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Research article

New non-traveling wave solutions for (3+1)-dimensional variable coefficients Date-Jimbo-Kashiwara-Miwa equation

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Abstract: In this paper, we investigate non-traveling wave solutions of the (3+1)-dimensional variable coefficients Date-Jimbo-Kashiwara-Miwa (VC-DJKM) equation, which describes the real physical phenomena owing to the inhomogeneities of media. By combining the extended homoclinic test approach with variable separation method, we obtain abundant new exact non-traveling wave solutions of the (3+1)-dimensional VC-DJKM equation. These results with a parabolic tail or linear tail reveal the complex structure of the solutions for (3+1)-dimensional VC-DJKM equation. Moreover, the tail in these solutions maybe give a prediction of physical phenomenon. When arbitrary functions contained in these non-traveling wave solutions are taken as some special functions, we can get the kink-type solitons, singular solitary wave solutions, and periodic solitary wave solutions, and so on. As the special cases of our work, the corresponding results of (3+1)-dimensional DJKM equation, (2+1)-dimensional DJKM equation, (2+1)-dimensional VC-DJKM equation are also given.

Keywords: extended homoclinic test approach; variable separation method; VC-DJKM equation; exact solutions
Mathematics Subject Classification: 35C99, 35G20, 37K10, 68W30

1. Introduction

As we known, a great number of significant problems such as physical, ecological science and engineering technology can be attributed to the research of higher-dimensional nonlinear partial differential equations (NLPDEs). It's especially important to seek the explicit analytic solutions of higher-dimensional NLPDEs so as to delve into the dynamic process described by the higher-dimensional NLPDEs models. However, in practical applications, most of real nonlinear mathematical physics equations possess variable coefficients. The exact solutions of the variable coefficients nonlinear partial differential equations have greater application values. Some properties of variable coefficients higher-dimensional NLPDEs have been studied [1–3].

Recently, many researchers have studied traveling wave solutions for higher-order and higher-dimensional NLPDEs. Arshad [4] applied modified extended mapping method to get bright and dark solitons, solitary wave and periodic solitary wave solutions of generalized higher order nonlinear Schrödinger equation in cubic quintic non Kerr medium. Using the generalized extended tanh method and the F-expansion method, Seadawy [5] derived exact solitary wave solutions of KP and modified KP equations. Özkan [6] applied the improved $\tan(\varphi/2)$-expansion method to obtain four types of solutions for the generalised Hirota-Satsuma coupled KdV equation and (2+1)-dimensional Nizhnik-Novikov-Veselov system. Iqbal [7] constructed some new solitary wave solutions (such as rational, trigonometric, hyperbolic, elliptic functions including dark, bright, periodic wave, and so on) of (2+1)-dimensional Nizhnik-Novikov-Vesselov equation by the extended modified rational expansion method. Traveling wave solutions are very special solutions of partial differential equations, which describe evolution of physical quantities. But partial differential equations are infinite dimensional systems, the solution space is infinite dimensional, and more solutions are non-traveling wave solutions. Deriving non-traveling exact wave solutions of nonlinear partial differential equations has recently received tremendous attention in mathematics and physics. Moreover, compared with the low-dimensional systems, higher-dimensional nonlinear partial differential equations have more complex behaviors. Shang [8, 9] have studied the non-traveling wave solutions of (3+1)-dimensional potential-YTSF equation and Calogero equation by combining the extended homoclinic approach with the method of separation of variables. Refs. [10–12] studied the non-traveling wave solutions for (2+1)-dimensional and (3+1)-dimensional nonlinear partial differential equations. Therefore, the study of non-traveling wave solutions for higher-dimensional nonlinear partial differential equations is valuable.

As one of the most important higher-dimensional variable coefficients NLPDEs, the new (3+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation with time-dependent coefficients

$$u_{xxxxy} + 4u_{xxy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} - \alpha u_{yyy} - 2\beta g(t)u_{xxt} + h(t)(au_x + bu_y + cu_z)_{xx} = 0 \tag{1.1}$$

describes the real physical phenomena owing to the inhomogeneities of media, where $u = u(x, y, z, t)$ denotes the wave amplitude, α, β, a, b, c are real constants, $g(t)$ is a smooth function, $g(t) \neq 0$, $h(t)$ is a function of t. For $g(t) = h(t) = 1$, Eq (1.1) reduces to (3+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation with constant coefficients

$$u_{xxxxy} + 4u_{xxy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} - \alpha u_{yyy} - 2\beta u_{xxt} + (au_x + bu_y + cu_z)_{xx} = 0. \tag{1.2}$$

Wazwaz [13] showed that Eq (1.1) and Eq (1.2) were completely integrable in the Painlevé sense and admitted multiple soliton solutions consisting of solitonic, singular, periodic solutions. When $g(t) = 1$, $h(t) = 0$, Eq (1.1) reduces to the (2+1)-dimensional Date-Jimbo-Kashiwara-Miwa (DJKM) equation

$$u_{xxxxy} + 4u_{xxy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} - \alpha u_{yyy} - 2\beta u_{xxt} = 0, \tag{1.3}$$

which describes the propagation of nonlinear dispersive waves in inhomogeneous media, where α and β are real constants. Refs. [14–17] obtained Lax pair, conservation laws, Wronskian and Grammian solutions, lump solutions, multi-shock wave solutions, complexiton solutions and soliton solutions of Eq (1.3). For $b = 1$, $a = c = 0$, Eq (1.1) reduces to the (2+1)-dimensional variable coefficients Date-Jimbo-Kashiwara-Miwa (VC-DJKM) equation

$$u_{xxxxy} + 4u_{xxy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} - \alpha u_{yyy} - 2\beta g(t)u_{xxt} + h(t)u_{xxy} = 0. \tag{1.4}$$

Kang [18] obtained the breather-kink wave solution, double-solitary wave solution and rogue wave solution of Eq (1.4) by implementing the homoclinic test method. Wazwaz [19] presented the multi-shock wave solutions and the multiple complex kink solutions of Eq (1.4). We can say that Eq (1.2), (1.3) and (1.4) are all particular forms of Eq (1.1). There are some researches about solutions of (2+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation. However, till date, to the best of our knowledge, few research has been conducted on the traveling or non-traveling wave solutions of (3+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation with constant coefficients and variable coefficients.

In this paper, we will study the new exact non-traveling wave solutions of the (3+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation with time-dependent coefficients (1.1). By utilizing the extended homoclinic test approach and variable separation method [8,9], we present sixteen kinds of non-traveling wave solutions, such as kink-like solutions, periodic solitary-like solutions and singular solitary-like solutions, and so on. When arbitrary functions in the non-traveling wave solutions are taken as special functions, we will get kink solitary solutions, singular solitary wave solutions and periodic solitary wave solutions. As the special cases of our work, the corresponding results of (3+1)-dimensional DJKM equation (1.2), (2+1)-dimensional DJKM equation (1.3), (2+1)-dimensional VC-DJKM equation (1.4) are also given. Meanwhile, we shed light on the structural characteristics of solutions by some graphics and explain the importance of our results in mathematics and physics.

2. Abundant new exact non-traveling wave solutions for the (3+1)-dimensional VC-DJKM equation

In this section, by combining the extended homoclinic test approach with the method of separation of variables [8,9], we derive abundant exact non-traveling wave solutions of Eq (1.1).

We first introduce a transformation

$$u(x, y, z, t) = \varphi(\xi, t) + q(y, t), \tag{2.1}$$

where $\xi = x + mz + \theta(y, t)$, $x, y, z \in R$, $t \in R^+$, $m \in R$ is an arbitrary constant, $\varphi(\xi, t), q(y, t)$ and $\theta(y, t)$ are functions to be determined later. Substituting (2.1) into (1.1) leads to the equation

$$\begin{aligned} &\theta_y\varphi_{\xi\xi\xi\xi\xi} + 6\theta_y\varphi_\xi\varphi_{\xi\xi\xi} + (2q_y - \alpha\theta_y^3 - 2\beta g(t)\theta_t + ah(t) + b\theta_y h(t) + cmh(t))\varphi_{\xi\xi\xi} + 6\theta_y\varphi_{\xi\xi}^2 \\ &- 3\alpha\theta_y\theta_{yy}\varphi_{\xi\xi} - \alpha\theta_{yyy}\varphi_\xi - \alpha q_{yyy} - 2\beta g(t)\varphi_{\xi\xi t} = 0. \end{aligned} \tag{2.2}$$

To simplify Eq (2.2), we let

$$2q_y - \alpha\theta_y^3 - 2\beta g(t)\theta_t + ah(t) + b\theta_y h(t) + cmh(t) = 0. \tag{2.3}$$

From (2.3), we get

$$q(y, t) = \int (\beta g(t)\theta_t + \frac{\alpha\theta_y^3 - (a + b\theta_y + cm)h(t)}{2})dy. \tag{2.4}$$

In order to further reduce Eq (2.2), we will discuss that $\theta(y, t)$ has two specific forms in what follows.

2.1. The multiplicative variable separable form of $\theta(y,t)$

In this case, $\theta(y,t)$ has the multiplicative variable separable form

$$\theta(y,t) = f(t)k(y), \tag{2.5}$$

where $f(t)$ and $k(y)$ are two smooth functions to be determined later. Substituting (2.5) into (2.4) yields

$$q(y,t) = \frac{\alpha}{2}f^3(t)\int (k'(y))^3 dy + \beta g(t)f'(t)\int k(y)dy - \frac{h(t)}{2}\int (a + bf(t)k'(y) + cm)dy. \tag{2.6}$$

Therefore, Eq (2.2) reduces to

$$\theta_y\varphi_{\xi\xi\xi\xi\xi\xi} + 6\theta_y\varphi_\xi\varphi_{\xi\xi\xi} + 6\theta_y\varphi_{\xi\xi}^2 - 3\alpha\theta_y\theta_{yy}\varphi_{\xi\xi} - \alpha\theta_{yyy}\varphi_\xi - \alpha q_{yyy} - 2\beta g(t)\varphi_{\xi\xi t} = 0. \tag{2.7}$$

In order to simply Eq (2.7), we let $k'(y)$=*Constant*. Without loss of generality, we take $k(y) = y$. Therefore, Eq (2.7) reduces to the following equation with variable coefficients

$$f(t)\varphi_{\xi\xi\xi\xi\xi\xi} + 6f(t)\varphi_\xi\varphi_{\xi\xi\xi} + 6f(t)\varphi_{\xi\xi}^2 - 2\beta g(t)\varphi_{\xi\xi t} = 0. \tag{2.8}$$

Furthermore, we need to transform Eq (2.8) to a partial differential equation with constant coefficients. Here, we introduce an appropriate variable transformation

$$\varphi(\xi,t) = \nu(\xi,\eta), \quad \eta = \int \frac{f(t)}{g(t)}dt. \tag{2.9}$$

Substituting (2.9) into (2.8), we get the following partial differential equation with constant coefficients

$$\nu_{\xi\xi\xi\xi\xi\xi} + 6\nu_\xi\nu_{\xi\xi\xi} + 6\nu_{\xi\xi}^2 - 2\beta\nu_{\xi\xi\eta} = 0. \tag{2.10}$$

Integrating (2.10) twice with respect to ξ and taking the integral constant to be zero, we get

$$\nu_{\xi\xi\xi} + 3\nu_\xi^2 - 2\beta\nu_\eta = 0. \tag{2.11}$$

In order to solving (2.11), we introduce a nonlinear function transformation

$$\nu = 2(ln\phi)_\xi, \tag{2.12}$$

where $\phi(\xi,\eta)$ is an undetermined real function. Substituting (2.12) into (2.11) leads to a bilinear equation

$$(D_\xi^4 - 2\beta D_\xi D_\eta)\phi\cdot\phi = 0, \tag{2.13}$$

where the bilinear operator D is defined as

$$D_\xi^m D_\eta^n f\cdot g = (\partial\xi - \partial\xi')^m(\partial\eta - \partial\eta')^n f(\xi,\eta)g(\xi',\eta')|_{(\xi',\eta')=(\xi,\eta)}.$$

In this section, we seek for the solution in the following form

$$\phi = k_1cos(\zeta_1) + k_2exp(\zeta_2) + exp(-\zeta_2), \tag{2.14}$$

where $\zeta_i = a_i\xi + b_i\eta$, $i = 1, 2$, $k_1, k_2 \in R$, $a_1, a_2, b_1, b_2 \in C$ are undetermined constants. Substituting (2.14) into (2.13) and equating all coefficients of $cos^2(\zeta_1)$, $cos(\zeta_1)exp(\zeta_2)$, $cos(\zeta_1)exp(-\zeta_2)$, $sin^2(\zeta_1)$, $sin(\zeta_1)exp(\zeta_2)$, $sin(\zeta_1)exp(-\zeta_2)$ and the constant term to zero yield a set of nonlinear algebraic equations as follows:

$$\begin{cases} k_1^2(4a_1^4 + 2\beta a_1 b_1) = 0, \\ k_1k_2(a_1^4 + a_2^4 - 6a_1^2a_2^2 + 2\beta a_1 b_1 - 2\beta a_2 b_2) = 0, \\ k_1(a_1^4 + a_2^4 - 6a_1^2a_2^2 + 2\beta a_1 b_1 - 2\beta a_2 b_2) = 0, \\ 2k_1^2(2a_1^4 + \beta a_1 b_1) = 0, \\ 2k_1k_2(2a_1a_2^3 - 2a_1^3a_2 - \beta a_1 b_2 - \beta a_2 b_1) = 0, \\ 2k_1(-2a_1a_2^3 + 2a_1^3a_2 + \beta a_1 b_2 + \beta a_2 b_1) = 0, \\ 8k_2(2a_2^4 - \beta a_2 b_2) = 0. \end{cases} \tag{2.15}$$

With the help of symbolic software such as Maple, we have the following results of (2.15).

Case 1:

$$\begin{cases} a_1 = a_1, \quad b_1 = b_1, \quad k_1 = 0, \\ a_2 = a_2, \quad b_2 = \dfrac{2a_2^3}{\beta}, \quad k_2 = k_2. \end{cases} \tag{2.16}$$

In this case, collecting (2.16), (2.14), (2.12), (2.9), (2.6), (2.5) with (2.1), one obtains Eq (1.1) admits exact solution given as

$$u(x, y, z, t) = 2a_2\frac{k_2exp(\zeta_2) - exp(-\zeta_2)}{k_2exp(\zeta_2) + exp(-\zeta_2)} + \frac{\beta g(t) f'(t)}{2}y^2 + \frac{\alpha f^3(t) - (a + bf(t) + cm)h(t)}{2}y, \tag{2.17}$$

where $\zeta_2 = a_2(x + mz + f(t)y) + \frac{2a_2^3}{\beta}\int \frac{f(t)}{g(t)}dt$.

In particular, solution (2.17) can be written as follows:

$$u_1(x, y, z, t) = 2a_2tanh(\zeta_2 + \frac{1}{2}\ln k_2) + \frac{\beta g(t) f'(t)}{2}y^2 + \frac{\alpha f^3(t) - (a + bf(t) + cm)h(t)}{2}y, \quad k_2 > 0, \tag{2.18}$$

$$u_2(x, y, z, t) = 2a_2coth(\zeta_2 + \frac{1}{2}\ln(-k_2)) + \frac{\beta g(t) f'(t)}{2}y^2 + \frac{\alpha f^3(t) - (a + bf(t) + cm)h(t)}{2}y, \quad k_2 < 0, \tag{2.19}$$

where $\zeta_2 = a_2(x + mz + f(t)y) + \frac{2a_2^3}{\beta}\int \frac{f(t)}{g(t)}dt$, a_2 is a real constant.

Case 2:

$$\begin{cases} a_1 = a_1, \quad b_1 = -\dfrac{2a_1^3}{\beta}, \quad k_1 = k_1, \\ a_2 = \pm ia_1, \quad b_2 = \mp\dfrac{2ia_1^3}{\beta}, \quad k_2 = 0. \end{cases} \tag{2.20}$$

In this case, collecting (2.20), (2.14), (2.12), (2.9), (2.6), (2.5) with (2.1), then Eq (1.1) has the exact solution given below

$$u(x, y, z, t) = 2a_1\frac{-k_1sin(\zeta_1) \mp iexp(-\zeta_2)}{k_1cos(\zeta_1) + exp(-\zeta_2)} + \frac{\beta g(t) f'(t)}{2}y^2 + \frac{\alpha f^3(t) - (a + bf(t) + cm)h(t)}{2}y, \tag{2.21}$$

where $\zeta_1 = a_1(x + mz + f(t)y) - \frac{2a_1^3}{\beta}\int \frac{f(t)}{g(t)}dt$ and $\zeta_2 = \pm i\zeta_1$.

In particular, solution (2.21) becomes

$$u_3(x,y,z,t) = 2a_1\frac{[1-(k_1+1)^2]sin(\zeta_1)cos(\zeta_1) \mp i(k_1+1)}{(k_1+1)^2cos^2(\zeta_1)+sin^2(\zeta_1)} + \frac{\beta g(t)f'(t)}{2}y^2 + \frac{\alpha f^3(t)-(a+bf(t)+cm)h(t)}{2}y, \quad a_1 \in R, \tag{2.22}$$

$$u_4(x,y,z,t) = 2k_3\frac{(k_1+1)sinh(\zeta_1^*) \mp cosh(\zeta_1^*)}{(k_1+1)cosh(\zeta_1^*) \mp sinh(\zeta_1^*)} + \frac{\beta g(t)f'(t)}{2}y^2 + \frac{\alpha f^3(t)-(a+bf(t)+cm)h(t)}{2}y, \quad a_1=-k_3i,\ k_3 \in R, \tag{2.23}$$

where $\zeta_1 = a_1(x+mz+f(t)y) - \frac{2a_1^3}{\beta}\int\frac{f(t)}{g(t)}dt$ and $\zeta_1^* = i\zeta_1$.

Case 3:

$$\begin{cases} a_1 = a_1, & b_1 = -\dfrac{2a_1^3}{\beta}, \quad k_1 = k_1, \\ a_2 = \pm ia_1, & b_2 = \mp\dfrac{2ia_1^3}{\beta}, \quad k_2 = k_2. \end{cases} \tag{2.24}$$

In this case, collecting (2.24), (2.14), (2.12), (2.9), (2.6), (2.5) with (2.1), then Eq (1.1) has exact solution expressed as

$$u = 2a_1\frac{-k_1sin(\zeta_1) \pm ik_2exp(\zeta_2) \mp iexp(-\zeta_2)}{k_1cos(\zeta_1)+k_2exp(\zeta_2)+exp(-\zeta_2)} + \frac{\beta g(t)f'(t)}{2}y^2 + \frac{\alpha f^3(t)-(a+bf(t)+cm)h(t)}{2}y, \tag{2.25}$$

where $\zeta_1 = a_1(x+mz+f(t)y) - \frac{2a_1^3}{\beta}\int\frac{f(t)}{g(t)}dt$ and $\zeta_2 = \pm i\zeta_1$.

In particular, solution (2.25) becomes

$$u_5(x,y,z,t) = 2a_1\frac{[(k_2-1)^2-(k_1+k_2+1)^2]sin(\zeta_1)cos(\zeta_1) \pm (k_1+k_2+1)(k_2-1)i}{(k_1+k_2+1)^2cos^2(\zeta_1)+(k_2-1)^2sin^2(\zeta_1)} + \frac{\beta g(t)f'(t)}{2}y^2 + \frac{\alpha f^3(t)-(a+bf(t)+cm)h(t)}{2}y, \quad a_1 \in R, \tag{2.26}$$

$$u_6(x,y,z,t) = 2k_3\frac{k_1sinh(\zeta_1^*) \pm 2\sqrt{k_2}sinh(\pm\zeta_1^*+\frac{1}{2}\ln k_2)}{k_1cosh(\zeta_1^*)+2\sqrt{k_2}cosh(\pm\zeta_1^*+\frac{1}{2}\ln k_2)} + \frac{\beta g(t)f'(t)}{2}y^2 + \frac{\alpha f^3(t)-(a+bf(t)+cm)h(t)}{2}y, \quad k_2>0,\ a_1=-k_3i,\ k_3 \in R, \tag{2.27}$$

$$u_7(x,y,z,t) = 2k_3\frac{k_1sinh(\zeta_1^*) \mp 2\sqrt{-k_2}cosh(\pm\zeta_1^*+\frac{1}{2}\ln(-k_2))}{k_1cosh(\zeta_1^*)-2\sqrt{-k_2}sinh(\pm\zeta_1^*+\frac{1}{2}\ln(-k_2))} + \frac{\beta g(t)f'(t)}{2}y^2 + \frac{\alpha f^3(t)-(a+bf(t)+cm)h(t)}{2}y, \quad k_2<0,\ a_1=-k_3i,\ k_3 \in R, \tag{2.28}$$

where $\zeta_1 = a_1(x+mz+f(t)y) - \frac{2a_1^3}{\beta}\int\frac{f(t)}{g(t)}dt$ and $\zeta_1^* = i\zeta_1$.

Case 4:

$$\begin{cases} a_1 = 0, & b_1 = 0, \quad k_1 = k_1, \\ a_2 = a_2, & b_2 = \dfrac{a_2^3}{2\beta}, \quad k_2 = 0. \end{cases} \tag{2.29}$$

In this case, collecting (2.29), (2.14), (2.12), (2.9), (2.6), (2.5) with (2.1), one obtains the solution of Eq (1.1) in the following form

$$u_8(x,y,z,t) = -2a_2\frac{exp(-\zeta_2)}{k_1+exp(-\zeta_2)} + \frac{\beta g(t)f'(t)}{2}y^2 + \frac{\alpha f^3(t)-(a+bf(t)+cm)h(t)}{2}y, \quad (2.30)$$

where $\zeta_2 = a_2(x+mz+f(t)y) + \frac{a_2^3}{2\beta}\int\frac{f(t)}{g(t)}dt$. In $u_1 - u_8$, $f(t)$ is an arbitrary first order derivable function. When taking the arbitrary function $f(t)$ as specific constant or function, we can derive rich exact non-traveling wave solutions for Eq (1.1). Moreover, when $a_1 = k_4 + ik_3$, we can get many other type solutions from (2.21) and (2.25), where k_3, k_4 are nonzero real numbers. Here, we omit the detail expression of these solutions.

2.2. The additive variable separable form of $\theta(y,t)$

In this case, $\theta(y,t)$ has an additive variable separable form

$$\theta(y,t) = f(t) + k(y), \quad (2.31)$$

where $f(t)$ and $k(y)$ are smooth functions to be determined later. Substituting (2.31) into (2.4) yields

$$q(y,t) = \frac{\alpha}{2}\int(k'(y))^3dy + \beta g(t)f'(t)\int dy - \frac{h(t)}{2}\int(a+bk'(y)+cm)dy. \quad (2.32)$$

Then, substituting (2.31) and (2.32) into (2.2), one obtains

$$k'\varphi_{\xi\xi\xi\xi\xi} + 6k'\varphi_\xi\varphi_{\xi\xi\xi} + 6k'\varphi_{\xi\xi}^2 - 3\alpha k'k''\varphi_{\xi\xi} - \alpha k'''\varphi_\xi - \alpha q_{yyy} - 2\beta g(t)\varphi_{\xi\xi t} = 0. \quad (2.33)$$

To simply Eq (2.33), we set that $k'(y)$ =*Constant*. Without loss of generality, we take $k(y) = y$. Therefore, Eq (2.33) becomes

$$\varphi_{\xi\xi\xi\xi\xi} + 6\varphi_\xi\varphi_{\xi\xi\xi} + 6\varphi_{\xi\xi}^2 - 2\beta g(t)\varphi_{\xi\xi t} = 0. \quad (2.34)$$

In order to solve Eq (2.34), we introduce an appropriate variable transformation

$$\varphi(\xi,t) = v(\xi,\eta), \quad \eta = \int\frac{1}{g(t)}dt \quad (2.35)$$

to transform Eq (2.34) to a partial differential equation with constant coefficients

$$v_{\xi\xi\xi\xi\xi} + 6v_\xi v_{\xi\xi\xi} + 6v_{\xi\xi}^2 - 2\beta v_{\xi\xi\eta} = 0. \quad (2.36)$$

Integrating (2.36) twice with respect to ξ yields

$$v_{\xi\xi\xi} + 3v_\xi^2 - 2\beta v_\eta = 0. \quad (2.37)$$

In the same way as solving Eq (2.11), we obtain the solutions of Eq (1.1) as follows

$$u_9(x,y,z,t) = 2a_2tanh(\zeta_2 + \frac{1}{2}\ln k_2) + (\beta g(t)f'(t) + \frac{\alpha-(a+b+cm)h(t)}{2})y, \quad k_2 > 0, \quad (2.38)$$

$$u_{10}(x,y,z,t) = 2a_2 coth(\zeta_2 + \frac{1}{2}\ln(-k_2)) + (\beta g(t)f'(t) + \frac{\alpha - (a+b+cm)h(t)}{2})y, \quad k_2 < 0, \tag{2.39}$$

where $\zeta_2 = a_2(x + y + mz + f(t)) + \frac{2a_2^3}{\beta}\int \frac{1}{g(t)}dt$.

$$\begin{aligned} u_{11}(x,y,z,t) &= 2a_1 \frac{[1-(k_1+1)^2]sin(\zeta_1)cos(\zeta_1) \mp i(k_1+1)}{(k_1+1)^2 cos^2(\zeta_1) + sin^2(\zeta_1)} \\ &+ (\beta g(t)f'(t) + \frac{\alpha - (a+b+cm)h(t)}{2})y, \quad a_1 \in R, \end{aligned} \tag{2.40}$$

$$\begin{aligned} u_{12}(x,y,z,t) &= 2k_3 \frac{(k_1+1)sinh(\zeta_1^*) \mp cosh(\zeta_1^*)}{(k_1+1)cosh(\zeta_1^*) \mp sinh(\zeta_1^*)} \\ &+ (\beta g(t)f'(t) + \frac{\alpha - (a+b+cm)h(t)}{2})y, \quad a_1 = -k_3 i, \ k_3 \in R, \end{aligned} \tag{2.41}$$

where $\zeta_1 = a_1(x + y + mz + f(t)) - \frac{2a_1^3}{\beta}\int \frac{1}{g(t)}dt$ and $\zeta_1^* = i\zeta_1$.

$$\begin{aligned} u_{13}(x,y,z,t) &= 2a_1 \frac{[(k_2-1)^2 - (k_1+k_2+1)^2]sin(\zeta_1)cos(\zeta_1) \pm (k_1+k_2+1)(k_2-1)i}{(k_1+k_2+1)^2 cos^2(\zeta_1) + (k_2-1)^2 sin^2(\zeta_1)} \\ &+ (\beta g(t)f'(t) + \frac{\alpha - (a+b+cm)h(t)}{2})y, \quad a_1 \in R, \end{aligned} \tag{2.42}$$

$$\begin{aligned} u_{14}(x,y,z,t) &= 2k_3 \frac{k_1 sinh(\zeta_1^*) \pm 2\sqrt{k_2} sinh(\pm\zeta_1^* + \frac{1}{2}\ln k_2)}{k_1 cosh(\zeta_1^*) + 2\sqrt{k_2} cosh(\pm\zeta_1^* + \frac{1}{2}\ln k_2)} \\ &+ (\beta g(t)f'(t) + \frac{\alpha - (a+b+cm)h(t)}{2})y, \quad k_2 > 0, \ a_1 = -k_3 i, \ k_3 \in R, \end{aligned} \tag{2.43}$$

$$\begin{aligned} u_{15}(x,y,z,t) &= 2k_3 \frac{k_1 sinh(\zeta_1^*) \mp 2\sqrt{-k_2} cosh(\pm\zeta_1^* + \frac{1}{2}\ln(-k_2))}{k_1 cosh(\zeta_1^*) - 2\sqrt{-k_2} sinh(\pm\zeta_1^* + \frac{1}{2}\ln(-k_2))} \\ &+ (\beta g(t)f'(t) + \frac{\alpha - (a+b+cm)h(t)}{2})y, \quad k_2 < 0, \ a_1 = -k_3 i, \ k_3 \in R, \end{aligned} \tag{2.44}$$

where $\zeta_1 = a_1(x + y + mz + f(t)) - \frac{2a_1^3}{\beta}\int \frac{1}{g(t)}dt$ and $\zeta_1^* = i\zeta_1$.

$$u_{16}(x,y,z,t) = -2a_2 \frac{exp(-\zeta_2)}{k_1 + exp(-\zeta_2)} + (\beta g(t)f'(t) + \frac{\alpha - (a+b+cm)h(t)}{2})y, \tag{2.45}$$

where $\zeta_2 = a_2(x+y+mz+f(t)) + \frac{a_2^3}{2\beta}\int \frac{1}{g(t)}dt$. In u_9-u_{16}, $f(t)$ is an arbitrary first order derivable function. When taking the arbitrary function $f(t)$ as specific constant or function, we can derive rich exact non-traveling wave solutions for Eq (1.1).

Remark 2.1 Especially, if $g(t) = h(t) = 1$, Eq (1.1) reduces to the (3+1)-dimensional DJMK equation with constant coefficients (1.2). In the same way of solving Eq (1.1), we can obtain sixteen kinds of non-traveling solutions of Eq (1.2) with $g(t) = h(t) = 1$ in (2.18), (2.19), (2.22), (2.23), (2.26), (2.27), (2.28), (2.30), and (2.38)–(2.45).

Remark 2.2 For $g(t) = 1$, $h(t) = 0$, Eq (1.1) reduces to the (2+1)-dimensional DJMK equation with constant coefficients (1.3). In a similar manner to solving Eq (1.1), we can obtain sixteen kinds of non-traveling solutions of Eq (1.3) with $g(t) = 1$, $h(t) = 0$, $m = 0$ in (2.18), (2.19), (2.22), (2.23), (2.26), (2.27), (2.28), (2.30), and (2.38)–(2.45).

Remark 2.3 For $b = 1$, $a = c = 0$, Eq (1.1) reduces to the (2+1)-dimensional VC-DJKM equation (1.4). In a manner similar to solving Eq (1.1), we can obtain sixteen kinds of non-traveling solutions of Eq (1.4) with $b = 1$, $a = c = m = 0$ in (2.18), (2.19), (2.22), (2.23), (2.26), (2.27), (2.28), (2.30), and (2.38)–(2.45).

3. Graphic analysis of solutions

In section 2, we combine the extended homoclinic test approach and variable separation method to get sixteen kinds of solutions. These solutions have a parabolic tail and a linear tail. The tails in these solutions maybe give a prediction of physical phenomenon and the free parameters in these solutions of Eq (1.1) have rich mathematical structures, which may be important for explaining some physical phenomena in variety of branches. According to the expression of solutions, the non-traveling solutions u_1, u_6, u_9 and u_{14} can be seen as kink-like type. u_2, u_4, u_7, u_{10}, u_{12} and u_{15} can be seen as singular solitary wave-like type. The non-traveling solutions u_3, u_5, u_{11} and u_{13} can be regarded as periodic solitary wave-like solutions. u_8 and u_{16} are single solitary wave-like type. The solutions u_1-u_8 have a parabolic tail. The solutions u_9-u_{16} possess a linear tail. These results reveal the complex structure of the solutions for the (3+1)-dimensional variable coefficients Date-Jimbo-Kashiwara-Miwa equation (1.1). Some cross sections of these solutions have solitary wave form. Here, through 3D graphic, we draw the cross sections of some solutions.

The representative sketches of the solutions in the form of u_1, u_2, u_3 and u_7 with a parabolic tail are presented in Figures 1–4 respectively.

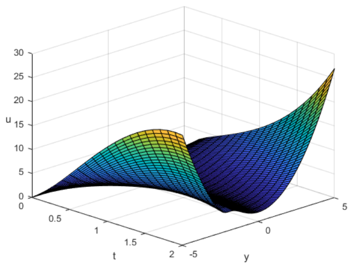

Figure 1. Kink-like solution u_1 as $a_2 = 1, k_2 = m = \alpha = \beta = 1, x = z = 0, f(t) = h(t) = g(t) = t, a = b = c = 1$.

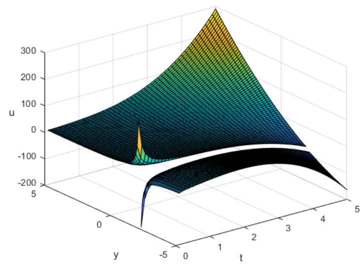

Figure 2. Singular solitary wave-like solution u_2 as $a_2 = 1$, $k_2 = -1, m = \alpha = \beta = 1, x = z = 0, f(t) = h(t) = g(t) = t$, $a = b = c = 1$.

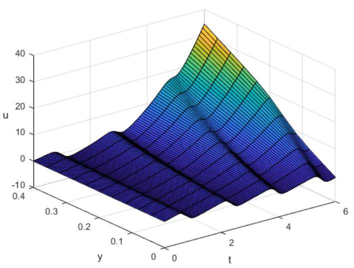

Figure 3. Periodic solitary wave-like solution u_3 as $a_1 = 1$, $a_2 = ia_1, k_1 = m = \alpha = \beta = 1, x = z = 0, f(t) = h(t) = g(t) = t, a = b = c = 1$.

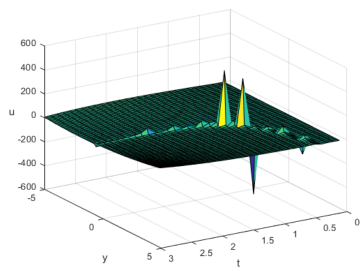

Figure 4. Singular solitary wave-like solution u_7 as $a_1 = -i$, $a_2 = ia_1 = 1, k_1 = m = \alpha = \beta = 1, k_2 = -1, x = z = 0$, $f(t) = h(t) = t, g(t) = t^2, a = b = c = 1$.

When $f(t)$, $g(t)$ and $h(t)$ are taken as suitable linear functions, u_9 and u_{14} become exact kink solutions, u_{10}, u_{12} and u_{15} become singular solitary wave solutions, u_{11} and u_{13} reduce to periodic solitary solutions, u_{16} becomes single wave solution. The representative sketches of the solutions in the form of u_9, u_{10}, u_{11} and u_{13} without a tail are presented in Figures 5–8 respectively.

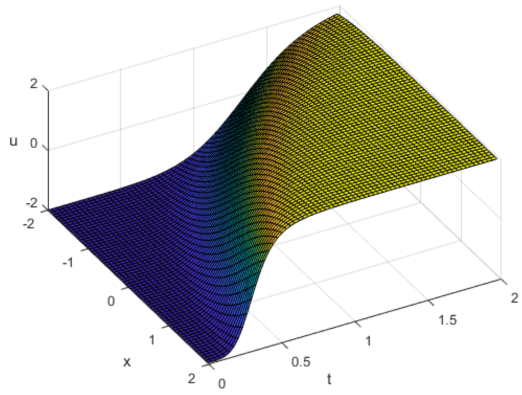

Figure 5. Kink solution u_9 as $a_2 = 1, k_2 = m = \alpha = \beta = 1$, $y = z = 0, f(t) = h(t) = t, g(t) = t, a = b = c = 1$.

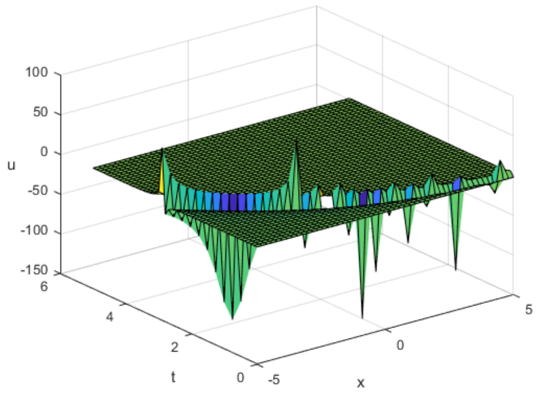

Figure 6. Singular solitary wave solution u_{10} as $a_2 = 1, k_2 = -1$, $m = \alpha = \beta = 1, y = z = 0, f(t) = h(t) = t, g(t) = t$, $a = b = c = 1$.

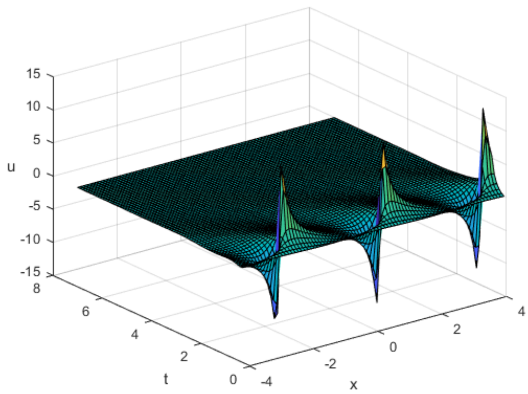

Figure 7. Periodic solitary wave solution u_{11} as $a_1 = 1, a_2 = ia_1$, $k_1 = m = \alpha = \beta = 1, y = z = 0, f(t) = h(t) = t, g(t) = t$, $a = b = c = 1$.

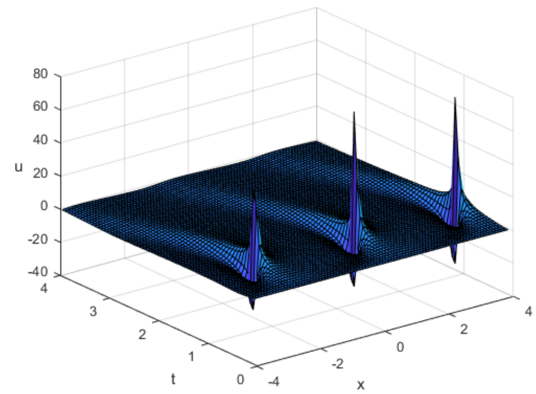

Figure 8. Periodic solitary wave solution u_{13} as $a_1 = 1$, $a_2 = ia_1, k_1 = k_2 = m = \alpha = \beta = 1, y = z = 0$, $f(t) = h(t) = t, g(t) = t, a = b = c = 1$.

In the above figures, Figure 2, Figure 4 and Figure 6 all express singular solitary wave type. u_2 is singular in a large interval. u_7 and u_{10} are singular in a small interval.

4. Conclusions

In conclusion, the extended homoclinic test approach (EHTA), which is based on the bilinear form of nonlinear partial differential equations, is a fairly effective method to seek solutions. Applying extended homoclinic test approach, four kinds of solutions, including some new types of special solutions such as breather type of soliton and two soliton, periodic type of soliton solutions and so on, can be obtained. Shang [8, 9] proposed the idea of combining variable separation method with the extended homoclinic test technique for solving higher-dimensional nonlinear partial differential equations. They got sixteen solutions for (3+1)-dimensional potential-YTSF equations. The method used in [8, 9] is more effective.

In this paper, by using extended homoclinic test approach and variable separation method, we obtain abundant exact non-traveling wave solutions of the (3+1)-dimensional variable coefficients Date-Jimbo-Kashiwara-Miwa (VC-DJKM) equation. Firstly, we apply the multi-linear variable separation approach to reduce (3+1)-dimensional VC-DJKM equation (1.1) to some (1+1)-dimensional nonlinear equation with variable coefficients. Then, by discussion on the type of function $\theta(y, t)$ and introducing an appropriate transformation, we simplify the variable coefficients nonlinear equation obtained above to a constant coefficients equation. Furthermore, with the help of Maple, we solve the simplified equation by the extended homoclinic test approach and obtain sixteen kinds of non-traveling exact solutions for the (3+1)-dimensional VC-DJKM equation (1.1). At last, we analyse the properties of solutions obtained in our paper by graphic and explain the importance of these solutions in mathematics and physics.

Especially, if $g(t)$, $h(t)$, a, b, c are taken some special value, Eq (1.1) reduces to the (3+1)-dimensional DJKM equation (1.2), (2+1)-dimensional DJKM equation (1.3) and (2+1)-dimensional VC-DJKM equation (1.4). In the same way to solving Eq (1.1), we can get abundant non-traveling solutions to these equations respectively. Moreover, $f(t)$ is an arbitrary first order derivable function in u_1-u_{16}. When taking the arbitrary function $f(t)$ as specific constant or function, we can derive rich exact non-traveling wave solutions for Eq (1.1). Also, if taking $f(t)$ = Constant, we can obtain abundant exact traveling wave solutions of Eq (1.1) with $g(t)$ = Constant. The results obtained in our work are the supplement and extension of results of the existing literatures. From our abundant results obtained in this paper, the methods applied here have been proved to be fairly effective method for seeking non-traveling wave solutions of higher-dimensional nonlinear partial differential equations. It is expected that our results are helpful for theoretical study of the associated higher-dimensional nonlinear partial differential equations in mathematical physics.

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Conflict of interest

The authors declare that they have no competing interests.

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