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Research article

The extended Burr-R class: properties, applications and modified test for censored data

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Abstract: This article introduces a new three-parameter Marshall-Olkin Burr-R (MOB-R) family which extends the generalize Burr-G class. Some of its general properties are discussed. One of its special models called the MOB-Lomax distribution is studied in detail for illustrative purpose. A modified chi-square test statistic is provided for right censored data from the MOB-L distribution. The model parameters are estimated via the maximum likelihood and simulation results are obtained to assess the behavior of the maximum likelihood approach. Applications to real data sets are provided to show the usefulness of the proposed MOB-Lomax distribution. The modified chi-square test statistic shows that the MOB-Lomax model can be used as a good candidate for analyzing real censored data.

Keywords: censored data; Lomax distribution; maximum likelihood estimation; modified chi-squared test; order statistics
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1. Introduction

Marshall Olkin in 1997 gave a general method to extend any existing distribution by adding one additional parameter called tiled parameter for the seek of goodness of fit. Later on many researchers adopted this method and proposed many new extended distributions, available in literature some of them are the Marshall Olkin Pareto distribution [7], Marshall Olkin beta distribution [14], Marshall Olkin extended Weibull distribution [10], Marshall Olkin exponentiated Burr XII distribution [11], Marshall Olkin additive Weibull distribution [2], Marshall Olkin alpha power family [22], Marshall Olkin Burr III family [1], and Marshall Olkin power generalized Weibull distribution [3], among many more.

In this study, we introduced a new family of distributions called Marshall Olkin Burr-R (MOB-R) family by inserting the generalized Burr-G (Arslan et al. [20]) in the Marshall Olkin family (Marshall and Olkin [16]), we obtain the following cdf and pdf of the MOB-R class as

$$G(x;\eta,\theta,k,\xi)=\frac{1-\left(1+\{-\log[1-R(x)]\}^{\theta}\right)^{-\rho}}{1-(1-\eta)\left(1+\{-\log[1-R(x)]\}^{\theta}\right)^{-\rho}} \tag{1.1}$$

and

$$g(x;\eta,\theta,\rho,\xi)=\frac{\theta\rho\,\eta\, r(x)\,\{-\log[1-R(x)]\}^{\theta-1}\left(1+\{-\log[1-R(x)]\}^{\theta}\right)^{-\rho-1}}{\{1-R(x)\}\left[1-(1-\eta)\left(1+\{-\log[1-R(x)]\}^{\theta}\right)^{-\rho}\right]^{2}}. \tag{1.2}$$

Its hazard rate function (hrf) has the form

$$h(x;\eta,\theta,\rho,\xi)=\frac{\theta\rho\, r(x)}{1-R(x)}\frac{\{-\log[1-R(x)]\}^{\theta-1}\left(1+\{-\log[1-R(x)]\}^{\theta}\right)}{\left[1-(1-\eta)\left(1+\{-\log[1-R(x)]\}^{\theta}\right)^{-\rho}\right]}. \tag{1.3}$$

The random variable (rv) X having the density (1.2)is denoted by X ~MOB-R(η,θ,ρ). The quantile function (qf), $Q_x(u)$, of the MOB-R family reduces to

$$Q_X(u)=R^{-1}\left(1-\exp\left[-\left\{\left(\frac{u-1}{1-\bar{\eta}u}\right)^{-\frac{1}{\rho}}-1\right\}^{\frac{1}{\theta}}\right]\right), \tag{1.4}$$

where $x=Q(u)$ follows the MOBG family with $U\sim uniform(0,1)$.

Motivation

The MOB-R family can be justified physically as follows. Cosider N independent components which are related by a series system and suppose that N is a rv with a probability mass function $P(N=n)=\delta(1-\delta)^n$, $0<\theta<1$ and $n=0,1,....$ Let $X_1,X_2,...,X_n$ represents the lifetimes of the components, which are assumed to be independently and identically distributed rv's with cdf $F(x;\theta,\rho,\xi)$. The rv $Y=\min(X_1,...,X_N)$ refers to time of the first failure and its cdf takes the form

$$G(y)=1-\sum_{n=0}^{\infty}P\left[\min(X_1,...,X_n)>Y\right]\,\delta(1-\delta)^n$$

$$= 1 - \delta \bar{F}(x) \sum_{n=0}^{\infty} \left\{ (1-\delta) \bar{F}(x) \right\}^n$$

$$= \frac{1 - \left(1 + \{-\log[1 - R(x)]\}^{\theta}\right)^{-\rho}}{1 - (1-\eta)\left(1 + \{-\log[1 - R(x)]\}^{\theta}\right)^{-\rho}},$$

where $R(x)$ is any baseline cdf, i.e $0 < X < \infty$ such as Lomax, Frechet, log-logistic and Weibull distributions.

The rest of the article is followed as. In Section 2, a useful linear mixture representation. The general properties of the MOB-R class are provided in Section 3. In Section 4, estimation of the MOB-R parameters is carried out by maximum likelihood. In Section 5, we study the MOB-Lomax model as a special model of the proposed family. In Section 6, a modified chi-square test statistic for right censored data is applied to validate the MOB-L model. In Section 7, we provide simulation results to check the performance of the maximum likelihood and to validate the test statistic. Real data applications are analyzed to show the flexibility of the MOB-Lomax model in Section 8. Further, a real censored data is analyzed to validate the considered test statistic. Some concluding remarks are reported in Section 9.

2. Useful expansion

This section provides an infinite linear mixture for the cdf and pdf of the MOB-R class in (1.1) and (1.2). By using following two series expansions

$$(1-z)^{-\rho} = \sum_{i=0}^{\infty} \binom{\rho + i - 1}{i} z^i.$$

The log-power expansion has the form

$$[\log(1+z)]^{\beta} = \beta \sum_{\rho=0}^{\infty} \binom{\rho - \beta}{\rho} \sum_{j=0}^{\rho} \frac{(-1)^j}{\beta - j} \binom{\rho}{j} \varrho_{j,\rho}\ z^{\rho},$$

where $\varrho_{j,\rho} = \frac{1}{\rho} \sum\limits_{m=1}^{\rho} (jm - \rho + m)\theta_m\ \varrho_{j,\rho-m},\ \ p_{j,0} = 1$ and $\theta_\rho = \frac{(-1)^\rho}{\rho+1}$.
The Eq (1.1) reduce to

$$G(x;\eta,\theta,\rho,\xi) = 1 - \sum_{m=0}^{\infty} a_m\ G(x)^m, \tag{2.1}$$

where $a_m = \eta \sum\limits_{i=0}^{\infty} \bar{\eta}^i \sum\limits_{j=0}^{\infty} \binom{\rho(i+1)+j-1}{j}, \theta\ j \binom{m-\theta j}{m} \sum\limits_{l=0}^{m} \frac{(-1)^{l+cj+m+j}}{cj-l} \binom{m}{l} \varrho_{l,m}$
and a_m are weights and $\sum a_m = 1$. Equation (2.1) can be written as

$$G(x;\eta,\theta,\rho,\xi) = \sum_{m=0}^{\infty} b_m\ H_m(x), \tag{2.2}$$

where $b_0 = 1 - a_0$, $b_m = -a_m$ and $H_m(x) = R(x)^m$.
Similarly

$$g(x; \eta, \theta, \rho, \xi) = \sum_{m=0}^{\infty} b_{m+1}\ h_{m+1}(x), \tag{2.3}$$

where $h_{m+1}(x) = (m+1)\ r(x)\ R(x)^m$. Equations (2.3) and (2.2) refer to the infinite linear representation for the cdf and pdf of the MOB-R class in terms of their baseline distributions, and they are helpful to derive the MOB-R properties.

3. General properties

In this section, we derive some properties of MOB-R family, such as moments, generating function (mgf), stochastic ordering, reliability parameter, and order statistics.

3.1. Moments

The rth moments of MOB-R family is derived using the following expression

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r\, g(x; \eta, \theta, \rho, \xi)\, dx.$$

Using mixture representation given in (2.3), we have

$$\mu_r' = \sum b_{m+1} \int_{-\infty}^{\infty} x^r\, h_{m+1}(x)\, dx.$$

$$\mu_r' = \sum b_{m+1}\, \Delta^r, \tag{3.1}$$

where $\Delta^r = \int_{-\infty}^{\infty} x^r\, h_{m+1}(x)\, dx$. Similarly, incomplete moments of MOB-R family can be calculated using the following formula

$$\mu_r^m = \sum b_{m+1} \int_0^m x^r h_{m+1}(x)\, dx. \tag{3.2}$$

Applications of Eq (3.2) are related to mean deviation, Zenga index, income quantiles such as Lorenz and Bonferroni curves, mean waiting time, and mean residual life.

The mgf of the MOB-G family can be obtained by using the following expression

$$M_0(t) = E(e^{t\,x}) = \int_{-\infty}^{\infty} e^{t\,x}\, g(x; \eta, \theta, \rho, \xi)\, dx. \tag{3.3}$$

Using mixture representation given in (2.3), we have

$$M_0(t) = \sum b_{m+1} \int_{-\infty}^{\infty} e^{t\,x} h_{m+1}(x)\, dx,$$

Note that these integrals only depends only for the arbitrary base line distribution.

3.2. Stress-strength analysis

In the field of reliability, stress-strength model has an important role which defines the life time of a component which has a random strength, say X_1, which is subject to an accidental stress, say X_2. The component will still work when $X_1 > X_2$. It has many application in engineering. Let X_1 and X_2 be two *rv*'s follow the MOB-R family i.e. X_1 ~MOB-R(θ_1, ρ_1, η_1) and X_2 ~MOB-R(θ_2, ρ_2, η_2) with a common shape and scale parameters.

$$R = P(X_1 < X_2) = \int_0^\infty f_1(x)\, F_2(x) dx. \tag{3.4}$$

Using Eqs (2.2) and (2.3), we can write

$$R = P(X_1 < X_2) = \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} b_{p+1} b_m \int_0^\infty h_{p+1}(x) H_m(x)\, dx, \tag{3.5}$$

where $h_{p+1}(x)$ and $H_m(x)$ are already defined in the previous section.

3.3. Stochastic ordering

The stochastic ordering is commonly used in showing the ordering mechanism in life time distribution. A *rv* X is stochastically greater than the *rv* Y if $F_X(x) \le F_Y(x)$ for all x's. Further, there are some important stochastic orderings namely, stochastic order, hazard rate order, mean residual order, likelihood ratio order, and reversed hazard rate order which are related to each other according to the following chain of stochastic orders

$$X \le_{rhr} Y \Leftarrow X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{st} Y \Rightarrow X \le_{mrl} Y$$

Further details about different stochastic orderings and their definitions can be explored in Shaked and Shanthikumar [23]. Let $X_1 \sim MOBG(\theta, \rho, \eta_1)$ and $X_2 \sim MOBG(\theta, \rho, \eta_2)$. Then, using likelihood ratio ordering defined by $\left[\frac{f(x)}{g(x)}\right]$, we can write

$$\frac{f(x)}{g(x)} = \left[\frac{1 - \bar{\eta}_1 \xi}{1 - \bar{\eta}_2 \xi}\right]^2,$$

where $\xi = \left(1 + (H_R(x))^\theta\right)^{-\rho}$ and $\xi' = \theta \rho\, h_R(x) H_R(x)^{\theta-1} \left(1 + (H_R(x))^\theta\right)^{-\rho-1}$. Therefore,

$$\frac{d}{dx}\frac{f(x)}{g(x)} = 2\left[\frac{1 - \bar{\eta}_1 \xi}{1 - \bar{\eta}_2 \xi}\right] \xi' \,\frac{\eta_1 - \eta_2}{(1 - \bar{\eta}_2 \xi)^2}.$$

From the above expression we see that $\frac{d}{dx}\frac{f(x)}{g(x)} < 0$, if $\eta_1 < \eta_2$, so the likelihood ratio ordering exists among the variables i.e. $X \le_{lr} Y$ and the remaining stochastic orderings follow simply from the above chain.

3.4. Order statistics

Let $X_1, ..., X_n$ be a random sample for MOB-R family. The ith order statistic, $X_{i:n}$, has the following pdf

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^i f(x) [F(x)]^{i+j-1} \tag{3.6}$$

Using Equations (2.2), (2.3) and the power series expansion in [13] (Pages 17–18), we have

$$= \sum_{j=0}^{n-i} \sum_{r,m=0}^{\infty} v_{j,r,m} h_{r+m}(x), \tag{3.7}$$

where $v_{j,r,m} = \frac{\rho\,(m+1)\, n! (-1)^j\, b_{r+1}\, e_{j+i-1:r}}{(i-1)!(n-i-j)!\, j!}(r+m+1)$, $h_{r+m}(x) = (r+m+1) g(x) G^{r+m}(x)$ is already defined in previous section.

4. Estimation

In this section, the estimation of the MOB-R parameters are obtained using the maximum likelihood (ML) method. Let $X_1, X_2, ..., X_n$ be a random sample from the MOB-R class. The log-likelihood function for $\Theta = (\eta, \theta, \rho, \xi)^T$ takes the form

$$\begin{aligned} \ell(\Theta) = n \log(\theta \rho \eta) + \sum_{i=1}^{n} \log\ [h_R(x_i)] + (\theta - 1) \sum_{i=1}^{n} \log\ [H_R(x_i)] \\ - (\rho + 1) \sum_{i=1}^{n} \log(1 + S_i) - 2 \sum_{i=1}^{n} \log[1 - \bar{\eta}(1 + S_i)^{-\rho}], \end{aligned} \tag{4.1}$$

where $S_i = \{-\log[1 - R(x_i)]\}^{\theta}$. The score vector elements take the forms

$$\begin{aligned} U_\eta &= \frac{n}{\eta} + 2 \sum_{i=1}^{n} \left[\frac{(1+S_i)^{-\rho}}{1 - \bar{\eta}(1+S_i)^{-\rho}} \right], \\ U_\theta &= \frac{n}{\theta} + (\rho+1) \sum_{i=1}^{n} \left(\frac{S'_{i:\theta}}{1+S_i} \right) + \sum_{i=1}^{n} \log\ [H_R(x_i)] + 2 \sum_{i=1}^{n} \left[\frac{\bar{\eta}\rho\,(1+S_i)^{-\rho-1}}{1 - \bar{\eta}(1+S_i)^{-\rho}} S'_{i:\theta} \right], \\ U_\rho &= \frac{n}{\rho} - \sum_{i=1}^{n} \log(1+S_i) + 2 \sum_{i=1}^{n} \left[\frac{\bar{\eta}(1+S_i)^{-\rho} \log(1+S_i)}{1 - \bar{\eta}(1+S_i)^{-\rho}} \right], \\ U_\xi &= \sum_{i=1}^{n} \left[\frac{h_R^{\xi}(x_i)}{h_R(x_i)} \right] - \sum_{i=1}^{n} \left(\frac{S'_{i:\xi}}{1+S_i} \right) + (\theta-1) \sum_{i=1}^{n} \left[\frac{H_R^{\xi}(x_i)}{H_R(x_i)} \right] - 2 \sum_{i=1}^{n} \left[\frac{\bar{\eta}\rho(1+S_i)^{-\rho-1}}{1 - \bar{\eta}(1+S_i)^{-\rho}} S'_{i:\xi} \right]. \end{aligned}$$

Equating the above equations by zero and solving them simultaneously yields the ML estimates.

5. The MOB-Lomax model

If X ∼Lomax (a, b), then its cdf takes the form $R(x) = 1 - \left(1 + \frac{x}{a}\right)^{-b}$. Using Eqs (1.2) and (1.1), we obtain the cdf and pdf of the MOB-Lomax (MOB-L) distribution as follows

$$G(x) = \frac{1 - \left\{1 + \left[b \log\left(1 + \frac{x}{a}\right)\right]^{\theta}\right\}^{-\rho}}{1 - \bar{\eta}\left\{1 + \left[b \log\left(1 + \frac{x}{a}\right)\right]^{\theta}\right\}^{-\rho}}$$

and

$$g(x) = \frac{b\,\theta\,\rho\,\eta\,\left[b \log\left(1 + \frac{x}{a}\right)\right]^{\theta-1}\left\{1 + \left[b \log\left(1 + \frac{x}{a}\right)\right]^{\theta}\right\}^{-\rho-1}}{a\left(1 + \frac{x}{a}\right)\left(1 - \bar{\eta}\left\{1 + \left[b \log\left(1 + \frac{x}{a}\right)\right]^{\theta}\right\}^{-\rho}\right)^{2}}.$$

The hrf of the MOB-L distribution reduces to

$$h(x) = \frac{b\,\theta\,\rho\,\eta\,\left[b \log\left(1 + \frac{x}{a}\right)\right]^{\theta-1}}{a\left(1 + \frac{x}{a}\right)\left\{1 + \left[b \log\left(1 + \frac{x}{a}\right)\right]^{\theta}\right\}\left(1 - \bar{\eta}\left\{1 + \left[b \log\left(1 + \frac{x}{a}\right)\right]^{\theta}\right\}^{-\rho}\right)^{2}}.$$

In Figure 1, the plots for pdf and hrf are presented for the MOB-L distribution for several values of parameters. As seen in Figure 1, MOB-L distribution is very flexible with skewed shapes.

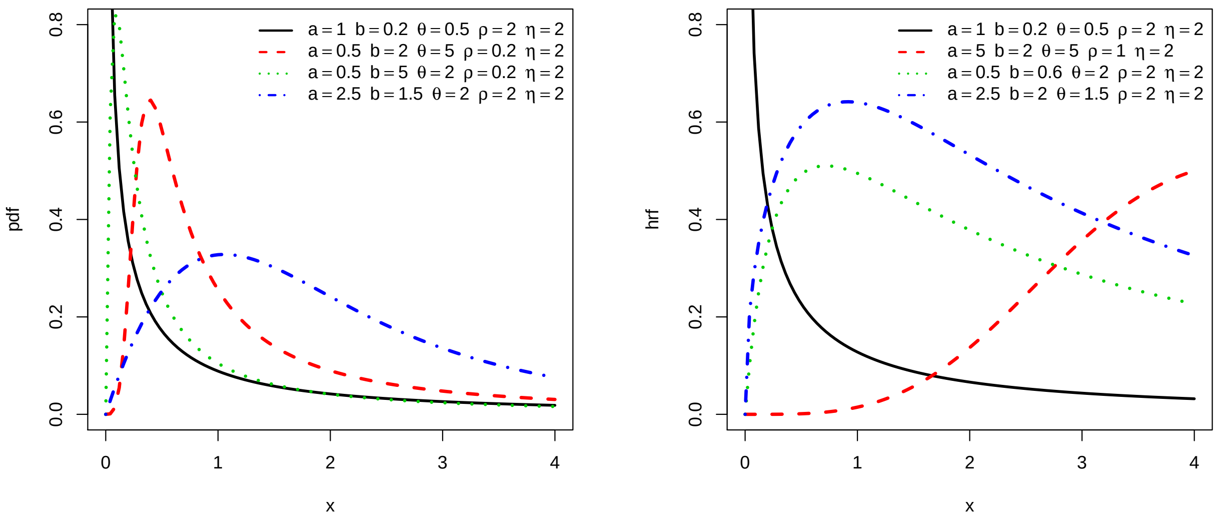

Figure 1. Plots of the pdf and hrf of the MOB-L distribution.

The mixture representations for the cdf and pdf of the MOB-L distribution follow from Equations (2.2) and (2.3) as

$$G(x) = \sum_{m=0}^{\infty} b_m \left\{1 - \left(1 + \frac{x}{a}\right)^{-b}\right\}^{m}$$

and

$$g(x) = \sum_{m=0}^{\infty} b_{m+1}(m+1)\frac{b}{a}\left(1+\frac{x}{a}\right)^{-b-1}\left\{1-\left(1+\frac{x}{a}\right)^{-b}\right\}^{m+1}.$$

The qf of the MOB-L distribution follows from Equation (1.4) as

$$Q_x(u) = a\left[(1-A)^{-\frac{1}{b}}-1\right],$$

where $A = 1-\exp\left[-\left\{\left(\frac{u-1}{1-\bar{\eta}}\right)^{-\frac{1}{\rho}}-1\right\}^{\frac{1}{\theta}}\right]$.

The r^{th} moment of the MOB-L model can be calculated using (3.1) as

$$\mu_r' = \sum_{m=0}^{\infty} b_{m+1}\ (m+1)\ a^r\ b\ \sum_{j=0}^{\infty}\binom{m+1}{j}(-1)^j\ B(r+1,\beta(j+1)-r)$$

Its r^{th} incomplete moment takes the form

$$m_r = \sum_{j=0}^{\infty} v_{j,m}\ a^r\ b\ B_{\frac{x}{a}}(r+1,\beta(j+1)-r), \tag{5.1}$$

where $v_{j,m} = \sum_{m=0}^{\infty}\ b_{m+1}\ (m+1)\ \binom{m+1}{j}(-1)^j$.

Setting $r=1$ in (5.1), the first incomplete moment reduces to

$$m_1 = \sum_{j=0}^{\infty} v_{j,m}\ a\ b\ B_{\frac{x}{a}}(2,\beta(j+1)-1).$$

The mgf of MOB-L distribution follows from Equation (3.3) as

$$M_0(t) = \sum_{i=0}^{\infty} v_{i,j,m}\ \Gamma(i+1)\left(\frac{-1}{t}\right)^{i+1},$$

where $v_{i,j,m} = \sum_{m,j=0}^{\infty}\ b_{m+1}\ (m+1)\ \binom{m+1}{j}\binom{\beta(j+1)+i}{i}(-1)^{i+j}$.

The pdf of i^{th} order statistic for the MOB-L model is

$$f_{i:n}(x) = \sum_{j=0}^{\infty}\sum_{r,m=0}^{\infty} v_{j,r,m}\left(1+\frac{x}{a}\right)^{-b-1}\left\{1-\left(1+\frac{x}{a}\right)^{-b}\right\}^{m+r+1},$$

where

$$v_{j,r,m} = \frac{\rho\ (m+1)\ n!\ (-1)^j\ b_{m+1}\ e_{j+i-1:r}}{(i-1)!\ (n-i-j)!\ j!}.$$

If we have two MOB-L distributions, such as MOB-L$(\eta_1,\theta_1,\rho_1,a,b_1)$ and MOB-L$(\eta_2,\theta_2,\rho_2,a,b_2)$, with a common parameter a then from Equation (3.5), we obtain the reliability function as

$$R = \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} b_{p+1}\,b_m\binom{p+1}{i}\binom{m+1}{j}\frac{(-1)^{i+j}\,(p+1)\,b_2}{\{b_2(i+1)+b_1 j\}}.$$

5.1. Estimation of the MOB-L Parameters

The log-likelihood function for the MOB-L model takes the form

$$\text{ł}(\Theta) = \log\left\{\frac{\theta\rho\, b\,\eta}{a}\right\} - \sum_{i=1}^{n}\log(d_i) + (\theta-1)\sum_{i=1}^{n}\log\{b\,\log(d_i)\}$$

$$-(\rho+1)\sum_{i=1}^{n}\log(1+B_i) - 2\sum_{i=1}^{n}\log\left[1-\bar{\eta}(1+B_i)^{-\rho}\right],$$

where $B_i = [b\log(d_i)]^{\theta}$ and $d_i = \left(1+\frac{x_i}{a}\right)$.
The components of score vector are

$$U_\eta = \frac{n}{\eta} - \sum_{i=1}^{n}\frac{(1+B_i)^{-\rho}}{1-\bar{\eta}(1+B_i)^{-\rho}},$$

$$U_\rho = \frac{n}{\rho} - \sum_{i=1}^{n}\log(1+B_i) - 2\sum_{i=1}^{n}\frac{\bar{\eta}\,(1+B_i)^{-\rho}\log(1+B_i)}{1-\bar{\eta}(1+B_i)^{-\rho}},$$

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n}\log\{b\,\log(d_i)\} - (\rho+1)\sum_{i=1}^{n}\frac{B_{i:\theta}}{1+B_i} - 2\sum_{i=1}^{n}\frac{\bar{\eta}\,\rho\,(1+B_i)^{-\rho-1}B'_{i:\theta}}{1-\bar{\eta}(1+B_i)^{-\rho}},$$

$$U_b = \frac{n}{b} + n\frac{(\theta-1)}{b} - (\rho+1)\sum_{i=1}^{n}\frac{B'_{i:b}}{1+B_i} - 2\sum_{i=1}^{n}\frac{\bar{\eta}\,\rho\,(1+B_i)^{-\rho-1}B'_{i:b}}{1-\bar{\eta}(1+B_i)^{-\rho}},$$

$$U_a = n\,a + \sum_{i=1}^{n}\frac{x_i}{a^2\left(1+\frac{x_i}{a}\right)} - (\theta-1)\sum_{i=1}^{n}\frac{x_i}{a^2\,(d_i)\{\log(d_i)\}} - (\rho+1)\sum_{i=1}^{n}\frac{B'_{i:a}}{1+B_i}$$

$$-2\sum_{i=1}^{n}\frac{\bar{\eta}\,\rho\,(1+B_i)^{-\rho-1}B'_{i:a}}{1-\bar{\eta}(1+B_i)^{-\rho}}.$$

The log-likelihood function can be maximized directly using the R-package (AdequecyModel). In AdequecyModel package, there are some maximization algorithms such as NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfarb-Shanno), BHHH (BerndtHall-Hall-Hausman), NM (Nelder-Mead), SANN (Simulated-Annealing) and limited memory quasi-Newton code for Bound-constrained optimization (L-BFGS-B). However, the MLEs here are computed using the BFGS algorithm.

6. Modified Chi-Square test for right censored data

In this section, we provide a modified chi-square test statistic for right censored data from the MOB-L distribution based on the modified chi-square type test which is proposed by Bagdonavicius et al. [8] and Bagdonavicius and Nikulin [9], for parametric models with right censored data. Using the maximum likelihood estimators (MLEs) for non-grouped data, this test statistic is also based on the differences between the numbers of observed failures and the numbers of expected failures in the chosen grouped intervals. Here, random grouping intervals are considered as data functions. Voinov et al. [24] developed the description of construction of this chi-square type test. The test statistic can be defined as follows.

Suppose that $X_1, X_2, ..., X_n$ is a random sample with right censoring from a parametric model, and a finite time τ.

The test statistic takes the form

$$Y_n^2 = \sum_{j=1}^{n} \frac{\left(U_j - e_j\right)^2}{U_j} + Q,$$

where U_j and e_j are the observed and expected numbers of failure in grouping intervals, and Q has the form

$$\begin{aligned}
Q &= W^T \widehat{G}^- W & \widehat{A}_j = U_j/n, \qquad U_j = \sum_{i:X_i \in I_j} \delta_i, \\
W &= (W_1, ..., W_s)^T, \quad \widehat{G} = [\widehat{g}_{ll'}]_{sxs}, & \widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{r} \widehat{C}_{lj} \widehat{C}_{l'j} \hat{A}_j^{-1}, \\
\widehat{C}_{lj} &= \frac{1}{n} \sum_{i:X_i \in I_j} \delta_i \frac{\partial}{\partial \xi} \ln h(x_i, \widehat{\xi}), & \widehat{i}_{ll'} = \frac{1}{n} \sum_{i=1}^{n} \delta_i \frac{\partial \ln h(x_i, \widehat{\xi})}{\partial \xi_l} \frac{\partial \ln h(x_i, \widehat{\xi})}{\partial \xi_{l'}}, \\
\widehat{W}_l &= \sum_{j=1}^{r} \widehat{C}_{lj} \widehat{A}_j^{-1} Z_j, \quad l, l' = 1, ..., s,
\end{aligned}$$

where $h(x_i, \hat{\xi})$ is the hrf $\xi = (b, \theta, \rho, \eta, \bar{\eta})$ and $\widehat{\xi}$ is the MLE of ξ on initial non-grouped data.

The limits a_j of r random grouping intervals $I_j = [a_{j-1}, a_j[$ are chosen such as the expected failure times to fall into these intervals which are the same for each $j = 1, .., r-1$, $\hat{a}_r = \max\left(X_{(l)}, \tau\right)$. The estimated $\hat{a}_j$ is

$$\hat{a}_j = H^{-1}\left(\frac{E_j - \sum_{l=1}^{i-1} H\left(x_l, \xi\right)}{n - i + 1}, \hat{\xi}\right), \qquad \hat{a}_r = \max\left(X_{(n),} \tau\right),$$

where $H(x)$ is the cdf of the considered distribution. This test statistic Y_n^2 follows a chi-square distribution.

The expected failure times e_j to fall into these intervals are $e_j = \frac{E_r}{r}$ for any j, with $E_r = \sum_{i=1}^{n} H\left(x_i, \hat{\gamma}\right)$.

The limit intervals a_j are considered as random variables such that the expected numbers of failures in each interval I_j are the same, so the expected numbers of failures e_j can be calculated by the following formula

$$E_j = \frac{-j}{k-1} \sum_{i=1}^{n} \ln\left(1 - \frac{1 - \left(1 + \varpi_i^{\theta}\right)^{-\rho}}{1 - \bar{\eta}(1 + \varpi_i^{\theta})^{-\rho}}\right), \quad j = 1, ..r-1.$$

6.1. Quadratic form Q

To calculate the quadratic form Q of the statistic Y_n^2, and as its distribution does not depend on the parameters, so we can use the estimated matrices $\hat{W}$ $\hat{C}$ and the estimated information matrix $\hat{I}$. The elements of $\hat{C}$ are defined by

$$\hat{C}_{lj} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \frac{\partial}{\partial \hat{\xi}_l} \ln h(x_i; \hat{\xi}),$$

where $\varpi_i = b \log(1 + \frac{x}{a})$ and

$$\ln h(x_i) = \ln(b\theta\rho\eta) + (\theta - 1) \ln \varpi_i - \ln a - \ln\left(1 + \frac{x}{a}\right) - \ln(1 + \varpi_i^\theta) - 2 \ln(1 - \bar{\eta}(1 + \varpi_i^\theta)^{-\rho})$$

The elements of $\hat{C}$ take the forms

$$\hat{C}_{1j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{1}{b} + \frac{\theta - 1}{b} - \frac{\theta\varpi_i^\theta}{b(1 + \varpi_i^\theta)} - \frac{2\bar{\eta}\theta\rho\varpi_i^\theta(1 + \varpi_i^\theta)^{-\rho-1}}{b(1 - \bar{\eta}(1 + \varpi_i^\theta)^{-\rho})}\right],$$

$$\hat{C}_{2j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{1}{\theta} + \frac{\ln \varpi_i}{1 + z_i^b} - \frac{2\bar{\eta}\rho\varpi_i^\theta \ln \varpi_i (1 + \varpi_i^\theta)^{-\rho-1}}{1 - \bar{\eta}(1 + \varpi_i^\theta)^{-\rho}}\right],$$

$$\hat{C}_{3j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{1}{\rho} - \frac{2\bar{\eta}(1 + \varpi_i^\theta)^{-\rho} \ln(1 + \varpi_i^\theta)}{1 - \bar{\eta}(1 + \varpi_i^\theta)^{-\rho}}\right],$$

$$\hat{C}_{4j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \frac{\delta_i}{\eta},$$

$$\hat{C}_{5j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{2(1 + \varpi_i^\theta)^{-\rho}}{1 - \bar{\eta}(1 + \varpi_i^\theta)^{-\rho}}\right]$$

and

$$\hat{W}_l = \sum_{j=1}^{r} \hat{C}_{lj} A_j^{-1} Z_j, \quad l = 1, .., m \qquad j = 1, .., r.$$

As the above components of the statistic have explicit forms, then we can obtain the test statistic for the MOB-L($\hat{\xi}$) distribution with unknown parameters and right censored data. This statistic, follows a chi-square distribution with r degrees of freedom, takes the form

$$Y_n^2\left(\hat{\xi}\right) = \sum_{j=1}^{r} \frac{\left(U_j - e_j\right)^2}{U_j} + \hat{W}^T \left[\hat{\imath}_{ll'} - \sum_{j=1}^{r} \hat{C}_{lj} \hat{C}_{l'j} \hat{A}_j^{-1}\right]^{-1} \hat{W}.$$

7. Simulations

In this section, we provide two simulation studies to assess the performance of the MLEs and to validate the test statistic $Y_n^2(\xi)$.

7.1. Simulation for the MLEs

Now, we will study the performance of the maximum likelihood in estimating the MOB-L parameters using simulations which are conducted for sample sizes $n = 50, 150, 300$, and for different parameter combinations (I: $a = 3, b = 4.5, \theta = 0.5, \rho = 4, \eta = 2$), (II: $a = 0.2, b = 0.8, \theta = 1.5, \rho = 8, \eta = 7$) and (III: $a = 0.2, b = 0.8, \theta = 0.5, \rho = 5, \eta = 5$). To obtain the average values of estimates (AEs), mean square errors (MSEs) and absolute biases (ABs) of the parameters, we generated $N = 3000$ samples from the MOB-L model using the R program.

The MSEs and ABs were determined by the following equations:

$$MSEs(\widehat{\boldsymbol{\vartheta}}) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta})^2, \quad ABs(\widehat{\boldsymbol{\vartheta}}) = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}|,$$

where $\boldsymbol{\vartheta} = (a, b, \theta, \rho, \eta)'$.

The simulation results are shown in Table 1. The small values of ABs and MSEs prove that the maximum likelihood performs very well in estimating the MOB-L parameters.

Table 1. Estimated AEs, ABs, and MSEs of the MLEs of the MOB-L parameters.

			I			II			III	
n	Parameters	AEs	ABs	MSEs	AEs	ABs	MSEs	AEs	ABs	MSEs
50	a	2.251	0.749	1.763	0.224	0.024	0.020	0.251	0.224	0.089
	b	5.010	0.510	0.351	0.885	0.085	0.076	0.921	0.194	0.100
	θ	0.735	0.235	0.272	2.222	0.722	0.541	0.823	0.624	0.320
	ρ	4.847	0.847	1.934	8.016	0.016	0.006	5.614	0.094	0.040
	η	2.593	2.407	1.240	6.989	0.015	1.038	5.824	0.324	0.192
150	a	2.549	0.451	0.651	0.157	0.015	0.018	0.230	0.222	0.077
	b	4.953	0.453	0.244	0.708	0.072	0.035	0.920	0.171	0.091
	θ	0.776	0.206	0.102	2.218	0.718	0.531	0.853	0.522	0.211
	ρ	4.631	0.631	1.113	7.980	0.08	0.002	5.714	0.082	0.034
	η	1.918	1.082	0.851	6.985	0.011	0.961	5.124	0.291	0.131
300	a	2.568	0.232	0.539	0.184	0.012	0.013	0.241	0.181	0.031
	b	4.996	0.406	0.210	0.767	0.033	0.017	0.811	0.123	0.052
	θ	0.754	0.154	0.082	2.177	0.677	0.490	0.639	0.031	0.021
	ρ	4.689	0.589	1.106	7.784	0.002	0.001	5.219	0.011	0.009
	η	2.062	1.009	0.818	7.023	0.003	0.910	5.277	0.091	0.035
500	a	2.988	0.202	0.511	0.204	0.008	0.002	0.204	0.125	0.010
	b	4.496	0.316	0.111	0.807	0.031	0.009	0.801	0.029	0.002
	θ	0.554	0.124	0.051	1.537	0.677	0.240	0.530	0.025	0.019
	ρ	4.029	0.511	0.096	7.989	0.002	0.001	5.111	0.011	0.008
	η	2.010	0.077	0.515	7.001	0.003	0.410	5.277	0.058	0.022

7.2. Simulation for test statistic Y^2

For testing the null hypothesis H_0 that right censored data are from MOB-L model, we calculate the test statistic $Y_n^2(\xi)$, defined above, using 10,000 simulated samples from the hypothesized distribution with different sizes $n = 15, 25, 50, 130, 350, 500, 1000$ using the package "bb solve algorithm" in the R software. Then, the empirical levels of significance are calculated, for $Y^2 > \chi_\varepsilon^2(r)$, with theoretical levels of significance $\varepsilon = 0.10,\ 0.05,\ 0.01$, and $r = 5$. The simulated levels of significance for $Y_n^2(\xi)$ of the MOB-L model are reported in Table 1.

Table 2. Simulated significance levels for $Y_n^2(\xi)$ statistic of the MOB-L model against their theoretical values.

$N = 10,000$	$n = 15$	$n = 25$	$n = 50$	$n = 130$	$n = 350$	$n = 500$	$n = 1000$
$\varepsilon = 1\%$	0.0059	0.0062	0.0067	0.0073	0.0086	0.0094	0.0109
$\varepsilon = 5\%$	0.0432	0.0455	0.0469	0.0478	0.0488	0.0492	0.0508
$\varepsilon = 10\%$	0.0922	0.0934	0.0954	0.0962	0.0979	0.0995	0.1001

The null hypothesis H_0 for which simulated samples are fitted by MOB-L distribution, is widely validated for different significance levels. Therefore, the proposed test statistic can be used to fit the data from the MOB-L distribution.

8. Data analysis

In this section, we provide two real data analysis to prove the importance and flexibility of the MOB-L distribution and another censored data to validate the MOB-L model using the modified test statistic.

8.1. Applications

In this section, we illustrate the performance and flexibility of the MOB-L distribution, as a sub-model of the MOB-R class, using two real-life data sets. The first data on 63 of strengths of 1.5 cm glass fibres which are used by [6] and [19]. The data are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.77, 1.84, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.48, 1.5, 1.55, 1.61, 1.62, 1.81, 2, 1.82, 2.01, 0.77, 1.61, 0.74, 1.04, 1.62, 1.66, 1.7, 1.64, 1.68, 1.73, 1.11, 1.28, 1.42, 1.5, 1.54, 1.3, 1.48, 1.51, 1.55, 1.61, 1.6, 0.84, 1.24, 1.63, 1.67, 1.7, 1.78, 1.89. The second data on 128 bladder cancer patients on their remission times (in months) which are reported in [15] and are analyzed by [4] and [5]. The data are: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 6.97, 9.02, 13.29, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 0.40, 2.26, 13.80, 25.74, 0.50, 2.46, 3.57, 5.06, 7.09, 9.22, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39,10.34, 3.64, 5.09, 7.26, 9.47, 14.24, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 10.75, 16.62, 43.01, 1.19, 2.75, 1.05, 2.69, 4.23, 5.41, 7.62, 4.26, 5.41, 7.63, 4.33, 5.49, 7.66, 11.25, 17.14, 17.12, 46.12, 1.26, 2.83, 79.05, 11.64, 17.36, 1.40, 3.02, 1.35, 2.87, 5.62, 7.87, 4.34, 5.71, 7.93, 11.79, 8.26, 11.98, 19.13, 1.76, 18.10, 1.46, 4.40, 5.85, 3.25, 4.50, 6.25, 6.54, 8.53, 12.03, 20.28, 2.02, 8.37, 12.02, 2.02, 3.31, 4.51, 3.36, 6.76, 2.07, 3.36, 12.07, 21.73, 6.93, 8.65, 12.63, 22.69.

The MOB-L model is compared with some existing models namely, Kumaraswamy-Lomax (Kw-L) [17], generalized-exponentiated exponential-Weibull (GE-EW), beta-Lomax (B-L) [17], generalized exponentiated-exponential (GEE), exponentiated-Weibull (EW) [21] and Lomax (L) distributions. The fitted competing models are assessed using the Anderson-Darling (AnD) and Cramer von Mises (CvM) measures.

The MLEs and the discrimination measures for all competing models are listed in Tables 3 and 4 for both data sets, respectively. It is clear from Tables 3 and 4, that the MOB-L model provides better fit for both data sets as compared with other competing models.

Table 3. Parameter estimates, AnD and CvM statistics for carbon fibres data.

Distribution	θ	ρ	η	a	b	AnD	CvM
MOB-L	1.92000	33.30000	20.99000	18.83000	2.15000	0.26360	0.04240
	(1.2500)	(1.0980)	(1.7870)	(0.7526)	(0.0946)		
GE-EW	0.15704	0.03692	3.22861	1.77021	-	0.37840	0.05954
	(0.3778)	(0.0389)	(0.6367)	(1.3850)			
Kw-L	103.18000	8.72000	-	3.90000	345.35000	0.58070	0.105900
	(31.2200)	(26.5700)	-	(0.6030)	(72.1100)		
B-L	181.89000	7.02000	-	7.57000	68.44000	1.33900	0.24740
	(38.4600)	(40.6400)	-	(1.3000)	(38.3300)		
L	109.20000	39.67000	-	-	-	1.36400	0.25160
	(19.5500)	(12.8070)					
GEE	0.26555	10.03650	7.23658	-	-	1.43415	0.26682
	(0.2162)	(2.5950)	(7.0528)				
EW	3.73666	0.01709	0.01402	-	-	0.40365	0.06479
	(0.4457)	(0.0213)	(0.0084)				

The hrf plots of the MOB-L model for the data sets are depicted in Figure 2. The TTT plots for the two data sets are shown in Figure 3. The TTT plot for glass fibres data is concave that refers to increasing failure rate, whereas the TTT plot for remission times data is concave then convex which refers to a unimodal hazard rate. As shown, Figures 2 and 3 are consistent, where the hrf of the MOB-L model is increasing for glass fibers data and unimodal for remission times data, hence we conclude that the MOB-L is a suitable distribution for fitting the two data sets. Further, the estimated pdf, cdf, sf and pp plots of the MOB-L model are displayed in Figures 4 and 5, for the two data sets.

8.2. Testing a real censored data using the Y_n^2 statistic

For now, we analyze the lymphoma data which represent times from diagnosis to death (in months) for 31 individuals with advanced non Hodgkin's lymphoma clinical symptoms, using the MOB-L model. This data have been analyzed by Matthews et al. [18] and Gijbels and Gurler [12]. Among these 31 observations 11 of the times are censored, because the patients were alive at the last time of follow-up. The data are: 2.5, 4.1, 4.6, 6.4, 6.7, 7.4, 7.6, 7.7, 7.8, 8.8, 13.3, 13.4, 18.3, 19.7, 21.9, 24.7, 27.5, 29.7, 30.1*, 32.9, 33.5, 35.4*, 37.7*, 40.9*, 42.6*, 45.4*, 48.5*, 48.9*, 60.4*, 64.4*, 66.4*. The * denotes a censored observation.

Table 4. Parameter estimates, AnD and CvM statistics for remission times data.

Distribution	θ	ρ	η	a	b	AnD	CvM
MOB-L	1.64953	0.08757	1.15492	32.19600	21.31120	0.09018	0.01391
	(0.0144)	(0.1735)	(0.8478)	(5.6221)	(1.8282)		
GE-EW	1×10^{-10}	1.30988	0.52009	3.74791	-	0.29907	0.04526
	(0.0983)	(1.9112)	(0.3223)	(3.3941)			
Kw-L	13.19000	0.53900	-	1.51800	8.28900	0.17240	0.02580
	(17.6800)	(2.7120)	-	(0.2667)	(47.4700)		
B-L	20.63000	0.08670	-	1.58500	54.60000	0.19230	0.02860
	(14.18000)	(0.31350)	-	(0.28360)	(19.9300)		
L	121.04100	13.94000	-	-	-	0.48730	0.08060
	(42.76000)	(15.39000)					
GEE	0.12117	1.21795	1.00156	-	-	0.71819	0.12840
	(0.1068)	(0.1877)	(0.8659)				
EW	1.04783	1.00500×10^{-7}	0.09389	-	-	0.96345	0.15430
	(0.31424)	(0.3013)	(0.1179)				

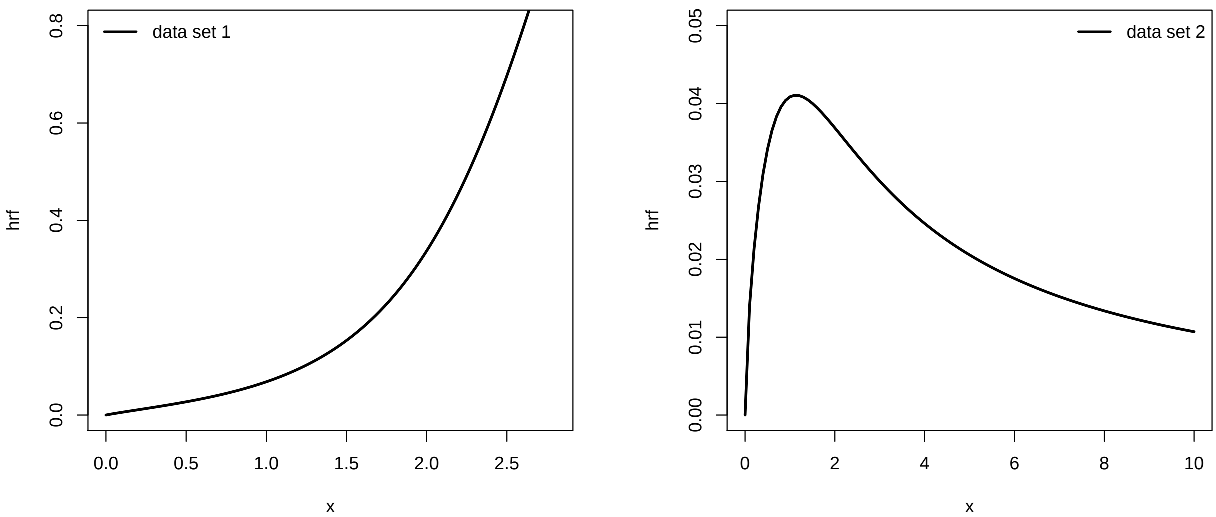

Figure 2. The hrf plots of the MOB-L model (left) for glass fibres data and (right) for remission times data.

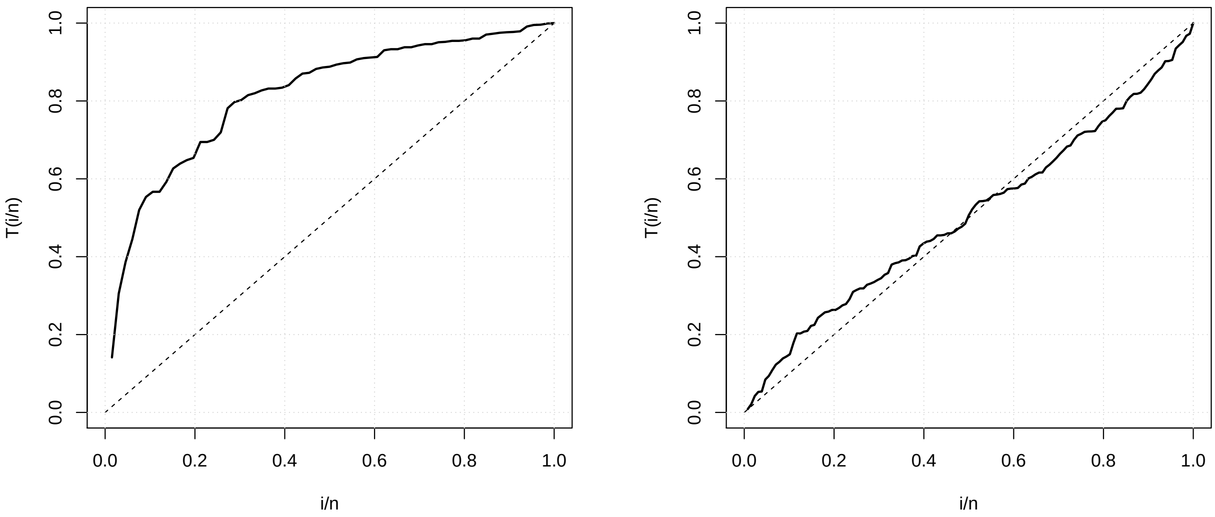

Figure 3. TTT plots (left) for glass fibres data and (right) for remission times data.

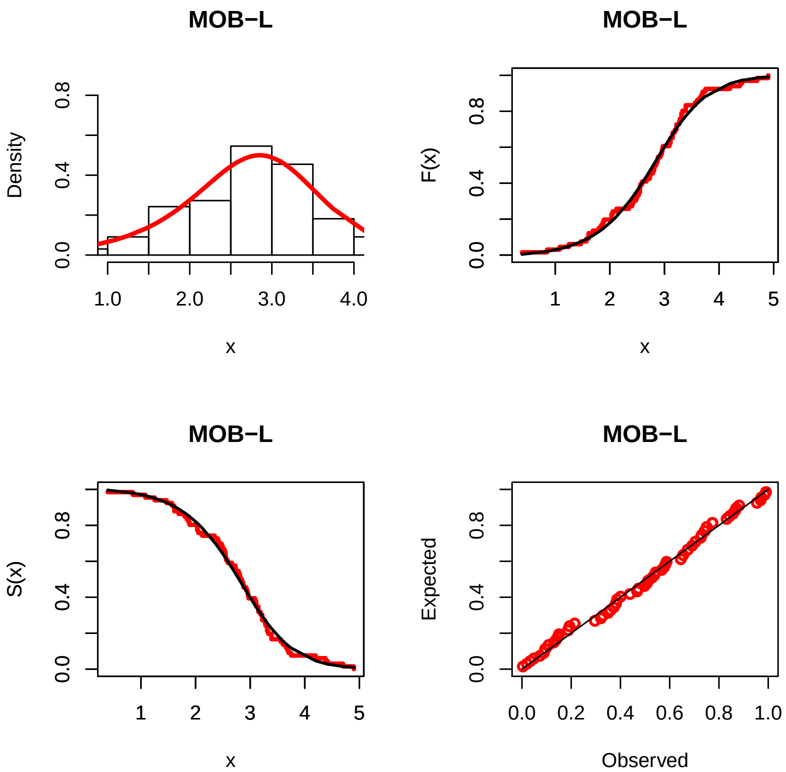

Figure 4. Estimated pdf, cdf, sf and pp of the MOB-L model for glass fibres data.

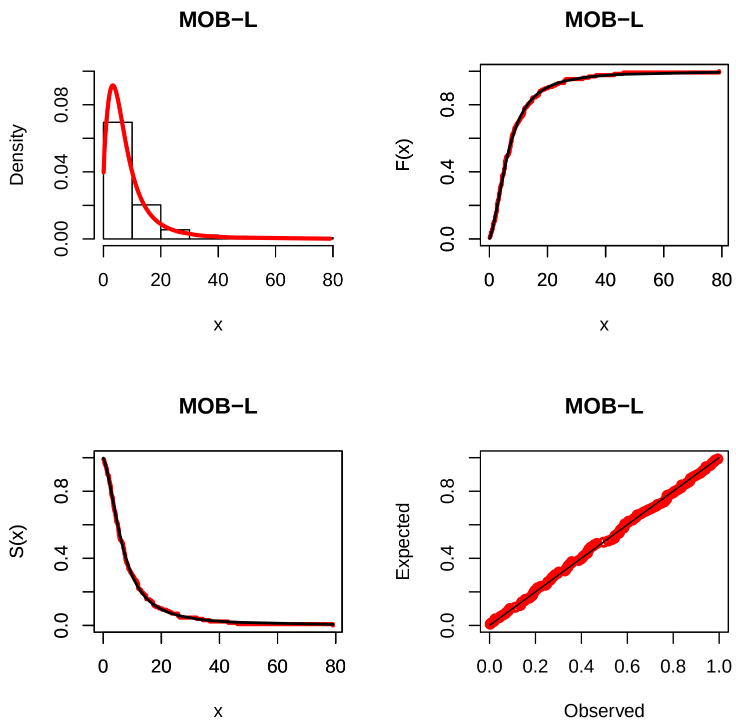

Figure 5. Estimated pdf, cdf, sf and pp of the MOB-L model for remission times data.

The test statistic provided in Section 7 is used to verify if these data can be modeled by the MOB-L distribution. To this end, we first calculate the MLEs of the MOB-L parameters

$$\hat{\xi} = (b, \theta, \rho, \eta, \bar{\eta})^T = (2.673, 1.983, 5.124, 4.286, 3.463)^T .$$

Data are grouped into $r = 5$ intervals I_j. The numerical result are listed in Table 5.

Table 5. The values of $\widehat{a}_j, e_j, U_j, \hat{C}_{1j}, \hat{C}_{2j}, \hat{C}_{3j}, \hat{C}_{4j}, \hat{C}_{5j}$.

$\hat{a}_j$	7.1	8.3	16.4	32.5	66.4
U_J	5	4	3	7	12
$\hat{C}_{1j}$	0.021	0.074	−0.526	−0.432	−0.236
$\hat{C}_{2j}$	−0.564	0.013	0.008	−0.718	−0.936
$\hat{C}_{3j}$	−0.536	−0.895	−0.921	−0.748	−1.103
$\hat{C}_{4j}$	1.166	0.933	0.699	1.399	0.466
$\hat{C}_{5j}$	0.852	0.763	0.236	0.974	0.125
e_j	2.1935	2.1935	2.1935	2.1935	2.1935

Then, we obtain the value of the test statistic Y_n^2 as

$$Y_n^2 = X^2 + Q = 4.623 + 2.635 = 7.258$$

For significance level $\varepsilon = 0.05$, the critical value $\chi_5^2 = 11.0705$ is superior than the value of $Y_n^2 = 7.258$, so we can conclude that the proposed MOB-L model fit these data very well.

9. Conclusions

In this paper, we present a new family called, Marshall-Olkin Burr-R family. Some general properties of this family are studied. The estimation of its parameters is carried out by the maximum likelihood approach. One special sub-model namely, Marshall-Olkin Burr-Lomax (MOB-L) distribution is discussed in detail. Two applications are used to check the performance of the MOB-L model. A modified chi-square test statistic for censored data is used to verify the validity of the MOB-Lomax distribution. This test statistic shows that the MOB-Lomax model can be used as a good candidate for analyzing real censored data. The work in the present paper can be extended in some ways. For example, bivariate Marshall-Olkin Burr family can be studied. Further, the parameters of the MOB-L distribution can be estimated using classical and Bayesian estimation methods and compare between them to determine the best estimation method.

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Conflict of interest

There is no conflict of interest declared by the authors.

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