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Research article

Lyapunov-type inequalities for Hadamard fractional differential equation under Sturm-Liouville boundary conditions

Youyu Wang*, **Lu Zhang and Yang Zhang**

Department of Mathematics, Tianjin University of Finance and Economics, Tianjin 300222, P. R. China

* **Correspondence:** Email: wang_youyu@163.com; Tel: +008602288186306.

Abstract: In this paper, we establish new Lyapunov-type inequalities for a Hadamard fractional differential equation under Sturm-Liouville boundary conditions. Our conclusions cover many results in the literature.

Keywords: Hadamard fractional derivative; Lyapunov-type inequality; boundary value problem; Green's function
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1. Introduction

For the second order Hill's equation,

$$u''(t) + r(t)u(t) = 0, \quad t \in (a, b), \tag{1.1}$$

where $r(t)$ is a positive continuous function on $[a, b]$, Lyapunov [1] obtain the interesting following result.

Theorem 1.1. *If Eq (1.1) has a nontrivial solution $u(t)$ satisfying $u(a) = u(b) = 0$, then*

$$\int_a^b r(t)\mathrm{d}t > \frac{4}{b-a}. \tag{1.2}$$

Because of the importance of Lyapunov inequality (1.2) in application, Lyapunov inequality has been extended in many directions. In the last few decades, with the increasing enthusiasm for the study of fractional differential equations, a large number of Lyapunov inequalities for fractional differential equations have appeared. For example, Ferreira [2] first obtained a Lyapunov-type inequality for Riemann-Liouville fractional differential equation, in 2014, Ferreira [3] developed a

Lyapunov-type inequality for the Caputo fractional differential equation. For more details on Lyapunov-type inequalities and their applications, we refer [4–15] and the references therein.

Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard [16]. Hadamard-type integrals arise in the formulation of many problems in mechanics such as in fracture analysis. For details and applications of Hadamard fractional derivative and integral, we refer the reader to the works in [17–22]. It must be noted that there are few papers studied Lyapunov-type inequality for Hadamard fractional differential equation. For instance, Ma et al. [23] developed a Lyapunov-type inequality for the Hadamard fractional boundary value problem in 2017.

Theorem 1.2. *If the Hadamard fractional boundary value problem*

$$({}^H D^{\alpha}_{1+}u)(t) + q(t)u(t) = 0, \quad 1 < t < e,\ 1 < \alpha \le 2, \tag{1.3}$$

$$u(1) = 0 = u(e), \tag{1.4}$$

has a nontrivial solution, where $q : [1, e] \to \mathbb{R}$ is a continuous function, then

$$\int_1^e |q(s)|\mathrm{d}s > \Gamma(\alpha)\lambda^{1-\alpha}(1-\lambda)^{1-\alpha}e^{\lambda}, \tag{1.5}$$

where $\lambda = \frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}$ and ${}^H D^{\alpha}_{a+}$ denotes the Hadamard fractional derivative of order α.

Recently, Dhar [24] and Laadjal et al. [25] generalized the Lyapunov-type inequality in Theorem 1.2 by replacing the interval $[1, e]$ with a general interval $[a, b](1 \le a < b)$,.

Theorem 1.3. *If the Hadamard fractional boundary value problem*

$$({}^H D^{\alpha}_{a+}u)(t) + q(t)u(t) = 0, \quad 1 \le a < t < b,\ 1 < \alpha \le 2, \tag{1.6}$$

$$u(a) = 0 = u(b), \tag{1.7}$$

has a nontrivial solution, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|\mathrm{d}s > 4^{\alpha-1}\Gamma(\alpha)a\left(\ln\frac{b}{a}\right)^{1-\alpha}. \tag{1.8}$$

Theorem 1.4. *If the Hadamard fractional boundary value problem*

$$({}^H D^{\alpha}_{a+}u)(t) + q(t)u(t) = 0, \quad 1 \le a < t < b,\ 1 < \alpha \le 2, \tag{1.9}$$

$$u(a) = 0 = u(b), \tag{1.10}$$

has a nontrivial solution, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|\mathrm{d}s > \Gamma(\alpha)\xi_1\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{1-\alpha}, \tag{1.11}$$

where

$$\xi_1 = \exp\left(\frac{1}{2}\left[[2(\alpha-1) + \ln ba] - \sqrt{4(\alpha-1)^2 + \ln^2\frac{b}{a}}\right]\right).$$

Very recently, J. Jonnalagadda and B. Debananda [26] obtained Lyapunov-type inequalities for Hadamard fractional boundary value problems associated with different sets of boundary conditions, the main results are as follows.

Theorem 1.5. *If the following fractional boundary value problem*

$$({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2, \tag{1.12}$$
$$l({}^H I_{a+}^{2-\alpha} u)(a) - m({}^H D_{a+}^{\alpha-1} u)(a) = 0, \tag{1.13}$$
$$nu(b) + p({}^H D_{a+}^{\alpha-1} u)(b) = 0, \tag{1.14}$$

has a nontrivial solution, then

$$\int_a^b (\ln \frac{s}{a})^{\alpha-2} |q(s)| \mathrm{d}s > \frac{A\Gamma(\alpha)}{[n(\ln \frac{b}{a})^{\alpha-1} + p\Gamma(\alpha)][l(\ln \frac{b}{a}) + m(\alpha-1)]}, \tag{1.15}$$

where $l, p \ge 0; m, n > 0$ *and*

$$A = ln\left(\ln \frac{b}{a}\right)^{\alpha-1} + mn(\alpha-1)\left(\ln \frac{b}{a}\right)^{\alpha-2} + lp\Gamma(\alpha) > 0.$$

Theorem 1.6. *If the following fractional boundary value problem*

$$({}^H D_{1+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 1 < t < T,\ 1 < \alpha \le 2, \tag{1.16}$$
$$({}^H I_{1+}^{2-\alpha} u)(1) + ({}^H I_{1+}^{2-\alpha} u)(T) = 0, \tag{1.17}$$
$$({}^H D_{1+}^{\alpha-1} u)(1) + ({}^H D_{1+}^{\alpha-1} u)(T) = 0, \tag{1.18}$$

has a nontrivial solution, then

$$\int_1^T (\ln \frac{s}{a})^{\alpha-2} |q(s)| \mathrm{d}s > \frac{4\Gamma(\alpha)}{(3-\alpha)\ln T}. \tag{1.19}$$

Inspired by papers [24–26], in this paper, we establish a few Lyapunov-type inequalities for Hadamard fractional differential equations

$$({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2, \tag{1.20}$$

with the following Sturm-Liouville multi-point and integral boundary conditions,

$$u(a) = 0, \quad \gamma u(b) + \delta u'(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \tag{1.21}$$

$$u(a) = 0, \quad \gamma u(b) + \delta u'(b) = \lambda \int_a^b h(s)u(s)\mathrm{d}s, \quad \lambda \ge 0, \tag{1.22}$$

where $\gamma \ge 0, \delta \ge 0, \gamma\delta > 0$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $\beta_i \ge 0 (i = 1, 2, \cdots, m-2)$ and $h : [a, b] \to [0, \infty)$ with $h \in L^1(a, b)$.

The main difficulty of this paper is to express the solution for boundary value problems (1.20) and (1.21) with Green's function. We solve this problem by properly decomposing coefficients. For convenience, we shall adopt the following notations and assumptions

$$\sigma = \sqrt{(\alpha-1)^2 + \left(\ln\sqrt{\frac{b}{a}}\right)^2},$$
$$\rho_1 = \gamma\left(\ln\frac{b}{a}\right)^{\alpha-1} + (\alpha-1)\frac{\delta}{b}\left(\ln\frac{b}{a}\right)^{\alpha-2} > 0,$$
$$\rho_2 = \gamma\left(\ln\frac{b}{a}\right)^{\alpha-1} + (\alpha-1)\frac{\delta}{b}\left(\ln\frac{b}{a}\right)^{\alpha-2} - \sum_{i=1}^{m-2}\beta_i\left(\ln\frac{\xi_i}{a}\right)^{\alpha-1} > 0,$$
$$\rho_3 = \gamma\left(\ln\frac{b}{a}\right)^{\alpha-1} + (\alpha-1)\frac{\delta}{b}\left(\ln\frac{b}{a}\right)^{\alpha-2} - \lambda\int_a^b\left(\ln\frac{t}{a}\right)^{\alpha-1}h(t)\mathrm{d}t > 0.$$

2. Preliminaries

In this paper, we shall use the following notations, definitions and some lemmas from the theory of fractional calculus in the sense of Hadamard. For more details, we refer to [27].

Definition 2.1. [27] The Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ for a continuous function $f : [a, b] \to \mathbb{R}$ is defined by

$$({}^H I^{\alpha}_{a+} f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s)\frac{\mathrm{d}s}{s}, \qquad \alpha > 0, \quad t \in [a, b].$$

Definition 2.2. [27] The Hadamard fractional derivative of order $\alpha \in \mathbb{R}_+$ for a continuous function $f : [a, b] \to \mathbb{R}$ is defined by

$$({}^H D^{\alpha}_{a+} f)(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} f(s)\frac{\mathrm{d}s}{s}, \qquad t \in [a, b],$$

where $n - 1 < \alpha < n, \quad n = [\alpha] + 1$.

Lemma 2.3. *[27] Let $\alpha > 0, n = [\alpha]+1$ and $0 < a < b < \infty$. if $u \in L^1(a, b)$ and $({}^H I^{n-\alpha}_{a+} u)(t) \in AC^n_\delta[a, b]$, then*

$$({}^H I^{\alpha}_{a+}\,{}^H D^{\alpha}_{a+} u)(t) = u(t) - \sum_{k=1}^{n}\frac{(\delta^{(n-k)}({}^H I^{n-\alpha}_{a+} u))(a)}{\Gamma(\alpha-k+1)}\left(\ln\frac{t}{a}\right)^{\alpha-k}.$$

where $AC^n_\delta[a, b] = \left\{\varphi : [a, b] \to \mathbb{C} : \delta^{(n-1)}\varphi \in AC[a, b], \delta = t\frac{d}{dt}\right\}$.

Lemma 2.4. *Assume that $r \in C[a, b]$. The Sturm-Liouville Hadamard fractional boundary value problem*

$$\begin{cases} ({}^H D^{\alpha}_{a+} u)(t) + r(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2, \\ u(a) = 0, \quad \gamma u(b) + \delta u'(b) = \sum_{i=1}^{m-2}\beta_i u(\xi_i), \end{cases}$$

has the unique solution

$$u(t) = \int_a^b G(t,s)r(s)\mathrm{d}s + \frac{1}{\rho_2}\left(\ln\frac{t}{a}\right)^{\alpha-1}\sum_{i=1}^{m-2}\beta_i\int_a^b G(\xi_i,s)r(s)\mathrm{d}s,$$

where $G(t,s)$ *is given by*

$$G(t,s) = \frac{1}{s\rho_1\Gamma(\alpha)}\begin{cases}\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]-\rho_1\left(\ln\frac{t}{s}\right)^{\alpha-1}, & a\le s\le t\le b,\\ \left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right], & a\le t\le s\le b.\end{cases}$$

Proof. Using Lemma 2.3, we have

$$u(t) = c_1\left(\ln\frac{t}{a}\right)^{\alpha-1} + c_2\left(\ln\frac{t}{a}\right)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}r(s)\frac{\mathrm{d}s}{s},$$

for some $c_1, c_2 \in \mathbb{R}$. Applying the boundary condition $u(a) = 0$, we have $c_2 = 0$, hence,

$$u(t) = c_1\left(\ln\frac{t}{a}\right)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}r(s)\frac{\mathrm{d}s}{s},$$

it is easy to obtain

$$\begin{aligned}u'(t) &= c_1(\alpha-1)\frac{1}{t}\left(\ln\frac{t}{a}\right)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_a^t(\alpha-1)\frac{1}{t}\left(\ln\frac{t}{s}\right)^{\alpha-2}r(s)\frac{\mathrm{d}s}{s}\\ &= \frac{\alpha-1}{t}\left[c_1\left(\ln\frac{t}{a}\right)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-2}r(s)\frac{\mathrm{d}s}{s}\right],\end{aligned}$$

the boundary condition $\gamma u(b) + \delta u'(b) = \sum_{i=1}^{m-2}\beta_i u(\xi_i)$ imply that,

$$\begin{aligned}&\gamma\left[c_1\left(\ln\frac{b}{a}\right)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-1}r(s)\frac{\mathrm{d}s}{s}\right]\\ &\quad+(\alpha-1)\frac{\delta}{b}\left[c_1\left(\ln\frac{b}{a}\right)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-2}r(s)\frac{\mathrm{d}s}{s}\right]\\ &=\sum_{i=1}^{m-2}\beta_i\left[c_1\left(\ln\frac{\xi_i}{a}\right)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_a^{\xi_i}\left(\ln\frac{\xi_i}{s}\right)^{\alpha-1}r(s)\frac{\mathrm{d}s}{s}\right],\end{aligned}$$

we obtain

$$c_1 = \frac{1}{\rho_2\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]r(s)\frac{\mathrm{d}s}{s} - \frac{1}{\rho_2\Gamma(\alpha)}\sum_{i=1}^{m-2}\beta_i\int_a^{\xi_i}(\ln\frac{\xi_i}{s})^{\alpha-1}r(s)\frac{\mathrm{d}s}{s},$$

by the relation

$$\frac{1}{\rho_2} = \frac{1}{\rho_1} + \frac{\sum_{i=1}^{m-2}\beta_i\left(\ln\frac{\xi_i}{a}\right)^{\alpha-1}}{\rho_1\rho_2},$$

we have

$$
\begin{aligned}
c_1 =&\frac{1}{\Gamma(\alpha)}\left(\frac{1}{\rho_1}+\frac{\sum_{i=1}^{m-2}\beta_i(\ln\frac{\xi_i}{a})^{\alpha-1}}{\rho_1\rho_2}\right)\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]r(s)\frac{\mathrm{d}s}{s}\\
&-\frac{1}{\rho_2\Gamma(\alpha)}\sum_{i=1}^{m-2}\beta_i\int_a^{\xi_i}\left(\ln\frac{\xi_i}{s}\right)^{\alpha-1}r(s)\frac{\mathrm{d}s}{s}\\
=&\frac{1}{\rho_1\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]r(s)\frac{\mathrm{d}s}{s}\\
&+\frac{1}{\rho_1\rho_2\Gamma(\alpha)}\sum_{i=1}^{m-2}\beta_i\int_a^b\left(\ln\frac{\xi_i}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]r(s)\frac{\mathrm{d}s}{s}\\
&-\frac{1}{\rho_2\Gamma(\alpha)}\sum_{i=1}^{m-2}\beta_i\int_a^{\xi_i}\left(\ln\frac{\xi_i}{s}\right)^{\alpha-1}r(s)\frac{\mathrm{d}s}{s}\\
=&\frac{1}{\rho_1\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]r(s)\frac{\mathrm{d}s}{s}+\frac{1}{\rho_2}\sum_{i=1}^{m-2}\beta_i\int_a^b G(\xi_i,s)r(s)\mathrm{d}s,
\end{aligned}
$$

therefore,

$$
\begin{aligned}
u(t) =&c_1\left(\ln\frac{t}{a}\right)^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}r(s)\frac{\mathrm{d}s}{s}\\
=&\frac{1}{\rho_1\Gamma(\alpha)}\int_a^b\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]r(s)\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\rho_2}\left(\ln\frac{t}{a}\right)^{\alpha-1}\sum_{i=1}^{m-2}\beta_i\int_a^b G(\xi_i,s)h(s)\mathrm{d}s-\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}r(s)\frac{\mathrm{d}s}{s}\\
=&\int_a^b G(t,s)h(s)\mathrm{d}s+\frac{1}{\rho_2}\left(\ln\frac{t}{a}\right)^{\alpha-1}\sum_{i=1}^{m-2}\beta_i\int_a^b G(\xi_i,s)r(s)\mathrm{d}s.
\end{aligned}
$$

The proof is complete. □

Lemma 2.5. *Assume that $g\in C[a,b]$. The Sturm-Liouville Hadamard fractional boundary value problem*

$$
\begin{cases}
({}^H D_{a+}^{\alpha}u)(t)+g(t)=0, \quad 0<a<t<b,\ 1<\alpha\le 2,\\
u(a)=0, \quad \gamma u(b)+\delta u'(b)=\lambda\int_a^b h(s)u(s)\mathrm{d}s, \quad \lambda\ge 0,
\end{cases}
$$

has the unique solution

$$
u(t)=\int_a^b G(t,s)g(s)\mathrm{d}s+\frac{\lambda}{\rho_3}\left(\ln\frac{t}{a}\right)^{\alpha-1}\int_a^b\left(\int_a^b G(t,s)g(s)ds\right)h(t)\mathrm{d}t,
$$

where $h:[a,b]\to[0,\infty)$ with $h\in L^1(a,b)$, $G(t,s)$ is defined in Lemma 2.4.

Proof. Using Lemma 2.3, we have

$$u(t) = c_1\left(\ln\frac{t}{a}\right)^{\alpha-1} + c_2\left(\ln\frac{t}{a}\right)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\frac{\mathrm{d}s}{s},$$

for some $c_1, c_2 \in \mathbb{R}$. Using the boundary condition $u(a) = 0$, we have $c_2 = 0$, therefore,

$$u(t) = c_1\left(\ln\frac{t}{a}\right)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\frac{\mathrm{d}s}{s},$$

and

$$u'(t) = \frac{\alpha-1}{t}\left[c_1\left(\ln\frac{t}{a}\right)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-2} g(s)\frac{\mathrm{d}s}{s}\right],$$

the boundary condition $\gamma u(b) + \delta u'(b) = \lambda\int_a^b h(t)u(t)\mathrm{d}t$ imply that,

$$\begin{aligned}&\gamma\left[c_1\left(\ln\frac{b}{a}\right)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-1} g(s)\frac{\mathrm{d}s}{s}\right]\\ &+(\alpha-1)\frac{\delta}{b}\left[c_1\left(\ln\frac{b}{a}\right)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-2} g(s)\frac{\mathrm{d}s}{s}\right] = \lambda\int_a^b h(t)u(t)\mathrm{d}t,\end{aligned}$$

we obtain

$$c_1 = \frac{1}{\rho_1\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s} + (\alpha-1)\frac{\delta}{b}\right]g(s)\frac{\mathrm{d}s}{s} + \frac{\lambda}{\rho_1}\int_a^b h(t)u(t)\mathrm{d}t,$$

therefore, the solution of the boundary value problem is

$$\begin{aligned}u(t) &= c_1\left(\ln\frac{t}{a}\right)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\frac{\mathrm{d}s}{s}\\ &=\frac{1}{\rho_1\Gamma(\alpha)}\int_a^b\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s} + (\alpha-1)\frac{\delta}{b}\right]h(s)\frac{\mathrm{d}s}{s}\\ &\quad+\frac{\lambda}{\rho_1}\left(\ln\frac{t}{a}\right)^{\alpha-1}\int_a^b h(t)u(t)\mathrm{d}t - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\frac{\mathrm{d}s}{s}\\ &=\int_a^b G(t,s)g(s)\mathrm{d}s + \frac{\lambda}{\rho_1}\left(\ln\frac{t}{a}\right)^{\alpha-1}\int_a^b h(t)u(t)\mathrm{d}t.\end{aligned}$$

Multiplying both side of above equation by $h(t)$ and integrating from a to b, we obtain

$$\int_a^b h(t)u(t)\mathrm{d}t = \int_a^b\left(\int_a^b G(t,s)g(s)ds\right)h(t)\mathrm{d}t + \frac{\lambda}{\rho_1}\int_a^b h(t)u(t)\mathrm{d}t\cdot\int_a^b\left(\ln\frac{t}{a}\right)^{\alpha-1}h(t)\mathrm{d}t,$$

and

$$\int_a^b h(t)u(t)\mathrm{d}t = \frac{\rho_1}{\rho_3}\int_a^b\left(\int_a^b G(t,s)g(s)\mathrm{d}s\right)h(t)\mathrm{d}t,$$

thus

$$u(t) = \int_a^b G(t,s)g(s)\mathrm{d}s + \frac{\lambda}{\rho_3}\left(\ln\frac{t}{a}\right)^{\alpha-1}\int_a^b\left(\int_a^b G(t,s)g(s)\mathrm{d}s\right)h(t)\mathrm{d}t,$$

which concludes the proof. □

Lemma 2.6. *The function G defined in Lemma 2.4 satisfies the following properties:*

1). $G(t,s) \geq 0 \quad on \quad [a,b] \times [a,b]$,

2). $\max_{t\in[a,b]} G(t,s) = G(s,s) = \dfrac{1}{s\rho_1\Gamma(\alpha)}\left(\ln\dfrac{s}{a}\right)^{\alpha-1}\left(\ln\dfrac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\dfrac{b}{s} + (\alpha-1)\dfrac{\delta}{b}\right].$

Proof. **(1).** Firstly, we define two functions as follows

$$g_1(t,s) = \frac{1}{s\rho_1\Gamma(\alpha)}\left[\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s} + (\alpha-1)\frac{\delta}{b}\right] - \rho_1\left(\ln\frac{t}{s}\right)^{\alpha-1}\right], \quad a \leq s \leq t \leq b,$$
$$g_2(t,s) = \frac{1}{s\rho_1\Gamma(\alpha)}\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s} + (\alpha-1)\frac{\delta}{b}\right], \quad a \leq t \leq s \leq b.$$

Obviously, $g_2(t,s) \geq 0$. Next, we consider function $g_1(t,s)$, differentiating $g_1(t,s)$ with respect to t, we get

$$\begin{aligned}\frac{\partial g_1(t,s)}{\partial t} =& \frac{\alpha-1}{st\rho_1\Gamma(\alpha)}\left[\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-2}[\gamma\ln\frac{b}{s} + (\alpha-1)\frac{\delta}{b}] - \rho_1\left(\ln\frac{t}{s}\right)^{\alpha-2}\right]\\ =& \frac{\alpha-1}{st\rho_1\Gamma(\alpha)}\left\{\gamma\left[\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-1} - \left(\ln\frac{t}{s}\right)^{\alpha-2}\left(\ln\frac{b}{a}\right)^{\alpha-1}\right]\right.\\ &\left. + (\alpha-1)\frac{\delta}{b}\left[\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-2} - \left(\ln\frac{t}{s}\right)^{\alpha-2}\left(\ln\frac{b}{a}\right)^{\alpha-2}\right]\right\},\end{aligned}$$

since $a \leq s \leq t \leq b$, we have

$$\left(\ln\frac{t}{a}\right)^{\alpha-2} < \left(\ln\frac{t}{s}\right)^{\alpha-2}, \quad \left(\ln\frac{b}{s}\right)^{\alpha-1} < \left(\ln\frac{b}{a}\right)^{\alpha-1},$$

which imply that

$$\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-1} - \left(\ln\frac{t}{s}\right)^{\alpha-2}\left(\ln\frac{b}{a}\right)^{\alpha-1} < 0.$$

The inequality

$$\begin{aligned}\ln\frac{t}{a}\ln\frac{b}{s} - \ln\frac{t}{s}\ln\frac{b}{a} &= \ln\frac{t}{a}\ln\frac{b}{s} - \ln\frac{t}{s}\left(\ln\frac{b}{s} + \ln\frac{s}{a}\right)\\ &= \ln\frac{b}{s}\left(\ln\frac{t}{a} - \ln\frac{t}{s}\right) - \ln\frac{s}{a}\ln\frac{t}{s}\\ &= \ln\frac{b}{s}\ln\frac{s}{a} - \ln\frac{s}{a}\ln\frac{t}{s} = \ln\frac{s}{a}\ln\frac{b}{t} \geq 0,\end{aligned}$$

implying that

$$\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-2} - \left(\ln\frac{t}{s}\right)^{\alpha-2}\left(\ln\frac{b}{a}\right)^{\alpha-2} < 0,$$

so we have $\frac{\partial g_1(t,s)}{\partial t} \leq 0$, this means that $g_1(b,s) \leq g_1(t,s) \leq g_1(s,s)$. On the other hand,

$$\begin{aligned} s\rho_1\Gamma(\alpha)g_1(b,s) =& \left(\ln\frac{b}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]-\rho_1\left(\ln\frac{b}{s}\right)^{\alpha-1} \\ =&(\alpha-1)\frac{\delta}{b}\left[\left(\ln\frac{b}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}-\left(\ln\frac{b}{s}\right)^{\alpha-1}\left(\ln\frac{b}{a}\right)^{\alpha-2}\right] \\ =&(\alpha-1)\frac{\delta}{b}\left(\ln\frac{b}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\ln\frac{b}{a}-\ln\frac{b}{s}\right] \\ =&(\alpha-1)\frac{\delta}{b}\left(\ln\frac{b}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-2}\ln\frac{s}{a}\geq 0, \end{aligned}$$

hence, $g_1(t,s) \geq g_1(b,s) \geq 0$ and $G(t,s) \geq 0$.

(2). By the above discussion, for function $g_1(t,s)$ satisfy $0 \leq g_1(t,s) \leq g_1(s,s) = G(s,s)$. It is easy to see that $0 \leq g_2(t,s) \leq g_2(s,s) = g_1(s,s) = G(s,s)$ for $a \leq t \leq s \leq b$. □

Lemma 2.7. *Assume* $0 < a \leq s \leq b$ *and* $1 < \alpha < 2$*, then*

$$0 \leq \frac{1}{s}\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1} \leq \frac{1}{\sqrt{ab}}\cdot\frac{(\alpha-1)^{\alpha-1}(\ln\frac{b}{a})^{2(\alpha-1)}e^{\sigma}}{(2e)^{\alpha-1}\,(\alpha-1+\sigma)^{\alpha-1}}.$$

Proof. Let

$$f(s) = \frac{1}{s}\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1},\ \ s\in[a,b].$$

Clearly $f(a) = f(b) = 0$ and $f(s) > 0$ on (a,b). By Rolle's Theorem, there exists $s^* \in (a,b)$ such that $f(s^*) = \max f(s)$ on (a,b), i.e., $f'(s^*) = 0$. Note that

$$f'(s) = \frac{1}{s^2}\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}\left[(\alpha-1)\frac{\ln\frac{b}{s}-\ln\frac{s}{a}}{\ln\frac{s}{a}\ln\frac{b}{s}}-1\right],$$

denote $x = \ln\frac{b}{s}$, $y = \ln\frac{s}{a}$, let $f'(s) = 0$, we obtain $(\alpha-1)(x-y) = xy$ and $x+y = \ln\frac{b}{a}$, by these two equalities, we get

$$(xy)^2 = (\alpha-1)^2(x-y)^2 = (\alpha-1)^2[(x+y)^2-4xy] = (\alpha-1)^2[(\ln\frac{b}{a})^2-4xy],$$

note the fact $xy > 0$, we have

$$\begin{aligned} xy =& (\alpha-1)\left[-2(\alpha-1)+\sqrt{4(\alpha-1)^2+\left(\ln\frac{b}{a}\right)^2}\right] \\ =& \frac{(\alpha-1)(\ln\frac{b}{a})^2}{2(\alpha-1)+\sqrt{4(\alpha-1)^2+\left(\ln\frac{b}{a}\right)^2}}, \end{aligned}$$

and

$$y = \frac{1}{2}\left(\ln\frac{b}{a}-\frac{xy}{\alpha-1}\right)$$

$$= \frac{1}{2}\left(\ln\frac{b}{a} + 2(\alpha-1) - \sqrt{4(\alpha-1)^2 + \left(\ln\frac{b}{a}\right)^2}\right)$$
$$= \ln\sqrt{\frac{b}{a}} + (\alpha-1) - \sqrt{(\alpha-1)^2 + \left(\ln\sqrt{\frac{b}{a}}\right)^2},$$

so,

$$s^* = ae^y = \frac{\sqrt{ab}e^{\alpha-1}}{e^{\sqrt{(\alpha-1)^2+\left(\ln\sqrt{\frac{b}{a}}\right)^2}}},$$

it is easy to show that $s^* \in (a,b)$, thus

$$\max f(s) = f(s^*) = \frac{1}{s^*}\left(\ln\frac{s^*}{a}\ln\frac{b}{s^*}\right)^{\alpha-1}$$
$$= \frac{1}{\sqrt{ab}} \cdot \frac{(\alpha-1)^{\alpha-1}\left(\ln\frac{b}{a}\right)^{2(\alpha-1)} e^{\sqrt{(\alpha-1)^2+\left(\ln\sqrt{\frac{b}{a}}\right)^2}}}{(2e)^{\alpha-1}\left(\alpha-1+\sqrt{(\alpha-1)^2+\left(\ln\sqrt{\frac{b}{a}}\right)^2}\right)^{\alpha-1}}.$$

which concludes the proof. □

3. Main results

We now present Lyapunov-type inequalities for the Sturm-Liouville Hadamard fractional boundary value problems (1.20) and (1.21).

Theorem 3.1. *If a nontrivial continuous solution of the Sturm-Liouville Hadamard fractional boundary value problem*

$$({}^H D^{\alpha}_{a+}u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2,$$
$$u(a) = 0, \quad \gamma u(b) + \delta u'(b) = \sum_{i=1}^{m-2}\beta_i u(\xi_i),$$

exists, where $q : [a,b] \to \mathbb{R}$ is a continuous function, and $\gamma \ge 0, \delta \ge 0, \gamma\delta > 0$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $\beta_i \ge 0 (i = 1,2,\cdots,m-2)$, then we have

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s} + (\alpha-1)\frac{\delta}{b}\right]|q(s)|\mathrm{d}s \ge \frac{\rho_1\rho_2}{\rho_2 + \sum_{i=1}^{m-2}\beta_i(\ln\frac{b}{a})^{\alpha-1}}\Gamma(\alpha). \tag{3.1}$$

Proof. Let $\mathbf{B} = C[a,b]$ be the Banach space endowed with norm $\|x\| = \sup_{t\in[a,b]}|x(t)|$. From Lemma 2.4, for all $t \in [a,b]$, we have

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds + \frac{1}{\rho_2}\left(\ln\frac{t}{a}\right)^{\alpha-1}\sum_{i=1}^{m-2}\beta_i\int_a^b G(\xi_i,s)q(s)u(s)\mathrm{d}s,$$

Now, an application of Lemma 2.6 yields

$$\begin{aligned} \|u\| &\leq \|u\|\left(\int_a^b G(s,s)|q(s)|\mathrm{d}s + \frac{1}{\rho_2}\sum_{i=1}^{m-2}\beta_i\left(\ln\frac{b}{a}\right)^{\alpha-1}\int_a^b G(s,s)|q(s)|\mathrm{d}s\right) \\ &= \|u\|\left(1+\frac{1}{\rho_2}\sum_{i=1}^{m-2}\beta_i\left(\ln\frac{b}{a}\right)^{\alpha-1}\right)\int_a^b G(s,s)|q(s)|\mathrm{d}s, \end{aligned}$$

Since u is non trivial, then $\|u\| \neq 0$, so

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}]|q(s)|\mathrm{d}s \geq \frac{\rho_1\rho_2}{\rho_2+\sum_{i=1}^{m-2}\beta_i\left(\ln\frac{b}{a}\right)^{\alpha-1}}\Gamma(\alpha).$$

from which inequality in (3.1) follows. □

Let $\beta_i = 0 (i = 1, 2, \cdots, m-2)$ in Theorem 3.1, we have

Corollary 3.2. *If a nontrivial continuous solution of the Sturm-Liouville Hadamard fractional boundary value problem*

$$\begin{aligned} &({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \leq 2, \\ &u(a) = 0, \quad \gamma u(b) + \delta u'(b) = 0, \end{aligned}$$

exists, where $q : [a,b] \to \mathbb{R}$ is a continuous function, and $\gamma \geq 0, \delta \geq 0, \gamma\delta > 0$, then we have

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]|q(s)|\mathrm{d}s \geq \Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-2}\left[\gamma\ln\frac{b}{a}+(\alpha-1)\frac{\delta}{b}\right]. \tag{3.2}$$

Let $\gamma = 1, \delta = 0$ or $\gamma = 0, \delta = 1$ in Corollary 3.2, we can obtain the following Lyapunov-type inequalities.

Corollary 3.3. *If a nontrivial continuous solution of the Hadamard fractional boundary value problem*

$$\begin{aligned} &({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \leq 2, \\ &u(a) = 0, \quad u(b) = 0, \end{aligned}$$

exists, where $q : [a,b] \to \mathbb{R}$ is a continuous function, then we have

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|\mathrm{d}s \geq \Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}. \tag{3.3}$$

This is Theorem 1 in [25].

Corollary 3.4. *If a nontrivial continuous solution of the Hadamard fractional boundary value problem*

$$\begin{aligned} &({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \leq 2, \\ &u(a) = 0, \quad u'(b) = 0, \end{aligned}$$

exists, where $q : [a,b] \to \mathbb{R}$ is a continuous function, then we have

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}|q(s)|\mathrm{d}s \geq \Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-2}. \tag{3.4}$$

Let $\gamma = 1, \delta = 0$ in Theorem 3.1, we have the following Lyapunov-type inequality.

Corollary 3.5. *If a nontrivial continuous solution of the Hadamard fractional multi-point boundary value problem*

$$\begin{aligned} &({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2, \\ &u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{aligned}$$

exists, where $q : [a, b] \to \mathbb{R}$ *is a continuous function,* $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $\beta_i \ge 0 (i = 1, 2, \cdots, m-2)$ *with* $0 \le \sum_{i=1}^{m-2} \beta_i < 1$*, then we have*

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1} |q(s)| \mathrm{d}s \ge \frac{(\ln\frac{b}{a})^{\alpha-1}[(\ln\frac{b}{a})^{\alpha-1} - \sum_{i=1}^{m-2}\beta_i(\ln\frac{\xi_i}{a})^{\alpha-1})]}{(\ln\frac{b}{a})^{\alpha-1} - \sum_{i=1}^{m-2}\beta_i(\ln\frac{\xi_i}{a})^{\alpha-1} + \sum_{i=1}^{m-2}\beta_i(\ln\frac{b}{a})^{\alpha-1}}\Gamma(\alpha). \tag{3.5}$$

Let $\gamma = 0, \delta = 1$ in Theorem 3.1, we have the following Lyapunov-type inequality.

Corollary 3.6. *If a nontrivial continuous solution of the Hadamard fractional multi-point boundary value problem*

$$\begin{aligned} &({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2, \\ &u(a) = 0, \quad u'(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{aligned}$$

exists, where $q : [a, b] \to \mathbb{R}$ *is a continuous function,* $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $\beta_i \ge 0 (i = 1, 2, \cdots, m-2)$ *with* $0 \le \sum_{i=1}^{m-2} \beta_i < 1$*, then we have*

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2} |q(s)| \mathrm{d}s \ge \frac{(\ln\frac{b}{a})^{\alpha-2}[\frac{\alpha-1}{b}(\ln\frac{b}{a})^{\alpha-2} - \sum_{i=1}^{m-2}\beta_i(\ln\frac{\xi_i}{a})^{\alpha-1})]}{\frac{\alpha-1}{b}(\ln\frac{b}{a})^{\alpha-2} - \sum_{i=1}^{m-2}\beta_i(\ln\frac{\xi_i}{a})^{\alpha-1} + \sum_{i=1}^{m-2}\beta_i(\ln\frac{b}{a})^{\alpha-1}}\Gamma(\alpha). \tag{3.6}$$

Let $\beta = \beta_1 \ge 0, \beta_2 = \beta_3 = \cdots = \beta_{m-2} = 0, \xi = \xi_1$ in Corollary 3.5, we obtain three-point Lyapunov-type inequality.

Corollary 3.7. *If a nontrivial continuous solution of the Hadamard fractional three-point boundary value problem*

$$\begin{aligned} &({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2, \\ &u(a) = 0, \quad u(b) = \beta u(\xi), \end{aligned}$$

exists, where $q : [a, b] \to \mathbb{R}$ *is a continuous function,* $a < \xi < b$, $0 \le \beta < 1$*, then we have*

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1} |q(s)| \mathrm{d}s \ge \frac{(\ln\frac{b}{a})^{\alpha-1}[(\ln\frac{b}{a})^{\alpha-1} - \beta(\ln\frac{\xi}{a})^{\alpha-1})]}{(\ln\frac{b}{a})^{\alpha-1} + \beta[(\ln\frac{b}{a})^{\alpha-1} - (\ln\frac{\xi}{a})^{\alpha-1}]}\Gamma(\alpha). \tag{3.7}$$

By the relation

$$\frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-2}[\gamma\ln\frac{b}{s} + (\alpha-1)\frac{\delta}{b}] = \frac{1}{s}\left(\ln\frac{s}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1} \cdot \frac{\gamma\ln\frac{b}{s} + (\alpha-1)\frac{\delta}{b}}{\ln\frac{b}{s}},$$

and applying Theorem 3.1 and Lemma 2.7, we easily get the following results.

Theorem 3.8. *If a nontrivial continuous solution of the Sturm-Liouville Hadamard fractional boundary value problem*

$$\begin{aligned}&({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2,\\ &u(a) = 0, \quad \gamma u(b) + \delta u'(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i),\end{aligned}$$

exists, where $q : [a,b] \to \mathbb{R}$ *is a continuous function, and* $\gamma \ge 0, \delta \ge 0, \gamma\delta > 0$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $\beta_i \ge 0 (i = 1, 2, \cdots, m-2)$, *then we have*

$$\int_a^b \frac{\gamma \ln \frac{b}{s} + (\alpha-1)\frac{\delta}{b}}{\ln \frac{b}{s}} |q(s)| \mathrm{d}s \ge \frac{\sqrt{ab}\rho_1\rho_2\Gamma(\alpha)}{\rho_2 + \sum_{i=1}^{m-2}\beta_i(\ln \frac{b}{a})^{\alpha-1}} \cdot \frac{(2e)^{\alpha-1}(\alpha-1+\sigma)^{\alpha-1}}{(\alpha-1)^{\alpha-1}(\ln \frac{b}{a})^{2(\alpha-1)}e^{\sigma}}. \tag{3.8}$$

A similar discussion can be made for Theorem 3.8. We omit the details here.

Next we give Lyapunov-type inequalities for Sturm-Liouville Hadamard fractional boundary value problems (1.20)–(1.22). The proof is essentially the same as that of Theorem 3.1 and Theorem 3.8. Therefore, we omit the proof.

Theorem 3.9. *If a nontrivial continuous solution of the Sturm-Liouville Hadamard fractional boundary value problem*

$$\begin{aligned}&({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2,\\ &u(a) = 0, \quad \gamma u(b) + \delta u'(b) = \lambda \int_a^b h(s)u(s)\mathrm{d}s, \quad \lambda \ge 0,\end{aligned}$$

exists, where $q : [a,b] \to \mathbb{R}$ *is a continuous function and* $h : [a,b] \to [0,\infty)$ *with* $h \in L^1(a,b)$, $\gamma \ge 0, \delta \ge 0, \gamma\delta > 0$, *then we have*

$$\int_a^b \frac{1}{s}\left(\ln \frac{s}{a}\right)^{\alpha-1}\left(\ln \frac{b}{s}\right)^{\alpha-2} [\gamma \ln \frac{b}{s} + (\alpha-1)\frac{\delta}{b}] |q(s)| \mathrm{d}s \ge \frac{\rho_1\rho_3}{\rho_3 + \lambda(\ln \frac{b}{a})^{\alpha-1}\int_a^b h(t)\mathrm{d}t}\Gamma(\alpha). \tag{3.9}$$

Theorem 3.10. *If a nontrivial continuous solution of the Sturm-Liouville Hadamard fractional boundary value problem*

$$\begin{aligned}&({}^H D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b,\ 1 < \alpha \le 2,\\ &u(a) = 0, \quad \gamma u(b) + \delta u'(b) = \lambda \int_a^b h(s)u(s)\mathrm{d}s, \quad \lambda \ge 0,\end{aligned}$$

exists, where $q : [a,b] \to \mathbb{R}$ *is a continuous function and* $h : [a,b] \to [0,\infty)$ *with* $h \in L^1(a,b)$, $\gamma \ge 0, \delta \ge 0, \gamma\delta > 0$, *then we have*

$$\int_a^b \frac{\gamma \ln \frac{b}{s} + (\alpha-1)\frac{\delta}{b}}{\ln \frac{b}{s}} |q(s)| \mathrm{d}s \ge \frac{\sqrt{ab}\rho_1\rho_3\Gamma(\alpha)}{\rho_3 + \lambda(\ln \frac{b}{a})^{\alpha-1}\int_a^b h(t)dt} \cdot \frac{(2e)^{\alpha-1}(\alpha-1+\sigma)^{\alpha-1}}{(\alpha-1)^{\alpha-1}(\ln \frac{b}{a})^{2(\alpha-1)}e^{\sigma}}. \tag{3.10}$$

Remark 3.11. The similar discussion can be made for Theorem 3.9 and Theorem 3.10. We omit the details here.

Remark 3.12. Comparisons with previous literatures, the conclusion of this paper has the following characteristics. Firstly, our results are new and include all the conclusions of [23–25]. Secondly, the result of Lemma 2.7 is better than that of Lemma 5 in [25].

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Conflict of interest

The authors declare no conflict of interest.

References

1. A. M. Lyapunov, Probleme général de la stabilité du mouvement, *Ann. Fac. Sci. Univ.*, **2** (1907), 27–247.
2. R. A. C. Ferreira, A Lyapunov-type inequality for a fractional boundary value problem, *Fract. Calc. Appl. Anal.*, **16**, (2013), 978–984.
3. R. A. C. Ferreira, On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function, *J. Math. Anal. Appl.*, **412** (2014), 1058–1063.
4. R. C. Brown, D. B. Hinton, *Lyapunov inequalities and their applications*, Dordrecht: Springer, 2000.
5. S. K. Ntouyas, B. Ahmad, T. P. Horikis, Recent developments of Lyapunov-type inequalities for fractional differential equations, In: *Differential and integral inequalities*, Cham: Springer, 2019, 619–686.
6. S. Dhar, Q. Kong, Lyapunov-type inequalities for third-order half-linear equations and applications to boundary value problems, *Nonlinear Anal.*, **110** (2014), 170–181.
7. S. Dhar, Q. Kong, Lyapunov-type inequalities for higher order half-linear differential equations, *Appl. Math. Comput.*, **273** (2016), 114–124.
8. X. Meng, M. Stynes, Green's functions, positive solutions, and a Lyapunov inequality for a Caputo fractional-derivative boundary value problem, *Fract. Calc. Appl. Anal.*, **22** (2019), 750–766.
9. M. Jleli, B. Samet, Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions, *Math. Inequal. Appl.*, **18** (2015), 443–451.
10. M. Jleli, L. Ragoub, B. Samet, Lyapunov-type inequality for a fractional differential equation under a Robin boundary conditions, *J. Funct. Space.*, **2015** (2015), 1–5.
11. A. Tiryaki, Recent development of Lyapunov-type inequalities, *Adv. Dyn. Syst. Appl.*, **5** (2010), 231–248.
12. Y. Wang, S. Liang, C. Xia, A Lyapunov-type inequality for a fractional differential equation under Sturm-Liouville boundary conditions, *Math. Inequal. Appl.*, **20** (2017), 139–148.
13. Y. Wang, Q. Wang, Lyapunov-type inequalities for fractional differential equations under multi-point boundary conditions, *J. Math. Inequal.*, **13** (2019), 611–619.
14. Y. Wang, Q. Wang, Lyapunov-type inequalities for nonlinear differential equation with Hilfer fractional derivative operator, *J. Math. Inequal.*, **12** (2018), 709–717.

15. Y. Wang, Q. Wang, Lyapunov-type inequalities for nonlinear fractional differential equation with Hilfer fractional derivative under multi-point boundary conditions, *Fract. Calc. Appl. Anal.*, **21** (2018), 833–843.
16. J. Hadamard, Essai sur l'etude des fonctions donnees par leur developpment de Taylor, *J. Math. Pure Appl.*, **8** (1892), 101–186.
17. P. L. Butzer, A. A. Kilbas, J. J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.*, **269** (2002), 387–400.
18. P. L. Butzer, A. A. Kilbas, J. J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.*, **269** (2002), 1–27.
19. P. L. Butzer, A. A. Kilbas, J. J. Trujillo, Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.*, **270** (2002), 1–15.
20. F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.*, **2012** (2012), 142.
21. A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.*, **38** (2001), 1191–1204.
22. A. A. Kilbas, J. J. Trujillo, Hadamard-type integrals as G-transforms, *Integr. Transf. Spec. F.*, **14** (2003), 413–427.
23. Q. Ma, C. Ma, J. Wang, A Lyapunov-type inequality for a fractional differential equation with Hadamard derivative, *J. Math. Inequal.*, **11** (2017), 135–141.
24. S. Dhar, On linear and non-linear fractional Hadamard boundary value problems, *Differ. Equ. Appl.*, **10** (2018), 329–339.
25. Z. Laadjal, N. Adjeroud, Q. Ma, Lyapunov-type inequality for the Hadamard fractional boundary value problem on a general interval $[a,b]$, *J. Math. Inequal.*, **13** (2019), 789–799.
26. J. Jonnalagadda, B. Debananda, Lyapunov-type inequalities for Hadamard type fractional boundary value problems, *AIMS Mathematics*, **5** (2020), 1127–1146.
27. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.

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