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Research article

Transportation inequalities for doubly perturbed stochastic differential equations with Markovian switching

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Abstract: In this paper, we focus on a class of doubly perturbed stochastic differential equations with Markovian switching. Using the Girsanov transformation argument we establish the quadratic transportation inequalities for the law of the solution of those equations with Markovian switching under the d_2 metric and the uniform metric d_∞.

Keywords: transportation inequalities; doubly perturbed SDEs; Girsanov transformation; Markovian switching
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1. Introduction

Consider the following doubly perturbed stochastic differential equations with Markovian switching

$$X(t) = x + \int_0^t f(s, X(s), r(s))ds + \int_0^t g(s, X(s), r(s))dB(s) + \alpha \max_{0\le s\le t} X(s) + \beta \min_{0\le s\le t} X(s). \tag{1.1}$$

where $r(t)$ be a right continuous Markov chain taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$, $f : [0, T] \times \mathbb{R} \times \mathbb{S} \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \times \mathbb{S} \to \mathbb{R}$ are some appropriate functions; W is a standard Brownian motion and α and β are two constants. There now exists a considerable body of literatures which are devoted to the study of "perturbed" stochastic equations. As the limit process from a weak polymers model, Carmona et al. [5] and Norris et al. [19] investigated the following doubly perturbed Brownian motion

$$x(t) = B(t) + \alpha \max_{0\le s\le t} x(s) + \beta \min_{0\le s\le t} x(s). \tag{1.2}$$

Many researchers have devoted themselves to studying this model due to its extensive applications, see Davis [7], Carmona, Petit and Yor [4], Perman and Werner [23], Chaumont and Doney [6], etc. Following them, Doney and Zhang [8] studied the following singly perturbed Skorohod equations

$$x(t) = x_0 + \int_0^t g(s, x(s))dB(s) + \int_0^t f(s, x(s))ds + \alpha \max_{0\le s\le t} x(s). \tag{1.3}$$

The authors proved the existence and uniqueness of the solution for (1.3) when the coefficients b, σ are the global Lipschitz. Mao et al. [18] discussed the following doubly perturbed stochastic differential equations

$$x(t) = x_0 + \int_0^t g(s, x(s))dB(s) + \int_0^t f(s, x(s))ds + \alpha \max_{0\le s\le t} x(s) + \beta \min_{0\le s\le t} x(s). \tag{1.4}$$

They showed the existence and uniqueness of the solution for (1.4) under some non-Lipschitz conditions. Hu and Ren [13] and Luo [16] extended the equation (1.4) to the neutral type and doubly perturbed jump-diffusion processes, respectively. Liu and Yang [15] established the existence and uniqueness of the solution for a class of doubly perturbed neutral jump-diffusion processes with Markovian switching.

On the other hand, in the past decade, a plenty of results have been published concerning Talagrand-type transportation cost inequalities (TCIs) on the path spaces of stochastic processes. Now let us consider the kinds of inequalities we will deal with. Let (E, d) be a metric space equipped with a σ-field $\mathscr{B}$ such that the distance $d(\cdot,\cdot)$ is $\mathscr{B}\otimes\mathscr{B}$-measurable. Given $p \ge 1$ and two probability measures μ and ν on E, the Wasserstein distance is defined by

$$W_p^d(\mu,\nu) = \inf_{\pi\in\mathscr{C}(\mu,\nu)} \left(\int\int d(x,y)^p d\pi(x,y) \right)^{1/p},$$

where $\mathscr{C}(\mu,\nu)$ denotes the totality of probability measures on the product space $E\times E$ with the marginals μ and ν. The relative entropy of ν with respect to μ is defined as

$$\mathbf{H}(\nu|\mu) = \begin{cases} \int \ln\frac{d\nu}{d\mu} d\nu, & \nu \ll \mu, \\ +\infty, & \text{otherwise.} \end{cases}$$

The probability measure μ satisfies the T^p transportation inequality on (E, d) if there exists a constant $C \ge 0$ such that for any probability measure ν,

$$W_p^d(\mu,\nu) \le \sqrt{2C\mathbf{H}(\nu|\mu)}.$$

As usual, we write $\mu \in T_p(C)$ for this relation. As is known to all, the cases "$p = 1$" and "$p = 2$" are of particular interest. $T_1(C)$ is related to concentration of measure phenomenon and well characterized, as it was shown by Djellout [9] using preliminary results obtained in Bobkov and Götze [1].

The property $T_2(C)$ is stronger than $T_1(C)$ and it has the dimension free tensorization property. The property $T_2(C)$ is closely linked with many other functional properties such as Poincaré inequality, logarithmic Sobolev inequality and Hamilton-Jacobi equations. For example, Otto and Villani [20] showed that in a smooth Riemannian setting, the logarithmic Sobolev inequality implies $T_2(C)$, and

$T_2(C)$ means the Poincaré inequality. Since Talagrand [25] established $T_2(C)$ for the Gaussian measure which was generalized in [11] to the framework of an abstract Wiener space, in the past decade, a plenty of results have been published concerning $T_2(C)$ on the path spaces of stochastic processes, see e.g. [9, 30, 31] for diffusion processes on $\mathbb{R}^d$, [21] for multidimensional semi-martingales, [26] for diffusion processes with history-dependent drift, [27, 28] for diffusion processes on Riemannian manifolds, [29] for SDEs driven by pure jump processes, and [17] for SDEs driven by both Gaussian and jump noises, [3] for Neutral functional SDEs and [24] for SDEs driven by a fractional Brownian motion, Li and Luo [14] and Boufoussi and Hajji [2] for stochastic delay evolution equations driven by fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and $H \in (0, 1/2)$ in infinite dimensional space, respectively.

However, to the best of our knowledge, there is no result on the transportation inequalities for stochastic differential equations with Markovian switching. Motivated by the need of hybrid system modeling and in connection with the above discussions, it is worthwhile to develop some techniques and methods to explore the transportation inequalities for doubly perturbed stochastic differential equations with Markovian switching. To this end, in this paper, we will investigate the properties $T_2(C)$ for law of the solution of doubly perturbed stochastic equations (1.1) with Markovian switching under the L^2 metric and the uniform metric. Because this kind of stochastic differential equations contain continuous-time Markov chains and doubly perturbed terms, the structure of this kind of stochastic differential equations become complex. Therefore, it is complicated to study the properties $T_2(C)$ for law of the solution of the considered equations.

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminaries. In Section 3, we state and prove our main results.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\{B(t)\}_{t\geq 0}$ be a one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\{r(t), t \in [0, +\infty)\}$ be a right continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$, where N is some positive integer, with generator $\Gamma = (\gamma_{i,j})_{N\times N}$ given by

$$\mathbb{P}(r(t+\triangle) = j | r(t) = i)) = \begin{cases} \gamma_{i,j}\triangle + o(\triangle), & \text{if } i \neq j, \\ 1 + \gamma_{i,j}\triangle + o(\triangle), & \text{if } i = j, \end{cases}$$

with $\triangle > 0$, where $\gamma_{i,j} \geq 0$ is the transition rate from i to j, if $i \neq j$; while $\gamma_{i,i} = -\sum_{j\neq i} \gamma_{i,j}$. We assume that Markov chain $r(\cdot)$ is independent of the $B(\cdot)$. It is known that almost every sample path of $r(t)$ is a right continuous step function with a finite number of simple jumps in any finite sub-interval of $[0, +\infty)$.

In order to obtain our main results, we impose the following three hypotheses.

Hypothesis 1. There exists some constant $K > 0$ such that for any fixed $t \in [0, +\infty)$, $i \in \mathbb{S}$ and $x, y \in \mathbb{R}$, the following inequality holds:

$$|f(t, x, i) - f(t, y, i)|^2 + |g(t, x, i) - g(t, y, i)|^2 \leq K|x - y|^2.$$

Hypothesis 2. For each $i \in \mathbb{S}$, $f(\cdot, 0, i)$, $g(\cdot, 0, i) \in L^2([0, T]; \mathbb{R})$ and for all $t \in [0, T]$ and some constant $\overline{K} > 0$, it follows that

$$|f(t, 0, i)|^2 + |g(t, 0, i)|^2 \leq \overline{K}.$$

Hypothesis 3. $|\alpha| + |\beta| < 1$.

Lemma 2.1. *If the Hypothesis 1-3 hold and the random variable x is independent of $B(t)$, $t \geq 0$ and $\mathbb{E}|x|^p < \infty$ for $p \in \mathbb{N}$, then there exists a unique $\mathcal{F}_t$-adapted solution $X(t)$, $t \geq 0$ for (1.1) such that for any $p \in \mathbb{N}$ and $T > 0$,*

$$\mathbb{E}\Big(\max_{0\leq t\leq T} |X(t)|^p\Big) \leq C,$$

where $C = C(T, p)$ and $\mathbb{E}$ denotes the expectation under probability measure $\mathbb{P}$.

Proof. We construct the solution by iteration. Let

$$X^0(t) = \frac{x}{1 - \alpha - \beta}, \quad 0 \leq t < \infty.$$

For $n \geq 1$ define $X^{n+1}(t)$ to be the unique adapted solution the following equation:

$$\begin{aligned} X^{n+1}(t) =& x + \int_0^t f(s, X^n(s), r(s))ds + \int_0^t g(s, X^n(s), r(s))dB(s) \\ &+ \alpha \max_{0\leq s\leq t} X^{n+1}(s) + \beta \min_{0\leq s\leq t} X^{n+1}(s). \end{aligned} \tag{2.1}$$

Then, by view of the Theorem 3.1 of [15], we know that $\{X^{n+1}(t), t \geq 0\}$ is Cauchy and $X(t) = \lim_{n\to\infty} X^n(t)$ is the unique $\mathcal{F}_t$-adapted solution for (1.1).

Now, by using (2.1) and $|\alpha| + |\beta| < 1$, we have for any $p \geq 2$,

$$\begin{aligned} \mathbb{E}\Big(\max_{0\leq t\leq T} |X(t)|^p\Big) \leq& 3^{p-1}\Big(\frac{1}{1 - |\alpha| - |\beta|}\Big)^p\Big(\mathbb{E}|x|^p + \mathbb{E}\Big[\max_{0\leq t\leq T}\Big|\int_0^t f(s, X(s), r(s))ds\Big|^p\Big] \\ &+ \mathbb{E}\Big[\max_{0\leq t\leq T}\Big|\int_0^t g(s, X(s), r(s))dB(s)\Big|^p\Big]\Big). \end{aligned}$$

Then, using similar arguments as in the proof of the Proposition 3.5 of [33] we can obtain our desired results. The proof is complete. □

Lemma 2.2. *If the Hypothesis 1-3 hold and the random variable x is independent of $B(t)$, $t \geq 0$ and $\mathbb{E}|x|^2 < \infty$, then for all $0 \leq s < t \leq T$,*

$$\mathbb{E}|X(t) - X(s)|^2 \leq C_1(t - s),$$

where C_1 is a positive constant.

Proof. For all $0 \leq s < t \leq T$, it follows from (1.1) that

$$\begin{aligned} X(t) - X(s) =& \int_s^t f(u, X(u), r(u))du + \int_s^t g(u, X(u), r(u))dB(u) \\ &+ \alpha \max_{0\leq u\leq t} X(u) + \beta \min_{0\leq u\leq t} X(u) - \alpha \max_{0\leq u\leq s} X(u) - \beta \min_{0\leq u\leq s} X(u). \end{aligned} \tag{2.2}$$

For $\beta \geq 0$, noticing that $\min_{0\leq u\leq t} X(u) \leq \min_{0\leq u\leq s} X(u)$, we have

$$\begin{aligned}X(t) - X(s) \leq &\int_s^t f(u, X(u), r(u))du + \int_s^t g(u, X(u), r(u))dB(u)\\ &+ \alpha \max_{0\leq u\leq t} X(u) - \alpha \max_{0\leq u\leq s} X(u).\end{aligned} \tag{2.3}$$

Then,

$$\begin{aligned}|X(t) - X(s)| \leq &\Big|\int_s^t f(u, X(u), r(u))du\Big| + \Big|\int_s^t g(u, X(u), r(u))dB(u)\Big|\\ &+ |\alpha|\Big|\max_{0\leq u\leq t} X(u) - \max_{0\leq u\leq s} X(u)\Big|.\end{aligned} \tag{2.4}$$

Next, let us consider the following two cases.
Case I. If $\max_{0\leq u\leq t} X(u) = \max_{0\leq u\leq s} X(u)$, then we get from (2.4) that

$$|X(t) - X(s)| \leq \Big|\int_s^t f(u, X(u), r(u))du\Big| + \Big|\int_s^t g(u, X(u), r(u))dB(u)\Big|. \tag{2.5}$$

Case II. If $\max_{0\leq u\leq t} X(u) > \max_{0\leq u\leq s} X(u)$, then there exists $\sigma \in (s, t]$ such that $X(\sigma) = \max_{0\leq u\leq t} X(u)$. So we get from (2.4) that

$$\begin{aligned}&|X(t) - X(s)|\\ \leq&\Big|\int_s^t f(u, X(u), r(u))du\Big| + \Big|\int_s^t g(u, X(u), r(u))dB(u)\Big| + |\alpha|\Big|X(\sigma) - \max_{0\leq u\leq s} X(u)\Big|\\ \leq&\Big|\int_s^t f(u, X(u), r(u))du\Big| + \Big|\int_s^t g(u, X(u), r(u))dB(u)\Big| + |\alpha||X(\sigma) - X(s)|\\ \leq&\Big|\int_s^t f(u, X(u), r(u))du\Big| + \Big|\int_s^t g(u, X(u), r(u))dB(u)\Big| + |\alpha| \max_{s\leq s'<t'\leq t} |X(t') - X(s')|.\end{aligned} \tag{2.6}$$

Thus, we obtain that

$$\max_{s\leq s'<t'\leq t} |X(t') - X(s')|^2 \leq \frac{2}{(1 - |\alpha|)^2}\Big(\Big|\int_s^t f(u, X(u), r(u))du\Big|^2 + \Big|\int_s^t g(u, X(u), r(u))dB(u)\Big|^2\Big). \tag{2.7}$$

Then, we have

$$\mathbb{E}|X(t) - X(s)|^2 \leq \frac{2}{(1 - |\alpha|)^2}\Big(\mathbb{E}\Big|\int_s^t f(u, X(u), r(u))du\Big|^2 + \mathbb{E}\Big|\int_s^t g(u, X(u), r(u))dB(u)\Big|^2\Big). \tag{2.8}$$

For $\beta < 0$, if $\min_{0\leq u\leq t} X(u) = \min_{0\leq u\leq s} X(u)$, then we have also that (2.4) holds. If $\min_{0\leq u\leq t} X(u) < \min_{0\leq u\leq s} X(u)$, then there exists $\delta \in (s, t]$ such that $X(\sigma) = \min_{0\leq u\leq t} X(u)$. Thus, we have

$$\begin{aligned}|\beta \min_{0\leq u\leq t} X(u) - \beta \min_{0\leq u\leq s} X(u)| =&|\beta|(\min_{0\leq u\leq s} X(u) - X(\sigma))\\ \leq&|\beta|(X(s) - X(\sigma)) \leq |\beta| \max_{s\leq s'<t'\leq t} |X(t') - X(s')|.\end{aligned} \tag{2.9}$$

Thus, we have

$$\mathbb{E}|X(t) - X(s)|^2 \leq \frac{2}{(1 - |\alpha| - |\beta|)^2}\Big(\mathbb{E}\Big|\int_s^t f(u, X(u), r(u))du\Big|^2 + \mathbb{E}\Big|\int_s^t g(u, X(u), r(u))dB(u)\Big|^2\Big). \tag{2.10}$$

Lastly, using the Lemma 2.1 and the Theorem 4.6.3 of [12] we can obtain our desired results. □

3. Main results

In this section, we discuss the TCIs for the law of the solution for (1.1).

Theorem 3.1. *Let the Hypothesis 1-3 hold and $\mathbb{P}_x$ be the law of $X(\cdot, x)$, solution process for (1.1). Assume further that g is bounded by $\widetilde{g} := \max_{0\leq t\leq T}|g(t, x(t), r(t))|$. Then the probability measure $\mathbb{P}_x$ satisfies $T_2(C)$ on the metric space $C([0, T]; \mathbb{R})$ with:*

(a) $C = \frac{3T\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}$ *with the metric*

$$d_\infty(\gamma_1, \gamma_2) := \max_{0\leq t\leq T}|\gamma_1 - \gamma_2|, \quad \gamma_1, \gamma_2 \in C([0, T]; \mathbb{R});$$

(b) $C = \frac{3T^2\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}$ *when using the metric*

$$d_2(\gamma_1, \gamma_2) = \left(\int_0^T |\gamma_1(t) - \gamma_2(t)|^2 dt\right)^{1/2}, \quad \gamma_1, \gamma_2 \in C([0, T]; \mathbb{R}).$$

Proof. Let $\mathbb{P}_x$ be the law of $X(\cdot, x)$ and $\mathbb{Q}$ be any probability measure such that $\mathbb{Q} \ll \mathbb{P}_x$. Define

$$\widetilde{\mathbb{Q}} := \frac{d\mathbb{Q}}{d\mathbb{P}_x}(X(\cdot, x))\mathbb{P}, \tag{3.1}$$

which is a probability measure on $(\Omega, \mathcal{F})$. Recalling the definition of entropy and adopting a measure-transformation argument we obtain from (3.1) that

$$\begin{aligned}
\mathbf{H}(\widetilde{\mathbb{Q}}|\mathbb{P}) &= \int_\Omega \ln\left(\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right)d\widetilde{\mathbb{Q}} = \int_\Omega \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}(X(\cdot, x))\right)\frac{d\mathbb{Q}}{d\mathbb{P}_x}(X(\cdot, x))d\mathbb{P} \\
&= \int_{C([0,T];\mathbb{R})} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}\right)\frac{d\mathbb{Q}}{d\mathbb{P}_x}d\mathbb{P}_x \\
&= \mathbf{H}(\mathbb{Q}|\mathbb{P}_x).
\end{aligned}$$

Following [10], there exists a predictable process $\{h(t)\}_{0\leq t\leq T} \in \mathbb{R}$ with $\int_0^T |h(s)|^2 ds < \infty$, $\mathbb{P}$-a.s., such that

$$\mathbf{H}(\widetilde{\mathbb{Q}}|\mathbb{P}) = \mathbf{H}(\mathbb{Q}|\mathbb{P}_x) = \frac{1}{2}\mathbb{E}_{\widetilde{\mathbb{Q}}}|h(t)|^2 dt.$$

Due to the Girsanov theorem, the process $(\widetilde{B}(t))_{t\in[0,T]}$ defined by

$$\widetilde{B}(t) = B(t) - \int_0^t h(s)ds$$

is a Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{Q}})$. Consequently, under the measure $\widetilde{\mathbb{Q}}$, the process $\{X(t, x)\}_{t\in[0,T]}$ satisfies

$$\begin{aligned}
X(t) =& x + \int_0^t f(s, X(s), r(s))ds + \int_0^t g(s, X(s), r(s))d\widetilde{B}(s) + \int_0^t g(s, X(s), r(s))h(s)ds \\
&+ \alpha \max_{0\leq s\leq t} X(s) + \beta \min_{0\leq s\leq t} X(s).
\end{aligned} \tag{3.2}$$

We now consider the solution Y (under $\widetilde{\mathbb{Q}}$) of the following equation:

$$Y(t) = x + \int_0^t f(s, Y(s), \tilde{r}(s))ds + \int_0^t g(s, Y(s), \tilde{r}(s))d\widetilde{B}(s) + \alpha \max_{0\le s\le t} Y(s) + \beta \min_{0\le s\le t} Y(s). \tag{3.3}$$

By the Lemma 2.1, under $\widetilde{\mathbb{Q}}$ the law of $Y(\cdot)$ is $\mathbb{P}_x$. Thus (X, Y) under $\widetilde{\mathbb{Q}}$ is a coupling of $(\widetilde{\mathbb{Q}}, \mathbb{P}_x)$, and it follows that

$$[W_2^{d_2}(\mathbb{Q}, \mathbb{P}_x)]^2 \le \mathbb{E}_{\widetilde{\mathbb{Q}}}(|d_2(X, Y)|^2) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\int_0^T |X(t) - Y(t)|^2 dt\right),$$

$$[W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_x)]^2 \le \mathbb{E}_{\widetilde{\mathbb{Q}}}(|d_\infty(X, Y)|^2) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\max_{0\le t\le T} |X(t) - Y(t)|^2\right).$$

where, $\mathbb{E}_{\widetilde{\mathbb{Q}}}$ denotes the expectation under probability measure $\widetilde{\mathbb{Q}}$.

We now estimate the distance between X and Y with respect to d_2 and d_∞.

Note from (3.2) and (3.3) that

$$\begin{aligned} X(t) - Y(t) = &\int_0^t [f(s, X(s), r(s)) - f(s, Y(s), \tilde{r}(s))]ds + \int_0^t g(s, X(s), r(s))h(s)ds \\ &+ \int_0^t [g(s, X(s), r(s)) - g(s, Y(s), \tilde{r}(s))]d\widetilde{B}(s) \\ &+ \alpha \max_{0\le s\le t} X(s) - \alpha \max_{0\le s\le t} Y(s) + \beta \min_{0\le s\le t} X(s) - \beta \min_{0\le s\le t} Y(s). \end{aligned} \tag{3.4}$$

Note that

$$|\max_{0\le s\le t} X(s) - \max_{0\le s\le t} Y(s)| \le \max_{0\le s\le t} |X(s) - Y(s)|$$

and

$$|\min_{0\le s\le t} X(s) - \min_{0\le s\le t} Y(s)| \le \max_{0\le s\le t} |X(s) - Y(s)|.$$

Then, by view of (3.4) we can obtain

$$\begin{aligned} &(1 - |\alpha|^2 - |\beta|^2)\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\max_{0\le s\le t} |X(s) - Y(s)|^2\Big) \\ \le &3\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\max_{0\le s\le t}\Big|\int_0^s [f(u, X(u), r(u)) - f(u, Y(u), \tilde{r}(u))]du\Big|^2\Big) \\ &+ 3\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\max_{0\le s\le t}\Big|\int_0^s g(u, X(u), r(u))h(u)du\Big|^2\Big) \\ &+ 3\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\max_{0\le s\le t}\Big|\int_0^t [g(u, X(u), r(u)) - g(u, Y(u), \tilde{r}(u))]d\widetilde{B}(u)\Big|^2\Big) \\ :=&3(I_1(t) + I_2(t) + I_3(t)). \end{aligned} \tag{3.5}$$

Firstly, applying the Hypothesis 1 and the Hölder's inequality, we have

$$\begin{aligned} I_1(t) \le &2\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\max_{0\le s\le t}\Big|\int_0^s [f(u, X(u), r(u)) - f(u, Y(u), r(u))]du\Big|^2\Big) \\ &+ 2\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\max_{0\le s\le t}\Big|\int_0^s [f(u, Y(u), r(u)) - f(u, Y(u), \tilde{r}(u))]du\Big|^2\Big) \\ \le &2KT\int_0^t \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\max_{0\le u\le s} |X(u) - Y(u)|^2\Big)ds + I_{12}(t). \end{aligned} \tag{3.6}$$

By the Hölder's inequality, we can obtain

$$\begin{aligned} I_{12}(t) \leq & 2T\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\int_0^t |f(s,Y(s),r(s)) - f(s,Y(s),\tilde{r}(s))|^2 ds\Big) \\ \leq & 2T\sum_{k=0}^{[t/\zeta]}\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} |f(s,Y(s),r(s)) - f(s,Y(t_k),r(s))|^2 ds \\ & + \int_{t_k}^{t_{k+1}} |f(s,Y(t_k),r(s)) - f(s,Y(t_k),\tilde{r}(s))|^2 ds \\ & + \int_{t_k}^{t_{k+1}} |f(s,Y(t_k),\tilde{r}(s)) - f(s,Y(s),\tilde{r}(s))|^2 ds\Big) \\ =: & \, 2T\sum_{k=0}^{[t/\zeta]}[I_{121}(t) + I_{122}(t) + I_{123}(t)], \end{aligned} \tag{3.7}$$

where $t_0 = 0 \leq \cdots \leq t_k \leq t_{t_{k+1}} \leq \cdots \leq t_{[t/\zeta]} \leq t_{[t/\zeta]+1} = t$ and $t_{k+1} - t_k = \zeta$, $k = 0, 1, \cdots, [t/\zeta] - 1$. Then, by the Hypothesis 1 and the Lemma 2.2, we have

$$I_{121}(t) \leq K\int_{t_k}^{t_{k+1}} \mathbb{E}_{\widetilde{\mathbb{Q}}}|Y(s) - Y(t_k)|^2 ds \leq K\int_{t_k}^{t_{k+1}} (s - t_k) ds \leq K\int_{t_k}^{t_{k+1}} \zeta ds, \tag{3.8}$$

and

$$I_{123}(t) \leq K\int_{t_k}^{t_{k+1}} \zeta ds. \tag{3.9}$$

Now, we estimate the term $I_{122}(t)$. Note that

$$\begin{aligned} I_{122}(t) \leq & 2\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\int_{t_k}^{t_{k+1}} |f(s,Y(t_k),r(s)) - f(s,Y(t_k),r(t_k))|^2 ds \\ & + \int_{t_k}^{t_{k+1}} |f(s,Y(t_k),r(t_k)) - f(s,Y(t_k),\tilde{r}(s))|^2 ds\Big) \\ =: & \, 2(I^1_{122}(t) + I^2_{122}(t)). \end{aligned} \tag{3.10}$$

By the Hypothesis 1-2 and the Lemma 2.2 we know that

$$\begin{aligned} I^1_{122}(t) = & \mathbb{E}_{\widetilde{\mathbb{Q}}}\sum_{i\in\mathbb{S}}\sum_{j\neq i}\int_{t_k}^{t_{k+1}} |f(s,Y(t_k),j) - f(s,Y(t_k),i)|^2 \mathcal{I}_{r(s)=j}\mathcal{I}_{r(t_k)=i} ds \\ \leq & K\mathbb{E}_{\widetilde{\mathbb{Q}}}\sum_{i\in\mathbb{S}}\sum_{j\neq i}\int_{t_k}^{t_{k+1}} [1 + |Y(t_k)|^2]\mathcal{I}_{r(t_k)=i}\mathbb{E}[\mathcal{I}_{r(s)=j}|Y(t_k), r(t_k) = i] ds \\ \leq & K\mathbb{E}_{\widetilde{\mathbb{Q}}}\sum_{i\in\mathbb{S}}\int_{t_k}^{t_{k+1}} [1 + |Y(t_k)|^2]\mathcal{I}_{\alpha(t_k)=i}\Big[\sum_{j\neq i} q_{ij}(Y(t_k))(s - t_k) + o(s - t_k)\Big] ds \\ \leq & K\int_{t_k}^{t_{k+1}} O(\zeta) ds, \end{aligned} \tag{3.11}$$

where $q_{ij}(\cdot)$ is the generator of $r(t)$.

On the other hand, by the Hypothesis 1-2 and the following estimate (see [22, 32]) that

$$\begin{aligned}&\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathcal{I}_{\tilde{r}(s)=j}|\tilde{r}(t_k)=i_1,r(t_k)=i,X(t_k),Y(t_k)]\\=&\sum_{l\in\mathbb{S}}\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathcal{I}_{\tilde{r}(s)=j},\mathcal{I}_{r(s)=l}|\tilde{r}(t_k)=i_1,r(t_k)=i,X(t_k),Y(t_k)]\\=&\sum_{l\in\mathbb{S}}q(i_1,i)q(j,l)(s-t_k)+o(s-t_k)=O(\zeta),\end{aligned}$$

we have

$$\begin{aligned}I_{122}^2(t)=&\mathbb{E}_{\widetilde{\mathbb{Q}}}\sum_{i\in\mathbb{S}}\sum_{j\neq i}\int_{t_k}^{t_{k+1}}|f(s,Y(t_k),j)-f(s,Y(t_k),i)|^2\mathcal{I}_{\tilde{r}(s)=j}\mathcal{I}_{r(t_k)=i}ds\\\leq&K\mathbb{E}_{\widetilde{\mathbb{Q}}}\sum_{i\in\mathbb{S}}\sum_{j\neq i}\int_{t_k}^{t_{k+1}}[1+|Y(t_k)|^2]\mathcal{I}_{\tilde{r}(t_k)=i_1}\mathcal{I}_{r(t_k)=i}\\&\times\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathcal{I}_{\tilde{r}(s)=j}|\tilde{r}(t_k)=i_1,r(t_k)=i,X(t_k),Y(t_k)]ds\\\leq&K\int_{t_k}^{t_{k+1}}O(\zeta)ds.\end{aligned}\tag{3.12}$$

From (3.7)–(3.12), letting $\zeta\to 0$, we have $I_{12}(t)\to 0$. Thus, combined with (3.6) we obtain

$$|I_1(t)|\leq 2KT\int_0^t\mathbb{E}_{\widetilde{\mathbb{Q}}}(\max_{0\leq u\leq s}|X(u)-Y(u)|^2)ds.\tag{3.13}$$

Since $h\in L^2([0,T];\mathbb{R})$, by the boundedness of g and the Hölder's inequality, we can obtain

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\max_{0\leq s\leq t}\Big|\int_0^s g(u,X(u),r(u))h(u)du\Big|^2\Big)\leq T\widetilde{g}^2\int_0^t\mathbb{E}_{\widetilde{\mathbb{Q}}}|h(s)|^2ds.\tag{3.14}$$

Next, we estimate the term $I_3(t)$. By using the Burkholder-Davis-Gundy's inequality and the assumptions on g we know that

$$\begin{aligned}I_3(t)\leq&4\mathbb{E}_{\widetilde{\mathbb{Q}}}\int_0^t|g(s,X(s),r(s))-g(s,Y(s),\tilde{r}(s))|^2ds\\\leq&8\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\int_0^t|g(s,X(s),r(s))-g(s,Y(s),r(s))|^2ds\Big)\\&+8\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\int_0^t|g(s,Y(s),r(s))-g(s,Y(s),\tilde{r}(s))|^2ds\Big)\\\leq&8K\int_0^t\mathbb{E}_{\widetilde{\mathbb{Q}}}(\max_{0\leq u\leq s}|X(u)-Y(u)|^2)ds+I_{32}(t).\end{aligned}$$

Then, being similar to $I_1(t)$, we also have

$$I_3(t)\leq 8K\int_0^t\mathbb{E}_{\widetilde{\mathbb{Q}}}(\max_{0\leq u\leq s}|X(u)-Y(u)|^2)ds.\tag{3.15}$$

Thus, combining (3.5), (3.13), (3.14) with (3.15) we have

$$\begin{aligned}&\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(\max_{0\le s\le t}|X(s)-Y(s)|^2\big)\\ \le&\frac{6KT+24K}{1-|\alpha|-|\beta|}\int_0^t\mathbb{E}_{\widetilde{\mathbb{Q}}}(\max_{0\le u\le s}|X(u)-Y(u)|^2du+\frac{3T\widetilde{g}^2}{1-|\alpha|-|\beta|}\int_0^t\mathbb{E}_{\widetilde{\mathbb{Q}}}|h(s)|^2ds.\end{aligned}\tag{3.16}$$

Now, the Gronwall's lemma implies that for any $t>0$,

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(\max_{0\le s\le t}|X(s)-Y(s)|^2\big)\le\frac{3T\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}\int_0^t\mathbb{E}_{\widetilde{\mathbb{Q}}}|h(s)|^2ds,$$

which implies that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}|X(t)-Y(t)|^2\le\frac{3T\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}\int_0^t\mathbb{E}_{\widetilde{\mathbb{Q}}}|h(s)|^2ds.$$

Hence, we may write that

$$d_\infty^2(X,Y)\le\frac{3T\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}\int_0^T\mathbb{E}_{\widetilde{\mathbb{Q}}}|h(s)|^2ds$$

and

$$[W_2^{d_\infty}(\mathbb{Q},\mathbb{P}_x)]^2\le 2C_{T,K}\mathbf{H}(\mathbb{Q}|\mathbb{P}_x)$$

with $C_{T,K}=\frac{3T\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}$.

Analogously for the metric d_2, we have by the Fubini's theorem

$$\begin{aligned}[W_2^{d_2}(\mathbb{Q},\mathbb{P}_x)]^2\le&\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\int_0^T|X(t)-Y(t)|^2dt\Big)=\int_0^T\mathbb{E}_{\widetilde{\mathbb{Q}}}(|X(t)-Y(t)|^2)dt\\ \le&\frac{3T\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}\int_0^T\int_0^t\mathbb{E}_{\widetilde{\mathbb{Q}}}|h(s)|^2dsdt\\ =&\frac{3T\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}\int_0^T\mathbb{E}_{\widetilde{\mathbb{Q}}}|h(s)|^2\Big(\int_s^T1\cdot dt\Big)ds\\ \le&\frac{3T^2\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}\int_0^T\mathbb{E}_{\widetilde{\mathbb{Q}}}|h(s)|^2ds.\end{aligned}$$

Thus, we can obtain

$$[W_2^{d_2}(\mathbb{Q},\mathbb{P}_x)]^2\le 2C_{T,K}\mathbf{H}(\mathbb{Q}|\mathbb{P}_x)$$

with $C_{T,K}=\frac{3T^2\widetilde{g}^2}{1-|\alpha|-|\beta|}e^{\frac{6KT+24K}{1-|\alpha|-|\beta|}}$. The proof is complete. □

Remark 3.1. *There are many interesting applications of the TCIs, e.g., in Tsirel'son-type inequality and Hoeffding-type inequality, see [9, 30, 31], and in concentration of empirical measure [17, 29].*

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Conflict of interest

The authors declare that they have no conflict of interest.

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