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Research article

The effect of multiplicative noise on the exact solutions of nonlinear Schrödinger equation

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Abstract: We consider in this paper the stochastic nonlinear Schrödinger equation forced by multiplicative noise in the Itô sense. We use two different methods as sine-cosine method and Riccati-Bernoulli sub-ODE method to obtain new rational, trigonometric and hyperbolic stochastic solutions. These stochastic solutions are of a qualitatively distinct nature based on the parameters. Moreover, the effect of the multiplicative noise on the solutions of nonlinear Schrödinger equation will be discussed. Finally, two and three-dimensional graphs for some solutions have been given to support our analysis.

Keywords: stochastic Schrödinger equation; multiplicative noise; exact solutions; sine-cosine method; Riccati-Bernoulli sub-ODE
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1. Introduction

Nonlinear complex phenomena arising in various fields of applied science such as fluid mechanics, chemical physics, solid state physics, plasma physics, biology, optics and geochemistry can be modelled into various nonlinear partial differential equations (NLPDEs) [1–9]. Recently, studying the nature of these models has attracted the attentions of many researches [10–21]. The nonlinear Schrödinger equation (NLS, for short) is one of the fundamental models of nonlinear waves. It has many applications for example in the theory of solids [22] and crystals [23], in laser beams [24] and in electromechanical systems [25].

Using stochastic processes in Schrödinger equations, thermal fluctuations or spontaneous emissions or general random disturbances can be modelled. Many authors studied the existence and uniqueness of the solution of stochastic Schrödinger equation with additive or multiplicative noise. The multiplicative noise case is investigated in [26–30], while the additive noise case is discussed in [31, 32]. For both cases are studied in [33, 34]. While, other authors are interested with numerical approximations of the solutions via effective schemes of stochastic Schrödinger equation for example [35–37].

In this article, we consider the following stochastic nonlinear Schrödinger equation with multiplicative noise in the Itô sense:

$$iu_t - u_{xx} + 2|u|^2 u - 2\rho^2 u + \sigma u \beta_t = 0, \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}, \tag{1.1}$$

where $u(t,x)$ is a complex-valued process, ρ and σ are constants, and $\beta_t = \frac{d\beta}{dt}$ is the time derivative of Browian motion $\beta(t)$. In this paper we consider the one-dimensional noise, because on one hand, this is the case where we are able to obtain the exact solutions. On the other hand, infinite-dimensional noise may lead to spatially unbounded solutions of Eq (1.1). At this point, it is convenient to provide a definition of $\beta(t)$. Brownian motion (also called one-dimensional Wiener process) is a stochastic process $\{\beta(t)\}_{t\geq 0}$ with the following properties: (i) $\beta(0) = 0$, (ii) β has continuous trajectories, (iii) The process $\{\beta(t)\}_{t\geq 0}$ has stationary, independent increments, (iv) For $s < t$ the stochastic variable $\beta(t) - \beta(s)$ has the normal distribution $N(0; t - s)$. The multiplicative noise in Eq (1.1) describes a process where the phase of the excitation is disturbed . In crystals, this type of noise corresponds to scattering of exciton by phonons due to thermal molecular vibrations.

In the current work, the Riccati-Bernoulli sub-ODE technique [38] and sine-cosine method are employed to obtain new solutions in different form of stochastic Schrödinger Eq (1.1). Moreover, we discuss the effect of multiplicative noise on these solutions. To the best of our knowledge, this article is the first one for finding the exact solutions for the stochastic Schrödinger Eq (1.1).

Our aim in the current work is to derive the exact solutions of stochastic nonlinear Schrödinger Eq (1.1) forced by a one- dimensional multiplicative white noise in the Itô sense by two various methods such as the Riccati-Bernoulli sub-ODE technique [38] and sine-cosine method. To the best of our knowledge, this article is the first one for finding the exact solutions for the stochastic Schrödinger Eq (1.1). Moreover, we discuss the effect of multiplicative noise on these solutions. The obtained solutions will be extremely helpful in future for further studies such as the improvement of biomedical, coastal water motions, industrial studies, quasi particle theory, space plasma and fiber applications.

This article is divided into the following sections. In the next section, we will obtain the stochastic exact solutions of stochastic nonlinear Schrödinger Eq (1.1) by using two different methods, while in section 3 we show the effect of multiplicative noise on the exact solution of nonlinear Schrödinger Eq (1.1). Finally, we introduce the conclusions of this paper.

2. The exact solutions of stochastic Schrödinger equation

In this section we will get the exact solutions of stochastic nonlinear Schrödinger Eq (1.1). Let us first use the following wave transformation

$$u(t,x) = \varkappa(\eta)e^{i\theta}, \quad \eta = k(x + 2\alpha t), \ \theta = \alpha x + \upsilon t + \sigma\beta(t), \tag{2.1}$$

where α is the speed of the wave solution $\varkappa(\eta)$, σ is the noise strength, k is a positive constant. By using (2.1) and

$$\begin{aligned}\frac{\partial u}{\partial t} &= (2\alpha k\varkappa' + i\upsilon\varkappa + i\sigma\beta_t)e^{i\theta},\\ \frac{\partial^2 u}{\partial x^2} &= (k^2\varkappa'' + 2i\alpha k\varkappa' - \alpha^2\varkappa)e^{i\theta},\end{aligned}$$

one can convert Eq (1.1) into the following ODE:

$$-k^2\varkappa'' + 2\varkappa^3 + A\varkappa = 0\,, \tag{2.2}$$

where

$$A = (\alpha^2 - 2\rho^2 - \nu).$$

In the following we apply two methods as the Riccati-Bernoulli sub-ODE method and sine-cosine method to obtain the solitary wave solution of Eq (2.2). And we, therefore, have stochastic exact solution of NLSE (1.1).

2.1. The Riccati-Bernoulli sub-ODE method

Consider the following Riccati-Bernoulli equation

$$\varkappa' = a_1\varkappa^{2-m} + a_2\varkappa + a_3\varkappa^m, \tag{2.3}$$

where a_1, a_2, a_3 and m are constants and $\varkappa = \varkappa(\eta)$.

Differentiating the Riccati-Bernoulli Eq (2.3) one time with respect to η, we obtain

$$\begin{aligned}\varkappa'' &= a_1a_2(3-m)\varkappa^{2-m} + a_1^2(2-m)\varkappa^{3-2m}\\ &\quad + ma_3^2\varkappa^{2m-1} + a_2a_3(m+1)\varkappa^m + (2a_1a_3 + a_2^2)\varkappa.\end{aligned} \tag{2.4}$$

By substituting (2.4) into (2.2), we have

$$\begin{aligned}&-k^2a_1a_2(3-m)\varkappa^{2-m} - k^2a_1^2(2-m)\varkappa^{3-2m} - mk^2a_3^2\varkappa^{2m-1} - a_2a_3k^2(m+1)\varkappa^m\\ &\quad + 2\varkappa^3 + (-2a_1a_3k^2 - a_2^2k^2 + A)\varkappa = 0\,.\end{aligned} \tag{2.5}$$

If we put $m = 0$, then Eq (2.5) will be become

$$(2 - 2a_1^2k^2)\varkappa^3 - 3a_1a_2k^2\varkappa^2 + (A - 2a_1a_3k^2 - a_2^2k^2)\varkappa - a_2a_3k^2 = 0. \tag{2.6}$$

Equating each coefficient of $\varkappa^i (i = 0, 1, 2, 3)$ to zero, we obtain the following algebraic equations

$$\begin{aligned}a_2a_3k^2 &= 0,\\ A - 2a_1a_3k^2 - a_2^2k^2 &= 0,\\ 3a_1a_2k^2 &= 0,\\ 2 - 2a_1^2k^2 &= 0.\end{aligned}$$

Solving the above equations, yields

$$a_1 = \pm\frac{1}{k},$$
$$a_2 = 0\,,$$

and

$$a_3 = \pm\frac{1}{2k}A.$$

Now, let us deduce the exact solutions of stochastic nonlinear Schrödinger Eq (1.1):

First case: If $m \neq 1$ and $A = (\alpha^2 - 2\rho^2 - \nu) = 0$, then the solution of (2.3) in this case takes the form

$$\varkappa(\eta) = (a_1(m-1)(\eta + C))^{\frac{1}{m-1}}\ . \tag{2.7}$$

Consequently, the exact solution of (1.1) is

$$u(t,x) = \varkappa(\eta)e^{i[\alpha x + \upsilon t + \sigma\beta(t)]} = [\pm(m-1)(x + 2\alpha t + C)]^{\frac{1}{m-1}} e^{i[\alpha x + \upsilon t + \sigma\beta(t)]}, \tag{2.8}$$

where C is the integration constant.

Second case: If $m \neq 1$ and $A = (\alpha^2 - 2\rho^2 - \nu) > 0$, then the solution of (2.3) in this case takes the form

$$\varkappa(\eta) = \Big(\pm \sqrt{\frac{A}{2}} \tan\Big((1-m)(x + 2\alpha t + C)\sqrt{\frac{A}{2}}\Big)\Big)^{\frac{1}{1-m}}, \tag{2.9}$$

and

$$\varkappa(\eta) = \Big(\mp \sqrt{\frac{A}{2}} \cot\Big((1-m)(x + 2\alpha t + C)\sqrt{\frac{A}{2}}\Big)\Big)^{\frac{1}{1-m}}\,. \tag{2.10}$$

Therefore, then the exact solution of (1.1) is

$$\begin{aligned} u(t,x) &= \varkappa(\eta)e^{i[\alpha x + \upsilon t + \sigma\beta(t)]} \\ &= e^{i[\alpha x + \upsilon t + \sigma\beta(t)]}\Big(\pm \sqrt{\frac{A}{2}} \tan\Big((1-m)(x + 2\alpha t + C)\sqrt{\frac{A}{2}}\Big)\Big)^{\frac{1}{1-m}}, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} u(t,x) &= \varkappa(\eta)e^{i[\alpha x + \upsilon t + \sigma\beta(t)]} \\ &= e^{i[\alpha x + \upsilon t + \sigma\beta(t)]}\Big(\mp \sqrt{\frac{A}{2}} \cot\Big((1-m)(x + 2\alpha t + C)\sqrt{\frac{A}{2}}\Big)\Big)^{\frac{1}{1-m}}\,. \end{aligned} \tag{2.12}$$

Third case: If $m \neq 1$ and $A = (\alpha^2 - 2\rho^2 - \nu) < 0$, then the solution of (2.3) in this case takes the form

$$\varkappa(\eta) = \Big(\mp \sqrt{\frac{-A}{2}} \tanh\Big((1-m)(x + 2\alpha t + C)\sqrt{\frac{-A}{2}}\Big)\Big)^{\frac{1}{1-m}}\,, \tag{2.13}$$

and

$$\varkappa(\eta) = \Big(\mp \sqrt{\frac{-A}{2}} \coth\Big((1-m)(x + 2\alpha t + C)\sqrt{\frac{-A}{2}}\Big)\Big)^{\frac{1}{1-m}}\,. \tag{2.14}$$

Consequently,then the exact solution of (1.1) is

$$u(t,x) \;=\; \varkappa(\eta)e^{i[\alpha x + \upsilon t + \sigma\beta(t)]}$$

$$= e^{i[\alpha x+\upsilon t+\sigma\beta(t)]}\Big(\mp\sqrt{\frac{-A}{2}}\tanh\Big((1-m)(x+2\alpha t+C)\sqrt{\frac{-A}{2}}\Big)\Big)^{\frac{1}{1-m}}, \tag{2.15}$$

and

$$\begin{aligned} u(t,x) &= \varkappa(\eta)e^{i[\alpha x+\upsilon t+\sigma\beta(t)]} \\ &= e^{i[\alpha x+\upsilon t+\sigma\beta(t)]}\Big(\mp\sqrt{\frac{-A}{2}}\coth\Big((1-m)(x+2\alpha t+C)\sqrt{\frac{-A}{2}}\Big)\Big)^{\frac{1}{1-m}}. \end{aligned} \tag{2.16}$$

2.2. Sine-Cosine method

While in this section we use the sine-cosine method [39–41]. Let the solution u take the form

$$\varkappa(\eta) = aY^m, \tag{2.17}$$

where

$$Y = \sin(b\eta)\text{or} \quad Y = \cos(b\eta). \tag{2.18}$$

Substituting Eq (2.17) into Eq (2.2) we have

$$-ab^2k^2[-m^2Y^m + m(m-1)Y^{m-2}] + 2a^3Y^{3m} + aAY^m = 0.$$

Rewriting the above equation

$$(aA + ab^2k^2m^2)Y^m - m(m-1)ab^2k^2Y^{m-2} + 2a^3Y^{3m} = 0. \tag{2.19}$$

Balancing the term of Y in Eq (2.19), we get

$$m-2 = 3m \quad \Longrightarrow \quad m = -1. \tag{2.20}$$

Substituting Eq (2.20) into Eq (2.19)

$$(aA + ab^2k^2)Y^{-1} + (2a^3 - 2ab^2k^2)Y^{-3} = 0. \tag{2.21}$$

Equating each coefficient of Y^{-1} and Y^{-3} to zero, we obtain

$$aA + ab^2k^2 = 0, \tag{2.22}$$

and

$$2a^3 - 2ab^2k^2 = 0. \tag{2.23}$$

We obtain by solving Eq (2.22) and Eq (2.23)

$$a = \pm\sqrt{-A} \quad \text{and} \quad b = \pm\frac{1}{k}\sqrt{-A}$$

There are two cases:

First case: If $A = (\alpha^2 - 2\rho^2 - \nu) < 0$, then in this case the solitary wave solution of Eq (2.2) takes the form

$$\varkappa_{1,1}(\xi) = \pm\sqrt{-A}[\sin(\frac{1}{k}\sqrt{-A}\xi)]^{-1} = \pm\sqrt{-A}\csc(\frac{1}{k}\sqrt{-A}\xi),$$

or

$$\varkappa_{1,2}(\xi) = \pm \sqrt{-A}\sec(\frac{1}{k}\sqrt{-A}\xi).$$

Therefore, the stochastic exact solutions of NLSE (1.1) is

$$u_{1,1}(t,x) = \pm e^{i[\alpha x+\upsilon t+\sigma\beta(t)]}\sqrt{-A}\csc[\frac{1}{k}\sqrt{-A}(x+2\alpha t)], \tag{2.24}$$

and

$$u_{1,2}(t,x) = \pm e^{i[\alpha x+\upsilon t+\sigma\beta(t)]}\sqrt{-A}\sec[\frac{1}{k}\sqrt{-A}(x+2\alpha t)]. \tag{2.25}$$

Second case: If $A > 0$, then in this case the solitary wave solutions of Eq (2.2) takes the form

$$\varkappa_{2,1}(\xi) = \pm i\sqrt{A}[\sin(\frac{i}{k}\sqrt{A}\xi)]^{-1} = \pm i\sqrt{A}[\sinh(\frac{1}{k}\sqrt{A}\xi)]^{-1} = \pm i\sqrt{A}\csc h(\frac{1}{k}\sqrt{A}\xi),$$

or

$$\varkappa_{2,2}(\xi) = \pm i\sqrt{A}[\cosh(\frac{1}{k}\sqrt{A}\xi)]^{-1} = \pm i\sqrt{A}\sec h(\frac{1}{k}\sqrt{A}\xi).$$

Therefore, the stochastic exact solutions of NLSE (1.1) are

$$u_{2,1}(t,x) = \pm e^{i[\alpha x+\upsilon t+\sigma\beta(t)]}\sqrt{A}\csc h[\frac{1}{k}\sqrt{A}(x+2\alpha t)], \tag{2.26}$$

and

$$u_{2,2}(t,x) = \pm e^{i[\alpha x+\upsilon t+\sigma\beta(t)]}\sqrt{A}\sec h[\frac{1}{k}\sqrt{A}(x+2\alpha t)]. \tag{2.27}$$

Substantially, it has been reported that the exact solutions of the nonlinear Schrödinger Eq (1.1) were gained in the explicit form, using sine-cosine and Riccati-Bernoulli sub-ODE methods. The difference between them is that they give different types of solutions. These solutions describe different phenomena in physics and applied science. The main advantages for these two methods over the most other methods is that they give various vital solutions with additional free parameters. Moreover, these methods are simple, sturdy and efficient. Indeed these two methods can be used to solve other models arising in physics.

3. The effect of the noise on the solutions

In this section we show the effect of multiplicative noise on the solution of Schrödinger Eq (1.1). In the following we introduce some graphical simulations for the fixed parameters $\alpha = 1.3$, $k = 1.3$; $p = 1.4$, $\mu = 2$, $\nu = 2.4$ and varying noise strength σ. The graphical simulations carried out with MATLAB package.

We see that the solution of NLSE (1.1) fluctuates and has a pattern if $\sigma = 0$ in the Figure 1. In Figures 2–4, we see that the pattern begins to destroy if the noise intensity σ increases. Finally, in Figure 5 we give 2-D graph of the solution of NLSE (1.1) with different values of the noise intensity σ.

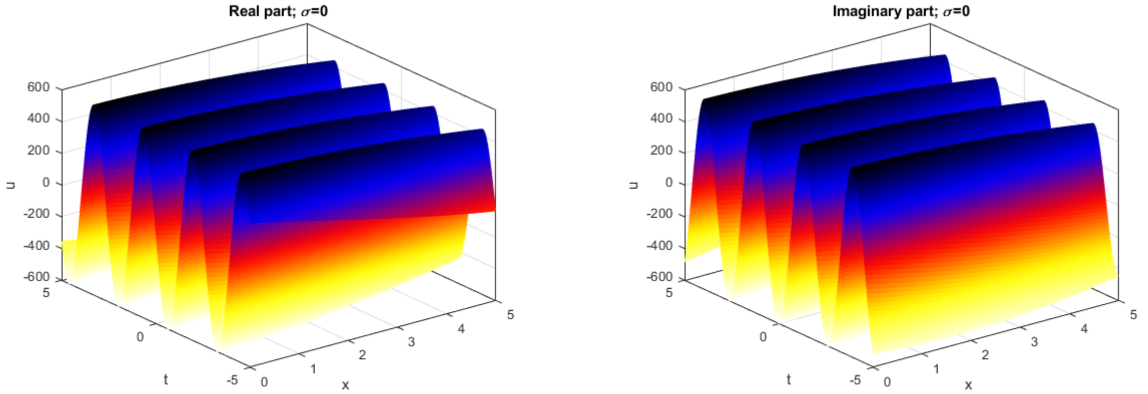

Figure 1. Graph of solution u in (2.11) with $\sigma = 0$.

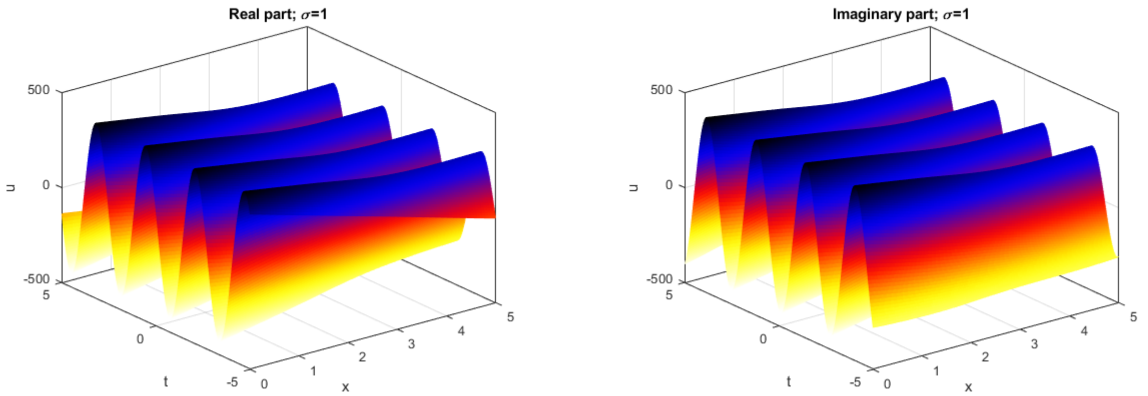

Figure 2. Graph of solution u in (2.11) with $\sigma = 1$.

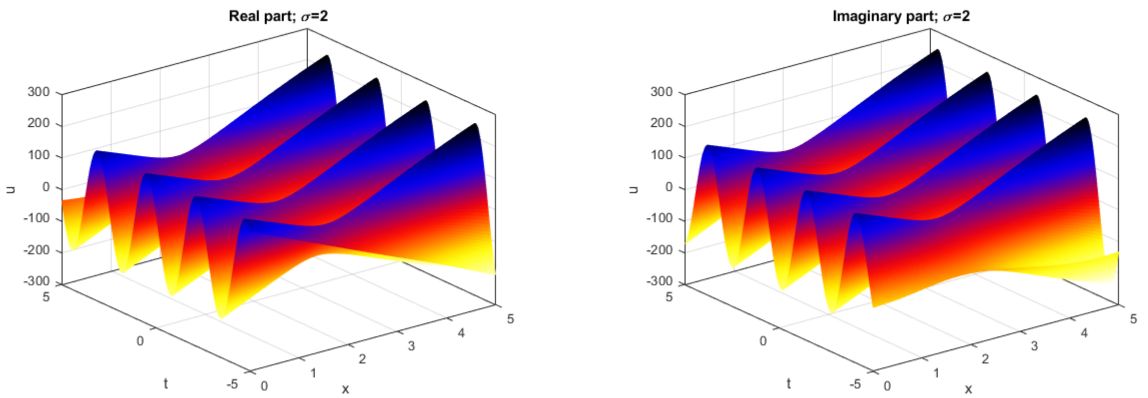

Figure 3. Graph of solution u in (2.11) with $\sigma = 2$.

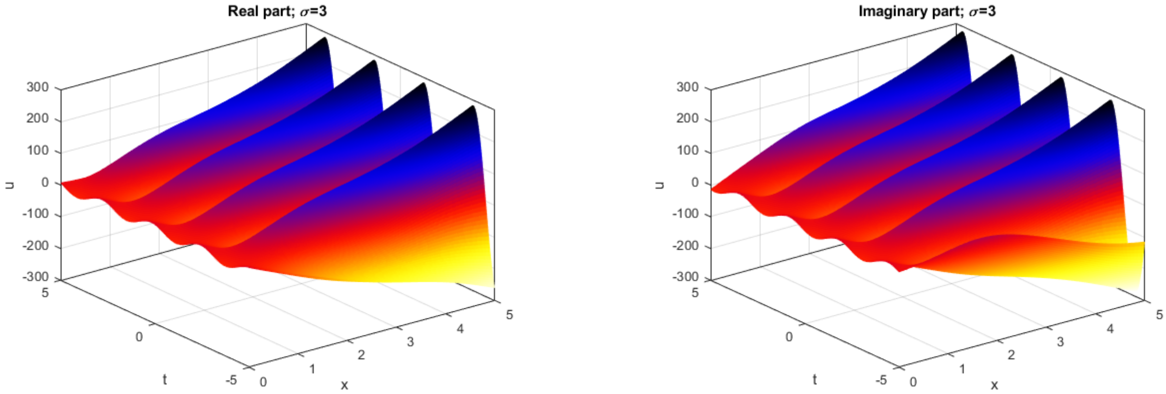

Figure 4. Graph of solution u in (2.11) with $\sigma = 3$.

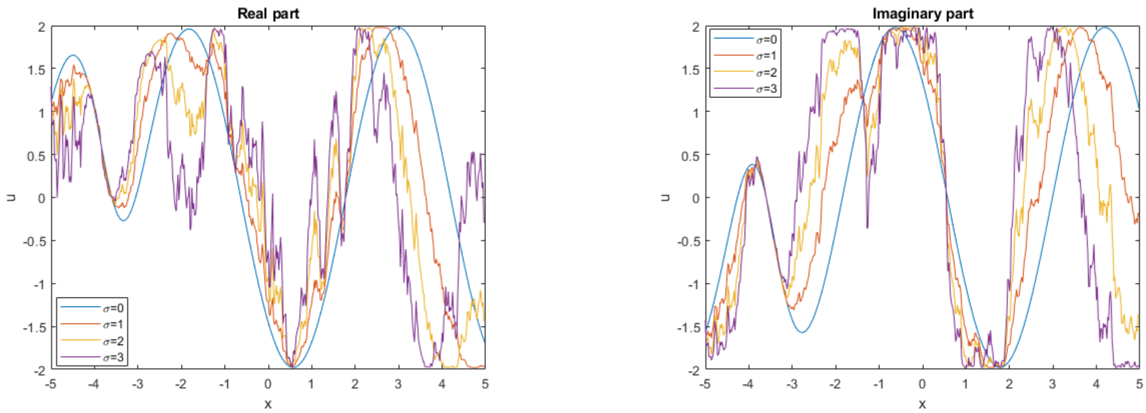

Figure 5. Graph of solution u in (2.11).

4. Conclusions

In this article we introduced a rich variety of families of wave solutions, to stochastic nonlinear Schrödinger equation with multiplicative noise in the Itô sense. These solutions are of significant importance in the explaining of some interesting complex physical phenomena. The proposed method is easy, concise, direct and effect tools that give interesting results. The obtained solutions will be extremely helpful in future for further studies such as the improvement of biomedical, coastal water motions, quasi particle theory, industrial studies, space plasma and fiber applications. Finally, we illustrated the effect of multiplicative noise on the solitary wave solution of Schördinger equation.

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Conflict of interest

The authors declare no conflict of interest.

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