \neq 0 \right)$$

with no double points. If function $f(x, y)$ is continuous on D , then

$$\iint_D f(x, y)\mathrm{d}x\, \mathrm{d}y = \int_a^b \mathrm{d}t \int_{L_t} f(x, y) \frac{\left| F_t' \right|}{\sqrt{F_x^2 + F_y^2}}\, \mathrm{d}s.$$

In addition, in Theorem 1, if the family of curves take the family of vertical lines, the family of horizontal lines and the family of lines passing through the origin respectively, three common calculation methods of double integral can be obtained.

Corollary 2. (1) The function $f(x, y)$ is continuous on the bounded closed region

$$D_x = \{(x, y) \mid a \leq x \leq b, y_1(x) \leq y \leq y_2(x)\},$$

then

$$\iint_{D_x} f(x, y)\mathrm{d}x\, \mathrm{d}y = \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y. \tag{2.5}$$

(2) The function $f(x, y)$ is continuous on the bounded closed region

$$D_y = \{(x, y) \mid c \leq y \leq d, x_1(y) \leq x \leq x_2(y)\},$$

then

$$\iint\limits_{D_y} f(x, y)\mathrm{d}x\, \mathrm{d}y = \int_c^d \mathrm{d}y \int_{x_1(y)}^{x_2(y)} f(x, y)\mathrm{d}x. \tag{2.6}$$

(3) The function $f(x, y)$ is continuous on the bounded closed region D . If D can be expressed as

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta)\}$$

with the polar transformation $x = r \cos\theta, y = r \sin\theta$, then

$$\iint\limits_D f(x, y)\mathrm{d}x\, \mathrm{d}y = \int_\alpha^\beta \mathrm{d}\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) \cdot r\, \mathrm{d}r.$$

Proof: Taking the family of vertical lines

$$L_t : x = t, a \leq t \leq b, y_1(t) \leq y \leq y_2(t),$$

we can know from (2.3)

$$\iint\limits_D f(x, y)\mathrm{d}x\, \mathrm{d}y = \int_a^b \mathrm{d}t \int_{y_1(t)}^{y_2(t)} f(t, y) \cdot 1\, \mathrm{d}y = \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y.$$

That is, the formula (2.5) is established. Similarly, taking the family of horizontal lines

$$L_t : y = t, c \le t \le d, x_1(t) \le x \le x_2(t),$$

we can get (2.6) from (2.1).

If we use

$$L_\theta : x = r\cos\theta, y = r\sin\theta,$$

where

$$\alpha \le \theta \le \beta, r_1(\theta) \le r \le r_2(\theta),$$

we can get $y = x\tan\theta$ and

$$y'_\theta = x\sec^2\theta = \frac{r}{\cos\theta}.$$

It follows that from (2.1)

$$\begin{aligned}\iint\limits_D f(x,y)\mathrm{d}x\,\mathrm{d}y &= \int_\alpha^\beta \mathrm{d}\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\cdot\frac{r}{|\cos\theta|}\cdot|\cos\theta|\mathrm{d}r\\ &= \int_\alpha^\beta \mathrm{d}\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\cdot r\,\mathrm{d}r.\end{aligned}$$

3. New techniques for calculating triple integrals

Theorem 2. If the bounded closed region Ω is generated by a family of smooth surfaces

$$\Sigma_t : z = z(x,y,t), a \le t \le b, (x,y) \in D_{xy}(t)$$

with no double points, where $D_{xy}(t)$ is the projection area of the surface Σ_t on xOy coordinate plane and $z'_t \ne 0$, and the function $f(x,y,z)$ is continuous on the closed region Ω , then

$$\iiint\limits_\Omega f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_a^b \mathrm{d}t \iint\limits_{D_{xy}(t)} f(x,y,z(x,y,t))\left|z'_t\right|\mathrm{d}x\,\mathrm{d}y, \tag{3.1}$$

$$\iiint\limits_\Omega f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_a^b \mathrm{d}t \iint\limits_{\Sigma_t} f(x,y,z)\frac{\left|z'_t\right|}{\sqrt{1+z'^2_x+z'^2_y}}\,\mathrm{d}S. \tag{3.2}$$

Proof: Let $\begin{cases} x = x, \\ y = y, \\ z = z(x,y,t), \end{cases}$ then the Jacobian determinant

$$\frac{\partial(x,y,z)}{\partial(x,y,t)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z'_x & z'_y & z'_t \end{vmatrix} = z'_t \ne 0.$$

And it transforms Ω in $O-xyz$ to

$$\Omega : a \le t \le b, (x,y) \in D_{xy}(t)$$

in $O-xyt$. By substitution method of triple integral, we can obtain formula (3.1). Since $\mathrm{d}S = \sqrt{1+z_x'^2+z_y'^2}\,\mathrm{d}x\,\mathrm{d}y$ on Σ_t , we know

$$\iint\limits_{\Sigma_t} f(x,y,z)\frac{\left|z_t'\right|}{\sqrt{1+z_x'^2+z_y'^2}}\,\mathrm{d}S = \iint\limits_{D_{xy}(t)} f(x,y,z(x,y,t))\left|z_t'\right|\mathrm{d}x\,\mathrm{d}y.$$

Substituting the above formula into (3.1) yields (3.2).

Corollary 3. (1) Let the bounded closed region Ω be generated by a family of smooth surfaces

$$\Sigma_t : x = x(y,z,t), a \le t \le b, (y,z) \in D_{yz}(t),$$

with no double points, where $D_{yz}(t)$ is the projection area of the surface Σ_t on yOz coordinate plane and $x_t' \neq 0$. If the function $f(x,y,z)$ is continuous on the closed region Ω , then

$$\begin{aligned} \iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_a^b \mathrm{d}t \iint\limits_{D_{yz}(t)} f(x(y,z,t),y,z)\left|x_t'\right|\mathrm{d}y\,\mathrm{d}z, \\ \iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_a^b \mathrm{d}t \iint\limits_{\Sigma_t} f(x,y,z)\frac{\left|x_t'\right|}{\sqrt{1+x_y'^2+x_z'^2}}\,\mathrm{d}S. \end{aligned} \tag{3.3}$$

(2) Let the bounded closed region Ω be generated by a family of smooth surfaces

$$\Sigma_t : y = y(x,z,t), a \le t \le b, (x,z) \in D_{xz}(t),$$

with no double points, where $D_{xz}(t)$ is the projection area of the surface Σ_t on xOz coordinate plane and $y_t' \neq 0$. If the function $f(x,y,z)$ is continuous on the closed region Ω , then

$$\begin{aligned} \iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_a^b \mathrm{d}t \iint\limits_{D_{xz}(t)} f(x,y(x,z,t),z)\left|y_t'\right|\mathrm{d}x\,\mathrm{d}z, \\ \iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_a^b \mathrm{d}t \iint\limits_{\Sigma_t} f(x,y,z)\frac{\left|y_t'\right|}{\sqrt{1+y_x'^2+y_z'^2}}\,\mathrm{d}S. \end{aligned} \tag{3.4}$$

Example 4. Calculate the triple integral $I = \iiint\limits_{\Omega}(x+y+z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where

$$\Omega : \frac{1}{2} \le x+y+z \le 1, x \ge 0, y \ge 0, z \ge 0,$$

as shown in Figure 8.

Method 1. As shown in Figure 9, Taking a family of triangle planes

$$\Sigma_t : z = t-x-y, \frac{1}{2} \le t \le 1, 0 \le x \le 1, 0 \le y \le t-x,$$

then $\left|z_t'\right| = 1$, $\sqrt{1+z_x'^2+z_y'^2} = \sqrt{3}$ hold, and the area of Σ_t is $\frac{\sqrt{3}}{2}t^2$. Substituting the above formula into (3.2) gives

$$I = \int_{\frac{1}{2}}^{1} \mathrm{d}t \iint\limits_{\Sigma_t} (x+y+z)\frac{\mathrm{d}S}{\sqrt{3}} = \frac{1}{\sqrt{3}}\int_{\frac{1}{2}}^{1} t\,\mathrm{d}t \iint\limits_{\Sigma_t} \mathrm{d}S = \frac{1}{\sqrt{3}}\int_{\frac{1}{2}}^{1} t\cdot\frac{\sqrt{3}}{2}t^2\cdot\mathrm{d}t = \frac{15}{128}.$$

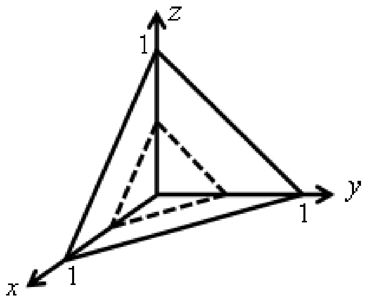

Figure 8. The closed region Ω.

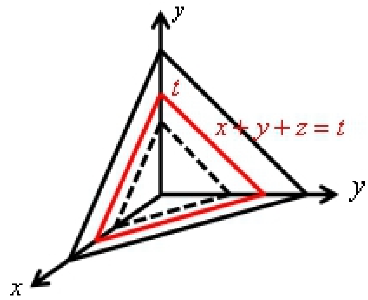

Figure 9. A family of triangle planes.

According to the additivity of the integral region, this integral can also be calculated as follows.
Method 2. Let

$$\Omega_1 = \{(x,y,z) \mid 0 \le z \le 1-x-y, 0 \le y \le 1-x, 0 \le x \le 1\},$$
$$\Omega_2 = \left\{(x,y,z) \mid 0 \le z \le \tfrac{1}{2}-x-y, 0 \le y \le \tfrac{1}{2}-x, 0 \le x \le \tfrac{1}{2}\right\},$$

then $\Omega = \Omega_1 - \Omega_2$. According to the symmetry of the triple integral, we have

$$\begin{aligned} I &= \iiint\limits_{\Omega} 3x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{\Omega_1} 3x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \iiint\limits_{\Omega_2} 3x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{1-x-y} 3x\,\mathrm{d}z - \iiint_0^{\frac{1}{2}} \mathrm{d}x \int_0^{\frac{1}{2}-x} \mathrm{d}y \int_0^{\frac{1}{2}-x-y} 3x\,\mathrm{d}z \\ &= \int_0^1 3x\,\mathrm{d}x \int_0^{1-x} (1-x-y)\mathrm{d}y - \int_0^{\frac{1}{2}} 3x\,\mathrm{d}x \int_0^{\frac{1}{2}-x} \left(\frac{1}{2}-x-y\right)\mathrm{d}y \\ &= \int_0^1 \frac{3}{2}\left(x-2x^2+x^3\right)\mathrm{d}x - \int_0^{\frac{1}{2}} \frac{3}{2}\left(\frac{1}{4}x-x^2+x^3\right)\mathrm{d}x \\ &= \frac{1}{8} - \frac{1}{128} = \frac{15}{128}. \end{aligned}$$

Example 5. Calculate the triple integral $I = \iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where

$$\Omega : x^2+y^2+z^2 \le 1, -z \le x+y \le 2z.$$

Method 1. Taking a family of spheres

$$\Sigma_t : z = \sqrt{t^2-x^2-y^2}, 0 \le t \le 1, -z \le x+y \le 2z,$$

then $\left|z_t'\right| = \frac{t}{\sqrt{t^2-x^2-y^2}}$, $\sqrt{1+z_x'^2+z_y'^2} = \frac{t}{\sqrt{t^2-x^2-y^2}}$ hold. In addition, the plane $x+y+z=0$ and the plane $2z-x-y=0$ are perpendicular to each other, so the area of Σ_t is πt^2 . Substituting the above results into (3.2) gives

$$I = \int_0^1 \mathrm{d}t \iint\limits_{\Sigma_t} \sqrt{x^2+y^2+z^2}\,\mathrm{d}S = \int_0^1 \mathrm{d}t \iint\limits_{\Sigma_t} t\,\mathrm{d}S = \int_0^1 t\cdot\pi t^2\,\mathrm{d}t = \pi\int_0^1 t^3\,\mathrm{d}t = \frac{\pi}{4}.$$

Method 2. Let a orthogonal transformation $u = \frac{1}{\sqrt{2}}(x-y), v = \frac{1}{\sqrt{3}}(x+y+z), w = \frac{1}{\sqrt{6}}(2z-x-y)$. And it transforms Ω in $O-xyz$ to

$$\Omega : u^2+v^2+w^2 \leq 1, v \geq 0, w \geq 0$$

in $O-uvw$, thus

$$I = \iiint\limits_{\Omega} \sqrt{u^2+v^2+w^2}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

Using spherical coordinates, we can obtain

$$I = \int_0^{\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 \rho \cdot \rho^2 \sin\varphi d\rho = \int_0^{\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \cdot \int_0^1 \rho^3 d\rho = \frac{\pi}{4}.$$

Example 6. Calculate the triple integral $I = \iiint\limits_{\Omega} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where

$$\Omega : 0 \leq z \leq \sqrt{x^2+y^2}, x^2+y^2 \leq 1, x \geq 0, y \geq 0.$$

Method 1. Taking a family of cylindrical surfaces

$$\Sigma_t : x^2+y^2 = t^2, x \geq 0, y \geq 0, 0 \leq z \leq t, 0 \leq t \leq 1$$

then the area of Σ_t is $\frac{1}{2}\pi t^2$. Because of $x = \sqrt{t^2-y^2}$, $\frac{|x'_t|}{\sqrt{1+x'^2_y+x'^2_z}} = 1$ holds. Substituting the above results into (3.3) gives

$$I = \int_0^1 \mathrm{d}t \iint\limits_{\Sigma_t} \sqrt{x^2+y^2}\,\mathrm{d}S = \int_0^1 \mathrm{d}t \iint\limits_{\Sigma_t} t\,\mathrm{d}S = \int_0^1 t \cdot \frac{1}{2}\pi t^2\,\mathrm{d}t = \frac{\pi}{8}.$$

Method 2. Taking a family of conic surfaces

$$\Sigma_t : z = t\sqrt{x^2+y^2}, x^2+y^2 \leq 1, x \geq 0, y \geq 0, 0 \leq t \leq 1.$$

then $z'_t = \sqrt{x^2+y^2}$ holds. The projection area on the xoy coordinate plane:

$$D_{xy}(t) : x^2+y^2 \leq 1, x \geq 0, y \geq 0.$$

Substituting the above results into (3.1) gives

$$I = \int_0^1 \mathrm{d}t \iint\limits_{D_{xy}(t)} \sqrt{x^2+y^2} \cdot \left|\sqrt{x^2+y^2}\right| \mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}t \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^1 r^3\,\mathrm{d}r = \frac{\pi}{8}.$$

Similarly, in Theorem 2, if we choose three kinds of special plane families, three common calculation methods of triple integral will be obtained.

Corollary 4. (1) If the function $f(x,y,z)$ is continuous on the bounded closed region

$$\Omega = \{(x,y,z) \mid a \leq z \leq b, (x,y) \in D_z\},$$

then

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = \int_c^d dz \iint\limits_{D_z} f(x,y,z)\mathrm{d}x\ \mathrm{d}y.$$

(2) The function $f(x,y,z)$ is continuous on the bounded closed region Ω . If Ω can be expressed as

$$\Omega = \{(r,\theta,z) \mid \alpha \le \theta \le \beta, r_1(\theta) \le r \le r_2(\theta), z_1(r,\theta) \le z \le z_2(r,\theta)\}$$

with the cylindrical coordinate transformation $x = r\cos\theta, y = r\sin\theta, z = z$, then

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \int_\alpha^\beta \mathrm{d}\theta \int_{r_1(\theta)}^{r_2(\theta)} \mathrm{d}r \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\ \mathrm{d}z.$$

(3) The function $f(x,y,z)$ is continuous on the bounded closed region Ω . If Ω can be expressed as

$$\Omega = \{(\theta,\varphi,\rho) \mid \alpha \le \theta \le \beta, \varphi_1(\theta) \le \varphi \le \varphi_2(\theta), \rho_1(\varphi,\theta) \le \rho \le \rho_2(\varphi,\theta)\}$$

with the spherical coordinate transformation $x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta, z = \rho\cos\varphi$, then

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \int_\alpha^\beta \mathrm{d}\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} \mathrm{d}\varphi \int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi \mathrm{d}\rho.$$

Proof: (1) In Theorem 2, we take plane family

$$\Sigma_t : \begin{cases} x = x, \\ y = y, & c \le t \le d, (x,y) \in D_{xy}(t) \\ z = z(x,y,t) = t \end{cases}$$

where $D_{xy}(t)$ is the projection area of the cross section of Σ_z and Ω on the xOy coordinate plane. It is easy to calculate $z'_t = 1$. Substituting this result into (3.1) gives

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = \int_c^d dz \iint_{D_z} f(x,y,z)\mathrm{d}x\ \mathrm{d}y.$$

(2) In Corollary 3, we take plane family

$$\Sigma_\theta : \begin{cases} x = r\cos\theta, \\ y = y(r,z,\theta) = r\sin\theta, \alpha \le \theta \le \beta, (r,z) \in D_{rz}(\theta) \\ z = z, \end{cases}$$

where

$$D_{rz}(\theta) = \{(r,z) \mid r_1(\theta) \le r \le r_2(\theta), z_1(r,\theta) \le z \le z_2(r,\theta)\}$$

is the projection area of the cross section of Σ_θ and Ω on the rOz plane. Obviously, $y = x\tan\theta$ is true. Thus $y'_\theta = \frac{x}{\cos^2\theta} = \frac{r}{\cos\theta}$. Substituting this result into (3.4) gives

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \int_\alpha^\beta \mathrm{d}\theta \iint\limits_{D_{xz}(\theta)} f(x, y(x,z,\theta), z) \cdot \frac{r}{|\cos\theta|}\mathrm{d}x\ \mathrm{d}z$$

Since $\frac{\partial(x,z)}{\partial(r,z)} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta$, $\mathrm{d}x\,\mathrm{d}z = |\cos\theta|\mathrm{d}r\,\mathrm{d}z$ holds. Thus

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \int_\alpha^\beta \mathrm{d}\theta \iint\limits_{D_{rz}(\theta)} f(r\cos\theta, r\sin\theta, z)\cdot r\,\mathrm{d}r\,\mathrm{d}z.$$
$$= \int_\alpha^\beta \mathrm{d}\theta \int_{r_1(\theta)}^{r_2(\theta)} \mathrm{d}r \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}z.$$

(3) In Corollary 3, we take plane family

$$\Sigma_\theta : \begin{cases} x = \rho\sin\varphi\cos\theta, \\ y = y(\rho,\varphi,\theta) = \rho\sin\varphi\sin\theta, \alpha \le \theta \le \beta, (\varphi,\rho) \in D_{\varphi\rho}(\theta) \\ z = \rho\cos\varphi, \end{cases}$$

where

$$D_{\varphi\rho}(\theta) = \{(\varphi,\rho) \mid \varphi_1(\theta) \le \varphi \le \varphi_2(\theta), \rho_1(\varphi,\theta) \le \rho \le \rho_2(\varphi,\theta)\}$$

is the projection area of the cross section of Σ_θ and Ω on the $\varphi O\rho$ plane. Obviously, $y = x\tan\theta$ is true. Thus $y'_\theta = \frac{x}{\cos^2\theta} = \rho\frac{\sin\varphi}{\cos\theta}$. Substituting this result into (3.4) gives

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \int_\alpha^\beta \mathrm{d}\theta \iint\limits_{D_{xz}(\theta)} f(x, y(x,z,\theta), z)\cdot \frac{\rho\sin\varphi}{|\cos\theta|}\mathrm{d}x\,\mathrm{d}z$$

Since $\frac{\partial(x,z)}{\partial(\rho,\varphi)} = \begin{vmatrix} \sin\varphi\cos\theta & \cos\varphi \\ \rho\cos\varphi\cos\theta & -\rho\sin\varphi \end{vmatrix} = \rho\cos\theta$, $\mathrm{d}x\,\mathrm{d}z = \rho|\cos\theta|\mathrm{d}\rho\mathrm{d}\varphi$ holds. Thus

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \int_\alpha^\beta \mathrm{d}\theta \iint\limits_{D_{\varphi\rho}(\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\cdot\rho\sin\varphi\mathrm{d}\varphi\mathrm{d}\rho.$$
$$= \int_\alpha^\beta \mathrm{d}\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} \mathrm{d}\varphi \int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\mathrm{d}\rho.$$

4. Another new scheme for calculating triple integrals

Theorem 3. Let the bounded closed region Ω be generated by a family of smooth surfaces

$$\Sigma_t : z = z(x,y,t), a \le t \le b, x_1(t) \le x \le x_2(t)$$

with no double points and $z'_t \neq 0$. Let Γ_{xt} be the curves where the planes, which pass through the point $(x,0,0)$ and are perpendicular to the X-axis, and surface Σ_t intersect. If the function $f(x,y,z)$ is continuous on the closed region Ω , then

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \iint\limits_{D} \mathrm{d}x\,\mathrm{d}t \int_{\Gamma_{xt}} f(x,y,z)\frac{|z'_t|}{\sqrt{1+z'^2_y}}\,\mathrm{d}s, \tag{4.1}$$

where $D = \{(t,x) \mid a \le t \le b, x_1(t) \le x \le x_2(t)\}$.

Proof: Let $x = x, y = y, z = z(x,y,t)$, then the Jacobian determinant

$$\frac{\partial(x,y,z)}{\partial(x,y,t)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z'_x & z'_y & z'_t \end{vmatrix} = z'_t \neq 0.$$

And it transforms Ω to

$$\Omega' = \{(x, y, z) \mid a \le t \le b, x_1(t) \le x \le x_2(t), y_1(x, t) \le y \le y_2(x, t)\}.$$

By substitution method of triple integral we can obtain

$$\begin{aligned}\iiint\limits_{\Omega} f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iiint\limits_{\Omega'} f(x, y, z(x, y, t)) \left|z_t'\right| \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &= \int_a^b \mathrm{d}t \int_{x_1(t)}^{x_2(t)} \mathrm{d}x \int_{y_1(x,t)}^{y_2(x,t)} f(x, y, z(x, y, t)) \left|z_t'\right| \mathrm{d}y.\end{aligned}$$

On the other hand,

$$\int\limits_{\Gamma_{xt}} f(x, y, z)\frac{\left|z_t'\right|}{\sqrt{1 + z_y'^2}}\,\mathrm{d}s = \int_{y_1(x,t)}^{y_2(x,t)} f(x, y, z(x, y, t)) \left|z_t'\right| \mathrm{d}y,$$

then

$$\iint\limits_{D} \mathrm{d}x\,\mathrm{d}t \int\limits_{\Gamma_{xt}} f(x, y, z)\frac{\left|z_t'\right|}{\sqrt{1 + z_y'^2}}\,\mathrm{d}s = \int_a^b \mathrm{d}t \int_{x_1(t)}^{x_2(t)} \mathrm{d}x \int_{y_1(x,t)}^{y_2(x,t)} f(x, y, z(x, y, t)) \left|z_t'\right| \mathrm{d}y.$$

thus

$$\iiint\limits_{\Omega} f(x, y, z)\mathrm{d}v = \iint\limits_{D} \mathrm{d}x\,\mathrm{d}t \int_{\Gamma_{xt}} f(x, y, z)\frac{\left|z_t'\right|}{\sqrt{1 + z_y'^2}}\,\mathrm{d}s.$$

Example 7. Calculate the triple integral $I = \iiint\limits_{\Omega} z\,\mathrm{d}v$, where $\Omega : x^2 + y^2 + z^2 \le 1, z \ge |y|$.

Method 1. Taking the surface family

$$\Sigma_t : z = \sqrt{t^2 - x^2 - y^2}(0 \le t \le 1).$$

For any given $t \in [0, 1]$, $x \in [-t, t]$ and

$$-\frac{1}{\sqrt{2}}\sqrt{t^2 - x^2} \le y \le \frac{1}{\sqrt{2}}\sqrt{t^2 - x^2}.$$

Let Γ_{xt} be the curves where the planes, which pass through the point $(x, 0, 0)$ and are perpendicular to the X-axis, and surface Σ_t intersect. That means

$$\Gamma_{xt} : \begin{cases} x = x, \\ z = \sqrt{t^2 - x^2 - y^2}. \end{cases}$$

Then $z_t' = \frac{t}{\sqrt{t^2-x^2-y^2}}$, thus

$$\int\limits_{\Gamma_{xt}} z \cdot \frac{\left|z_t'\right|}{\sqrt{1 + z_y'^2}}\,\mathrm{d}s = \int_{-\frac{1}{\sqrt{2}}\sqrt{t^2-x^2}}^{-\frac{1}{\sqrt{2}}\sqrt{t^2-x^2}} t\,\mathrm{d}y = \sqrt{2}t\sqrt{t^2 - x^2}.$$

Substituting the above formula into (4.1) gives

$$I = \iint\limits_{D} \sqrt{2}t\sqrt{t^2 - x^2}\,\mathrm{d}x\,\mathrm{d}t,$$

where $D = \{(t, x) \mid 0 \le t \le 1, -t \le x \le t\}$. So,

$$I = \int_0^1 \mathrm{d}t \int_{-t}^{t} \sqrt{2}t\sqrt{t^2 - x^2}\,\mathrm{d}x = \frac{1}{\sqrt{2}}\pi \int_0^1 t^3\,\mathrm{d}t = \frac{\sqrt{2}}{8}\pi.$$

Method 2. According to the symmetry of the integral, it is easy to see that

$$I = \iiint\limits_{\Omega} z\,\mathrm{d}v = 2\iiint\limits_{\Omega'} z\,\mathrm{d}v,$$

where

$$\Omega' : x^2 + y^2 + z^2 \le 1, z \ge y \ge 0.$$

And the projection area of Ω' on xOy plane is

$$D_{xy} = \left\{(x, y) \mid x^2 + 2y^2 \le 1, y \ge 0\right\},$$

thus

$$I = 2\iint\limits_{D_{xy}} dxdy \int_{y}^{\sqrt{1-x^2-y^2}} zdz = \iint\limits_{D_{xy}} \left(1 - x^2 - 2y^2\right) dxdy.$$

Using the generalized polar transformation, the above integral can be calculated as follows:

$$I = \int_0^{\pi} d\theta \int_0^1 \left(1 - r^2\right) \frac{1}{\sqrt{2}} rdr = \frac{\sqrt{2}}{8}\pi.$$

Corollary 5. (1) The bounded closed region Ω is generated by a family of smooth surfaces

$$\Sigma_t : z = z(x, y, t), a \le t \le b, y_1(t) \le y \le y_2(t)$$

with no double points and $z_t' \neq 0$. Let Γ_{yt} be the curves where the planes, which pass through the point $(0, y, 0)$ and are perpendicular to the Y-axis, and surface Σ_t intersect. If the function $f(x, y, z)$ is continuous on the closed region Ω , then

$$\iiint\limits_{\Omega} f(x, y, z)\mathrm{d}v = \iint\limits_{D} \mathrm{d}y\,\mathrm{d}t \int\limits_{\Gamma_{yt}} f(x, y, z) \frac{\left|z_t'\right|}{\sqrt{1 + z_x'^2}}\,\mathrm{d}s,$$

where $D = \{(t, y) \mid a \le t \le b, y_1(t) \le y \le y_2(t)\}$.

(2) The bounded closed region Ω is generated by a family of smooth surfaces

$$\Sigma_t : y = y(x, z, t), a \le t \le b, x_1(t) \le x \le x_2(t)$$

with no double points and $y'_t \neq 0$. Let Γ_{xt} be the curves where the planes, which pass through the point $(x, 0, 0)$ and are perpendicular to the X-axis, and surface Σ_t intersect. If the function $f(x, y, z)$ is continuous on the closed region Ω , then

$$\iiint_{\Omega} f(x,y,z)\mathrm{d}v = \iint_{D} \mathrm{d}x\,\mathrm{d}t \int_{\Gamma_{xt}} f(x,y,z)\frac{|y'_t|}{\sqrt{1+y'^2_z}}\,\mathrm{d}s, \tag{4.2}$$

where $D = \{(t, x) \mid a \le t \le b, x_1(t) \le x \le x_2(t)\}$.

In fact, there are several other cases of Theorem 3, which are not repeated here.

Method 3 of Example 6. Taking a family of cylindrical surfaces

$$\Sigma_t : y = \sqrt{t^2 - x^2}, \quad D = \{(t, x) \mid 0 \le t \le 1, 0 \le x \le t\}.$$

then $\frac{|y'_t|}{\sqrt{1+y'^2_z}} = \frac{t}{\sqrt{t^2-x^2}}$ holds. Let Γ_{xt} be $\begin{cases} x = x, \\ y = \sqrt{t^2 - x^2}, 0 \le x \le t \\ z = t, \end{cases}$. And the length of the straight-line segment Γ_{xt} is t . Substituting the above results into (4.2) gives

$$I = \iint_{D} \mathrm{d}x\,\mathrm{d}t \int_{\Gamma_{xt}} \sqrt{x^2+y^2}\cdot\frac{t}{\sqrt{t^2-x^2}}\,\mathrm{d}s = \iint_{D} \mathrm{d}x\,\mathrm{d}t \int_{\Gamma_{xt}} \frac{t^2}{\sqrt{t^2-x^2}}\,\mathrm{d}s = \iint_{D} \frac{t^2}{\sqrt{t^2-x^2}}\cdot t\,\mathrm{d}x\,\mathrm{d}t = \frac{\pi}{8}.$$

Corollary 6. Let the function $f(x, y, z)$ be continuous on the bounded closed region

$$\Omega = \{(x,y,z) \mid z_1(x,y) \le z \le z_2(x,y), (x,y) \in D\},$$

then

$$\iiint_{\Omega} f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{D} \mathrm{d}x\,\mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z,$$

where D is the projection area of Ω on the xOy coordinate plane.

Proof: In the Corollary 5(2), if we take

$$\Sigma_t : y = y(x,z,t) = t,$$

Γ_{xt} is straight lines Γ_{xy} parallel to the Z-axis. Notice

$$y'_t = 1, y'_z = 0,\ \mathrm{d}s = \mathrm{d}z,$$

It follows from (4.2) that

$$\iiint_{\Omega} f(x,y,z)\mathrm{d}v = \iint_{D} \mathrm{d}x\,\mathrm{d}y \int_{\Gamma_{xy}} f(x,y,z)\frac{1}{\sqrt{1+0^2}}\,\mathrm{d}s = \iint_{D} \mathrm{d}x\,\mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z.$$

5. Conclusions

In this paper we establish several formulas for converting the double integral to a line integral with respect to arc length, and the triple integral to a surface integral with respect to area or a line integral with respect to arc length. Some commonly used calculation methods are special cases of our methods. Examples show that the methods presented here are simple and effective in computing certain complex double integrals and triple integrals.

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Conflict of interest

The authors declare that they have no conflict of interest.

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