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Research article

Random attractors for stochastic discrete long wave-short wave resonance equations driven by fractional Brownian motions

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Abstract: We study the dynamical behavior of the solutions of stochastic discrete long wave-short wave resonance equations driven by fractional Brownian motions with Hurst parameter $H \in (\frac{1}{2}, 1)$. And then we prove that the random dynamical system has a unique random equilibrium, which constitutes a singleton sets random attractor.

Keywords: stochastic discrete long wave-short wave resonance equation; random attractor; fractional Brownian motion
Mathematics Subject Classification: 37H05, 60G22, 60H15

1. Introduction

The long wave-short wave resonance equation has been deeply explored by many experts and scholars in recent years, see [13, 17, 20, 25–27]. The focus of recent research has gradually shifted from deterministic to stochastic long wave-short wave equations. For instance, in reference [29], the authors have proved that the compact kernel part of the long wave-short wave equation on an infinite lattice is upper semi-continuous. The random attractor of the stochastic discrete long wave-short wave resonance equation on an infinite lattice has been obtained by reference [28]. Many studies are obtained by using Itö theorem under standard Brownian motion (see [14]). We find that the method of standard Brownian motion is no longer applicable here when we study fractional Brownian motion. The most significant difference between fractional Brownian motion and standard Brownian motion is that it has no independent increment, so we can not found the Markov process. However, we have known that some theories of stochastic dynamical systems can also explain non-Markov processes through references [10, 21]. Therefore, we obtain a new technique to study fractional Brownian motion, which is inspired by reference [12].

In recent years, many experts and scholars have carried out extensive and in-depth research on lattice dynamic system because of its more and more critical role in biology, electrical engineering, laser system and other fields. For example, the travelling wave solutions of the lattice dynamical system are studied in [2]. Many people also have studied the dynamics of lattice dynamical systems, see [3, 6, 23, 26, 30] and related references. Besides, a large number of authorities have studied the specific equations in the lattice system. For instance, In reference [18], the author has explored the stochastic dynamical system of the stochastic 3D Navier-Stokes equations and obtained the random attractor of this system. As we know, the study of attractors has received considerable attention, see [19, 20]. Recently, a lot of work has been done on the existence of random attractors, see [4, 8, 11, 12, 15, 22, 28, 30].

In this article, we discuss the stochastic discrete long wave-short wave resonance equation driven by fractional Brownian motions. For $\forall\ n \in \mathbf{Z},\ t > 0$, we have

$$\begin{cases} i(\dfrac{du_n}{dt} + \alpha u_n) - (Au)_n - u_n v_n = f_n(t) + a_n \dfrac{d\beta_n^H(t)}{dt}, \\ \dfrac{dv_n}{dt} + \mu v_n + \gamma (B(|u|^2))_n = g_n(t) + b_n \dfrac{d\beta_n^H(t)}{dt}, \\ u(0) = u_n(0) = (u_{n0})_{n\in\mathbf{Z}},\ \ v(0) = v_n(0) = (v_{n0})_{n\in\mathbf{Z}}, \end{cases} \tag{1.1}$$

where $u_n = u_n(t)$ is a complex-valued function and $v_n = v_n(t)$ is a real-valued function, α, γ and μ are positive constants. $\mathbf{Z}$ is the set of integers, and i is the unit of imaginary numbers. It is well known that i satisfies the equation $i^2 = -1$. $a = (a_n)_{n\in\mathbf{Z}} \in \ell^2$, $b = (b_n)_{n\in\mathbf{Z}} \in \ell^2$, $f = (f_n)_{n\in\mathbf{Z}} \in \ell^2$ and $g = (g_n)_{n\in\mathbf{Z}} \in \ell^2$, where $\ell^2 \in \mathbb{C}_b(\mathbb{R}, \ell^2)$ is the space composed of all bounded continuous functions from $\mathbf{R}$ into ℓ^2. $\{\beta_n^H(t) : n \in \mathbf{Z}\}$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$, A and B are linear operators. We will give their detailed definitions in the second section.

The goal of this article is to study the stochastic discrete long wave-short wave resonance equation driven by fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and to obtain the random attractor of the system (1.1). The method of this paper mainly comes from literature [10]. This method has used a pathwise Riemann-Stieltjes integral (see [24]) to define the random integral of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. Furthermore, by using fractional Ornstein-Uhlenbeck process and Gronwall lemma, a pullback absorption set of the stochastic dynamical system is obtained. In this paper, we need to consider two problems, one is the path differentiability of the solution of the system (1.1), and the other is the property of the random attractor of the solution. To solve the first problem, we use the indivisibility of the trajectory of the system (1.1) to find the difference between any two solutions. Furthermore, we can obtain the path differentiability of solutions (see [5]). To solve the second problem, the existence of a unique stochastic equilibrium set were researched by using the stationarity of fractional Ornstein-Uhlenbeck process. Combining the above two points, we can get that the random attractor of the system (1.1) is a singleton sets random attractor.

The rest of this article is organized as follows. In Section 2, we introduce some symbols and spaces. Furthermore, we review some facts about fractional Brownian motion and random dynamical systems. In Section 3, we prove that the system (1.1) has a global solution by transforming the equation form and making a priori estimation. In Section 4, we show that (1.1) generates a stochastic dynamical system ϕ. And then the system ϕ developed by Eq (1.1) has a unique stochastic equilibrium, which constitutes a singleton sets random attractor.

2. Preliminary

In this portion, we introduce some symbols used in the following and describe some theorems associated with fractional Brownian motion (fBm) and random attractors (see [3, 7]).

Firstly, we introduce some symbolic representations and some spaces related to this article.

$$L^2 = \{u = (u_k)_{k\in\mathbf{Z}}, u_k \in \mathbf{C} : \sum_{k\in\mathbf{Z}} |u_k|^2 < \infty\},$$
$$\ell^2 = \{v = (v_k)_{k\in\mathbf{Z}}, u_k \in \mathbf{R} : \sum_{k\in\mathbf{Z}} |v_k|^2 < \infty\},$$

where $\mathbf{R}$ is the set of real numbers, and $\mathbf{C}$ is the set of imaginary numbers. Both L^2 and ℓ^2 are Hilbert spaces. They have the inner product and norm as

$$(u, v) = \mathrm{Re} \sum_{k\in\mathbf{Z}} u_k\overline{v_k},\ \|u\|^2 = (u, u) = \sum_{k\in\mathbf{Z}} |u_k|^2,\ \ \forall u \in L^2,\ v \in \ell^2,$$

where $\overline{v_k}$ means the conjugate of v_k. Assume that $\Phi = (\Phi(t)) = (u(t), v(t))$ for all $t \geq 0$. Let $\mathbb{H}$ denote $\mathbb{H} = L^2 \times \ell^2$. Then we have norm $\|\cdot\|_{\mathbb{H}}$ as follows

$$\|\Phi\|^2_{\mathbb{H}} = \|u\|^2 + \|v\|^2,$$

where $u = (u_k)_{k\in\mathbf{Z}} \in L^2$ and $v = (v_k)_{k\in\mathbf{Z}} \in \ell^2$. Next, we give the definition of these linear operators A, B and B^* in the system (1.1)

$$(Au)_k = -u_{k-1} + 2u_k - u_{k+1},\ (Bu)_k = -u_k + u_{k+1},\ (B^*u)_k = u_{k-1} - u_k,$$

where $u = (u_k)_{k\in\mathbf{Z}} \in L^2$. By simple calculation, we can get

$$A = BB^* = B^*B, (B^*u, u') = (u', Bu),\ \forall u,\ u' \in L^2.$$

Obviously, $(Au, u) \geq 0$ holds. Moreover, for $u = (u_k)_{k\in\mathbf{Z}} \in L^2$, the L^2 norm of Bu satisfies the following inequality([12])

$$\|Bu\|^2 \leq 4\|u\|^2. \tag{2.1}$$

Secondly, we review some knowledge of the fractional Brownian motion (fBm). Let $\beta^H(t)$, $t \in \mathbf{R}$ be a continuous centred Gaussian process, assuming that H is the Hurst parameter and $H \in (0, 1)$. Now we give the definition of the two-sided one-dimensional fractional Brownian motion β^H

$$\mathbf{E}\beta^H(t)\beta^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}),\ \ t, s \in \mathbf{R}.$$

Next, we discuss the value of Hurst parameter H. When $H = \frac{1}{2}$, β^H becomes the standard Brownian motion, and the random attractor of the stochastic discrete long wave-short wave equation is obtained in reference [28]. When $H \neq \frac{1}{2}$, the fractional Brownian motion does not satisfy the Markov process. But we have

$$\mathbf{E}|\beta^H(t) - \beta^H(s)|^2 = |t - s|^{2H}.$$

When $0 < H < \frac{1}{2}$, the generalized Stieltjes integral can not be used to define random integral, which makes it very difficult to deal with the stochastic dynamical systems. So we only consider the case of Hurst parameter $\frac{1}{2} < H < 1$ in this article.

According to the definition of $\beta^H(t)$, we can use Kolmogorov's theorem not only to know that $\beta^H(t)$ is continuous, but also that the path is Hölder continuous of any order $H' \in (0, H)$. For details, see [16]. In this paper, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbf{R}})$ is an ergodic metric dynamical system, where $\mathcal{F}$ is a relevant sigma-algebra, $\Omega = C_0(\mathbf{R}, \ell^2)$ denotes a continuous function space with value ℓ^2 on $\mathbf{R}$ with an open compact topology, and $\{\theta_t\}_{t\in\mathbf{R}}$ is the group of Wiener transformations on Ω given by

$$\{\theta_t\}\omega(s) = \omega(s + t) - \omega(t),$$

where every $s, t \in \mathbf{R}$. Consequently, we can get the following equation

$$\begin{aligned} &\beta^H(\cdot, \omega) = \omega(\cdot), \\ &\beta^H(\cdot, \theta_s\omega) = \beta^H(\cdot + s, \omega) - \beta^H(s, \omega) = \omega(\cdot + s) - \omega(s). \end{aligned} \tag{2.2}$$

In light of Definition 2.1 in reference [12], it can be seen that system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbf{R}})$ exists a measurable mapping $\theta : (\mathbf{R} \times \Omega, \mathcal{B}(\mathbf{R} \otimes \mathcal{F}) \to (\Omega, \mathcal{F})$.

Finally, we review some definitions of random attractors. For the convenience of readers, we will make a brief description. We should note that in the following definitions, the symbols $(X, \|\cdot\|_X)$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbf{R}})$ represent separable Hilbert space and metric dynamical system, respectively.

Definition 2.1. (see [12]) Let ϕ be a random dynamical system on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbf{R}})$. If $\phi := \phi(t, \omega)$ is continuous for every $t \geq 0,\ \omega \in \Omega$ and satisfies

$$\begin{aligned} &\phi(0, \omega,) = Id_X, \\ &\phi(t + s, \omega) = \phi(t, \theta_s\omega, \phi(s, \omega)), \end{aligned}$$

where $s, t \geq 0,\ \omega \in \Omega$. Then the system ϕ is called a $(\mathcal{B} \times \mathbb{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping.

Definition 2.2. We have $\mathcal{D} = \{D(\omega), \omega \in \Omega\}$ as a domain containing the set of all nonempty subsets $D(\omega)$ in X. For every $\omega \in \Omega$, it holds

$$D \in \mathcal{D} \ and\ D'(\omega) \subset D(\omega).$$

Then $D' \in \mathcal{D}$. We say that the domain $\mathcal{D}$ satisfies the inclusion property.

Definition 2.3. (see [1]) Let $\mathcal{A} = \{A(\omega), \omega \in \Omega\}$ be composed of the nonempty measurable compact subsets $\mathcal{A}(\omega)$ of X. We have

$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega),\ \forall\ t \in \mathbf{R}^+,$$

where $\mathbf{R}^+$ is the set of positive real numbers. Then $\mathcal{A}(\omega)$ of X is referred to as a ϕ-invariant.

Definition 2.4. (see [1]) There is a family $\mathcal{A}$ as defined in Definition 2.3. If the pathwise pullback attracting satisfying

$$\lim_{t\to\infty} d_X(\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega), \mathcal{A}(\omega))) = 0,$$

for all $D \in \mathcal{D}$. Then $\mathcal{A}(\omega)$ of X become a random attractor. Here d_X is the Hausdorff semi-distance on X.

Definition 2.5. Let $\{D(\omega)\}$ be a random bounded set. For all $\beta > 0$, we have

$$\lim_{t\to\infty} e^{-\beta t} d(D(\theta_{-t}\omega)) = 0, \quad \omega \in \Omega,$$

where $d(D) = \sup_{b\in D} \|b\|_X$. Then $\{D(\omega)\}$ is tempered with regard to $(\theta_t)_{t\in\mathbf{R}}$.

In this article, we will always use random tempered sets to give the attracting universe.

Definition 2.6. Let u: $\Omega \to X$ be a random variable. Suppose u is invariant under the action of stochastic dynamical system $\phi(t, \omega)$, i.e.,

$$\phi(t, \omega)u(\omega) = u(\theta_t\omega), \ for\ all\ t \geq 0\ and\ \omega \in \Omega.$$

Then u is called stochastic equilibrium.

Definition 2.7. (see [7]) We have the nonempty measurable sets $D(\omega) \in \mathcal{D}$. For all $T_D \geq 0$, a family $\mathcal{B}$ composed of $B(\omega)$, we have

$$\phi(t, \theta_{-t}\omega, D(\phi_{-t}\omega)) \subset B(\omega), \ \forall t \geq T_D(\omega), \ \omega \in \Omega.$$

Then a family $\mathcal{B}$ is called random absorbing.

Theorem 2.8. (see [1, 7]) Suppose that the system ϕ as defined in Definition 2.1 is asymptotically compact in X and $\mathcal{B}$ be a closed random tempered absorbing family. If there exists $\omega \in \Omega$ such that

$$\mathcal{A}(\omega) = \bigcap_{t>0} \overline{\bigcup_{\tau\geq t} \phi(\tau, \theta_{-\tau}, B(\theta_{-\tau}\omega))}.$$

Then $\mathcal{A}(\omega)$ is a random attractor of system ϕ.

We have to note that when the random variable u^* has the relation $\mathcal{A}(\omega) = \{u^*(\omega)\}$. In other words, the random attractor includes singleton sets. Then $u^*(t) = u^*(\theta_t\omega)$ is called a stationary stochastic process.

3. The existence of the global solution

In this section, We mainly divide into three steps to prove the existence of global solution of system (1.1). Firstly, we transform the system (1.1) with initial conditions into a pathwise Riemann-Stieltjes integral equation in $\mathbb{H}$

$$\begin{cases} u(t) = u_0 + \int_0^t [-iu(s)v(s) - \alpha u(s) - iAu(s) - if(s)]ds + W_1(t), \\ v(t) = v_0 + \int_0^t [-\mu v(s) - \gamma(B(|u(s)|^2)) + g(s)]ds + W_2(t), \\ u(0) = u_n(0) = (u_{n0})_{n\in\mathbf{Z}}, \ \ v(0) = v_n(0) = (v_{n0})_{n\in\mathbf{Z}}, \end{cases} \tag{3.1}$$

where

$$W_1(t) := W_1(t, \omega) = \sum_{i\in\mathbf{Z}} a_n\omega_i(t)e^i,$$

$$W_2(t) := W_2(t,\omega) = \sum_{i\in\mathbf{Z}} b_n\omega_i(t)e^i.$$

$(e^i)_{i\in\mathbf{Z}} \in \ell^2$ represent the element with a value of 1 at position i and 0 at rest. And then we make a priori estimation of the transformed system (3.1). Finally, we prove that the system (3.1) has the global solution.

Moreover, through [21,22], we can be known that if $(\widetilde{\rho}_i(\omega))_{i\in\mathbf{Z}} \in \ell^2$ are some positive constants and for $\forall\omega \in \Omega$ such that

$$\|W_j(t)\|^2 \leq 2\max\{\|a\|^2, \|b\|^2\}\|\widetilde{\rho}(\omega)\|^2(1+|t|^4),\ j = 1,2, \tag{3.2}$$

Then the fractional Brownian motions are well-defined.

Now let's start with a priori estimation of the pathwise Riemann-Stieltjes integral Eq (3.1).

Lemma 3.1. Suppose that $f(t) = (f_k(t))_{k\in\mathbf{Z}} \in C_b(\mathbf{R}, L^2)$. Then we obtain that the system (3.1) exists solutions and satisfies

$$\|u\|^2 \leq e^{-\alpha t}\|u_0\|^2 + \frac{1}{\alpha}\|f\|^2,\ \forall t \geq 0,\ \omega \in \Omega, \tag{3.3}$$

where $\|f\| = \sup_{t\in\mathbf{R}} |f(t)|^2$.

Here, we omit the proof (see Lemma 3.1 of [28] for details). By using similar methods in references [12,21], we can obtain the following two lemmas.

Lemma 3.2. Suppose that $f \in C(\mathbf{R}, L^2)$ and $T > 0$. Then we can obtain that there exists a pathwise solution $\Phi = (\Phi(t))_{t\geq 0} = (u(t), v(t))_{t\geq 0}$ for system (3.1). And the solution satisfies

$$\begin{aligned}\sup_{t\in[0,T]} \|\Phi(t)\|^2_{\mathbb{H}} \leq & C\|\Phi(0)\|^2_{\mathbb{H}} + \sup_{t\in[0,T]}(\|W_1(t)\|^2 + \|W_1(t)\|^2) \\ & + \int_0^T (\|W_1(t)\|^2 + \|AW_1(t)\|^2 + \|W_2(t)\|^2 + \|f\|^2 + \|g\|^2 + \|W_1(t)\|^4)ds,\end{aligned}$$

where C is a positive constant independent of T.

Lemma 3.3. The solution of Eq (3.1) can be rewritten as determinants of a continuous random dynamical system $\phi :\ \mathbf{R}^+ \times \Omega \times \mathbb{H} \to \mathbb{H}$ as follows

$$\phi(t,\omega,\Phi_0) = \Phi_0 + \int_0^t G(\Phi(s))ds + \eta(t,\omega)\ \ \forall\ t \geq 0, \tag{3.4}$$

where $G(\Phi(t)) = L\Phi(t) + F(\Phi(t))$ and

$$L = \begin{pmatrix} -\alpha - iA & 0 \\ 0 & -\mu \end{pmatrix},\ F = \begin{pmatrix} -iuv - if \\ -\gamma(B(|u|^2)) + g(t) \end{pmatrix},\ \eta(t,\omega) = \begin{pmatrix} W_1(t,\omega) \\ W_2(t,\omega)) \end{pmatrix}.$$

4. Existence of Random attractors

In this part, we not only prove that there exists a random attractor for the system ϕ defined in Lemma 3.3, but also show that the random attractor is composed of the singleton sets.

We study the fractional Ornstein-Uhlenbeck equations as follows

$$du(t) = -\alpha u(t)dt + W_1(t), \tag{4.1}$$
$$dv(t) = -\mu v(t)dt + dW_2(t), \tag{4.2}$$

where α, μ are positive constants and $W_1(t), W_2(t)$ denote one-dimensional fractional Brownian motions with Hurst parameter $H \in (\frac{1}{2}, 1)$. According to the simple calculation, we can get the solutions

$$\begin{aligned} u(t) &= u_0 e^{-\alpha t} + e^{-\alpha t}\int_0^t e^{\alpha s} dW_1(s), \\ v(t) &= v_0 e^{-\mu t} + e^{-\mu t}\int_0^t e^{\mu s} dW_2(s). \end{aligned} \tag{4.3}$$

By taking the pullback limit for the above time t, we obtain a stochastic stationary solution

$$\begin{aligned} \overline{u}(t) &= e^{-\alpha t}\int_{-\infty}^t e^{\alpha s} dW_1(s), \\ \overline{v}(t) &= e^{-\mu t}\int_{-\infty}^t e^{\mu s} dW_2(s),\ t \in \mathbf{R}. \end{aligned} \tag{4.4}$$

This solution $(\overline{u}(t), \overline{v}(t))$ is called fractional Ornstein-Uhlenbeck solution.

Lemma 4.1. Suppose that $(\rho_i(\omega))_{i\in\mathbf{Z}}$ and $(\varrho_i(\omega))_{i\in\mathbf{Z}} \in \ell^2$ are two positive constants, and for each $\omega \in \Omega$, it holds

$$(\rho^2(\omega)) = 16\sum_{i\in\mathbf{Z}} a_i^2\rho_i^2(\omega)),\ (\varrho^2(\omega)) = 16\sum_{i\in\mathbf{Z}} b_i^2\varrho_i^2(\omega)).$$

Then the Riemann-Stieltjes integrals of the system (4.4) are well defined on $\mathbb{H}$. Furthermore, we also establish

$$\begin{aligned} ||e^{-\alpha t}\int_{-\infty}^t e^{\alpha s} dW_1(s)|| &\le \rho(\omega)(1 + |t|^2), \\ ||e^{-\mu t}\int_{-\infty}^t e^{\mu s} dW_2(s)|| &\le \varrho(\omega)(1 + |t|^2), \end{aligned}$$

where all $\omega \in \Omega$, $t \in \mathbf{R}$.

Proof. We omit the proof here because we can draw this conclusion directly from [9].

Next, we give the main conclusion of this paper and prove it in detail.

Theorem 4.2. Suppose that $f, g \in C_b(\mathbf{R}, L^2)$. Then the system ϕ has a unique stochastic equilibrium, and it constitutes a singleton sets random attractor of the system ϕ.

Proof. Expressing two solutions of system (1.1) with $\Phi = (u(t), v(t))$ and $\overline{\Phi} = (\overline{u}(t), \overline{v}(t))$. For all $t \ge 0$, their difference satisfies pathwise differentiability. So according to (3.4), we obtain that

$$\Phi(t) - \overline{\Phi}(t) = \Phi_0 - \overline{\Phi}_0 + \int_0^t [L(\Phi(s) - \overline{\Phi}(s)) + F(\Phi(s)) - F(\overline{\Phi}(s))]ds.$$

Moreover, according to the integrand function on the right side of the above equation is path continuous, we can get that $\Phi(t) - \overline{\Phi}(t)$ is path differentiable by the basic theorem of calculus. It's easy to get

$$\frac{d}{dt}(\Phi(t) - \overline{\Phi}(t)) = L(\Phi(t) - \overline{\Phi}(t)) + F(\Phi(t)) - F(\overline{\Phi}(t)). \tag{4.5}$$

Taking the inner product of (4.5) and $\Phi(t) - \overline{\Phi}(t)$ on $\mathbb{H}$, we get

$$\begin{aligned}\frac{d}{dt}\|\Phi(t) - \overline{\Phi}(t)\|_{\mathbb{H}}^2 =& 2(\Phi(t) - \overline{\Phi}(t), L(\Phi(t) - \overline{\Phi}(t)))_{\mathbb{H}} \\ &+ 2(\Phi(t) - \overline{\Phi}(t), F(\Phi(t)) - F(\overline{\Phi}(t)))_{\mathbb{H}} \\ \leq& -2\kappa\|\Phi(t) - \overline{\Phi}(t)\|_{\mathbb{H}}^2,\end{aligned}$$

where $\kappa = \min\{\alpha, \gamma, \mu\}$. Thus we have

$$\lim_{t\to\infty}\|\Phi(t) - \overline{\Phi}(t)\|_{\mathbb{H}}^2 \leq \lim_{t\to\infty} e^{-2\beta t}\|\Phi_0 - \overline{\Phi}_0\|_{\mathbb{H}}^2 = 0.$$

The above equation shows that the solutions of system ϕ converge pathwise forward concerning time t.

Next, we need to find the convergence value of the solution. According to the path continuity of $\Phi(t)$ and $\overline{\Phi}(t)$, we assert that $\Phi(t) - \overline{\Phi}(t)$ is path differentiable. When $t \geq 0$, it satisfies the following path differentiable Eq (4.5). In other words, we can write (4.5) in the following equivalent form

$$\begin{aligned}&\frac{d}{dt}(u(t) - \overline{u}(t)) = -\alpha(u(t) - \overline{u}(t)) - iu(t)v(t) - iAu(t) - if(t), \\ &\frac{d}{dt}(v(t) - \overline{v}(t)) = -\mu(v(t) - \overline{v}(t)) - \gamma B(|u(t)|^2) + g(t).\end{aligned} \tag{4.6}$$

Let's take the inner product of $\mathbb{H}$ with $u(t) - \overline{u}(t)$ and $v(t) - \overline{v}(t)$ respectively, and add their real parts together. we have

$$\begin{aligned}&\frac{d}{dt}(\|u(t) - \overline{u}(t)\|^2 + \|v(t) - \overline{v}(t)\|^2) \\ =& -2\alpha\|u(t) - \overline{u}(t)\|^2 - 2\mu\|v(t) - \overline{v}(t)\|^2 \\ &- 2\mathrm{Im}(-u(t)v(t) - Au(t) - f(t), u(t) - \overline{u}(t)) \\ &+ 2(-\gamma B(|u(t)|^2) + g(t), v(t) - \overline{v}(t)).\end{aligned} \tag{4.7}$$

Using (2.1) and Young's inequality, we have the following estimates

$$\begin{aligned}2\mathrm{Im}(u(t)v(t), u(t) - \overline{u}(t)) \leq& \frac{4}{\alpha}\|u\|^2\|v\|^2 + \frac{\alpha}{4}\|u(t) - \overline{u}\|^2 \\ \leq& \frac{2}{\alpha}(\|u\|^4 + \|v\|^4) + \frac{\alpha}{4}\|u(t) - \overline{u}\|^2, \\ 2\mathrm{Im}(Au(t), u(t) - \overline{u}(t)) \leq& \frac{4}{\alpha}\|Au\|^2\|v\|^2 + \frac{\alpha}{4}\|u(t) - \overline{u}\|^2 \\ \leq& \frac{32}{\alpha}\|\overline{u}\|^2 + \frac{\alpha}{2}\|u(t) - \overline{u}\|^2,\end{aligned}$$

$$2\mathrm{Im}(f,u(t)-\overline{u}(t))\leq \frac{4}{\alpha}\|f\|^2\|v\|^2+\frac{\alpha}{4}\|u(t)-\overline{u}\|^2,$$
$$-2\gamma(B(|u|^2),v(t)-\overline{v}(t))\leq \frac{4\gamma}{\mu}\|B(|u|^2)\|^2+\frac{\gamma\mu}{4}\|v(t)-\overline{v}\|^2$$
$$\leq \frac{16\gamma}{\mu}(\|u\|^4+\|v\|^4)+\frac{\gamma\mu}{4}\|v(t)-\overline{v}\|^2,$$
$$2(g,v(t)-\overline{v}(t))\leq \frac{4}{\mu}\|g\|^2+\frac{\mu}{4}\|v(t)-\overline{v}\|^2.$$

We recall that $\kappa = \min\{\alpha,\mu,\gamma\}$. For some terms on the right side of (4.7), utilizing the above inequalities, we get

$$\frac{d}{dt}(\|u(t)-\overline{u}(t)\|^2+\|v(t)-\overline{v}\|^2)\leq -\kappa(\|u(t)-\overline{u}(t)\|^2+\|v(t)-\overline{v}\|^2)$$
$$+C_2(\|f\|^2+\|g\|^2+\|\overline{u}(t)\|^2), \tag{4.8}$$

where $C_2=C_2(\alpha,\mu,\|v\|)$. Then we get

$$\frac{d}{dt}(\|\Phi(t)-\overline{\Phi}(t)\|^2_{\mathbb{H}})\leq -\kappa\|\Phi(t)-\overline{\Phi}(t)\|^2_{\mathbb{H}}+C_2(\|f\|^2+\|g\|^2+\|\overline{u}(t)\|^2). \tag{4.9}$$

By using Gronwall lemma, we get

$$\|\Phi(t)-\overline{\Phi}(t)\|^2_{\mathbb{H}}\leq e^{-\kappa t}\|\Phi_0(\omega)-\overline{\Phi}_0(\omega)\|^2_{\mathbb{H}}$$
$$+C_2e^{-\kappa t}\int_0^t e^{\kappa s}(\|f(s)\|^2+\|g(s)\|^2+\|\overline{u}(s)\|^2)ds. \tag{4.10}$$

In the following description, we usually write $\overline{\Phi}(\cdot)$ (or $\overline{u}$, $\overline{v}$, Φ) as $\overline{\Phi}(\omega)$ (or $\overline{u}(\omega)$, $\overline{v}(\omega)$, $\Phi(\omega)$) to better illustrate the correlation between these variables and ω. We find the family of balls with $\overline{\Phi}_0(\omega)$ as the centre and $r(\omega)$ as the radius. Its radius $r(\omega)$ is expressed as follows

$$r(\omega):=\sqrt{1+C_2\int_{-\infty}^0 e^{\kappa s}(\|f(s)(\omega)\|^2+\|g(s)(\omega)\|^2+\|\overline{u}(s)(\omega)\|^2)ds}. \tag{4.11}$$

Then we infer that it is a family of pullback absorbing of the system (1.1). The random radius $r(\omega)$ is well defined by the properties of $f,g\in C_b(\mathbf{R},L^2)$ and Lemma 4.1. By replacing ω by $\theta_{-t}\omega$, we obtain from (4.10) that

$$\|\Phi(\theta_{-t}\omega)-\overline{\Phi}(\theta_{-t}\omega)\|^2_{\mathbb{H}}$$
$$\leq e^{-\kappa t}\|\Phi_0(\theta_{-t}\omega)-\overline{\Phi}_0(\theta_{-t}\omega)\|^2_{\mathbb{H}}+C_2\int_0^t e^{\kappa(s-t)}(\|f(s)(\theta_{-t}\omega)\|^2$$
$$+\|g(s)(\theta_{-t}\omega)\|^2+\|\overline{u}(s)(\theta_{-t}\omega)\|^2)ds,$$
$$\leq e^{-\kappa t}\|\Phi_0(\theta_{-t}\omega)-\overline{\Phi}_0(\theta_{-t}\omega)\|^2_{\mathbb{H}}+C_2\int_{-t}^0 e^{\kappa s}(\|f(s)(\theta_{-t}\omega)\|^2$$
$$+\|g(s)(\theta_{-t}\omega)\|^2+\|\overline{u}(s)(\theta_{-t}\omega)\|^2)ds. \tag{4.12}$$

We also have

$$\overline{u}(s)(\theta_{-t}\omega) = \overline{u}_0(s)(\theta_{s-t}\omega) = \overline{u}(s-t)(\omega).$$

If we take the above formula into (4.12), we obtain that $(\overline{u}(t))_{t\in\mathbf{R}}$ and $(\overline{v}(t))_{t\in\mathbf{R}}$ are stationary processes. This conclusion holds when $t \to \infty$.

According to Lemma 4.1, we get

$$e^{-\kappa t}\|(\overline{\Phi}_0(\theta_{-t}\omega)\|^2_{\mathbb{H}} = e^{-\kappa t}\|(\overline{\Phi}(-t)(\omega)\|^2_{\mathbb{H}} \to 0 \text{ as } t \to \infty.$$

Then we can obtain the pullback absorption as follows

$$\|\Phi(\theta_{-t}\omega)\|^2_{\mathbb{H}} = \|(\overline{\Phi}_0(\omega)\|^2_{\mathbb{H}} + r^2(\omega),\ \forall t \geq T_{D(\omega)}. \tag{4.13}$$

Therefore, we can get the existence of stationary random process $\overline{\Phi}(t)(\omega) := \overline{\Phi}_0(\theta_t\omega)$. This random process $\overline{\Phi}(t)(\omega)$ is a random equilibrium because it attracts all solutions along the pathwise either forward or pullback. Then we define a singleton set $\mathcal{A}$, which is composed of random equilibria and expressed as $\mathcal{A} = \{A(\omega), \omega \in \Omega\} = \{\overline{\Phi}_0(\omega)\}$. According to Definition 2.4 and its absorption, we can obtain that $\mathcal{A}$ has compactness, invariance and attraction. So we can deduce that the singleton set $\mathcal{A}$ is a random attractor. We have completed the proof of Theorem 4.2.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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