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Research article

p-moment exponential stability of second order differential equations with exponentially distributed moments of impulses

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Abstract: Differential equations of second order with impulses at random moments are set up and investigated in this paper. The main characteristic of the studied equations is that the impulses occur at random moments which are exponentially distributed random variables. The presence of random variables in the ordinary differential equation leads to a total change of the behavior of the solution. It is not a function as in the case of deterministic equations, it is a stochastic process. It requires combining of the results in Theory of Differential Equations and Probability Theory. The initial value problem is set up in appropriate way. Sample path solutions are defined as a solutions of ordinary differential equations with determined fixed moments of impulses. P-moment generalized exponential stability is defined and some sufficient conditions for this type of stability are obtained. The study is based on the application of Lyapunov functions. The results are illustrated on examples.

Keywords: second order differential equations; impulses at random moments; exponentially distribution; P-moment generalized exponential stability; Lyapunov functions
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1. Introduction

It is well known that linear and nonlinear ordinary second order differential equations (ODEs) are adequate apparatus to model many phenomena in physics, biology, chemistry, biophysics, mechanics, medicine, aerodynamics, economy, atomic energy, control theory, information theory, population dynamics, electrodynamics of complex media, and so on. One of the main question concerning qualitative behaviors of the solutions to ODEs of second order, is the stability (see, for example, [2, 4] and the references cited therein).

At the same time, many real world phenomena are characterizing by a special type of changes of the state of the process under investigation. If these changes act on a negligible small time, i.e. they act impulsively, then the dynamic of the state variable is modeling adequately by impulses. Impulsive differential equations have many applications in engineering, science and finance. As a ubiquitous phenomenon, pulses exist in mechanical systems with impacts, optimal control models in economics, and the transfers of satellite orbit. Impulsive control method has been widely used in many cases (see, for example, [10] about output formation-containment problem of interacted heterogeneous linear systems and the impulsive control method, paper [11] about delayed impulsive positive system model and a copositive Lyapunov-Krasovskii functional, [8] for multiagent neural networks). In the case when the impulses occur at random times, then the model requires the time of impulses to be considered as random variables. When there is uncertainty in the behavior of the state of the investigated process an appropriate model is usually a stochastic differential equation where one or more of the terms in the differential equation are stochastic processes, and this usually results with the solution being a stochastic process ([6, 13, 14, 15]). But there are some real world phenomena the dynamic of the state variable is changing deterministically between two consecutive instantaneous changes at uncertain moments. In this case appropriate models are impulsive differential equations with random impulses. The presence of random variables usually determine that the solutions of these equations are stochastic processes. We note that impulsive differential equations with random impulsive moments differs from stochastic differential equations with jumps. Deterministic differential equations with random impulses were considered, for example, in [1, 3, 9, 12]).

The main goal of the paper is to study stability properties of solutions of second order impulsive differential equation when the waiting time between two consecutive impulses is exponentially distributed. Note, the existence of second order differential equations with impulses are studied by several author (see for example, for deterministic impulses [16], for random impulses [5, 18, 17] and for stochastic equations with random impulses [7]). In this paper the p-moment generalized exponential stability of the zero solution is defined and some sufficient conditions are obtained. The results are illustrated on an example.

2. Random impulses in second order differential equations

Initially, we will give a brief overview of second order differential equations with deterministic impulses.

Let the increasing sequence of nonnegative points $\{T_k\}_{k=0}^{\infty}$ be given and $\lim\limits_{k\to\infty}\{T_k\} = \infty$. Consider the initial value problem for the system of *second order impulsive differential equations* (IDE) with fixed points of impulses

$$\begin{aligned}
&x'' = f(t, x(t), x'(t)) \quad \text{for} \quad t \in (T_k, T_{k+1}], \quad k = 0, 1, 2, \dots, \\
&x(T_k + 0) = I_k(x(T_k - 0)), \quad x'(T_k + 0) = J_k(x'(T_k - 0)) \quad \text{for} \quad k = 1, 2, \dots, \\
&x(0) = x_0, \quad x'(0) = x_1
\end{aligned} \tag{2.1}$$

where $x, x_0, x_1 \in \mathbb{R}^n$, $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k, J_k : \mathbb{R}^n \to \mathbb{R}^n$.

The solution of IDE (2.1) depends not only on the initial values (x_0, x_1) but on the moments of impulses T_k, $k = 1, 2, \dots$ and we will denote it by $x(t; x_0, x_1, \{T_k\})$. We will assume that $x(T_k; x_0, x_1, \{T_k\}) = \lim\limits_{t\to T_k-0} x(t; x_0, x_1, \{T_k\})$ and $x'(T_k; x_0, x_1, \{T_k\}) = \lim\limits_{t\to T_k-0} x'(t; x_0, x_1, \{T_k\})$ for any $k = 1, 2, \dots$.

We will assume the following conditions are satisfied:

H1. *$f(t,0,0)=0$ for $t\geq 0$, and $I_k(0)=0$, $J_k(0)=0$, $k=1,2,\dots$.*

H2. *For any initial values (t_0,x_0,x_1) the ODE $x''=f(t,x,x')$ with $x(t_0)=x_0$, $x'(t_0)=x_1$ has a unique solution $x(t)=x(t;t_0,x_0,x_1)$ defined for $t\geq t_0$.*

Let the probability space $(\Omega,\mathcal{F},P)$ be given. Let $\{\tau_k\}_{k=1}^{\infty}$ be a sequence of random variables defined on the sample space Ω.

Assume $\sum\limits_{k=1}^{\infty}\tau_k=\infty$ with probability 1.

Remark 1. *The random variables τ_k will define the time between two consecutive impulsive moments of the impulsive differential equation with random impulses.*

We will assume the following condition is satisfied

H3. *The random variables $\{\tau_k\}_{k=1}^{\infty}$ are independent exponentially distributed random variables with the same rate parameter λ , i.e. $\tau_k\in Exp(\lambda)$.*

Define the increasing sequence of random variables $\{\xi_k\}_{k=0}^{\infty}$ such that $\xi_0=0$ and $\xi_k=\sum\limits_{i=1}^{k}\tau_i$, $k=1,2,\dots$.

Remark 2. *The random variable ξ_n will be called the waiting time and it gives the arrival time of n-th impulses in the impulsive differential equation with random impulses.*

Let the points t_k be arbitrary values of the random variables τ_k, $k=1,2,\dots$ correspondingly. Define the increasing sequence of points $T_k=\sum\limits_{i=1}^{k}t_i$, $k=1,2,3\dots$ that are values of the random variables ξ_k and consider the initial value problem for the system of IDE with fixed points of impulses (2.1). The set of all solutions $x(t;x_0,x_1,\{T_k\})$ of IDE (2.1) for any values t_k of the random variables τ_k, $k=1,2,\dots$ generates a stochastic process with state space $\mathbb{R}^n$. We denote it by $x(t;x_0,x_1,\{\tau_k\})$ and we will say that it is a solution of the initial value problem (IVP) for differential equations with random moments of impulses (RIDE) formally written by

$$\begin{aligned}
&x''(t)=f(t,x(t),x'(t)) \quad \text{for} \quad \xi_k<t<\xi_{k+1},\ k=0,1,\dots,\\
&x(\xi_k+0)=I_k(x(\xi_k-0)),\quad x'(\xi_k+0)=J_k(x'(\xi_k-0)) \quad \text{for}\ \ k=1,2,\dots,\\
&x(0)=x_0,\quad x'(0)=x_1.
\end{aligned}\tag{2.2}$$

Definition 1. *Let t_k be a value of the random variable τ_k, $k=1,2,3,\dots$ and $T_k=\sum\limits_{i=1}^{k}t_i$, $k=1,2,\dots$. Then the solution $x(t;x_0,x_1,\{T_k\})$ of the IVP for the IDE with fixed points of impulses (2.1) is called* a sample path solution *of the IVP for the RIDE (2.2).*

Remark 3. *We note that if condition (H2) is satisfied then the sample path solution of the IVP for the RIDE (2.2) exists for all $t\geq T_0$ provided that the times between two consecutive impulses t_k are such that $\sum\limits_{k=1}^{\infty}t_k=\infty$.*

Definition 2. *A stochastic process $x(t;x_0,x_1,\{\tau_k\})$ with an uncountable state space $\mathbb{R}^n$ is said to be a solution of the IVP for the system of RIDE (2.2) if for any values t_k of the random variable τ_k, $k=1,2,3,\dots$ and $T_k=\sum\limits_{i=1}^{k}t_i, k=1,2,\dots$ the corresponding function $x(t;x_0,x_1,\{T_k\})$ is a sample path solution of the IVP for RIDE (2.2).*

Example 1. *Case 1*. (*differential equations without any type of impulses*). Consider the following IVP for the scalar DE

$$x''(t) = 0 \quad \text{for} \quad t \geq 0,$$
$$x(0) = x_0 \neq 0, \quad x'(0) = 0, \tag{2.3}$$

where x_0 is a given constant.

The solution of IVP (2.3) is $x(t; x_0) = x_0,\ t \geq 0$.

Case 2. (*impulsive differential equations with fixed points of impulses*). Let the points $\{T_k\}_{k=1}^{\infty}$ are initially given and consider the following IVP for the scalar differential equation with impulses at the points $T_k,\ k = 1, 2, \ldots$:

$$\begin{aligned} &x''(t) = 0 \quad \text{for} \quad t \geq 0,\ \ t \neq T_k, \\ &x(T_k + 0) = ax'(T_k - 0),\ \ x(T_k + 0) = bx'(T_k - 0) \quad \text{for}\ \ k = 1, 2, \ldots, \\ &x(0) = x_0 \neq 0, \quad x'(0) = 0, \end{aligned} \tag{2.4}$$

where a, b, x_0 are given constants.

The solution of IVP (2.4) is the piecewise continuous function

$$x(t; x_0) = x_0(a + b(t - T_k)) \prod_{i=1}^{k-1} (a + b(T_i - T_{i-1})), \quad t \in (T_k, T_{k+1}]. \tag{2.5}$$

The behavior of $x(t; x_0)$ depends significantly on the amplitudes a, b of the impulses. It is obviously the behavior of the solution is totally changed because of the presence of impulses (compare Case 1 and Case 2).

Case 3. (*differential equation with random points of impulses*). Let the random variables $\tau_k \in Exp(\lambda),\ k = 1, 2, \ldots$ be given and $\xi_0 = 0$ and $\xi_k = \sum_{i=1}^{k} \tau_i, \quad k = 1, 2, \ldots,\ k = 1, 2, \ldots$. Consider the following special case of the IVP for RIDE (2.2)

$$\begin{aligned} &x'' = 0 \quad \text{for} \quad \xi_k < t < \xi_{k+1},\ k = 0, 1, \ldots, \\ &x(\xi_k + 0) = ax'(\xi_k - 0),\ \ x(\xi_k + 0) = bx'(\xi_k - 0) \quad \text{for}\ \ k = 1, 2, \ldots, \\ &x(0) = x_0 \neq 0, \quad x'(0) = 0 \end{aligned} \tag{2.6}$$

where $x_0 \in \mathbb{R}$, a, b are constants and the random variables ξ_k are defined above.

Let for any $k = 1, 2, \ldots$ the point t_k be an arbitrary value of the random variable τ_k and consider the points $T_k = \sum_{i=0}^{k} t_i$, $k = 1, 2, 3 \ldots$, i.e. any T_k is a value of the random variable ξ_k. Consider the IVP for the corresponding (2.4) with impulses at points T_k, $k = 1, 2, \ldots$, i.e. deterministic points of impulses. This solution of (2.4) is given by (2.5) and it depends on both initial value x_0 and the moments of impulses T_k, i.e. on the initially chosen arbitrary values t_k of the random variables τ_k, $k = 1, 2, \ldots$. When these values t_k, $k = 1, 2, \ldots$ are changed, we obtain a new solution of (2.4) given by (2.5). Then, the set of all solutions of the IVP (2.4) for any values t_k of the random variables τ_k generates a stochastic process which will be denoted by $x(t; x_0, \{\tau_k\})$ and for which according to (2.5) the equality will be true: $x(t; x_0, \{\tau_k\}) = a^k x_0$ for $\xi_k < t \leq \xi_{k+1}$

$$x(t; x_0, \{\tau_k\}) = x_0(a + b(t - \xi_k)) \prod_{i=1}^{k} (a + b\tau_i), \quad \xi_k < t \leq \xi_{k+1},\ k = 0, 1, \ldots\ . \tag{2.7}$$

3. Exponentially distributed moments of impulses

For any $t \geq 0$ consider the events

$$S_k(t) = \{\omega \in \Omega : \ \xi_k(\omega) < t < \xi_{k+1}(\omega)\},$$

where the random variables $\xi_k, \ k = 1, 2, \dots$ are defined as above.

In the case of exponentially distributed random variables $\tau_k, \ k = 1, 2, \dots$ we will use the following result:

Lemma 3.1. *([1]) Let condition (H3) be satisfied.*

Then the probability that there will be exactly k impulses until time t, $t \geq 0$, is given by $P(S_k(t)) = \frac{\lambda^k t^k}{k!} e^{-\lambda t}$.

In our further research we will use the formula for the solution of the initial value problem for a scalar linear first order differential equation with random moments of impulses(see [1])

$$\begin{aligned} & u' = -mu, \quad \text{for } \ \xi_k < t < \xi_{k+1},, \ k = 0, 1, \dots, \\ & u(\xi_k + 0) = C_k u(\xi_k - 0), \quad \text{for } \ k = 1, 2, \dots, \\ & u(0) = u_0, \end{aligned} \tag{3.1}$$

where $u_0 \in \mathbb{R}$, $m > 0$ and $C_k \neq 1, \ (k = 1, 2, \dots)$ are real constants.

Lemma 3.2. *([1]). Let the condition (H3) be fulfilled and there exists a positive constants C such that $\sum\limits_{k=0}^{\infty} \prod_{i=1}^{k} |C_i| = C$.*

Then the solution of the IVP for the linear RIDE (3.1) is

$$u(t; u_0, \{\tau_k\}) = u_0 \Big(\prod_{i=1}^{k} C_i \Big) e^{\sum_{i=1}^{k} -m\tau_i} e^{-m(t-\xi_k)} \quad \textit{for } \ \xi_k < t < \xi_{k+1}, \ k = 1, 2, \dots \tag{3.2}$$

and the expected value of the solution satisfies the inequality

$$E(|u(t; u_0, \{\tau_k\})|) \leq |u_0| e^{-mt} e^{-\lambda t} \sum_{k=0}^{\infty} \prod_{i=1}^{k} \Big(|C_i| \Big) \frac{\lambda^k t^k}{k!} \leq |u_0| C e^{-mt}. \tag{3.3}$$

4. Main result

In this paper we will use the class $\Lambda(J, \Delta)$ of Lyapunov functions $V(t, x, y) : J \times \Delta \times \Delta \to \mathbb{R}_+$, which are continuous differentiable on $J \times \Delta \times \Delta$ and locally Lipschitzian with respect to its second and third arguments, where $J \subset \mathbb{R}_+$ and $\Delta \subset \mathbb{R}^n, \ 0 \in \Delta$.

Definition 3. *Let $p > 0$. Then the trivial solution ($x_0 = 0$) of the RIDE (2.2) is said to be p-moment generalized exponentially stable if for any $x_0, x_1 \in \mathbb{R}^n$ there exist constants $\alpha, \mu > 0$ and an increasing function $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$E[||x(t; x_0, x_1, \{\tau_k\})||^p] < \alpha \ m(\max\{||x_0||, ||x_1||\}^p) \ e^{-\mu t}, \quad \textit{for all } t > 0,$$

where $x(t; x_0, x_1, \{\tau_k\})$ is the solution of the IVP for the RIDE (2.2).

Remark 4. *In the case $m(u) \equiv u$ the p-moment generalized exponential stability is reduced to the known in the literature p-moment exponential stability.*

Remark 5. *We note that the two-moment exponential stability for stochastic equations is known as exponential stability in mean square.*

In this section we will use Lyapunov functions to obtain sufficient conditions for the p-moment exponential stability of the trivial solution of the nonlinear impulsive random system impulses occurring at random moments (2.2).

Theorem 4.1. *Let the following conditions be satisfied:*

1. Conditions (H1), (H2), (H3) hold.

2. The function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ and

(i) there exist positive constants $a, p > 0$ and an increasing function $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$a\|(z,y)\|^p \leq V(t,z,y) \leq m(\|(z,y)\|^p)$$

for $t \in \mathbb{R}_+,\ z, y \in \mathbb{R}^n$ where $\|(z,y)\| = \max\{\|z\|, \|y\|\}$.

(ii) there exists a constant $K \geq 0$ such that:

$$\frac{\partial}{\partial t}V(t,z,y) + \sum_{i=1}^{n} \frac{\partial}{\partial z_i}V(t,z,y)y_i + \sum_{i=1}^{n} \frac{\partial}{\partial y_i}V(t,z,y)f_i(t,z,y) \leq -KV(t,z,y),$$

for $t \in \mathbb{R}_+,\ z, y \in \mathbb{R}^n$ where $z = (z_1, z_2, \dots, z_n), y = (y_1, y_2, \dots, y_n)$.

(iii) for any $k = 1, 2, \dots$ there exist constants $C_k > 0,\ k = 1, 2, \dots,$ such that $\sum_{k=0}^{\infty} \prod_{i=1}^{k} C_i = C < \infty$ and

$$V(t, I_k(z), J_k(y)) \leq C_k V(t,z,y) \quad \text{for } t \geq 0,\ z, y \in \mathbb{R}^n.$$

Then the trivial solution of the RIDE (2.2) is p-moment generalized exponentially stable, i.e. the inequality

$$E(\|x(t; x_0, x_1, \{\tau_k\})\|^p) \leq \frac{C}{a} m(\|(x_0, x_1)\|^p)e^{-Kt}, \quad t \geq 0$$

holds, where $x(t; x_0, x_1, \{\tau_k\})$ is a solution of the IVP for the RIDE (2.2).

Proof. Let $(x_0, x_1) \in \mathbb{R}_+ \times \mathbb{R}^n$ be an arbitrary initial value and the stochastic process $x_\tau(t) = x(t; x_0, x_1, \{\tau_k\})$ be a solution of the IVP for the RIDE (2.2).

Let t_k be arbitrary values of the random variables τ_k, $k = 1, 2, \dots$ and $T_k = T_0 + \sum_{i=1}^{k} t_i, k = 1, 2, \dots$. Consider the sample path solution $x(t) = x(t; x_0, x_1, \{T_k\})$ of IVP (2.2), i.e. a solution of (2.1). Substitute $y = x',\ z = x$ in (2.1) and obtain the following system of impulsive differential equation

$$\begin{aligned}
&z'(t) = y(t)\\
&y' = f(t, z(t), y(t)) \quad \text{for} \quad t \in (T_k, T_{k+1}],\ \ k = 0, 1, 2, \dots,\\
&z(T_k + 0) = I_k(z(T_k - 0)), \quad y(T_k + 0) = J_k(y(T_k - 0)) \quad \text{for } \ k = 1, 2, \dots,\\
&z(0) = x_0, \quad y(0) = x_1.
\end{aligned} \tag{4.1}$$

The couple of functions $(z(t), y(t)),\ t \geq 0$, with $z(t) = x(t), y(t) = x'(t)$ is a solution of (4.1).

Let $v(t) = V(t, z(t), y(t))$ for $t \geq 0$.

Then $v'(t) = \frac{\partial}{\partial t} V(t, z(t), y(t)) + \frac{\partial}{\partial z} V(t, z(t), y(t)) y(t) + \frac{\partial}{\partial y} V(t, z(t), y(t)) f(t, z(t), y(t)) \leq -KV(t, z(t), y(t)) = -Kv(t), t \neq T_k$.

For any $k = 1, 2, \dots$ we get $v(T_k+0) = V(T_k+0, z(T_k+0), y(T_k+0)) = V(T_k+0, I_k(z(T_k)), J_k(y(T_k))) \leq C_k V(T_k, z(T_k), y(T_k)) = C_k v(T_k)$.

Thus, function $v(t)$ satisfies the linear impulsive differential inequalities with fixed points of impulses

$$\begin{aligned}
& v'(t) \leq -K\, v(t) \quad \text{for } \ t \geq 0,\ t \neq T_k,\ k = 1, 2, \dots, \\
& v(T_k + 0) \leq C_k v(T_k), \quad k = 1, 2, \dots, \\
& v(0) = V(0, x_0, x_1).
\end{aligned} \tag{4.2}$$

The function $v(t)$ is a sample path solution of the corresponding to (3.1) with $u_0 = V(0, x_0, x_1)$ whose solution is the generated stochastic process $v_\tau(t)$ with state space $\mathbb{R}^n$. According to Lemma 3.2 the expected value of the corresponding stochastic process satisfies

$$E\Big(v_\tau(t)\Big) \leq V(0, x_0, x_1) C e^{-Kt}, \quad t \geq 0.$$

Thus, applying condition $2(i)$ we obtain for the solution $x_\tau(t)$ of IVP (2.1)

$$\begin{aligned}
& E(\|x_\tau(t)\|^p) = \frac{1}{a} E(a\|x_\tau(t)\|^p) \leq \frac{1}{a} E(a\|(z_\tau(t), y_\tau(t))\|^p) \\
& \leq \frac{1}{a} E(V(t, x_\tau(t))) = \frac{1}{a} E(v_\tau(t)) \\
& \leq \frac{C}{a} V(0, x_0, x_1) e^{-Kt} \leq \frac{C}{a} m(\|(x_0, x_1)\|^p) e^{-Kt}, \quad t \geq 0.
\end{aligned} \tag{4.3}$$

Inequality (4.3) proves the p-moment generalized exponential stability of zero solution with $\alpha = \frac{C}{a}$, $\mu = K$. □

Remark 6. *The mean square exponential stability of differential equations with exponentially distributed impulsive times is studied in [1] but by quadratic Lyapunov functions. This restriction is not required in this paper (see, Example 2). Also, the applied Lyapunov function could directly depend on the time which makes the assumptions on the right side part of the equation less restrictive.*

Corollary 1. *Let the conditions of Theorem 4.1 be satisfied with* $m(u) \equiv bu,\ b > a$ *and replacing 2(ii) by:*

(ii) there exists a constant* $L \geq 0$ *such that:*

$$\frac{\partial}{\partial t} V(t, z, y) + \sum_{i=1}^{n} \frac{\partial}{\partial z_i} V(t, z, y) y_i + \sum_{i=1}^{n} \frac{\partial}{\partial y_i} V(t, z, y) f_i(t, z, y) \leq -L\|(z, y)\|^p,$$

for $t \in \mathbb{R}_+,\ z, y \in \mathbb{R}^n$;

Then the trivial solution of the RIDE (2.2) is p-moment generalized exponentially stable.

Proof. In this case the inequality (4.2) will be true with replacing K by $\frac{K}{b}$ and thus,

$$\begin{aligned}
&E(\|x_\tau(t)\|^p) = \frac{1}{a}E(a\|x_\tau(t)\|^p) \le \frac{1}{a}E(a\|(z_\tau(t), y_\tau(t))\|^p)\\
&\le \frac{1}{a}E(V(t, x_\tau(t))) = \frac{1}{a}E(v_\tau(t))\\
&\le \frac{C}{a}V(0, x_0, x_1)e^{-\frac{K}{b}t} \le \frac{Cb}{a}\|(x_0, x_1)\|^p e^{-\frac{K}{b}t}, \quad t \ge 0.
\end{aligned} \tag{4.4}$$

□

Remark 7. *Note conditions 2(i) and 2(ii) guarantee the exponential stability of the corresponding to (2.1) ODE. Therefore, if additionally the condition (H3) and 2(iii) are satisfied, the presence of random impulses in the equation does not change on average the stability of the solution.*

5. Applications

We will illustrate the obtained sufficient conditions on some particular examples with impulses at random times.

First, we will consider the application of a Lyapunov function which does not depend on the time variable t. Because we would like to emphasize on the influence of random impulses on the behavior of the solutions and in connection with better graphical illustrations we will consider a scalar nonlinear second order differential equation with random impulses.

Example 1. Consider the scalar equation

$$\begin{aligned}
&x''(t) = -x'(t) - x(t) \qquad \text{for} \quad \xi_k < t < \xi_{k+1},\ k = 0, 1, \dots,\\
&x(\xi_k + 0) = -0.5x(\xi_k - 0), \quad x'(\xi_k + 0) = -0.5x'(\xi_k - 0) \quad \text{for} \ k = 1, 2, \dots,\\
&x(0) = x_0, \quad x'(0) = x_1.
\end{aligned} \tag{5.1}$$

Then $f(t, z, y) = -y - z$ and $I_k(y) = -0.5y,\ J_k(y) = -0.5y$.

Note that the ordinary differential equation $x''(t) = -x'(t) - x(t),\ x(0) = A, x'(0) = B$ has an explicit solution

$$x(t) = \frac{1}{3}e^{-0.5t}\Big(3A\cos(0.5\sqrt{3}t) + \sqrt{3}A\sin(0.5\sqrt{3}t) + 2\sqrt{3}B\sin(0.5\sqrt{3}t)\Big)$$

and it is stable (see Figure 1 for various initial data: $x_0 = -1, x_1 = 3$, $x_0 = 1, x_1 = 2$ and $x_0 = 1.5, x_1 = -3$.)

Consider the function $V(t, x, y) = y^2 + x^2 + xy$. The following inequalities

$$V(t, x, y) = 0.5(x + y)^2 + 0.5x^2 + 0.5y^2 \ge 0.5(x^2 + y^2)$$

and

$$V(t, x, y) \le 1.5x^2 + 1.5y^2 \le 1.5\max\{|x|^2, |y|^2\} = m(\|x, y\|^2)$$

hold, i.e. condition 2(i) is satisfied with $p = 2$, $a = 0.5$ and $m(u) = 1.5u$.

Then we get

$$\begin{aligned}
&\frac{\partial}{\partial t}V(t,z,y) + \frac{\partial}{\partial z}V(t,z,y)y + \frac{\partial}{\partial y}V(t,z,y)f(t,z,y) = 2zy + y^2 + (z+2y)f(t,z,y) \\
&= 2zy + y^2 - (z+2y)(y+z) = 2zy + y^2 - zy - z^2 - 2y^2 - 2zy \\
&= -zy - z^2 - y^2 = -V(t,z,y),
\end{aligned} \tag{5.2}$$

i.e. condition 2 (ii) is satisfied with $K = 1$.

Also,

$$\begin{aligned}
&V(t, I_k(z), J_k(y)) \\
&= (-0.9y)^2 + (-0.9z)^2 + (-0.9x)(-0.9y) \\
&= 0.81y^2 + 0.81z^2 + 0.811zy = 0.81V(t,z,y)
\end{aligned} \tag{5.3}$$

with $\sum\limits_{k=0}^{\infty}\prod\limits_{i=1}^{k} 0.25 = \frac{1}{1-0.0.81} = C < \infty$, i.e. condition 2(iii) is satisfied.

According to Theorem 4.1 the zero solution of (5.1) is mean square generalized stable with $m(u) = e^{u^2}$, $\alpha = \frac{4}{3}$, $\mu = K = 1$, i.e. the

$$E[|x(t; x_0, x_1, \{\tau_k\})|^2] < 4\|(x_0, x_1)\|^2 e^{-t}, \quad \text{for all } t > 0 \tag{5.4}$$

holds.

We would like to note that the inequality (5.4) is about the expected value (mean) of the norm of the stochastic process which is a solution of (5.1) and for a sample path solution the inequality (5.4) could not be satisfied but on average it is true.

To illustrate the behavior of the zero solution of (5.1) with impulses occurring at random times, we consider several sample path solutions. We fix the initial values as $x_0 = 1$, $x_1 = 3$ and choose different values of each random variable $t_k, k = 1, 2, \ldots$, and graph the sample path solutions on the interval $[0, 20]$ (see Figures 5 and 6 combining the 3 particular sample path solutions) in the following way:

- values of random variables τ_1, τ_2, τ_3, respectively, $t_1 = 2, t_2 = 5, t_3 = 10$, i.e. impulses at the points $T_1 = 2, T_2 = 7, T_3 = 17$ (see Figure 2);
- values of random variables τ_1, τ_2, τ_3, respectively, $t_1 = 5, t_2 = 4, t_3 = 6$, i.e. impulses at the points $T_1 = 5, T_2 = 9, T_3 = 15$ (see Figure 3);
- values of random variables τ_1, τ_2, τ_3, respectively, $t_1 = 3, t_2 = 3, t_3 = 2, t_4 = 6$, i.e. impulses at the points $T_1 = 3, T_2 = 6, T_3 = 8, T_4 = 14$(see Figure 4).

Note all solutions coincide until the first point of impulse, i.e. the first particular value of the random variable τ_1 (see Figure 5).

From Figures 2–6 it could be seen the particular sample path solutions approach zero.

□

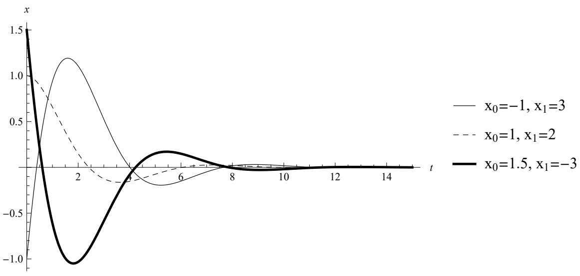

Figure 1. Graphs of the solutions of ODE with various initial values.

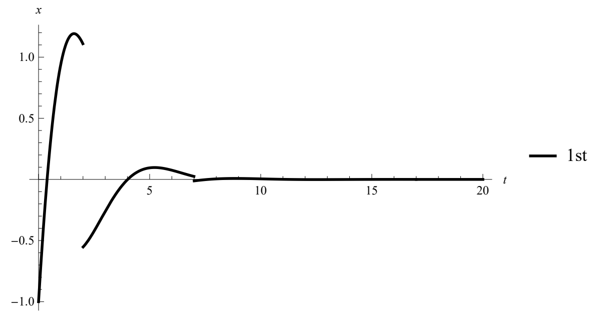

Figure 2. Graph of a particular sample path solution of (5.1) with $t_1 = 2, t_2 = 5, t_3 = 10$.

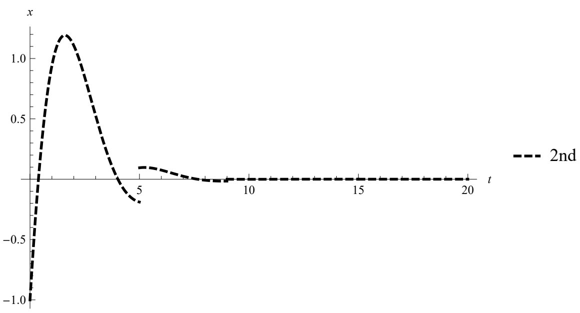

Figure 3. Graph of a particular sample path solution of (5.1) with $t_1 = 5, t_2 = 4, t_3 = 6$.

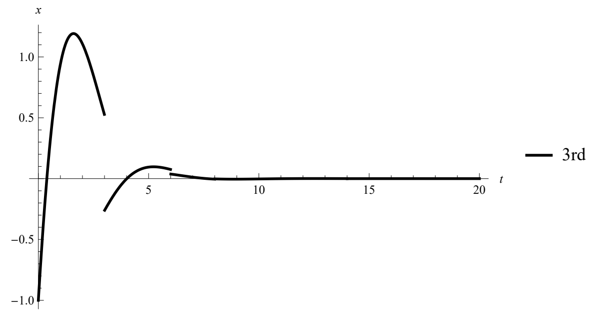

Figure 4. Graph of a particular sample path solution of (5.1) with $t_1 = 3, t_2 = 3, t_3 = 2, t_4 = 6$.

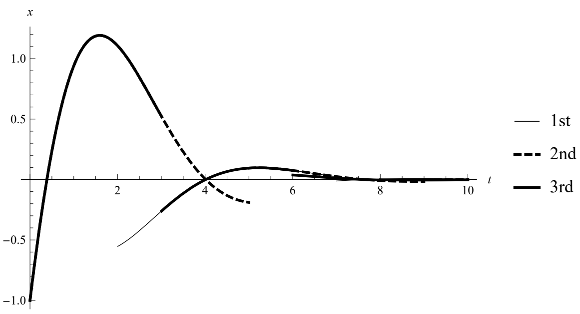

Figure 5. Graphs of some particular sample path solutions of (5.1) on $[0, 10]$.

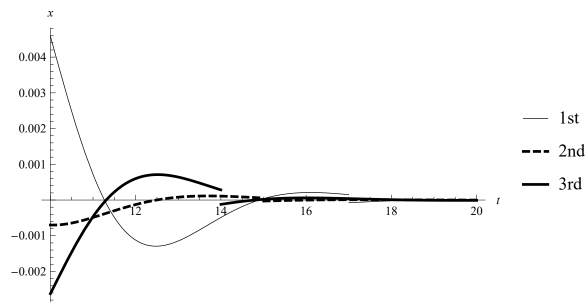

Figure 6. Graphs of some particular sample path solutions of (5.1) on $[10, 20]$.

Now we will consider the application of a Lyapunov function depending on the time variable t on a system of nonlinear second order differential equations with random impulses.

Example 2. Consider the system of two second order differential equations with impulses at random time:

$$
\begin{aligned}
&x_1''(t) = -x_2(t)x_2'(t)x_1'(t) - \frac{x_1'(t)}{2(e^{-t}+1)} \qquad \text{for} \quad \xi_k < t < \xi_{k+1},\ k = 0,1,\dots,\\
&x_2''(t) = -x_1(t)x_1'(t)x_2'(t) - \frac{x_2'(t)}{2(e^{-t}+1)} \qquad \text{for} \quad \xi_k < t < \xi_{k+1},\ k = 0,1,\dots,\\
&x_1(\xi_k+0) = ax_1(\xi_k-0), \quad x_1'(\xi_k+0) = bx_1'(\xi_k-0) \quad \text{for} \ k = 1,2,\dots,\\
&x_2(\xi_k+0) = cx_2(\xi_k-0), \quad x_2'(\xi_k+0) = dx_2'(\xi_k-0) \quad \text{for} \ k = 1,2,\dots,\\
&x_1(0) = x_{01}, \quad x_1'(0) = x_{11} \quad x_2(0) = x_{02}, \quad x_2'(0) = x_{12},
\end{aligned}
\tag{5.5}
$$

where $a, b, c, d \in (-1, 1)$.

Then $f_1(t, z_1, z_2, y_1, y_2) = -z_2y_2y_1 - \frac{y_1}{2(e^{-t}+1)}$ and $f_2(t, z_1, z_2, y_1, y_2) = -z_1y_1y_2 - \frac{y_2}{2(e^{-t}+1)}$

Consider $V(t, z_1, z_2, y_1, y_2) = (e^{-t}+1)\Big(e^{z_1^2}y_1^2 + e^{z_2^2}y_2^2\Big)$. Applying that $e^{x^2} \geq x^2$ we get

$$V(t, z_1, z_2, y_1, y_2) \geq \max\{z_1^2, y_1^2\} + \max\{z_2^2, y_2^2\} \geq \max\{z_1^2 + z_2^2, y_1^2 + y_2^2\}$$

and

$$V(t, z_1, z_2, y_1, y_2) \leq 2e^{z_1^2+z_2^2}(y_1^2 + y_2^2) \leq 2e^{z_1^2+z_2^2}e^{y_1^2+y_2^2} \leq 2e^{2\max\{z_1^2+z_2^2, y_1^2+y_2^2\}},$$

i.e. condition 2(i) is satisfied with $m(u) = \sqrt{2}e^{u^2}$ and $p = 2$.

About the derivative we get

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial z_1}y_1 + \frac{\partial V}{\partial z_2}y_2 + \frac{\partial V}{\partial y_1}f(t, z_1, z_2, y_1, y_2) + \frac{\partial V}{\partial y_2}f(t, z_1, z_2, y_1, y_2)\\
&= -e^{-t}\Big(e^{z_1^2}y_2^2 + e^{z_2^2}y_1^2\Big)\\
&+ 2(e^{-t}+1)\Big(z_1e^{z_1^2}y_2^2y_1 + z_2e^{z_2^2}y_1^2y_2 + y_1e^{z_2^2}f_1(t, z, y) + y_2e^{z_1^2}f_2(t, z, y)\Big)\\
&= -e^{-t}\Big(e^{z_1^2}y_2^2 + e^{z_2^2}y_1^2\Big) - e^{z_1^2}y_2^2 - e^{z_2^2}y_1^2,
\end{aligned}
\tag{5.6}
$$

i.e. condition 2(ii) is satisfied with $K = 1$.

Also, $V(t, I_k(z), J_k(y)) = (e^{-t}+1)\Big(e^{a^2z_1^2}c^2y_1^2 + e^{b^2z_2^2}d^2y_2^2\Big) < (e^{-t}+1)\Big(e^{z_1^2}y_1^2 + e^{z_2^2}y_2^2\Big)$,i.e. condition 2(iii) is satisfied with $C = 1$.

According to Theorem 4.1 the zero solution of (5.5) is mean square generalized stable, i.e.

$$E(\|x(t; x_0, x_1, \{\tau_k\})\|^p) \leq 2e^{2\max\{x_{01}^2+x_{02}^2, x_{11}^2+x_{12}^2\}}e^{-t}, \quad t \geq 0.$$

To illustrate the behavior of the zero solution of (5.5) with impulses occurring at random times, we consider several sample path solutions. We fix the initial values as $x_{01} = 3$, $x_{11} = -1$, $x_{02} = -1$, $x_{12} = 0.01$ and choose different values of each random variable t_k, $k = 1, 2, \dots$, and graph the sample path solutions on the interval $[0, 30]$ (see Figures 7 and Figure 8) in the following way:

- values of random variables τ_1, τ_2, τ_3, respectively, $t_1 = 3, t_2 = 3, t_3 = 6$, i.e. impulses at points $T_1 = 3, T_2 = 6, T_3 = 12$;
- values of random variables τ_1, τ_2, τ_3, respectively, $t_1 = 4, t_2 = 4, t_3 = 8$, i.e. impulses at points $T_1 = 4, T_2 = 8, T_3 = 16$;

- values of random variables τ_1, τ_2, τ_3, respectively, $t_1 = 5, t_2 = 5, t_3 = 5$, i.e. impulses at $T_1 = 5, T_2 = 10, T_3 = 15$.
- values of random variables $\tau_1, \tau_2, \tau_3, \tau_4$, respectively, $t_1 = 6, t_2 = 6, t_3 = 6, t_4 = 6$, i.e. impulses at $T_1 = 6, T_2 = 12, T_3 = 18, T_4 = 24$.

From Figures 7 and 8 it could be seen the particular sample path solutions approach zero.

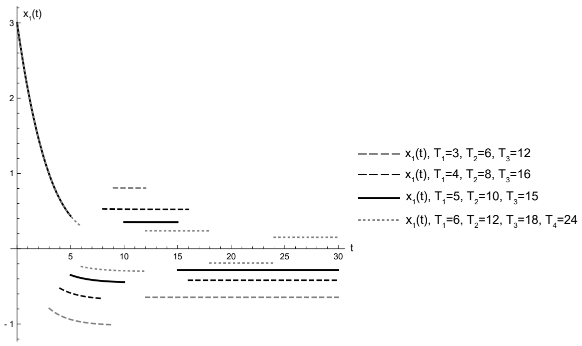

Figure 7. Graphs of first component $x_1(t)$ of some particular sample path solutions of (5.5) on $[0, 30]$.

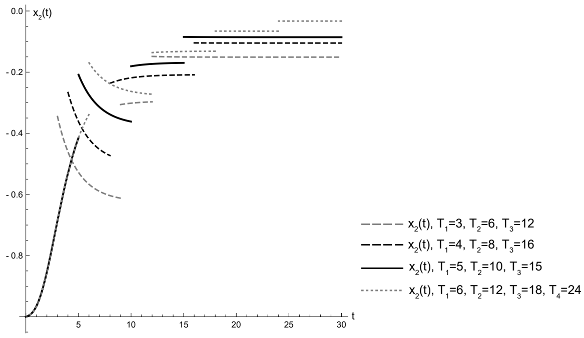

Figure 8. Graphs of the second component $x_2(t)$ of some particular sample path solutions of (5.5) on $[0, 30]$.

□

6. Conclusions

The main goal of the paper is to set up in appropriate way a second order nonlinear differential equation with impulses occurring at random times. The time between two consecutive impulses is exponentially distributed. The p-moment generalized exponential stability of the zero solution of the studied system is defined and some sufficient conditions are obtained. In this way a mathematical apparatus for more adequate modeling of some real World phenomena is given. For the set up problem some other properties different than stability could be also studied. In this study results from Theory of Differential Equations and Probability Theory have to be combined.

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