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Langevin differential equation in frame of ordinary and Hadamard fractional derivatives under three point boundary conditions

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Abstract: In this paper, we study a type of Langevin differential equations within ordinary and Hadamard fractional derivatives and associated with three point local boundary conditions

$$\mathcal{D}_1^\alpha \left(\mathrm{D}^2 + \lambda^2\right) x(t) = f\left(t, x(t), \mathcal{D}_1^\alpha \left[x\right](t)\right),$$

$\mathrm{D}^2 x(1) = x(1) = 0$, $x(e) = \beta x(\xi)$, for $t \in (1, e)$ and $\xi \in (1, e]$, where $0 < \alpha < 1$, $\lambda, \beta > 0$, $\mathcal{D}_1^\alpha$ denotes the Hadamard fractional derivative of order α, D is the ordinary derivative and $f : [1, e] \times C([1, e], \mathbb{R}) \times C([1, e], \mathbb{R}) \to C([1, e], \mathbb{R})$ is a continuous function. Systematical analysis of existence, stability and solution's dependence of the addressed problem is conducted throughout the paper. The existence results are proven via the Banach contraction principle and Schaefer fixed point theorem. We apply Ulam's approach to prove the Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability of solutions for the problem. Furthermore, we investigate the dependence of the solution on the parameters. Some illustrative examples along with graphical representations are presented to demonstrate consistency with our theoretical findings.

Keywords: Hadamard fractional derivative; fractional Langevin equation; discretization method; continuous dependence of solution
Mathematics Subject Classification: 26A33, 34A08, 34A12, 34B15

1. Introduction

In the recent years, it has been realized that fractional calculus has an important role in various scientific fields. Fractional differential equations (FDE), which is a consequence of the development of fractional calculus, have attracted the attention of many researchers working in different disciplines ([28]). Scientific literature has witnessed the appearance of several kinds of fractional derivatives, such as the Riemann-Liouville fractional derivative, Caputo fractional derivative, Hadamard fractional derivative, Grünwald-Letnikov fractional derivative and Caputo-Fabrizio etc (for more details, see [11, 13, 16, 21, 22, 44, 48, 51, 54]). It is worthy mentioning here that almost all researches have been conducted within Riemann-Liouville or Caputo fractional derivatives, which are the most popular fractional differential operators.

J. Hadamard suggested a construction of fractional integro-differentiation which is a fractional power of the type $\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}$. This construction is well suited to the case of the half-axis and is invariant relative to dilation ([53, p. 330]). The dilation is interpreted in various forms in relation to the field of application. Furthermore, Riemann-Liouville fractional integro-differentiation is formally a fractional power $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}$ of the differentiation operator $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ and is invariant relative to translation if considered on the whole axis. On the other hand, the investigations in terms of Hadamard or Grünwald-Letnikov fractional derivatives are comparably considered seldom.

The boundary value problems defined by FDE have been extensively studied over the last years. Particularly, the study of solutions of fractional differential and integral equations is the key topic of applied mathematics research. Many interesting results have been reported regarding the existence, uniqueness, multiplicity and stability of solutions or positive solutions by means of some fixed point theorems, such as the Krasnosel'skii fixed point theorem, the Schaefer fixed point theorem and the Leggett-Williams fixed point theorem. However, most of the considered problems have been treated in the frame of fractional derivatives of Riemann-Liouville or Caputo types ([12, 14, 15, 41, 45]). The qualitative investigations with respect to Hadamard derivative have gained less attention compared to the analysis in terms of Riemann-Liouville and Caputo settings. Recent results on Hadamard FDE can be consulted in ([1, 4, 5, 7, 10, 17, 42, 43, 52]).

The physical phenomena in fluctuating environments are adequately described using the so called Langevin differential equation (LDE) which was proposed by Langevin himself in [31, 1908] to give an elaborated interpretation of Brownian motion. Indeed, LDE is a powerful tool for the study of dynamical properties of many interesting systems in physics, chemistry and engineering ([9, 32, 57]). The generalized LDE was introduced later by Kubo in [29, 1966], where a fractional memory kernel was incorporated into the equation to describe the fractal and memory properties. Since then the investigation of the generalized LDE has become a hot research topic. As a result, various generalizations of LDE have been offered to describe dynamical processes in a fractal medium. One such generalization is the generalized LDE which incorporates the fractal and memory properties with a disruptive memory kernel. This gives rise to study fractional Langevin equation ([36]). As the intensive development of fractional derivative, a natural generalization of the LDE is to replace the ordinary derivative by a fractional derivative to yield fractional Langevin equation (FLE). The FLE was introduced by Mainardi and Pironi in earlier 1990s ([40]). Afterwards, different types of FLE were introduced and studied in [2, 3, 8, 19, 30, 34, 37–39, 47, 50, 60–62]. In [3], the authors studied a nonlinear LDE involving two fractional orders in different intervals with three-point boundary conditions. The study of FLE in frame of Hadamard derivative has comparably been seldom; see the papers [27, 56] in which the authors discussed Sturm-Liouville and Langevin equations via Caputo-Hadamard fractional derivatives and systems of FLE of Riemann-Liouville and Hadamard types, respectively.

In the paper by Kiataramkul *et al.* [27]: Generalized Sturm–Liouville and Langevin equations via Hadamard fractional derivatives with anti-periodic boundary conditions. In particular, the authors initiate the study of the existence and uniqueness of solutions for the generalized Sturm-Liouville and Langevin fractional differential equations of Caputo-Hadamard type ([21]), with two-point nonlocal anti-periodic boundary conditions, by applying the Banach contraction mapping principle. Moreover, two existence results are established via Leray-Schauder nonlinear alternative and Krasnosleskii's fixed point theorem. In addition, the article by W. Sudsutad *et al.* [56]: Systems of fractional Langevin equations of Riemann-Liouville and Hadamard types subject to the nonlocal Hadamard and standard Riemann-Liouville with multi-point and multi-term fractional integral boundary conditions, respectively. In particular, the authors also studied the existence and uniqueness results of solutions for coupled and uncoupled systems are obtained by Banach's contraction mapping principle, Leray-Schauder's alternative.

In the present work, we study the existence, uniqueness and stability of solutions for the following FLE with Hadamard fractional derivatives involving local boundary conditions

$$\begin{cases} \mathcal{D}_1^{\alpha}\left(\mathrm{D}^2+\lambda^2\right)x(t)=f(t,x(t),\mathcal{D}_1^{\alpha}[x](t)), & t\in(1,e),\\ \mathrm{D}^2x(1)=x(1)=0, & \\ x(e)=\beta x(\xi), & \xi\in(1,e], \end{cases} \tag{1.1}$$

where $0<\alpha<1$, $\lambda,\beta>0$, such that

$$\sin\lambda(e-1)\neq\beta\sin\lambda(\xi-1),$$

$\mathcal{D}_1^{\alpha}$ denotes the Hadamard fractional derivative of order α, D is the ordinary derivative and

$$f:[1,e]\times C([1,e],\mathbb{R})\times C([1,e],\mathbb{R})\to C([1,e],\mathbb{R}),$$

is a continuous function.

Our approach is new and is totally different from the ones obtained in [27, 56] in the sense that different fractional derivatives, ordinary and Hadamard fractional order, are accommodated. Different boundary conditions are associated to problem (1.1) such as three point local boundary conditions and associating different fixed point theorems. It is worthwhile to mention that the nonlinear term f in papers [27, 56] is independent of fractional derivative of unknown function $x(t)$. But the opposite case is more difficult and complicated. The dependence of the solution on the parameters is discussed, which has not been investigated in [27, 56]. It is worth mentioning here that Ulam and generalized Ulam-Hyers-Rassias stability results have not been considered in [27, 56]. Furthermore, the presented work illustrates a numerical simulation obtained through a discretization methods for the evaluation of the Hadamard derivative.

Our method differs from that used by [27, 56] in our emphasis on the Schaefer fixed point theorem is utilized to investigate existence results for problem (1.1). We also employ the generalization

Gronwall inequality techniques to prove the Ulam stability for problem (1.1), and we use important classical and fractional techniques such as: integration by parts in the settings of Hadamard fractional operators, right Hadamard fractional integral, method of variation of parameters, mean value theorem, Dirichlet formula, differentiating an integral, incomplete Gamma function and discretization methods. To the best of the authors' knowledge, there is no work in literature which treats local boundary value problems on mixed type ordinary differential equations involving the Hadamard fractional derivative using the above mentioned techniques.

The rest of the paper is organized as follows: In Section 2, we introduce some notations, definitions and lemmas that are essential in our further analysis. In Section 3, we systemically analyze problem (1.1). An equivalent integral equation is constructed for problem (1.1) and some infra structure are furnished for the use of fixed point theorems. The main results of existence and stability are discussed in Sections 4 and 5, respectively. We prove the main results via the implementation of some fixed point theorems and Ulam's approach. We study the solution's dependence on parameters in Section 6. Indeed, we give an affirmative response to the question on how the solution varies when we change the order of differential operator, the initial values or the nonlinear term f. In Section 7, some illustrative examples along with graphical representations are presented to prove consistency with our theoretical findings.

2. Fundamental definitions, lemmas and remarks

In this section we introduce notations, lemmas, definitions and preliminary facts which are used throughout this paper. In terms of the familiar Gamma function $\Gamma(t)$, the incomplete Gamma function $\gamma(\alpha, t)$ and its complement $\Gamma(\alpha, t)$ are defined by (see, for details, [16, 20])

$$\gamma(\alpha, t) = \int_0^t \tau^{\alpha-1} e^{-\tau} \mathrm{d}\tau, \Re e(t) > 0, \ \left|\arg(t)\right| < \pi,$$

and

$$\Gamma(\alpha, t) = \int_t^\infty \tau^{\alpha-1} e^{-\tau} \mathrm{d}\tau,$$

for all complex t. For fixed α, $\gamma(\alpha, t)$ is an increasing function of t with $\lim_{t\to\infty} \gamma(\alpha, t) = \Gamma(\alpha)$. The classical Riemann-Liouville fractional integral of order α for suitable function x is defined as

$$J_a^\alpha [x](t) := J_{a+}^\alpha [x(\tau)](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\, \mathrm{d}\tau, \tag{2.1}$$

for $0 < a < t$ and $\Re e(\alpha) > 0$. The corresponding left-sided Riemann-Liouville fractional derivative of order α is defined by

$$D_a^\alpha [x](t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} x(\tau)\, \mathrm{d}\tau, \tag{2.2}$$

for $\alpha \in [n-1, n)$. However, the left and right Hadamard fractional integrals of order $\Re e(\alpha) > 0$, for suitable function x, introduced essentially by $\mathcal{J}$. Hadamard fractional integral in [18, 1892], are defined by

$$\mathcal{J}_{a_+}^\alpha [x](t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} x(\tau) \frac{\mathrm{d}\tau}{\tau}, \tag{2.3}$$

and

$$\mathcal{J}_{b^-}^{\alpha}[x](t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\ln \frac{\tau}{t}\right)^{\alpha-1} x(\tau) \frac{d\tau}{\tau}, \tag{2.4}$$

respectively. Definition (2.3) is based on the generalisation of the nth integral

$$\mathcal{J}_a^n[x](t) = \int_a^t \frac{d\tau_1}{\tau_1} \int_a^{\tau_1} \frac{d\tau_2}{\tau_2} \cdots \int_a^{\tau_{n-1}} x(\tau_n) \frac{d\tau_n}{\tau_n}$$
$$\equiv \frac{1}{\Gamma(n)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-1} x(\tau) \frac{d\tau}{\tau},$$

where $n = [\Re e(\alpha)] + 1$ and $[\Re e(\alpha)]$ means the integer part of $\Re e(\alpha)$. Hadamard also proposed [18,53] a definition of the fractional integral as

$$\mathcal{J}_a^{\alpha}[x](t) = \frac{t^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(ts)\, ds. \tag{2.5}$$

It should be emphasized that expression (2.5) contains $x(ts)$ in place of $x(s)$. Therefore we can consider the term $s > 0$ as a variable that describes dilation. As a consequence, using the change of variables $\tau = ts$, would results in the definition of the classical Riemann-Liouville fractional integral. It should be noted that in order to describe the change of dilation we can use the operator Υ_s (see [53, p. 330]) such that $(\Upsilon_s x)(t) = x(\exp(ts))$ where $s > 0$. It is known that the dilation of Euclidean geometric figures changes in size while the shape is unchanged. The connection

$$\mathcal{J}_a^{\alpha}[x] = \Upsilon_s^{-1} J_a^{\alpha} \Upsilon_s [x], \qquad (\Upsilon_s x)(t) = x(\exp(ts)), \tag{2.6}$$

allows us to extend various properties of operators J_a^{α} to the case of operators $\mathcal{J}_a^{\alpha}$. It is directly checked that such connections for the operators (2.5) and (2.1) are given by the relations (2.6). The corresponding left-sided Hadamard fractional derivative of order α is defined by

$$\mathcal{D}_a^{\alpha}[x](t) = \delta^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-\alpha-1} x(\tau) \frac{d\tau}{\tau}, \tag{2.7}$$

where $\alpha \in [n-1, n)$ and $\delta^n = (t\mathrm{D})^n$ is the so-called δ-derivative and $\mathrm{D} \equiv \frac{d}{dt}$.

Firstly, from the above definitions, we see the difference between Hadamard derivative and the Riemann–Liouville one. As a clarification, the aforementioned derivatives differ in the sense that the kernel of the integral in the definition of the Hadamard derivative contains a logarithmic function, while the Riemann-Liouville integral contains a power function. On the other hand, the Hadamard derivative is viewed as a generalization of the operator $(t\mathrm{D})^n$, while the Riemann–Liouville derivative is considered as an extension of the classical Euler differential operator $(\mathrm{D})^n$. Secondly, we observe that formally the relationship between Hadamard-type derivatives and Riemann-Liouville derivatives is given by the change of variable $t \to \ln(t)$, leading to the logarithmic kernel.

Supposedly one can reduce the theorems and results to the corresponding ones of Hadamard-type derivatives by a simple change of variables and functions. It is possible to reduce a formula by such a change of operations but not the precise hypotheses under which a formula is valid. As an illustration, the function $x(t) = \sin t$ is obviously uniformly continuous, but not ln-uniformly continuous on $\mathbb{R}_+$, while the function $x(t) = \sin(\ln t)$ is ln-uniformly continuous but not uniformly continuous on $\mathbb{R}_+$.

However, the two notions are equivalent on every bounded interval $[a, b]$ with $a > 0$. Besides, the Hadamard derivative (also integral) starts at the initial time a which is bigger than zero, but the Riemann–Liouville derivative (also integral) often begins at the origin (or any other real number). Under certain precise conditions, an equivalence could be obtained between a problem involving Hadamard derivative to another defined using a Riemann Liouville derivative.

Lemma 2.1. *[28] Let $\Re e(\alpha) > 0$, $n = [\Re e(\alpha)] + 1$ and $x \in C[a, +\infty) \cap L^1[a, +\infty)$, then the Hadamard fractional differential equation $\mathcal{D}_a^\alpha [x](t) = 0$, has a solution*

$$x(t) = \sum_{k=1}^{n} c_k \left(\ln \frac{t}{a}\right)^{\alpha-k}.$$

Further, the following formulas hold

$$\begin{cases} \mathcal{J}_a^\alpha \mathcal{D}_a^\alpha [x](t) = x(t) - \sum_{k=1}^{n} c_k \left(\ln \frac{t}{a}\right)^{\alpha-k}, \\ \mathcal{D}_a^\alpha \mathcal{J}_a^\alpha [x](t) = x(t), \end{cases} \tag{2.8}$$

where $c_k \in \mathbb{R}, (k = 1, 2, \ldots, n)$ are arbitrary constants.

Lemma 2.2. *([21]) If $0 < \alpha < 1$, then*

$$\mathcal{D}_a^\alpha [x](t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{-\alpha} \delta [x(\tau)] \frac{\mathrm{d}\tau}{\tau} + \frac{x(a)}{\Gamma(1-\alpha)} \left(\ln \frac{t}{a}\right)^{-\alpha}.$$

Theorem 2.3. *([6]) Consider the continuous function $x : [a, b] \to \mathbb{R}$ belongs to $C^2[a, +\infty)$ and let $\Delta T = \frac{1}{n} \ln \frac{b}{a}$ for $n \geq 1$. Denote the time and space grid by*

$$t_N = a \exp(N\Delta T) = a \sqrt[n]{\left(\frac{b}{a}\right)^N}, \tag{2.9}$$

and $x_N = x(t_N)$ for $N \in \{0, 1, 2, \cdots, n\}$. Then for all $N \in \{1, 2, \cdots, n\}$,

$$\mathcal{D}_a^\alpha [x](t_N) = \tilde{\mathcal{D}}_a^\alpha [x](t_N) + O(\Delta T),$$

where

$$\tilde{\mathcal{D}}_a^\alpha [x](t_N) = \frac{x(a)}{\Gamma(1-\alpha)} \left(\ln \frac{t_N}{a}\right)^{-\alpha} + \zeta \sum_{k=1}^{N} (\tau_{N-k+1}^\alpha) \frac{x(t_k) - x(t_{k-1})}{\exp(k\Delta T)} .t_k,$$

and $\lim_{\Delta T \to 0} O(\Delta T) = 0$, here $\left(\tau_k^\alpha\right) = k^{1-\alpha} - (k-1)^{1-\alpha}$ and

$$\zeta = \frac{(\Delta T)^{1-\alpha}}{a[1 - \exp(-\Delta T)]\Gamma(2-\alpha)}.$$

Lemma 2.4. *([26]) If $\alpha, \beta > 0$, then the following equality holds*

$$\mathcal{J}_a^\alpha \left[\tau^\beta\right](t) = \frac{\beta^{-\alpha} t^\beta}{\Gamma(\alpha)} \gamma \left(\alpha, \beta \ln \frac{t}{a}\right),$$

where $a > 0$ is the starting point in the interval. In particular, for $a = 0$,

$$\mathcal{J}_0^\alpha \left[\tau^\beta\right](t) = \beta^{-\alpha} t^\beta.$$

The following discussion is essential for our further investigation.

Remark 2.5. *If $\alpha, \beta > 0$, for $t \in [1, e]$. Then*

i) It is easy to verify that

$$\mathcal{J}_1^\alpha \left[\tau^\beta\right](t) \le \beta^{-\alpha} \frac{(\beta \ln t)^\alpha}{\alpha \Gamma(\alpha)} t^\beta = \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} t^\beta.$$

ii) The function $\mathcal{J}_1^\alpha [\sin \lambda (t-1)]$ is continuous as a result of the continuity of sin *function. Furthermore and according to* (2.3)*, we have*

$$\mathcal{J}_1^\alpha [\sin \lambda (\tau - 1)](t) \le \mathcal{J}_1^\alpha [1](t) = \frac{1}{\Gamma(1+\alpha)} (\ln t)^\alpha.$$

Note that

$$\mathcal{J}_1^\alpha [\sin \lambda (\tau - 1)](1) = \lim_{t \to 1^+} \left| \mathcal{J}_1^\alpha [\sin \lambda (\tau - 1)](t) \right| = 0. \tag{2.10}$$

iii) From Lemma 2.2, we have

$$\begin{aligned}
\mathcal{D}_1^\alpha [\sin \lambda (\tau - 1)](t) &= \mathcal{J}_1^{1-\alpha} [\delta \sin \lambda (\tau - 1)](t) + \frac{0}{\Gamma(1+\alpha)} . (\ln t)^{-\alpha} \\
&= \frac{\lambda}{\Gamma(1-\alpha)} \int_1^t \left(\ln \frac{t}{\tau} \right)^{-\alpha} \tau \cos \lambda (\tau - 1) \frac{\mathrm{d}\tau}{\tau} \\
&\le \lambda \mathcal{J}_1^{1-\alpha} [\tau](t) \le \lambda \frac{\gamma(1-\alpha, \ln t)}{\Gamma(1-\alpha)} t \\
&\le \frac{\lambda t}{\Gamma(2-\alpha)} (\ln t)^{1-\alpha}.
\end{aligned}$$

Remark 2.6. *If $\alpha > 0$, for $t \in [1, e]$. Then, using the elementary inequality $(\ln s)^\alpha \le s^\alpha$, we obtain the inequality*

$$0 \le \rho_\alpha(t) = \int_1^t (\ln s)^\alpha \, \mathrm{d}s \le \frac{1}{\alpha+1} \max_{t \in [1,e]} \left\{ t^{\alpha+1} - 1 \right\} = \frac{e^{\alpha+1}}{\alpha+1}. \tag{2.11}$$

Utilizing the particular case of the Fubini's theorem, one can deduce that

$$\int_1^t \mathcal{J}_1^\alpha [x](s) \, \mathrm{d}s = \frac{1}{\Gamma(\alpha)} \int_1^t \rho_{\alpha-1}\left(\frac{t}{s} \right) x(s) \mathrm{d}s. \tag{2.12}$$

Indeed, interchanging the order of integration with the help of (2.3) *and* (2.4) *and it follows that*

$$\begin{aligned}
\int_1^t \mathcal{J}_{1^+}^\alpha [x](s) \, \mathrm{d}s &= \int_1^t s \times \mathcal{J}_1^\alpha [x](s) \frac{\mathrm{d}s}{s} \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t s \times \left(\int_1^s \left(\ln \frac{s}{\tau} \right)^{\alpha-1} x(\tau) \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{d}s}{s}. \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t x(\tau) \left(\int_\tau^t s \left(\ln \frac{s}{\tau} \right)^{\alpha-1} \frac{\mathrm{d}s}{s} \right) \frac{\mathrm{d}\tau}{\tau} \\
&= \int_1^t x(\tau) \mathcal{J}_{t^-}^\alpha [\tau](s) \, \mathrm{d}s.
\end{aligned}$$

If we take $v = \frac{s}{\tau}$, then

$$\int_1^t x(\tau)\,\mathcal{J}_t^{\alpha}[\tau](s)\,\mathrm{d}s = \frac{1}{\Gamma(\alpha)}\int_1^t x(\tau)\left(\int_1^{\frac{t}{\tau}} (\ln v)^{\alpha-1}\,\mathrm{d}v\right)\mathrm{d}\tau$$
$$= \frac{1}{\Gamma(\alpha)}\int_1^t x(\tau)\rho_{\alpha-1}\left(\frac{t}{\tau}\right)\mathrm{d}\tau.$$

Following [48], we bring a formula generalizing the well-known rule of differentiating an integral with respect to its upper limit which serves also as a parameter of the integrand

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{p(t)}^{q(t)} G(t,\tau)\,\mathrm{d}\tau = \int_{p(t)}^{q(t)} \frac{\partial}{\partial t}G(t,\tau)\,\mathrm{d}\tau + G(t,q(t))\frac{\mathrm{d}q(t)}{\mathrm{d}t} - G(t,p(t))\frac{\mathrm{d}p(t)}{\mathrm{d}t}. \tag{2.13}$$

From (2.7), we have for $\alpha \in (0,1)$ and $t \in (a,b)$ that

$$\mathcal{D}_a^{\alpha}\left[\int_a^s G(s,\tau)\,\mathrm{d}\tau\right](t) = \frac{1}{\Gamma(1-\alpha)} t\frac{d}{dt}\int_a^t \left(\ln\frac{t}{s}\right)^{-\alpha}\left[\int_a^s G(s,\tau)\,\mathrm{d}\tau\right]\frac{\mathrm{d}s}{s}.$$

Interchanging the order of integration and applying Dirichlet formula, we obtain

$$\begin{aligned}
\mathcal{D}_a^{\alpha}\Big[\int_a^s G(s,\tau)\,\mathrm{d}\tau\Big](t) &= \frac{1}{\Gamma(1-\alpha)} t\frac{d}{dt}\int_a^t \left(\ln\frac{t}{s}\right)^{-\alpha}\left[\int_a^s G(s,\tau)\,\mathrm{d}\tau\right]\frac{\mathrm{d}s}{s}\\
&= t\frac{d}{dt}\int_a^t\left(\frac{1}{\Gamma(1-\alpha)}\int_\tau^t\left(\ln\frac{t}{s}\right)^{-\alpha} G(s,\tau)\frac{\mathrm{d}s}{s}\right)\mathrm{d}\tau\\
&= t\frac{d}{dt}\int_a^t \mathcal{J}_\tau^{1-\alpha}[G(s,\tau)](t)\,\mathrm{d}\tau\\
&= \int_a^t t\frac{\partial}{\partial t}\mathcal{J}_\tau^{1-\alpha}[G(s,\tau)](t)\,\mathrm{d}\tau + t\lim_{\tau\to t-a}\mathcal{J}_\tau^{1-\alpha}[G(s,\tau)](t)\\
&= \int_a^t \mathcal{D}_\tau^{\alpha}[G(s,\tau)](t)\,\mathrm{d}\tau + t\lim_{\tau\to t-a}\mathcal{J}_\tau^{1-\alpha}[G(s,\tau)](t).
\end{aligned}$$

In particular, we get

$$\begin{aligned}
\mathcal{D}_a^{\alpha}\left[\int_a^s G(s,\tau)h(\tau)\mathrm{d}\tau\right](t) = &\int_a^t \mathcal{D}_\tau^{\alpha}[G(s,\tau)](t)\,h(\tau)\mathrm{d}\tau\\
&+ t\lim_{\tau\to t-a}\left(h(\tau)\mathcal{J}_\tau^{1-\alpha}[G(s,\tau)](t)\right).
\end{aligned} \tag{2.14}$$

To simplify the presentation, we let

$$f_x(t) = f(t, x(t), \mathcal{D}_1^{\alpha}[x](t)), \qquad g(t-s) = \sin\lambda(t-s). \tag{2.15}$$

In virtue of equation (2.14), we deduce that

$$\begin{aligned}
\mathcal{D}_1^{\alpha}\Big(\int_1^s g(s-\tau)\mathcal{J}_1^{\alpha}[f_x](\tau)\,\mathrm{d}\tau\Big)(t) = &\int_1^t \mathcal{D}_\tau^{\alpha}[g(s-\tau)](t)\,\mathcal{J}_1^{\alpha}[f_x](\tau)\,\mathrm{d}\tau\\
&+ t\lim_{\tau\to t-1}\left(\mathcal{J}_1^{\alpha}[f_x](\tau)\,\mathcal{J}_\tau^{1-\alpha}[G(s,\tau)](t)\right).
\end{aligned}$$

Applying a suitable shift in the fractional operators with lower terminal τ, we deduce the next property [23, 24].

Property 2.1. *Let* $0 < \alpha < 1$, $\mathcal{D}_1^{\alpha}[g] \in L^1(1,e)$ *and* $\mathcal{J}_1^{\alpha}[f_x] \in C(1,e)$. *Then we have*

$$\mathcal{D}_1^{\alpha}\Big(\int_1^t g(t-s)\mathcal{J}_1^{\alpha}[f_x](s)\,\mathrm{d}s\Big)(t) = \int_1^t \mathcal{D}_1^{\alpha}[g(s-1)](\tau)\mathcal{J}_1^{\alpha}[f_x](t-\tau+1)\,\mathrm{d}\tau + t\mathcal{J}_1^{\alpha}[f_x](t)\lim_{\tau\to 1^+}\Big(\mathcal{J}_1^{1-\alpha}[g(s-1)](\tau)\Big).$$

In the literature, we can read the following Schaefer fixed point theorem.

Lemma 2.7. *[21,55] Let E be a Banach space and assume that* $\Psi : E \to E$ *is a completely continuous operator. If the set*

$$\Lambda = \Big\{x \in E : x = \mu\Psi x : 0 < \mu < 1\Big\},$$

is bounded, then Ψ *has a fixed point in E.*

The next result is a generalization of Gronwall inequality due to Pachpatte ([46]).

Lemma 2.8. *Let* $u \in C(I,\mathbb{R}^+)$, $\tilde{a}(t,s)$, $\tilde{b}(t,s) \in C(D,\mathbb{R}^+)$ *and* $\tilde{a}(t,s)$, $\tilde{b}(t,s)$ *are nondecreasing in t for each* $s \in I$, *where* $I = [\tilde{\alpha},\tilde{\beta}]$, $\mathbb{R}^+ = [0,\infty)$,

$$D = \{(t,s) \in I \times I : \tilde{\alpha} \le s \le t \le \tilde{\beta}\},$$

and suppose that

$$u(t) \le k + \int_{\tilde{\alpha}}^{t} \tilde{a}(t,s)u(s)\,\mathrm{d}s + \int_{\tilde{\alpha}}^{\tilde{\beta}} \tilde{b}(t,s)u(s)\,\mathrm{d}s,$$

for $t \in I$, *where* $k \ge 0$ *is a constant. If*

$$p(t) = \int_{\tilde{\alpha}}^{\tilde{\beta}} \tilde{b}(t,s)\exp\left(\int_{\tilde{\alpha}}^{s} \tilde{a}(s,\tau)d\tau\right)\mathrm{d}s < 1,$$

for $t \in I$, *then*

$$u(t) \le \frac{k}{1-p(t)}\exp\left(\int_{\tilde{\alpha}}^{t} \tilde{a}(t,s)\,\mathrm{d}s\right).$$

The following hypotheses will be used in the sequel:

H1: There exist a constant $N_i > 0$ $(i = 1,2)$ such that

$$|f(t,x_1,\tilde{x}_1) - f(t,x_2,\tilde{x}_2)| \le N_1|x_1 - x_2| + N_2|\tilde{x}_1 - \tilde{x}_2|,$$

for each $t \in [1,e]$ and all $x_i, \tilde{x}_i \in \mathbb{R}$.

H2: There exists a constant $L > 0$ such that $|f(t,x,\tilde{x})| \le L$, for each $t \in [1,e]$ and all $x, \tilde{x} \in \mathbb{R}$.

3. The nonlinear boundary value problem (1.1)

In order to study the nonlinear problem (1.1), we first consider the associated linear problem and obtain its solution:

$$\mathcal{D}_1^{\alpha}\left(\mathrm{D}^2 + \lambda^2\right)[x](t) = h(t),$$

for $0 < \alpha \le 1$, where h is a continuous function on $[1,e]$.

Lemma 3.1. *The general solution of the linear differential equation*

$$\left(\mathrm{D}^2 + \lambda^2\right) x(t) = \tilde{x}(t), \tag{3.1}$$

for $t \in [1, e]$*, is given by*

$$x(t) = \frac{1}{\lambda} \int_1^t \sin \lambda (t - s) \tilde{x}(s) \, \mathrm{d}s + c_1 \cos \lambda t + c_2 \sin \lambda t,$$

where c_1, c_2 *are unknown arbitrary constants.*

Proof. Assume that $x(t)$ satisfies (3.1), then the method of variation of parameters implies the desired results. □

Lemma 3.2. *Let* $0 < \alpha < 1,\ h \in C([1, e], \mathbb{R})$*. Then the unique solution of the linear problem*

$$\begin{cases} \mathcal{D}_1^{\alpha} [\tilde{x}](t) = h(t), \\ \tilde{x}(1) = 0, \end{cases} \tag{3.2}$$

for $t \in (1, e)$*, is equivalent to the integral equation*

$$\tilde{x}(t) = \mathcal{J}_1^{\alpha} [h](t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{\tau}\right)^{\alpha - 1} h(\tau) \frac{\mathrm{d}\tau}{\tau}. \tag{3.3}$$

Proof. Applying Lemma 2.1, we may reduce (3.2)-a to an equivalent integral equation

$$\tilde{x}(t) = \mathcal{J}_1^{\alpha} [h](t) + c_0 (\ln t)^{\alpha - 1},$$

where $c_0 \in \mathbb{R}$. In view of the boundary condition $\tilde{x}(1) = 0$, we have $c_0 = 0$, thus (3.3) holds. □

Lemma 3.3. *Let* $h \in C([1, e], \mathbb{R}), \alpha \in (0, 1]$ *and* $1 < \xi < e$*. Then the fractional problem*

$$\begin{cases} \mathcal{D}_1^{\alpha} \left(\mathrm{D}^2 + \lambda^2\right) [x](t) = h(t), \\ x(1) = \mathrm{D}^2 [x](1) = 0, \qquad x(e) = \beta x(\xi), \end{cases} \tag{3.4}$$

has a unique solution given by

$$\begin{aligned} x(t) = {} & \frac{1}{\lambda} \int_1^t \sin \lambda (t - s) \left[\frac{1}{\Gamma(\alpha)} \int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha - 1} h(\tau) \frac{\mathrm{d}\tau}{\tau}\right] \mathrm{d}s \\ & + \frac{\beta}{\Delta} \sin \lambda (t - 1) \int_1^{\xi} \sin \lambda (\xi - s) \times \left[\frac{1}{\Gamma(\alpha)} \int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha - 1} h(\tau) \frac{\mathrm{d}\tau}{\tau}\right] \mathrm{d}s \\ & - \frac{1}{\Delta} \sin \lambda (t - 1) \int_1^{e} \sin \lambda (e - s) \left[\frac{1}{\Gamma(\alpha)} \int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha - 1} h(\tau) \frac{\mathrm{d}\tau}{\tau}\right] \mathrm{d}s, \end{aligned} \tag{3.5}$$

where

$$\Delta = \lambda (\sin \lambda (e - 1) - \beta \sin \lambda (\xi - 1)) \neq 0. \tag{3.6}$$

Proof. Assuming
$$\left(\mathrm{D}^2+\lambda^2\right)[x](t)=\tilde{x}(t)$$
and then applying Lemma 3.1 when $0<\alpha<1$, we get

$$x(t)=\frac{1}{\lambda}\int_1^t \sin\lambda(t-s)\,\tilde{x}(s)\,\mathrm{d}s+c_1\cos\lambda t+c_2\sin\lambda t.$$

By the boundary condition $x(1)=0$ and privous equation, we conclude that

$$c_1\cos\lambda=-c_2\sin\lambda. \tag{3.7}$$

On the other hand, $x(e)=\beta x(\xi)$, combining with

$$x(e)=\frac{1}{\lambda}\int_1^e \sin\lambda(e-s)\,\tilde{x}(s)\,\mathrm{d}s+c_1\cos\lambda e+c_2\sin\lambda e,$$

and
$$x(\xi)=\frac{1}{\lambda}\int_1^\xi \sin\lambda(\xi-s)\,\tilde{x}(s)\,\mathrm{d}s+c_1\cos\lambda\xi+c_2\sin\lambda\xi,$$

yield
$$c_2=\frac{\cos\lambda}{\Delta}\left(\beta\int_1^\xi \sin\lambda(\xi-s)\,\tilde{x}(s)\,\mathrm{d}s-\int_1^e \sin\lambda(e-s)\,\tilde{x}(s)\,\mathrm{d}s\right),$$
where Δ is given by (3.6). If $\lambda=\frac{(2k+1)\pi}{2}$, $k=0,1,\ldots$, then $c_2=0$, and by (3.7), we get

$$\begin{aligned}c_1=&\frac{2}{(2k+1)\pi}\left[\cos\frac{(2k+1)\pi e}{2}-\beta\cos\frac{(2k+1)\pi\xi}{2}\right]^{-1}\\&\times\left[\beta\int_1^\xi \sin\frac{(2k+1)\pi}{2}(\xi-s)\,\tilde{x}(s)\,\mathrm{d}s\right.\\&\left.-\int_1^e \sin\frac{(2k+1)\pi}{2}(e-s)\,\tilde{x}(s)\,\mathrm{d}s\right],\end{aligned}$$

otherwise, we find

$$c_1=-\frac{\sin\lambda}{\Delta}\left(\beta\int_1^\xi \sin\lambda(\xi-s)\,\tilde{x}(s)\,\mathrm{d}s-\int_1^e \sin\lambda(e-s)\,\tilde{x}(s)\,\mathrm{d}s\right).$$

The above two expressions of c_1 are equivalent for the particular choice of λ. Substituting these values of c_1 and c_2 in (3.7) and applying Lemma 3.2, we finally obtain (3.5). So, the unique solution of problem (3.4) is given by (3.5). Conversely, let $x(t)$ be given by formula (3.5), operating D^2 on both sides and using (2.13), we get

$$\begin{aligned}\mathrm{D}^2x(t)=&-\lambda\int_1^t \sin\lambda(t-s)\left[\frac{1}{\Gamma(\alpha)}\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\frac{\mathrm{d}\tau}{\tau}\right]\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}h(\tau)\frac{\mathrm{d}\tau}{\tau}\\&-\frac{\beta\lambda^2}{\Delta}\sin\lambda(t-1)\int_1^\xi \sin\lambda(\xi-s)\left[\frac{1}{\Gamma(\alpha)}\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\frac{\mathrm{d}\tau}{\tau}\right]\mathrm{d}s\end{aligned}$$

$$+\frac{\lambda^2}{\Delta}\sin\lambda(t-1)\int_1^e \sin\lambda(e-s)\left[\frac{1}{\Gamma(\alpha)}\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\frac{\mathrm{d}\tau}{\tau}\right]\mathrm{d}s.$$

Hence

$$\left(\mathrm{D}^2+\lambda^2\right)[x](t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}h(\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Operating $\mathcal{D}_1^\alpha$ on the above relation and using (2.8), we obtain the first equation of (3.4). Further, it is easy to get that all conditions in (3.4) are satisfied. The proof is completed. □

By virtue of Lemma 3.3, we get the following result.

Lemma 3.4. *Let* $0<\alpha<1$, $\lambda>0$. *Then the problem* (1.1) *is equivalent to the integral equation*

$$x(t)=\frac{1}{\lambda}\int_1^t g(t-s)\mathcal{J}_1^\alpha[f_x](s)\,\mathrm{d}s \tag{3.8}$$
$$+\frac{1}{\Delta}g(t-1)\left[\beta\int_1^\xi g(\xi-s)\mathcal{J}_1^\alpha[f_x](s)\,\mathrm{d}s-\int_1^e g(e-s)\mathcal{J}_1^\alpha[f_x](s)\,\mathrm{d}s\right].$$

For convenience, we define the following functions

$$\phi_x(t)=\int_1^t g(t-s)\mathcal{J}_1^\alpha[f_x](s)\,\mathrm{d}s \tag{3.9}$$

and

$$H_x(\xi,\beta)=\frac{1}{\Delta}\left(\beta\phi_x(\xi)-\phi_x(e)\right). \tag{3.10}$$

Then, the integral equation (3.8) can be written as

$$x(t)=\frac{1}{\lambda}\phi_x(t)+H_x(\xi,\beta)\,g(t-1). \tag{3.11}$$

From the expressions of (3.5) and (3.8), we can see that if all conditions in Lemmas 3.3 and 3.4 are satisfied, then the solution is a continuous solution of the boundary value problem (1.1). Let $C=C([1,e],\mathbb{R})$ be a Banach space of all continuous functions defined on $[1,e]$ endowed with the usual supremum norm. Consider the space defined by

$$E=\left\{x:x\in C,\mathcal{D}_1^\alpha[x]\in C\right\},$$

equipped with the norm $\|x\|_E=\|x\|+\left\|\mathcal{D}_1^\alpha[x]\right\|$, then $(E,\|.\|_E)$ is a Banach space. On this space, by virtue of Lemma 3.4, we may define the operator $\Psi:E\to E$ by

$$\Psi x(t)=\frac{1}{\lambda}\phi_x(t)+H_x(\xi,\beta)\,g(t-1),$$

where $g(t-1)$, $\phi_x(t)$ and $H_x(\xi,\beta)$ defined by (2.15), (3.9) and (3.10) respectively. Then

$$\mathcal{D}_1^\alpha[\Psi x](t)=\frac{1}{\lambda}\mathcal{D}_1^\alpha[\phi_x](t)+H_x(\xi,\beta)\,\mathcal{D}_1^\alpha[g(t-1)]. \tag{3.12}$$

By virtue of Property 2.1 and Eq (2.10) in Remark 2.5, we get the following

$$\mathcal{D}_1^{\alpha}\left[\phi_x\right](t)=\int_1^t \mathcal{D}_1^{\alpha}\left[g\left(\tau-1\right)\right](s)\,\mathcal{J}_1^{\alpha}\left[f_x\right](t-s+1)\,\mathrm{d}s. \tag{3.13}$$

The continuity of the functional f would imply the continuity of Ψx and $\mathcal{D}_1^{\alpha}[\Psi x]$. Hence the operator Ψ maps the Banach space E into itself. This operator will be used to prove our main results. Next section, we employ fixed point theorems to prove the main results of this paper. In view of Lemma 3.4, we transform problem (1.1) as

$$x=\Psi x, \qquad x\in E. \tag{3.14}$$

Observe that problem (1.1) or (3.8) has solutions if the operator Ψ in (3.14) has fixed points. For computational convenience, we set the notations:

$$0\leq\rho(t):=\frac{1}{\lambda}\rho_{\alpha}(t)+\frac{1}{|\Delta|}\left(\beta\rho_{\alpha}(\xi)+\rho_{\alpha}(e)\right)\leq M_{\rho}, \tag{3.15}$$

and

$$0\leq\sigma_{\alpha}(t):=\int_1^t s\left(\ln s\right)^{1-\alpha}\left(\ln\left(t-s+1\right)\right)^{\alpha}\mathrm{d}s\leq M_{\sigma}, \tag{3.16}$$

where

$$\begin{aligned} M_{\rho}&:=\frac{1}{\lambda}+\frac{1}{|\Delta|}\left(\beta+1\right),\\ M_{\sigma}&:=\max\left\{\int_1^t s^{2-\alpha}\left(t-s+1\right)^{\alpha}\mathrm{d}s : t\in[1,e]\right\}, \end{aligned} \tag{3.17}$$

and

$$Q\geq\frac{1}{\Gamma(1+\alpha)}\left[M_{\rho}+\frac{1}{\Gamma(2-\alpha)}\left(M_{\sigma}+\lambda\frac{\beta\rho_{\alpha}(\xi)+\rho_{\alpha}(e)}{|\Delta|}e^{2-\alpha}\right)\right]. \tag{3.18}$$

4. Results of existence and uniqueness

In this section, we establish the existence and uniqueness results via fixed point theorems.

Theorem 4.1. *Assume that $f:[1,e]\times C\times C\to C$ is a continuous function that satisfies (H1). If we suppose*

$$N=\max\left\{N_1,N_2\right\}, \qquad NQ<1, \tag{4.1}$$

where Q is defined in (3.18)*, then problem* (3.14) *has a unique solution in E.*

Proof. To prove this theorem, we need to prove that the operator Ψ has a fixed point in E. So, we shall prove that Ψ is a contraction mapping on E. For any $x,\tilde{x}\in E$ and for each $t\in[1,e]$, we have

$$|\Psi\tilde{x}(t)-\Psi x(t)|\leq\frac{1}{\lambda}\left|\phi_{\tilde{x}}(t)-\phi_x(t)\right|+\left|H_{\tilde{x}}(\xi,\beta)-H_x(\xi,\beta)\right|\left|g\left(t-1\right)\right|, \tag{4.2}$$

where $x(t)$ and $\tilde{x}(t)$ are defined in Lemma 3.4. From assumption (H1) and Eqs (3.9) and (4.1), we obtain

$$\begin{aligned} |\phi_{\tilde{x}}(t) - \phi_x(t)| &= \left| \int_1^t g(t-s) \mathcal{J}_1^\alpha [f_{\tilde{x}}(\tau) - f_x(\tau)](s)\,\mathrm{d}s \right| \\ &\le \sup_{t\in[1,e]} |f_{\tilde{x}}(t) - f_x(t)| \left| \int_1^t \mathcal{J}_1^\alpha [1]\,\mathrm{d}s \right| \\ &\le \sup_{t\in[1,e]} \frac{N\left(|\tilde{x}(t) - x(t)| + \left|\mathcal{D}_1^\alpha \tilde{x}(t) - \mathcal{D}_1^\alpha x(t)\right|\right)}{\Gamma(1+\alpha)} \int_1^t (\ln s)^\alpha\,\mathrm{d}s \\ &\le \frac{\rho_\alpha(t)}{\Gamma(1+\alpha)} N(M_1 + M_2), \end{aligned} \tag{4.3}$$

where $\rho_\alpha(t)$ is given by (2.11) and

$$\begin{aligned} M_1 &= \sup_{t\in[1,e]} |\tilde{x}(t) - x(t)|, \\ M_2 &= \sup_{t\in[1,e]} \left|\mathcal{D}_1^\alpha \tilde{x}(t) - \mathcal{D}_1^\alpha x(t)\right|. \end{aligned}$$

Similarly, we can obtain $|\phi_{\tilde{x}}(\xi) - \phi_x(\xi)|$ and $|\phi_{\tilde{x}}(e) - \phi_x(e)|$. Then

$$\begin{aligned} |H_{\tilde{x}}(\xi,\beta) - H_x(\xi,\beta)| &\le \frac{1}{|\Delta|} \left[\beta |\phi_{\tilde{x}}(\xi) - \phi_x(\xi)| + |\phi_{\tilde{x}}(e) - \phi_x(e)|\right] \\ &\le \frac{\beta\rho_\alpha(\xi) + \rho_\alpha(e)}{|\Delta|\,\Gamma(1+\alpha)} N(M_1 + M_2). \end{aligned} \tag{4.4}$$

Linking (4.2), (4.3) and (4.4), for every $x,\ \tilde{x} \in E$, we get

$$|\Psi\tilde{x}(t) - \Psi x(t)| \le \frac{\rho(t)}{\Gamma(1+\alpha)} N(M_1 + M_2),$$

where $\rho(t)$ is given by (3.15). Consequently, it yields that

$$\|\Psi\tilde{x} - \Psi x\| \le Q_1 N\left(\|\tilde{x} - x\| + \left\|\mathcal{D}_1^\alpha \tilde{x} - \mathcal{D}_1^\alpha x\right\|\right), \tag{4.5}$$

with

$$Q_1 \ge \max\left\{\frac{\rho(t)}{\Gamma(1+\alpha)} : t \in [1,e]\right\}. \tag{4.6}$$

On the other hand, we observe that

$$\begin{aligned} \left|\mathcal{D}_1^\alpha [\Psi\tilde{x}](t) - \mathcal{D}_1^\alpha [\Psi x](t)\right| \le{}& \frac{1}{\lambda} \left|\mathcal{D}_1^\alpha [\phi_{\tilde{x}}](t) - \mathcal{D}_1^\alpha [\phi_x](t)\right| \\ &+ |H_{\tilde{x}}(\xi,\beta) - H_x(\xi,\beta)| \left|\mathcal{D}_1^\alpha [g(t-1)]\right|. \end{aligned} \tag{4.7}$$

By (3.13), we have

$$\left|\mathcal{D}_1^\alpha [\phi_{\tilde{x}}](t) - \mathcal{D}_1^\alpha [\phi_x](t)\right| = \left| \int_1^t [\mathcal{D}_1^\alpha [g(t-1)](s)] [(\mathcal{J}_1^\alpha [f_{\tilde{x}} - f_x])(t-s+1)]\,\mathrm{d}s \right| \tag{4.8}$$

$$\leq \sup_{t\in[1,e]} |f_{\tilde{x}}(t) - f_x(t)| \int_1^t \left|\mathcal{D}_1^\alpha [g(t-1)](s)\right| \mathcal{J}_1^\alpha [1](t-s+1)\, \mathrm{d}s.$$

Taking into account that

$$\begin{aligned} R_{11}(t) &= \int_1^t \left|\mathcal{D}_1^\alpha [g(t-1)](s)\right| \mathcal{J}_1^\alpha [1](t-s+1)\, \mathrm{d}s \\ &\leq \int_1^t \lambda \left|\frac{\gamma(1-\alpha, \ln s)}{\Gamma(1-\alpha)} s\right| \frac{(\ln(t-s+1))^\alpha}{\Gamma(1+\alpha)} \mathrm{d}s \\ &\leq \frac{\lambda}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \int_1^t s(\ln s)^{1-\alpha} (\ln(t-s+1))^\alpha\, \mathrm{d}s \\ &\leq \frac{\lambda}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \sigma_\alpha(t), \end{aligned} \tag{4.9}$$

we have,

$$\left|\mathcal{D}_1^\alpha [\phi_{\tilde{x}}](t) - \mathcal{D}_1^\alpha [\phi_x](t)\right| \leq \frac{\lambda \sigma_\alpha(t)}{\Gamma(2-\alpha)\Gamma(1+\alpha)} N(M_1 + M_2), \tag{4.10}$$

where $\sigma_\alpha(t)$ is given by (3.16). Therefore, from (4.7), (4.7) and (4.10), we have

$$\left|\mathcal{D}_1^\alpha [\Psi\tilde{x}](t) - \mathcal{D}_1^\alpha [\Psi x](t)\right| \leq \frac{N(M_1 + M_2)}{\Gamma(1+\alpha)} \left(\frac{\sigma_\alpha(t)}{\Gamma(2-\alpha)} + R_{12}(t)\right), \tag{4.11}$$

where

$$\begin{aligned} R_{12}(t) &= \frac{\beta\rho_\alpha(\xi) + \rho_\alpha(e)}{|\Delta|} \frac{\lambda t\, |\gamma(1-\alpha, \ln t)|}{\Gamma(1-\alpha)} \\ &\leq \frac{\lambda(\beta\rho_\alpha(\xi) + \rho_\alpha(e))}{|\Delta|\,\Gamma(2-\alpha)} (\ln t)^{1-\alpha}\, t. \end{aligned} \tag{4.12}$$

This gives

$$\left\|\mathcal{D}_1^\alpha [\Psi\tilde{x}] - \mathcal{D}_1^\alpha [\Psi x]\right\| \leq Q_2 N\left(\|\tilde{x} - x\| + \left\|\mathcal{D}_1^\alpha \tilde{x} - \mathcal{D}_1^\alpha x\right\|\right), \tag{4.13}$$

with

$$Q_2 \geq \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \max\left\{\sigma_\alpha(t) + \lambda \frac{\beta\rho_\alpha(\xi) + \rho_\alpha(e)}{|\Delta|} (\ln t)^{1-\alpha}\, t : t \in [1, e]\right\}. \tag{4.14}$$

By (4.5) and (4.13), we can write

$$\|\Psi\tilde{x} - \Psi x\|_E \leq QN\, \|\tilde{x} - x\|_E\,, \tag{4.15}$$

with $Q \geq Q_1 + Q_2$. Combining (4.1) with (4.15), we conclude that Ψ is contractive on E. As a consequence of Banach fixed point theorem, we deduce that Ψ has a unique fixed point which is a solution of our problem in E. □

Corollary 4.2. *Let the assumptions of the Theorem 4.1 be fulfilled. If we suppose that* (4.1) *holds, with Q is defined as*

$$Q = \frac{1}{\Gamma(1+\alpha)} \left[M_\rho + \frac{1}{\Gamma(2-\alpha)} \left(M_\sigma + \lambda \frac{\beta+1}{|\Delta|} e\right)\right], \tag{4.16}$$

then, problem (3.14) *has a unique solution in E.*

Let $B_r \subset E$ be bounded, i.e., there exists a positive constant $r > 0$ such that $\|x\|_E < r$ for all $x \in B_r$. then B_r is a closed ball in the Banach space E, hence it is also a Banach space. The restriction of Ψ on B_r is still a contraction by Theorem 4.1. Then, problem (3.14) has a unique solution in B_r if $\Psi(B_r) \subset B_r$.

Theorem 4.3. *Assume that $f : [1,e] \times C \times C \to C$ is a continuous function that satisfies (H1). If we suppose that* (4.1) *holds, with Q is defined in* (3.18)*, then problem* (3.14) *has a unique solution in B_r.*

Proof. Now we show that $\Psi(B_r) \subset B_r$, that is $\|\Psi x\|_E \le r$ whenever $\|x\|_E \le r$. Denoting

$$L_b = N_1 \sup_{t\in[1,e]} |x(t)| + N_2 \sup_{t\in[1,e]} \left|\mathcal{D}_1^\alpha x(t)\right| + L_0,$$

where $L_0 = \max\{|f(t,0,0| : t \in [1,e]\}$. Observe that

$$|f_x(t)| = |f_x(t) - f_0(t) + f_0(t)| \le |f_x(t) - f_0(t)| + |f_0(t)| \le L_b.$$

So, we have

$$\begin{aligned} |\phi_x(t)| &= \left|\int_1^t g(t-s)\mathcal{J}_1^\alpha [f_x - f_0 + f_0](s)\,\mathrm{d}s\right| \\ &\le \sup_{t\in[1,e]} (|f_x(t) - f_0(t)| + |f_0(t)|) \int_1^t \mathcal{J}_1^\alpha [1](s)\,\mathrm{d}s \\ &\le \frac{\rho_\alpha(t)}{\Gamma(1+\alpha)} L_b \end{aligned}$$

and

$$|H_x(\xi,\beta)| \le \frac{\beta\rho_\alpha(\xi) + \rho_\alpha(e)}{\Gamma(1+\alpha)|\Delta|} L_b. \tag{4.17}$$

Then $|\Psi x(t)| \le \frac{\rho(t)}{\Gamma(1+\alpha)} L_b$. Therefore,

$$\|\Psi x\| \le Q_1 \left(N\left(\|x\| + \left\|\mathcal{D}_1^\alpha x\right\|\right) + L_0\right), \tag{4.18}$$

where Q_1 is given by (4.6). On the other hand, we have

$$\left|\mathcal{D}_1^\alpha [\Psi x](t)\right| \le \frac{\left|\mathcal{D}_1^\alpha [\phi_x](t)\right|}{\lambda} + \frac{\beta|\phi_x(\xi)| + |\phi_x(e)|}{|\Delta|} \left|\mathcal{D}_1^\alpha (g(t-1))\right|. \tag{4.19}$$

Thanks to (H1), it yields that

$$\begin{aligned} \left|\mathcal{D}_1^\alpha [\phi_x](t)\right| &\le \left|\int_1^t \mathcal{D}_1^\alpha [g(t-1)](s)\mathcal{J}_1^\alpha [f_x - f_0 + f_0](t-s+1)\,\mathrm{d}s\right| \\ &\le L_b\left[\int_1^t [\mathcal{D}_1^\alpha [g(t-1)](s)]\mathcal{J}_1^\alpha [1](t-s+1)\,\mathrm{d}s\right]. \end{aligned}$$

This gives

$$\left|\mathcal{D}_1^\alpha [\phi_x](t)\right| \le \frac{\lambda\sigma_\alpha(t)}{\Gamma(1+\alpha)\Gamma(2-\alpha)} L_b, \tag{4.20}$$

where $\sigma_\alpha(t)$ is given by (3.16). Consequently, by (4.17), (4.19) and (4.20), we have

$$\left\|\mathcal{D}_1^\alpha[\Psi x]\right\| \leq Q_2\left(N\left(\|x\| + \left\|\mathcal{D}_1^\alpha x\right\|\right) + L_0\right), \tag{4.21}$$

where Q_2 is given by (4.14). Using (4.18) and (4.21), we obtain

$$\|\Psi x\|_E \leq Q\left(N\left(\|x\| + \left\|\mathcal{D}_1^\alpha x\right\|\right) + L_0\right),$$

and we find that $\|\Psi x\|_E \leq Q(Nr + L_0) \leq r$, where we choose $r \geq L_0\left(Q^{-1} - N\right)^{-1}$. Hence, the operator Ψ maps bounded sets into bounded sets in B_r, therefore Ψ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. □

Corollary 4.4. *Assume that $f : [1,e] \times C \times C \to C$ is a continuous function that satisfies (H1). If we suppose $N = \max\{N_1, N_2\}$ and*

$$L_0\left(Q^{-1} - N\right)^{-1} < r,$$

where Q is defined in (4.16)*. Then, problem* (3.14) *has a unique solution in B_r.*

Our second result will use the Scheafer fixed point theorem.

Theorem 4.5. *The problem* (3.14) *has at least one solution defined on E, whenever assumption (H2) be hold.*

Proof. The proof will be given in several steps.
Step 1: We show that Ψ is continuous. Let us consider a sequence $\{x_n\} \in E$ converging to x. For each $t \in [1,e]$, we have

$$|\Psi x_n(t) - \Psi x(t)| \leq \frac{1}{\lambda}\left|\phi_{x_n}(t) - \phi_x(t)\right| + \left|H_{x_n}(\xi,\beta) - H_x(\xi,\beta)\right| |g(t-1)|,$$

where

$$\begin{aligned} \left|\phi_{x_n}(t) - \phi_x(t)\right| &= \left|\int_1^t g(t-s)\mathcal{J}_1^\alpha[f_{x_n} - f_x](s)\,\mathrm{d}s\right| \\ &\leq \frac{\rho_\alpha(t)}{\Gamma(1+\alpha)}\left|f_{x_n}(t) - f_x(t)\right|. \end{aligned} \tag{4.22}$$

Similarly, we can obtain

$$\left|H_{x_n}(\xi,\beta) - H_x(\xi,\beta)\right| \leq \frac{\beta\rho_\alpha(\xi) + \rho_\alpha(e)}{|\Delta|\Gamma(1+\alpha)}\left|f_{x_n}(t) - f_x(t)\right|. \tag{4.23}$$

Thus, from (4.19), (4.22) and (4.23), we have

$$|\Psi x_n(t) - \Psi x(t)| \leq \frac{\rho(t)}{\Gamma(1+\alpha)}\left|f_{x_n}(t) - f_x(t)\right|. \tag{4.24}$$

If $(t,x) \in [1,e] \times E$, $x_n \to x$ as $n \to \infty$ and f is continuous, then (4.24) gives

$$\|\Psi x_n - \Psi x\| \to 0, \tag{4.25}$$

as $n \to \infty$. On the other hand, from (4.11) we observe that

$$\left|\mathcal{D}_1^\alpha [\Psi x_n](t) - \mathcal{D}_1^\alpha [\Psi x](t)\right| \le \frac{1}{\Gamma(1+\alpha)}\left(\frac{\sigma_\alpha(t)}{\Gamma(2-\alpha)} + R_{12}\right)\left|f_{x_n}(\tau) - f_x(\tau)\right|,$$

where R_{12} is given by (4.12). Thus

$$\left\|\mathcal{D}_1^\alpha [\Psi x_n] - \mathcal{D}_1^\alpha [\Psi x]\right\| \to 0, \tag{4.26}$$

as $n \to \infty$. Since the convergence of a sequence implies its boundedness, therefore, there exists $r > 0$ such that $\|x_n\| \le r, \|x\| \le r$ and hence f is uniformly continuous on the compact set

$$\left\{(t, x(t), \mathcal{D}_1^\alpha [x](t)) : t \in [1, e], \|x\| \le r_1, \left\|\mathcal{D}_1^\alpha [x]\right\| \le r_2\right\}.$$

By (4.25) and (4.26), we can write $\|[\Psi x_n] - [\Psi x]\|_E \to 0$ as $n \to \infty$. This shows that Ψ is continuous.
Step 2: Now we show that the operator $\Psi : E \to E$ maps bounded sets into bounded sets in E. Let $B_r \subset E$ be bounded, i.e., there exists a positive constant $r > 0$ such that $\|x\|_E \le r$ for all $x \in B_r$. Let

$$L = \max\left\{\left|f(t, x(t), \mathcal{D}_1^\alpha [x](t))\right| : t \in [1, e], 0 < \|x\| \le r, \left\|\mathcal{D}_1^\alpha [x]\right\| \le r\right\},$$

then, for $x \in B_r$, we have

$$|\phi_x(t)| = \left|\int_1^t g(t-s)\mathcal{J}_1^\alpha [f_x](s)\,\mathrm{d}s\right| \le \frac{\rho_\alpha(t)}{\Gamma(1+\alpha)}L \tag{4.27}$$

and

$$|H_x(\xi, \beta)| \le \frac{\beta\rho_\alpha(\xi) + \rho_\alpha(e)}{|\Delta|\Gamma(1+\alpha)}L. \tag{4.28}$$

Then from (4.27) and (4.28), we get $|\Psi x(t)| \le \frac{\rho(t)}{\Gamma(1+\alpha)}L$. Therefore,

$$\|\Psi x\| \le Q_1 L. \tag{4.29}$$

According to Property 2.1, we should have

$$\begin{aligned}\left|\mathcal{D}_1^\alpha [\phi_x](t)\right| &= \left|\int_1^t \mathcal{D}_1^\alpha [g(t-1)](s)\mathcal{J}_1^\alpha [f_x](t-s+1)\,\mathrm{d}s\right| \\ &\le \frac{\lambda\sigma_\alpha(t)}{\Gamma(1+\alpha)\Gamma(2-\alpha)}L.\end{aligned} \tag{4.30}$$

Consequently, by (3.12), (4.28) and (4.30), we have

$$\left\|\mathcal{D}_1^\alpha \Psi x\right\| \le Q_2 L. \tag{4.31}$$

Using (4.29) and (4.31), we obtain $\|\Psi x\|_E \le QL$. Hence, the operator Ψ maps bounded sets into bounded sets in E. Next we show that Ψ maps bounded sets into equicontinuous sets of B_r.
Step 3: In this step, we show that $\Psi(B_r)$ is equicontinuity. Let $t_1, t_2 \in [1, e]$ such that $t_1 < t_2$. Then we obtain

$$|\Psi x(t_2) - \Psi x(t_1)| \le \frac{1}{\lambda}|\phi_x(t_2) - \phi_x(t_1)| + |H_x(\xi, \beta)|\,|g(t_2-1) - g(t_1-1)|. \tag{4.32}$$

We can show that

$$|\phi_x(t_2)-\phi_x(t_1)| = \left|\int_1^{t_2} g(t_2-s)\mathcal{J}_1^\alpha[f_x](s)\mathrm{d}s - \int_1^{t_1} g(t_1-s)\mathcal{J}_1^\alpha[f_x](s)\mathrm{d}s\right| \tag{4.33}$$

$$\begin{aligned}
&\le \int_1^{t_1} |g(t_2-s)-g(t_1-s)|\left|\mathcal{J}_1^\alpha[f_x](s)\right|\mathrm{d}s + \int_{t_1}^{t_2} |g(t_2-s)|\left|\mathcal{J}_1^\alpha[f_x](s)\right|\mathrm{d}s\\
&\le L\int_1^{t_1} |g(t_2-s)-g(t_1-s)|\mathcal{J}_1^\alpha[1](s)\,\mathrm{d}s + L\int_{t_1}^{t_2} |g(t_2-s)|\mathcal{J}_1^\alpha[1](s)\,\mathrm{d}s\\
&\le L\int_1^{t_1}\left|\lambda\int_{t_1}^{t_2}\cos\lambda(\tau-s)\,\mathrm{d}\tau\right|\mathcal{J}_1^\alpha[1](s)\,\mathrm{d}s + L\int_{t_1}^{t_2}\mathcal{J}_1^\alpha[1](s)\,\mathrm{d}s\\
&\le \frac{L}{\Gamma(1+\alpha)}\left[\lambda|t_2-t_1|\int_1^{t_1}(\ln s)^\alpha\,\mathrm{d}s + \int_{t_1}^{t_2}(\ln s)^\alpha\,\mathrm{d}s\right].
\end{aligned}$$

Hence

$$|\phi_x(t_2)-\phi_x(t_1)| \le \frac{L}{\Gamma(2+\alpha)}\left[\lambda|t_2-t_1|\left|t_1^{\alpha+1}-1\right| + \left|t_2^{\alpha+1}-t_1^{\alpha+1}\right|\right]. \tag{4.34}$$

It is easy to find that

$$|g(t_2-1)-g(t_1-1)| = \left|\lambda\int_{t_1}^{t_2}\cos\lambda(\tau-1)\,\mathrm{d}\tau\right| \le \lambda|t_2-t_1|.$$

Therefore by (4.28), (4.32), (4.33) and (4.34) we have

$$\begin{aligned}
|\Psi x(t_2)-\Psi x(t_1)| &\le \frac{1}{\lambda}|\phi_x(t_2)-\phi_x(t_1)| + |H_x(\xi,\beta)||g(t_2-1)-g(t_1-1)| \qquad (4.35)\\
&\le \frac{L}{\Gamma(2+\alpha)}\left[|t_2-t_1|\left|t_1^{\alpha+1}-1\right| + \frac{1}{\lambda}\left|t_2^{\alpha+1}-t_1^{\alpha+1}\right|\right]\\
&\quad + \lambda L\frac{\beta\rho_\alpha(\xi)+\rho_\alpha(e)}{|\Delta|\Gamma(1+\alpha)}|t_2-t_1|.
\end{aligned}$$

We have also,

$$\begin{aligned}
\left|\mathcal{D}_1^\alpha[\Psi x](t_2)-\mathcal{D}_1^\alpha[\Psi x](t_1)\right| &\le \frac{1}{\lambda}\left|\mathcal{D}_1^\alpha[\phi_x](t_2)-\mathcal{D}_1^\alpha[\phi_x](t_1)\right| \qquad (4.36)\\
&\quad + |H_x(\xi,\beta)|\left|\mathcal{D}_1^\alpha[g(t_2-1)]-\mathcal{D}_1^\alpha[g(t_1-1)]\right|.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\left|\mathcal{D}_1^\alpha[\phi_x](t_2)-\mathcal{D}_1^\alpha[\phi_x](t_1)\right| &\le \left|\int_1^{t_2}\mathcal{D}_1^\alpha[g(\tau-1)](s)\mathcal{J}_1^\alpha[f_x](t_2-s+1)\,\mathrm{d}s\right.\\
&\quad \left.-\int_1^{t_1}\mathcal{D}_1^\alpha[g(\tau-1)](s)\mathcal{J}_1^\alpha[f_x](t_1-s+1)\,\mathrm{d}s\right|\\
&\le \int_1^{t_1}\left|\mathcal{D}_1^\alpha[g(\tau-1)](s)\right|\left|\mathcal{J}_1^\alpha[f_x](t_2-s+1)-\mathcal{J}_1^\alpha[f_x](t_1-s+1)\right|\mathrm{d}s\\
&\quad + \int_{t_1}^{t_2}\left|\mathcal{D}_1^\alpha[g(\tau-1)](s)\right|\mathcal{J}_1^\alpha[f_x](t_2-s+1)\,\mathrm{d}s.
\end{aligned}$$

We find that

$$\begin{aligned}
&\left|\mathcal{J}_1^{\alpha}[f_x]\ (t_2-s+1)-\mathcal{J}_1^{\alpha}[f_x](t_1-s+1)\right| \\
&\quad \le L\left|\mathcal{J}_1^{\alpha}[1](t_2-s+1)-\mathcal{J}_1^{\alpha}[1](t_1-s+1)\right| \\
&\quad = \frac{L}{\Gamma(\alpha)}\left|\int_1^{t_2-s+1}\left(\ln\frac{t_2-s+1}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau}-\int_1^{t_1-s+1}\left(\ln\frac{t_1-s+1}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau}\right| \\
&\quad = \frac{L}{\Gamma(\alpha)}\int_1^{t_1-s+1}\left|\left(\ln\frac{t_2-s+1}{\tau}\right)^{\alpha-1}-\left(\ln\frac{t_1-s+1}{\tau}\right)^{\alpha-1}\right|\frac{d\tau}{\tau} \\
&\qquad + \frac{L}{\Gamma(\alpha)}\int_{t_1-s+1}^{t_2-s+1}\left(\ln\frac{t_2-s+1}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} \\
&\quad \le \frac{L}{\Gamma(1+\alpha)}\left[2\left(\ln\frac{t_2-s+1}{t_1-s+1}\right)^{\alpha}+(\ln t_2-s+1)^{\alpha}-(\ln t_1-s+1)^{\alpha}\right].
\end{aligned} \tag{4.37}$$

Note that

$$\left|\mathcal{J}_1^{\alpha}[f_x](t_2-s+1)-\mathcal{J}_1^{\alpha}[f_x](t_1-s+1)\right|,$$

is independent of x. Therefore

$$\begin{aligned}
\left|\mathcal{D}_1^{\alpha}[\phi_x](t_2)-\mathcal{D}_1^{\alpha}[\phi_x](t_1)\right| \le \frac{L}{\Gamma(1+\alpha)}\int_1^{t_1}&\left[2\left(\ln\frac{t_2-s+1}{t_1-s+1}\right)^{\alpha}\right. \\
&\left.+(\ln(t_2-s+1))^{\alpha}-(\ln(t_1-s+1))^{\alpha}\right]ds \\
&+\frac{\lambda L(\sigma_\alpha(t_2)-\sigma_\alpha(t_1))}{\Gamma(2-\alpha)\Gamma(1+\alpha)}.
\end{aligned} \tag{4.38}$$

In accordance with (4.35), (4.36), (4.37) and (4.38), we deduce that

$$\|\Psi x(t_2)-\Psi x(t_1)\|+\left\|\mathcal{D}_1^{\alpha}[\Psi x](t_2)-\mathcal{D}_1^{\alpha}[\Psi x](t_1)\right\|\to 0,$$

as $|t_2-t_1|\to 0$. Hence the sets of functions $\{\Psi x(t): x\in B_r\}$ and

$$\left\{\mathcal{D}_1^{\alpha}[\Psi x](t): x\in B_r\right\},$$

are bounded in B_r and equicontinuous on $[1,e]$. Thus, by the Arzelá–Ascoli Theorem, the mapping Ψ is completely continuous on E.

Step 4: In the last step, it remains to show that the set defined by

$$\Lambda=\left\{x\in E: x=\mu\Psi x \text{ for some } 0<\mu<1\right\},$$

is bounded. Let x be a solution. Then, for $t\in[1,e]$ and using the computations in proving that Ψ is bounded, we have $|x(t)|=|\mu(\Psi x)(t)|$. Let $x\in\Lambda$, $x=\mu\Psi x$ for some $0<\mu<1$. Thus, by (4.29), for each $t\in[1,e]$, we have

$$\|x\|=\mu\|\Psi x\|\le\|\Psi x\|\le Q_1L, \tag{4.39}$$

for $\mu\in(0,1)$. On the other hand, by (4.31), we have

$$\left\|\mathcal{D}_1^{\alpha}[x]\right\|=\mu\left\|\mathcal{D}_1^{\alpha}[\Psi x]\right\|\le Q_2L. \tag{4.40}$$

It follows from (4.39) and (4.40) that $\|x\|_E = \|x\| + \left\|\mathcal{D}_1^\alpha [x]\right\| \le QL < \infty$, where Q defined by (3.18). This implies that the set Λ is bounded independently of $\mu \in (0,1)$. Therefore, Λ is bounded. As a conclusion of Schaefer fixed point theorem, we deduce that Ψ has at least one fixed point, which is a solution of (3.14). The proof is completed. □

Remark 4.6. *Let $\varepsilon > 0$ and choose a number $\eta > 0$ such that*

$$\varepsilon > \frac{2L}{\lambda\Gamma(2+\alpha)} \min\left\{\kappa_1(\eta), \kappa_2(\eta)\right\} + 2\lambda L \frac{\beta\rho_\alpha(\xi) + \rho_\alpha(e)}{|\Delta|\Gamma(1+\alpha)}\eta, \tag{4.41}$$

where

$$\kappa_1(\eta) = \lambda\eta\left(e^{\alpha+1} - 1\right) + e(\alpha+1)\eta^\alpha,$$
$$\kappa_2(\eta) = \lambda\eta\left(\eta^{\alpha+1} - 1\right) + \left((2\eta)^{\alpha+1} - 1\right).$$

Assume $t_1, t_2 \in [1, e]$; $t_1 < t_2$ such that $t_2 - t_1 \le \eta$. It obvious that $t_2 > \eta$ and there are two possibilities for t_1 and η.
Case 1: *For $\eta \le t_1 < t_2 \le e$, by means of mean value theorem of differentiation implies that there exists $t \in (t_1, t_2)$, such that*

$$t_2^{\alpha+1} - t_1^{\alpha+1} = (\alpha+1)(t_2 - t_1)t^\alpha \le \eta(\alpha+1)tt^{\alpha-1} \le e(\alpha+1)\eta^\alpha,$$

whence, we obtain

$$\begin{aligned}|\phi_x(t_2) - \phi_x(t_1)| &\le \frac{L}{\Gamma(2+\alpha)}\left[\lambda\eta\left(e^{\alpha+1} - 1\right) + e(\alpha+1)\eta^\alpha\right] \\ &= \frac{L}{\Gamma(2+\alpha)}\kappa_1(\eta).\end{aligned}$$

Case 2: *For $1 \le t_1 < \eta < t_2 \le e$ and so $t_2 < 2\eta$. These imply that*

$$\begin{aligned}|\phi_x(t_2) - \phi_x(t_1)| &\le \frac{L}{\Gamma(2+\alpha)}\left[\lambda\eta\left(\eta^{\alpha+1} - 1\right) + \left((2\eta)^{\alpha+1} - 1\right)\right] \\ &= \frac{L}{\Gamma(2+\alpha)}\kappa_2(\eta).\end{aligned}$$

Combining Case 1 and 2, we obtain

$$|\phi_x(t_2) - \phi_x(t_1)| \le \frac{L}{\Gamma(2+\alpha)} \min\left\{\kappa_1(\eta), \kappa_2(\eta)\right\}.$$

Now, it is obvious by (4.35) *and* (4.41) *that $|\Psi x(t_2) - \Psi x(t_1)| \le \frac{\varepsilon}{2}$. A similar argument can be applied to obtain*

$$\left|\mathcal{D}_1^\alpha[\Psi x](t_2) - \mathcal{D}_1^\alpha[\Psi x](t_1)\right| \le \frac{\varepsilon}{2}.$$

Corollary 4.7. *Suppose that the conditions of Theorem 4.5 hold, with Q is defined as* (4.16)*. Then, problem* (3.14) *has at least one solution defined on* $[1, e]$.

5. Results of stability

For the study of Hyers-Ulam-Rassias and generalized Ulam-Hyers-Rassias stabilities of problem (1.1) on a compact interval $[1, e]$, we adopt the following definitions [22, 35, 49, 58, 59].

Definition 5.1. *Problem* (1.1) *is Ulam-Hyers-Rassias stable with respect to* $\varphi \in C([1,e], \mathbb{R}^+)$ *if for each* $\epsilon > 0$ *and each solution* $\tilde{x}$ *of the inequality*

$$\left|\mathcal{D}_1^\alpha\left(\mathrm{D}^2+\lambda^2\right)\tilde{x}(t) - f(t, \tilde{x}(t), \mathcal{D}_1^\alpha[\tilde{x}](t))\right| \leq \epsilon\varphi(t), \tag{5.1}$$

for $t \in [1, e]$*, there exists a real number* $c_\varphi > 0$ *and a solution* x *of problem* (1.1) *such that*

$$|\tilde{x}(t) - x(t)| \leq \epsilon c_\varphi \varphi(t),$$

for $t \in [1, e]$*. Particularly, in the case that* φ *is identity function on* $[1, e]$*, problem* (1.1) *is called Ulam-Hyers stable. Moreover, if there exists* $\psi \in C(\mathbb{R}^+, \mathbb{R}^+), \psi(0) = 0$*, such that* $|\tilde{x}(t) - x(t)| \leq \psi(\epsilon)$*, for* $t \in [1, e]$*, then problem* (1.1) *is called generalized Ulam-Hyers stable.*

Definition 5.2. *Problem* (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to a function* $\varphi \in C([1,e], \mathbb{R}^+)$ *if for each solution* $\tilde{x}$ *of the inequality*

$$\left|\mathcal{D}_1^\alpha\left(\mathrm{D}^2+\lambda^2\right)\tilde{x}(t) - f(t, \tilde{x}(t), \mathcal{D}_1^\alpha[\tilde{x}](t))\right| \leq \varphi(t), \tag{5.2}$$

for $t \in [1, e]$*, there exist a real number* $c_\varphi > 0$ *and a solution* x *of problem* (1.1) *such that* $|\tilde{x}(t) - x(t)| \leq c_\varphi\varphi(t)$*, for* $t \in [1, e]$*.*

Remark 5.3. *A function* $\tilde{x} \in C$ *is a solution of the inequality* (5.1) *if and only if there exists a function* $h \in C$ *(which depends on* $\tilde{x}$*) such that for all* $t \in [1, e]$*,*

(i) $|h(t)| \leq \epsilon\varphi(t)$*.*
(ii) $\mathcal{D}_1^\alpha\left(\mathrm{D}^2+\lambda^2\right)\tilde{x}(t) = f(t, \tilde{x}(t), \mathcal{D}_1^\alpha[\tilde{x}](t)) + h(t)$*.*

Lemma 5.4. *Let* $0 < \alpha < 1$*, if* $\tilde{x} \in C$ *is a solution of the inequality* (5.1)*(or* (5.2)*) then* $\tilde{x}$ *is a solution of the following integral inequality*

$$\left|\tilde{x}(t) - \frac{1}{\lambda}\phi_{\tilde{x}}(t) - H_{\tilde{x}}(\xi, \beta)\, g(t-1)\right| \leq \epsilon\omega(t), \tag{5.3}$$

for $t \in [1, e]$*, where*

$$\tilde{x}(1) = 0 = \mathrm{D}^2\tilde{x}(1), \qquad \tilde{x}(e) = \beta\tilde{x}(\xi), \tag{5.4}$$

for $\xi \in (1, e]$ *and*

$$\omega(t) = \frac{1}{\lambda}\int_1^t \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s + \frac{\beta}{|\Delta|}\int_1^\xi \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s + \frac{1}{|\Delta|}\int_1^e \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s.$$

Proof. By using Remark 5.3-ii, we have

$$\mathcal{D}_1^\alpha\left(\mathrm{D}^2+\lambda^2\right)\tilde{x}(t) = f(t, \tilde{x}(t), \mathcal{D}_1^\alpha[\tilde{x}](t)) + h(t).$$

In accordance with Lemma 3.4, we deduce $\tilde{x}(t) = \frac{1}{\lambda}\phi_{\tilde{x},h}(t) + H_{\tilde{x},h}(\xi,\beta)\, g(t-1)$, where

$$\phi_{\tilde{x},h}(t) = \int_1^t g(t-s)\, \mathcal{J}_1^\alpha [f_{\tilde{x}} + h](s)\, \mathrm{d}s$$

and $H_{\tilde{x},h}(\xi,\beta) = \frac{1}{\Delta}\left(\beta\phi_{\tilde{x},h}(\xi) - \phi_{\tilde{x},h}(e)\right)$. Hence

$$\begin{aligned}\tilde{x}(t) = {} & \frac{1}{\lambda}\phi_{\tilde{x}}(t) + H_{\tilde{x}}(\xi,\beta)\, g(t-1) + \frac{1}{\lambda}\int_1^t g(t-s)\, \mathcal{J}_1^\alpha [h](s)\, \mathrm{d}s \\ & + g(t-1)\frac{\beta}{\Delta}\int_1^\xi g(\xi-s)\, \mathcal{J}_1^\alpha [h](s)\, \mathrm{d}s - \frac{1}{\Delta} g(t-1)\int_1^e g(e-s)\, \mathcal{J}_1^\alpha [h](s)\, \mathrm{d}s.\end{aligned}$$

Accordingly, we easily deduce equation (5.3). □

By virtue of Remark 5.3-i, it can be easily seen that

$$\begin{aligned}\left|\phi_{\tilde{x},h}(t) - \phi_x(t)\right| & \le \left|\phi_{\tilde{x}}(t) - \phi_x(t)\right| + \left|\int_1^t g(t-s)\, \mathcal{J}_1^\alpha [h](s)\, \mathrm{d}s\right| \\ & \le \left|\phi_{\tilde{x}}(t) - \phi_x(t)\right| + \epsilon\int_1^t \mathcal{J}_1^\alpha [\varphi](s)\, \mathrm{d}s.\end{aligned}$$

Similar arguments can be applied as in (4.3) to deduce that

$$\left|\phi_{\tilde{x}}(t) - \phi_x(t)\right| \le \frac{N\rho_\alpha(t)}{\Gamma(1+\alpha)}\|\tilde{x} - x\|_E .$$

Now, it is obvious that

$$\left|\phi_{\tilde{x},h}(t) - \phi_x(t)\right| \le \frac{N\rho_\alpha(t)}{\Gamma(1+\alpha)}\|\tilde{x} - x\|_E + \epsilon\int_1^t \mathcal{J}_1^\alpha [\varphi](s)\, \mathrm{d}s, \tag{5.5}$$

and

$$\begin{aligned}\left|H_{\tilde{x},h}(\xi,\beta) - H_x(\xi,\beta)\right| & \le \frac{1}{|\Delta|}\left(\beta\left|\phi_{\tilde{x},h}(\xi) - \phi_x(\xi)\right| + \left|\phi_x(e) - \phi_{\tilde{x},h}(e)\right|\right) \\ & \le \frac{N}{|\Delta|\,\Gamma(1+\alpha)}\left(\beta\rho_\alpha(\xi) + \rho_\alpha(e)\right)\|\tilde{x} - x\|_E \\ & \quad + \frac{\epsilon}{|\Delta|}\left(\beta\int_1^\xi \mathcal{J}_1^\alpha [\varphi](s)\, \mathrm{d}s + \int_1^e \mathcal{J}_1^\alpha [\varphi](s)\, \mathrm{d}s\right).\end{aligned} \tag{5.6}$$

Theorem 5.5. *Assume that the conditions of Theorem 4.1 and* (5.1)*,* (5.4) *hold. Then, problem* (1.1) *is Ulam-Hyers-Rassias stable with respect to positive constant functions. Particularly, problem* (1.1) *is Ulam-Hyers stable and generalized Ulam-Hyers stable.*

Proof. Using Theorem 4.1, there exists a unique solution $x \in C$ of problem (1.1) that is given by integral equation (3.11). Let $\tilde{x} \in C$ be any solution of the inequality (5.1), then by (5.5) and (5.6), we have

$$|\tilde{x}(t) - x(t)| = \left|\frac{1}{\lambda}\phi_{\tilde{x},h}(t) + H_{\tilde{x},h}(\xi,\beta)\, g(t-1) - \frac{1}{\lambda}\phi_x(t) - H_x(\xi,\beta)\, g(t-1)\right|$$

$$\leq \frac{1}{\lambda}\left|\phi_{\tilde{x},h}(t)-\phi_{x}(t)\right|+\left|H_{\tilde{x},h}(\xi,\beta)-H_{x}(\xi,\beta)\right||g(t-1)|$$
$$\leq \frac{N\rho_{\alpha}(t)}{\Gamma(1+\alpha)}\|\tilde{x}-x\|_{E}+\epsilon\Big[\frac{1}{\lambda}\int_{1}^{t}\mathcal{J}_{1}^{\alpha}[\varphi](s)\,\mathrm{d}s$$
$$+\frac{\beta}{|\Delta|}\int_{1}^{\xi}\mathcal{J}_{1}^{\alpha}[\varphi](s)\,\mathrm{d}s+\frac{1}{|\Delta|}\int_{1}^{e}\mathcal{J}_{1}^{\alpha}[\varphi](s)\,\mathrm{d}s\Big],$$

where $x(t)$ and $\rho(t)$ are as in (3.8) and (3.15). Thus

$$\|\tilde{x}-x\|_{E}\leq \frac{\epsilon\Gamma(1+\alpha)}{\Gamma(1+\alpha)-NM_{\rho}}\Big[\frac{1}{\lambda}\int_{1}^{e}\mathcal{J}_{1}^{\alpha}[\varphi](s)\,\mathrm{d}s$$
$$+\frac{\beta}{|\Delta|}\int_{1}^{\xi}\mathcal{J}_{1}^{\alpha}[\varphi](s)\,\mathrm{d}s+\frac{1}{|\Delta|}\int_{1}^{e}\mathcal{J}_{1}^{\alpha}[\varphi](s)\,\mathrm{d}s\Big],$$

since by (3.18), we get $0<\frac{NM_{\rho}}{\Gamma(1+\alpha)}<1$. Then, for each $t\in[1,e]$,

$$|\tilde{x}(t)-x(t)|\leq\epsilon\Big[\frac{1}{\lambda}\int_{1}^{t}\mathcal{J}_{1}^{\alpha}[\varphi](s)\,\mathrm{d}s+\frac{\beta}{|\Delta|}\left(1+\frac{N\rho_{\alpha}(t)}{\Gamma(1+\alpha)-NM_{\rho}}\right)\int_{1}^{\xi}\mathcal{J}_{1}^{\alpha}[\varphi](s)\,\mathrm{d}s$$
$$+\left(\frac{1}{|\Delta|}+\frac{1}{\lambda}\frac{N\rho_{\alpha}(t)}{\Gamma(1+\alpha)-NM_{\rho}}+\frac{1}{|\Delta|}\frac{N\rho_{\alpha}(t)}{\Gamma(1+\alpha)-NM_{\rho}}\right)\int_{1}^{e}\mathcal{J}_{1}^{\alpha}[\varphi](s)\,\mathrm{d}s\Big].$$

Accordingly, to satisfy the inequality $|\tilde{x}(t)-x(t)|\leq\epsilon c_{\varphi}\varphi(t)$, we have to pose that φ is a constant function on $[1,e]$. Hence, if $\varphi(t)=c>0$, $t\in[1,e]$, then any finite positive constant

$$c_{\varphi}\geq\frac{c\rho_{\alpha}(e)}{\lambda\Gamma(1+\alpha)}+\frac{c\beta\rho_{\alpha}(\xi)}{|\Delta|\Gamma(1+\alpha)}\left[1+\frac{N\rho_{\alpha}(e)}{\Gamma(1+\alpha)-NM_{\rho}}\right]$$
$$+\frac{c\rho_{\alpha}(e)}{\Gamma(1+\alpha)}\left[\frac{1}{|\Delta|}+\frac{1}{\lambda}\frac{N\rho_{\alpha}(t)}{\Gamma(1+\alpha)-NM_{\rho}}+\frac{1}{|\Delta|}\frac{N\rho_{\alpha}(e)}{\Gamma(1+\alpha)-NM_{\rho}}\right],$$

will satisfy the problem. Thus, the fractional boundary value problem (1.1) is Ulam-Hyers-Rassias with respect to a constant function. The Ulam-Hyers stability can be obtained by putting $\varphi=1$, and hence generalized Ulam-Hyers stable with ψ as identity function. □

In the next result, we prove the (generalized) Ulam-Hyers-Rassias stability in terms of a function.

Theorem 5.6. *Assume that the conditions of Theorem 4.1 and* (5.1) *hold. Then, problem* (1.1) *is Ulam-Hyers-Rassias stable with respect to* φ *provided that*

$$\varphi(t)\geq\frac{\beta}{|\Delta|\Gamma(\alpha)}\int_{1}^{\xi}\rho_{\alpha-1}\left(\frac{\xi}{s}\right)\varphi(s)\,\mathrm{d}s+\frac{1}{|\Delta|\Gamma(\alpha)}\int_{1}^{e}\rho_{\alpha-1}\left(\frac{e}{s}\right)\varphi(s)\,\mathrm{d}s \tag{5.7}$$
$$+\frac{1}{\lambda\Gamma(\alpha)}\int_{1}^{t}\rho_{\alpha-1}\left(\frac{t}{s}\right)\varphi(s)\,\mathrm{d}s,$$

and $\sup_{t\in[1,e]}p(t)<1$, *where*

$$p(t)=\int_{1}^{e}\left(\frac{\beta N_{1}}{|\Delta|\Gamma(\alpha)}\rho_{\alpha-1}\left(\frac{\xi}{s}\right)+\frac{\beta N_{2}}{|\Delta|}+\frac{N_{1}}{|\Delta|\Gamma(\alpha)}\rho_{\alpha-1}\left(\frac{e}{s}\right)+\frac{N_{2}}{|\Delta|}\right)$$
$$\times\exp\left(\frac{N_{1}}{\lambda\Gamma(\alpha)}\int_{\alpha}^{s}\rho_{\alpha-1}\left(\frac{s}{\tau}\right)d\tau+\frac{N_{2}}{\lambda}(s-1)\right)ds.$$

Proof. Let us denote by $x \in C$ the unique solution of the problem (1.1). Let $\tilde{x} \in C$ be a solution of the inequality (5.1), with

$$\tilde{x}(1) = x(1), \ \tilde{x}(e) = x(e). \tag{5.8}$$

By modifying the estimate (5.5), we have

$$\begin{aligned} \left|\phi_{\tilde{x},h}(t) - \phi_x(t)\right| \leq N_1 \int_1^t \mathcal{J}_1^{\alpha}[|\tilde{x} - x|](s)\,\mathrm{d}s \\ + N_2 \int_1^t \mathcal{J}_1^{\alpha}\mathcal{D}_1^{\alpha}[|\tilde{x} - x|](s)\,\mathrm{d}s + \int_1^t \mathcal{J}_1^{\alpha}[h](s)\,\mathrm{d}s. \end{aligned} \tag{5.9}$$

The above inequality implies

$$\begin{aligned} \left|H_{\tilde{x},h}(\xi,\beta) - H_x(\xi,\beta)\right| &\leq \frac{1}{|\Delta|}\left[\beta\left|\phi_{\tilde{x},h}(\xi) - \phi_x(\xi)\right| + \left|\phi_{\tilde{x},h}(e) - \phi_x(e)\right|\right] \\ &\leq \frac{\beta N_1}{|\Delta|}\int_1^{\xi} \mathcal{J}_1^{\alpha}[|\tilde{x} - x|](s)\,\mathrm{d}s + \frac{\beta N_2}{|\Delta|}\int_1^{\xi} |\tilde{x}(s) - x(s)|\,\mathrm{d}s \\ &\quad + \frac{N_1}{|\Delta|}\int_1^{e} \mathcal{J}_1^{\alpha}[|\tilde{x} - x|](s)\,\mathrm{d}s + \frac{N_2}{|\Delta|}\int_1^{e} |\tilde{x}(s) - x(s)|\,\mathrm{d}s \\ &\quad + \frac{\beta}{|\Delta|}\int_1^{\xi} \mathcal{J}_1^{\alpha}[h](s)\,\mathrm{d}s + \frac{1}{|\Delta|}\int_1^{e} \mathcal{J}_1^{\alpha}[h](s)\,\mathrm{d}s. \end{aligned} \tag{5.10}$$

Taking into account (2.12), (5.9) and (5.10), lead to

$$\begin{aligned} |\tilde{x}(t) - x(t)| &\leq \frac{1}{\lambda}\left|\phi_{\tilde{x},h}(t) - \phi_x(t)\right| + \left|H_{\tilde{x},h}(\xi,\beta) - H_x(\xi,\beta)\right| |g(t-1)| \\ &\leq \frac{N_1}{\lambda}\int_1^t \mathcal{J}_1^{\alpha}[|\tilde{x} - x|](s)\,\mathrm{d}s + \frac{\beta N_1}{|\Delta|}\int_1^{\xi} \mathcal{J}_1^{\alpha}[|\tilde{x} - x|](s)\,\mathrm{d}s \\ &\quad + \frac{N_1}{|\Delta|}\int_1^{e} \mathcal{J}_1^{\alpha}[|\tilde{x} - x|](s)\,\mathrm{d}s + \frac{N_2}{\lambda}\int_1^{t} |\tilde{x}(s) - x(s)|\,\mathrm{d}s \\ &\quad + \frac{\beta N_2}{|\Delta|}\int_1^{\xi} |\tilde{x}(s) - x(s)|\,\mathrm{d}s + \frac{N_2}{|\Delta|}\int_1^{e} |\tilde{x}(s) - x(s)|\,\mathrm{d}s \\ &\quad + \frac{1}{\lambda}\int_1^{t} \mathcal{J}_1^{\alpha}[h](s)\,\mathrm{d}s + \frac{\beta}{|\Delta|}\int_1^{\xi} \mathcal{J}_1^{\alpha}[h](s)\,\mathrm{d}s + \frac{1}{|\Delta|}\int_1^{e} \mathcal{J}_1^{\alpha}[h](s)\,\mathrm{d}s \\ &\leq \int_1^t \left(\frac{N_1}{\lambda\Gamma(\alpha)}\rho_{\alpha-1}\left(\frac{t}{s}\right) + \frac{N_2}{\lambda}\right)|\tilde{x}(s) - x(s)|\,\mathrm{d}s \\ &\quad + \int_1^{\xi} \left(\frac{\beta N_1}{|\Delta|\,\Gamma(\alpha)}\rho_{\alpha-1}\left(\frac{\xi}{s}\right) + \frac{\beta N_2}{|\Delta|}\right)|\tilde{x}(s) - x(s)|\,\mathrm{d}s \\ &\quad + \int_1^{e} \left(\frac{N_1}{|\Delta|\,\Gamma(\alpha)}\rho_{\alpha-1}\left(\frac{e}{s}\right) + \frac{N_2}{|\Delta|}\right)|\tilde{x}(s) - x(s)|\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_1^{e} \left(\frac{\beta}{|\Delta|}\rho_{\alpha-1}\left(\frac{\xi}{s}\right) + \frac{1}{|\Delta|}\rho_{\alpha-1}\left(\frac{e}{s}\right) + \frac{1}{\lambda}\rho_{\alpha-1}\left(\frac{t}{s}\right)\right)h(s)\,\mathrm{d}s. \end{aligned}$$

Hence

$$|\tilde{x}(t)-x(t)| \leq k+\int_1^t \tilde{a}(t,s)\,|\tilde{x}(s)-x(s)|\,\mathrm{d}s+\int\limits_1^e \tilde{b}(t,s)\,|\tilde{x}(s)-x(s)|\,\mathrm{d}s,$$

where

$$k=\frac{1}{\Gamma(\alpha)}\int_1^e\left(\frac{\beta}{|\Delta|}\rho_{\alpha-1}\left(\frac{\xi}{s}\right)+\frac{1}{|\Delta|}\rho_{\alpha-1}\left(\frac{e}{s}\right)+\frac{1}{\lambda}\rho_{\alpha-1}\left(\frac{e}{s}\right)\right)h(s)\,\mathrm{d}s,$$

and

$$\tilde{a}(t,s)=\left(\frac{N_1}{\lambda\Gamma(\alpha)}\rho_{\alpha-1}\left(\frac{t}{s}\right)+\frac{N_2}{\lambda}\right),$$
$$\tilde{b}(t,s)=\frac{\beta N_1}{|\Delta|\,\Gamma(\alpha)}\rho_{\alpha-1}\left(\frac{\xi}{s}\right)+\frac{\beta N_2}{|\Delta|}+\frac{N_1}{|\Delta|\,\Gamma(\alpha)}\rho_{\alpha-1}\left(\frac{e}{s}\right)+\frac{N_2}{|\Delta|}.$$

In virtue of Lemma 2.8, we deduce that

$$|\tilde{x}(t)-x(t)| \leq \frac{k}{1-p(t)}\exp\left(\int_1^t a(t,s)ds\right),$$

for $t\in[1,e]$. Problem (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \geq \frac{1}{\epsilon}|h|$, φ must satisfy the inequality (5.7). In this case, we get $|\tilde{x}(t)-x(t)| \leq c_\varphi\epsilon\varphi(t)$, where

$$c_\varphi=\max_{t\in[1,e]}\left[\frac{1}{1-p(t)}\exp\left(\int_1^t\left(\frac{N_1}{\lambda\Gamma(\alpha)}\rho_{\alpha-1}\left(\frac{t}{s}\right)+\frac{N_2}{\lambda}\right)ds\right)\right].$$

This completes the proof. □

Theorem 5.7. *Assume the conditions of Theorem 4.1 and* (5.2) *hold. Then, problem* (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to φ provided for any $t\in[1,e]$ that*

$$\begin{aligned}\varphi(t) \geq{}& \frac{1}{\lambda}\int_1^t \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s+\frac{\beta}{|\Delta|}\int_1^\xi \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s+\frac{1}{|\Delta|}\int_1^e \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s \\
&+\frac{1}{\Gamma(1+\alpha)-NM_\rho}\left[N\,\|\varphi\|\,\rho(t)\left(\frac{\rho_\alpha(e)}{\lambda}+\frac{\beta\rho_\alpha(\xi)}{|\Delta|}+\frac{\rho_\alpha(e)}{|\Delta|}\right)\right] \\
&+\frac{1}{\left(\Gamma(1+\alpha)-NM_\rho\right)}\Big[\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right) \\
&-N\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)\Big]^{-1} \\
&\times N^2\,\|\varphi\|\left(\frac{\rho_\alpha(e)}{\lambda}+\frac{\beta\rho_\alpha(\xi)}{|\Delta|}+\frac{\rho_\alpha(e)}{|\Delta|}\right)\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)\rho(t) \\
&+N\rho(t)\left[\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)-N\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)\right]^{-1} \\
&\times\Big[\int_1^t s(\ln s)^{1-\alpha}\mathcal{J}_1^\alpha[\varphi](t-s+1)\,\mathrm{d}s \\
&+\frac{\lambda\beta e^{2-\alpha}}{|\Delta|}\int_1^\xi \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s+\frac{\lambda e^{2-\alpha}}{|\Delta|}\int_1^e \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s\Big].\end{aligned} \tag{5.11}$$

Proof. Let us denote by $x \in C([1,e],\mathbb{R})$ the unique solution of the problem (1.1). Let $\bar{x} \in C$ be a solution of the inequality (5.2), with (5.8). It follows

$$\begin{aligned}|\phi_{\bar{x}}(t)-\phi_x(t)| &= \left|\int_1^t g(t-s)\mathcal{J}_1^\alpha[f_{\bar{x}}-f_x](s)\,\mathrm{d}s\right| \\ &\le N_1\int_1^t \mathcal{J}_1^\alpha[|\bar{x}-x|](s)\,\mathrm{d}s + N_2\int_1^t \mathcal{J}_1^\alpha\left[\left|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right|\right](s)\,\mathrm{d}s \\ &\le N_1\int_1^t \mathcal{J}_1^\alpha[|\bar{x}-x|](s)\,\mathrm{d}s + \frac{N_2\rho_\alpha(t)}{\Gamma(1+\alpha)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\|.\end{aligned}$$

On the other hand, we have, for each $t \in [1,e]$,

$$\begin{aligned}|H_{\bar{x}}(\xi,\beta)-H_x(\xi,\beta)| &\le \frac{1}{|\Delta|}\left[\beta|\phi_{\bar{x}}(\xi)-\phi_x(\xi)| + |\phi_{\bar{x}}(e)-\phi_x(e)|\right] \\ &\le \frac{N_2(\rho_\alpha(e)+\beta\rho_\alpha(\xi))}{|\Delta|\Gamma(1+\alpha)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\| \\ &\quad + \frac{\beta N_1}{|\Delta|}\int_1^\xi \mathcal{J}_1^\alpha[|\bar{x}-x|](s)\,\mathrm{d}s + \frac{N_1}{|\Delta|}\int_1^e \mathcal{J}_1^\alpha[|\bar{x}-x|](s)\,\mathrm{d}s.\end{aligned}$$

Hence by Lemma 5.4, for each $t \in [1,e]$, we get

$$\begin{aligned}|\bar{x}(t)-x(t)| &\le \left|\bar{x}(t)-\frac{1}{\lambda}\phi_{\bar{x}}(t)-H_{\bar{x}}(\xi,\beta)g(t-1)\right| + \frac{1}{\lambda}|\phi_{\bar{x}}(t)-\phi_x(t)| \\ &\quad + |H_{\bar{x}}(\xi,\beta)-H_x(\xi,\beta)||g(t-1)| \\ &\le \omega(t) + \frac{N_1}{\lambda}\int_1^t \mathcal{J}_1^\alpha[|\bar{x}-x|](s)\,\mathrm{d}s + \frac{\beta N_1}{|\Delta|}\int_1^\xi \mathcal{J}_1^\alpha[|\bar{x}-x|](s)\,\mathrm{d}s \\ &\quad + \frac{N_1}{|\Delta|}\int_1^e \mathcal{J}_1^\alpha[|\bar{x}-x|](s)\,\mathrm{d}s + \frac{N_2\rho_\alpha(t)}{\lambda\Gamma(1+\alpha)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\| \\ &\quad + \frac{N_2(\rho_\alpha(e)+\beta\rho_\alpha(\xi))}{|\Delta|\Gamma(1+\alpha)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\| \\ &\le \omega(t) + \frac{N_1\rho(t)}{\Gamma(1+\alpha)}\|\bar{x}-x\| + \frac{N_2\rho(t)}{\Gamma(1+\alpha)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\| \\ &\le \Omega + \frac{N_1M_\rho}{\Gamma(1+\alpha)}\|\bar{x}-x\| + \frac{N_2M_\rho}{\Gamma(1+\alpha)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\|,\end{aligned}$$

where

$$\Omega = \frac{\|\varphi\|\rho_\alpha(e)}{\lambda} + \frac{\beta\|\varphi\|\rho_\alpha(\xi)}{|\Delta|} + \frac{\|\varphi\|\rho_\alpha(e)}{|\Delta|}.$$

Then (see (3.17))

$$\|\bar{x}-x\| \le \frac{\Omega\Gamma(1+\alpha)}{\Gamma(1+\alpha)-NM_\rho} + \frac{NM_\rho}{\Gamma(1+\alpha)-NM_\rho}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\|.$$

Accordingly, we get

$$|\bar{x}(t)-x(t)| \le \omega(t) + \frac{\Omega N\rho(t)}{\Gamma(1+\alpha)-NM_\rho} + \frac{N\rho(t)}{\Gamma(1+\alpha)-NM_\rho}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\|.$$

We are going now to get an estimate for $\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\|$. It is obvious that

$$\begin{aligned}\left|\mathcal{D}_1^\alpha[\bar{x}](t)-\mathcal{D}_1^\alpha[x](t)\right| \leq \frac{1}{\lambda}&\left|\mathcal{D}_1^\alpha[\phi_{\bar{x},h}](t)-\mathcal{D}_1^\alpha[\phi_x](t)\right| \\ &+\left|H_{\bar{x},h}(\xi,\beta)-H_x(\xi,\beta)\right|\left|\mathcal{D}_1^\alpha[g(t-1)]\right|,\end{aligned}$$

where

$$\begin{aligned}\left|\mathcal{D}_1^\alpha[\phi_{\bar{x},h}](t)-\mathcal{D}_1^\alpha[\phi_x](t)\right| &= \left|\int_1^t \mathcal{D}_1^\alpha[g(t-1)](s)\,\mathcal{J}_1^\alpha[f_{\bar{x}}-f_x+h](t-s+1)\,\mathrm{d}s\right| \\ &\leq N_1\int_1^t\left|\mathcal{D}_1^\alpha[g(t-1)](s)\right|\mathcal{J}_1^\alpha[|\bar{x}-x|](t-s+1)\,\mathrm{d}s \\ &\quad+N_2\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\|\int_1^t\left|\mathcal{D}_1^\alpha[g(t-1)](s)\right|\mathcal{J}_1^\alpha[1](t-s+1)\,\mathrm{d}s \\ &\quad+\int_1^t\left|\mathcal{D}_1^\alpha[g(t-1)](s)\right|\mathcal{J}_1^\alpha[|h|](t-s+1)\,\mathrm{d}s \\ &\leq \frac{\lambda N\Omega\sigma_\alpha(t)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)} \\ &\quad+\frac{\lambda N\sigma_\alpha(t)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\| \\ &\quad+\frac{\lambda}{\Gamma(2-\alpha)}\int_1^t s(\ln s)^{1-\alpha}\,\mathcal{J}_1^\alpha[|h|](t-s+1)\,\mathrm{d}s.\end{aligned}$$

Also, we get

$$\begin{aligned}\left|H_{\bar{x},h}(\xi,\beta)-H_x(\xi,\beta)\right| &\leq \frac{\beta}{|\Delta|}\left(|\phi_{\bar{x}}(\xi)-\phi_x(\xi)|+\int_1^\xi\mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s\right) \\ &\quad+\frac{1}{|\Delta|}\left(|\phi_{\bar{x}}(e)-\phi_x(e)|+\int_1^e\mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s\right) \\ &\leq \frac{\Omega N(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|\left(\Gamma(1+\alpha)-NM_\rho\right)}+\frac{N(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|\left(\Gamma(1+\alpha)-NM_\rho\right)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\| \\ &\quad+\frac{\beta}{|\Delta|}\int_1^\xi\mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s+\frac{1}{|\Delta|}\int_1^e\mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s.\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}\left|\mathcal{D}_1^\alpha[\bar{x}](t)-\mathcal{D}_1^\alpha[x](t)\right| &\leq \frac{1}{\lambda}\left|\mathcal{D}_1^\alpha[\phi_{\bar{x},h}](t)-\mathcal{D}_1^\alpha[\phi_x](t)\right| \\ &\quad+\left|H_{\bar{x},h}(\xi,\beta)-H_x(\xi,\beta)\right|\left|\mathcal{D}_1^\alpha[g(t-1)]\right| \\ &\leq \frac{N\Omega\left(\sigma_\alpha(t)+\lambda t(\ln t)^{1-\alpha}\,\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}\right)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)} \\ &\quad+\frac{N\left(\sigma_\alpha(t)+\lambda t(\ln t)^{1-\alpha}\,\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}\right)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\|\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(2-\alpha)}\int_1^t s(\ln s)^{1-\alpha}\mathcal{J}_1^\alpha[|h|](t-s+1)\,\mathrm{d}s\\
&+\frac{\lambda t}{\Gamma(2-\alpha)}(\ln t)^{1-\alpha}\frac{\beta}{|\Delta|}\int_1^\xi \mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s\\
&+\frac{\lambda t}{\Gamma(2-\alpha)}(\ln t)^{1-\alpha}\frac{1}{|\Delta|}\int_1^e \mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s\\
\le\ &\frac{N\Omega\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)}\\
&+\frac{N\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)}\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\|\\
&+\frac{1}{\Gamma(2-\alpha)}\int_1^t s(\ln s)^{1-\alpha}\mathcal{J}_1^\alpha[|h|](t-s+1)\,\mathrm{d}s\\
&+\frac{\lambda\beta t}{|\Delta|\Gamma(2-\alpha)}(\ln t)^{1-\alpha}\int_1^\xi \mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s\\
&+\frac{\lambda t}{|\Delta|\Gamma(2-\alpha)}(\ln t)^{1-\alpha}\int_1^e \mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\|\mathcal{D}_1^\alpha[\bar{x}]-\mathcal{D}_1^\alpha[x]\right\|\le\ &\frac{N\Omega\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)-N\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)}\\
&+\frac{\left(\Gamma(1+\alpha)-NM_\rho\right)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)-N\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)}\\
&\times\left(\int_1^e s(\ln s)^{1-\alpha}\mathcal{J}_1^\alpha[|h|](e-s+1)\,\mathrm{d}s\right.\\
&\left.+\frac{\lambda\beta e^{2-\alpha}}{|\Delta|}\int_1^\xi \mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s+\frac{\lambda e^{2-\alpha}}{|\Delta|}\int_1^e \mathcal{J}_1^\alpha[|h|](s)\,\mathrm{d}s\right).
\end{aligned}
$$

If $|h|\le\varphi$, we get

$$
\begin{aligned}
|\bar{x}(t)-x(t)|\le\ &\frac{1}{\lambda}\int_1^t \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s+\frac{\beta}{|\Delta|}\int_1^\xi \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s+\frac{1}{|\Delta|}\int_1^e \mathcal{J}_1^\alpha[\varphi](s)\,\mathrm{d}s\\
&+\frac{\Omega N\rho(t)}{\Gamma(1+\alpha)-NM_\rho}+\frac{N\rho(t)}{\Gamma(1+\alpha)-NM_\rho}\\
&\times\left[\frac{N\Omega\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)-N\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)}\right.\\
&\left.+\frac{\left(\Gamma(1+\alpha)-NM_\rho\right)}{\Gamma(2-\alpha)\left(\Gamma(1+\alpha)-NM_\rho\right)-N\left(M_\sigma+\lambda\frac{(\beta\rho_\alpha(\xi)+\rho_\alpha(e))}{|\Delta|}e^{2-\alpha}\right)}\right]
\end{aligned}
$$

$$\times\left[\int_1^t s(\ln s)^{1-\alpha}\mathcal{J}_1^{\alpha}[\varphi](t-s+1)\,\mathrm{d}s\right.$$
$$\left.+\frac{\lambda\beta e^{2-\alpha}}{|\Delta|}\int_1^{\xi}\mathcal{J}_1^{\alpha}[\varphi](s)\,\mathrm{d}s+\frac{\lambda e^{2-\alpha}}{|\Delta|}\int_1^{e}\mathcal{J}_1^{\alpha}[\varphi](s)\,\mathrm{d}s\right].$$

Hence by the given condition (5.11), the equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to φ. □

6. Dependence of solution on the parameters

For f Lipschitz in the second and the third variables, the solution's dependence on the order of the differential operator, the boundary values and the nonlinear term f are discussed in this section. We show that the solutions of two equations with neighbouring orders will (under suitable conditions on their right hand sides f) lie close to one another.

Theorem 6.1. *Suppose that the conditions of Theorem 4.1 hold. Let $x(t)$, $x_\epsilon(t)$ be the solutions, respectively, of problems* (1.1) *and*

$$\mathcal{D}_1^{\alpha-\epsilon}\left(\mathrm{D}^2+\lambda^2\right)x(t)=f\left(t,x(t),\mathcal{D}_1^{\alpha}[x](t)\right),\tag{6.1}$$

for $t\in(0,1)$ and $\epsilon>0$, with the boundary conditions (1.1)*-b, where $0<\alpha-\epsilon<\alpha<1$. Then there exists a constant $k_\epsilon>0$ such that*

$$\|x-x_\epsilon\|_E\le k_\epsilon\|f\|_*,\tag{6.2}$$

where $\|f\|_*=\sup_\epsilon\left\|f_{x_\epsilon}\right\|$ and $f_{x_\epsilon}(t):=f\left(t,x_\epsilon(t),\mathcal{D}_1^{\alpha}[x_\epsilon](t)\right)$.

Proof. By Lemma 3.4 and equation (3.11), we can obtain

$$x_\epsilon(t)=\frac{1}{\lambda}\phi_{x_\epsilon}(t)+H_{x_\epsilon}(\xi,\beta)\,g(t-1),$$

is the solution of (6.1) with the boundary conditions in (1.1), where

$$\phi_{x_\epsilon}(t)=\int_1^t g(t-s)\mathcal{J}_1^{\alpha-\epsilon}[f_{x_\epsilon}](s)\,\mathrm{d}s$$

and $H_{x_\epsilon}(\xi,\beta)=\frac{1}{\Delta}\left(\beta\phi_{x_\epsilon}(\xi)+\phi_{x_\epsilon}(e)\right)$. Then

$$\begin{aligned}\left|\phi_{x_\epsilon}(t)-\phi_x(t)\right|&=\left|\int_1^t g(t-s)\mathcal{J}_1^{\alpha-\epsilon}[f_{x_\epsilon}](s)\,\mathrm{d}s-\int_1^t g(t-s)\mathcal{J}_1^{\alpha}[f_x](s)\,\mathrm{d}s\right|\\&\le\left|\int_1^t g(t-s)\mathcal{J}_1^{\alpha}[f_{x_\epsilon}-f_x](s)\,\mathrm{d}s\right|\\&\quad+\left|\int_1^t g(t-s)\left[\mathcal{J}_1^{\alpha-\epsilon}[f_{x_\epsilon}](s)-\mathcal{J}_1^{\alpha}[f_{x_\epsilon}](s)\right]\mathrm{d}s\right|\\&\le\left\|f_{x_\epsilon}-f_x\right\|\int_1^t|g(t-s)|\mathcal{J}_1^{\alpha}[1](s)\,\mathrm{d}s\end{aligned}$$

$$+\left\|f_{x_\epsilon}\right\|\int_1^t |g(t-s)|\left|\mathcal{J}_1^{\alpha-\epsilon}[1](s)-\mathcal{J}_1^{\alpha}[1](s)\right|\mathrm{d}s$$
$$\leq \frac{N\|x-x_\epsilon\|_E}{\Gamma(1+\alpha)}\int_1^t(\ln s)^\alpha\,\mathrm{d}s+\left\|f_{x_\epsilon}\right\|\int_1^t\left|\frac{(\ln s)^{\alpha-\epsilon}}{\Gamma(1+\alpha-\epsilon)}-\frac{(\ln s)^{\alpha}}{\Gamma(1+\alpha)}\right|\mathrm{d}s.$$

This leads to

$$\left|\phi_{x_\epsilon}(t)-\phi_x(t)\right|\leq\frac{N}{\Gamma(1+\alpha)}\rho_\alpha(t)\|x-x_\epsilon\|_E+\varrho_\epsilon(t)\left\|f_{x_\epsilon}\right\|,$$

with

$$\varrho_\epsilon(t)=\int_1^t\left|\frac{(\ln s)^{\alpha-\epsilon}}{\Gamma(1+\alpha-\epsilon)}-\frac{(\ln s)^{\alpha}}{\Gamma(1+\alpha)}\right|\mathrm{d}s.$$

In a similar manner, we can get

$$\left|H_{x_\epsilon}(\xi,\beta)-H_x(\xi,\beta)\right|\leq\frac{N}{\Gamma(1+\alpha)}\frac{1}{|\Delta|}\left[\beta\rho_\alpha(\xi)+\rho_\alpha(e)\right]\|x-x_\epsilon\|_E$$
$$+\frac{1}{|\Delta|}\left[\beta\varrho_\epsilon(\xi)+\varrho_\epsilon(e)\right]\left\|f_{x_\epsilon}\right\|.$$

Then

$$|x(t)-x_\epsilon(t)|\leq\frac{N}{\Gamma(1+\alpha)}\rho(t)\|x-x_\epsilon\|_E+\varrho(t)\left\|f_{x_\epsilon}\right\|,\tag{6.3}$$

with $\varrho(t)=\frac{1}{\lambda}\varrho_\epsilon(t)+\frac{1}{|\Delta|}\left[\beta\varrho_\epsilon(\xi)+\varrho_\epsilon(e)\right]$. On the other hand,

$$\left|\mathcal{D}_1^\alpha[x_\epsilon](t)-\mathcal{D}_1^\alpha[x](t)\right|\leq\frac{1}{\lambda}\left|\mathcal{D}_1^\alpha[\phi_{x_\epsilon}](t)-\mathcal{D}_1^\alpha[\phi_x](t)\right|$$
$$+\left|H_{x_\epsilon}(\xi,\beta)-H_x(\xi,\beta)\right|\left|\mathcal{D}_1^\alpha[g(t-1)]\right|.$$

By (4.8), we have

$$\left|\mathcal{D}_1^\alpha[\phi_{x_\epsilon}](t)-\mathcal{D}_1^\alpha[\phi_x](t)\right|=\left|\int_1^t\mathcal{D}_1^\alpha[g(t-1)](s)\left[\mathcal{J}_1^{\alpha-\epsilon}[f_{x_\epsilon}]-\mathcal{J}_1^{\alpha}[f_x]\right](t-s+1)\,\mathrm{d}s\right|$$
$$=\left|\int_1^t\mathcal{D}_1^\alpha[g(t-1)](s)\right.$$
$$\left.\times\left[\mathcal{J}_1^{\alpha-\epsilon}[f_{x_\epsilon}]-\mathcal{J}_1^{\alpha}[f_x]+\mathcal{J}_1^{\alpha}[f_{x_\epsilon}]-\mathcal{J}_1^{\alpha}[f_{x_\epsilon}]\right](t-s+1)\,\mathrm{d}s\right|.$$

$$\left|\mathcal{D}_1^\alpha[\phi_{x_\epsilon}](t)-\mathcal{D}_1^\alpha[\phi_x](t)\right|\leq\left|\int_1^t\mathcal{D}_1^\alpha[g(t-1)](s)\,\mathcal{J}_1^{\alpha}[f_{x_\epsilon}-f_x](t-s+1)\,\mathrm{d}s\right|$$
$$+\left|\int_1^t\mathcal{D}_1^\alpha[g(t-1)](s)\left(\mathcal{J}_1^{\alpha-\epsilon}[f_{x_\epsilon}]-\mathcal{J}_1^{\alpha}[f_{x_\epsilon}]\right)(t-s+1)\,\mathrm{d}s\right|$$
$$\leq N\|x-x_\epsilon\|_E\int_1^t\left|\mathcal{D}_1^\alpha[g(t-1)](s)\right|\mathcal{J}_1^{\alpha}[1](t-s+1)\,\mathrm{d}s$$
$$+\left\|f_{x_\epsilon}\right\|\int_1^t\left|\mathcal{D}_1^\alpha[g(t-1)](s)\right|\left|\left(\mathcal{J}_1^{\alpha-\epsilon}[1]-\mathcal{J}_1^{\alpha}[1]\right)(t-s+1)\right|\mathrm{d}s.$$

Then

$$\left|\mathcal{D}_1^{\alpha}\left[\phi_{x_\epsilon}\right](t)-\mathcal{D}_1^{\alpha}\left[\phi_x\right](t)\right| \leq N\sigma_\alpha(t)\|x-x_\epsilon\|_E+\upsilon_\epsilon(t)\left\|f_{x_\epsilon}\right\|,$$

with

$$\upsilon_\epsilon(t)=\int_1^t\left|\mathcal{D}_1^{\alpha}\left[g(t-1)\right](s)\right|\left|\mathcal{J}_1^{\alpha-\epsilon}[1]-\mathcal{J}_1^{\alpha}[1](t-s+1)\right|\mathrm{d}s. \tag{6.4}$$

Then the expression above becomes

$$\begin{aligned}\left|\mathcal{D}_1^{\alpha}[x_\epsilon](t)-\mathcal{D}_1^{\alpha}[x](t)\right| &\leq \frac{1}{\lambda}N\sigma_\alpha(t)\|x-x_\epsilon\|_E+\upsilon_\epsilon(t)\left\|f_{x_\epsilon}\right\| \\ &\quad+\frac{N}{\Gamma(1+\alpha)}\frac{1}{|\Delta|}\left[\beta\rho_\alpha(\xi)+\rho_\alpha(e)\right]\|x-x_\epsilon\|_E \\ &\quad+\frac{1}{|\Delta|}\left[\beta\varrho_\epsilon(\xi)+\varrho_\epsilon(e)\right]\left\|f_{x_\epsilon}\right\|\left|\mathcal{D}_1^{\alpha}\left[g(t-1)\right]\right| \\ &\leq N\left[\frac{1}{\lambda}\sigma_\alpha(t)+\frac{1}{\Gamma(1+\alpha)}\frac{1}{|\Delta|}\left[\beta\rho_\alpha(\xi)+\rho_\alpha(e)\right]\left|\mathcal{D}_1^{\alpha}\left[g(t-1)\right]\right|\right]\|x-x_\epsilon\|_E \\ &\quad+\left[\frac{1}{\lambda}\upsilon_\epsilon(t)+\frac{1}{|\Delta|}\left[\beta\varrho_\epsilon(\xi)+\varrho_\epsilon(e)\right]\left|\mathcal{D}_1^{\alpha}\left[g(t-1)\right]\right|\right]\left\|f_{x_\epsilon}\right\|.\end{aligned}$$

Then

$$\left|\mathcal{D}_1^{\alpha}[x_\epsilon](t)-\mathcal{D}_1^{\alpha}[x](t)\right| \leq NC_{22}(t)\|x-x_\epsilon\|_E+C_{33}(t)\left\|f_{x_\epsilon}\right\|, \tag{6.5}$$

with

$$\begin{aligned}C_{22}(t)&=\left[\frac{1}{\lambda}\sigma_\alpha(t)+\frac{1}{\Gamma(1+\alpha)}\frac{1}{|\Delta|}\left[\beta\rho_\alpha(\xi)+\rho_\alpha(e)\right]\left|\mathcal{D}_1^{\alpha}\left[g(t-1)\right]\right|\right], \\ C_{33}(t)&=\left[\frac{1}{\lambda}\upsilon_\epsilon(t)+\frac{1}{|\Delta|}\left[\beta\varrho_\epsilon(\xi)+\varrho_\epsilon(e)\right]\left|\mathcal{D}_1^{\alpha}\left[g(t-1)\right]\right|\right].\end{aligned} \tag{6.6}$$

Moreover, from (6.3), (6.5), we deduce that

$$\begin{aligned}&|x(t)-x_\epsilon(t)|+\left|\mathcal{D}_1^{\alpha}[x_\epsilon](t)-\mathcal{D}_1^{\alpha}[x](t)\right| \\ &\qquad\leq N\left[\frac{1}{\Gamma(1+\alpha)}\rho(t)+C_{22}(t)\right]\|x-x_\epsilon\|_E+\left[\varrho(t)+C_{33}(t)\right]\left\|f_{x_\epsilon}\right\|.\end{aligned}$$

Finally, we get the inequality

$$\|x-x_\epsilon\|_E \leq \frac{\sup_{t\in[1,e]}\left[\varrho(t)+C_{33}(t)\right]}{1-N\sup_{t\in[1,e]}\left[\frac{1}{\Gamma(1+\alpha)}\rho(t)+C_{22}(t)\right]}\|f\|_*,$$

which is exactly the required inequality (6.2), where

$$k_\epsilon=\frac{\sup_{t\in[1,e]}\left[\varrho(t)+C_{33}(t)\right]}{1-N\sup_{t\in[1,e]}\left[\frac{1}{\Gamma(1+\alpha)}\rho(t)+C_{22}(t)\right]}. \tag{6.7}$$

□

Theorem 6.2. *Suppose that the conditions of Theorem 4.1 hold. Let $x(t), x_\epsilon(t)$ be the solutions, respectively, of the problems* (1.1) *and*

$$\mathcal{D}_1^\alpha\left(\mathrm{D}^2+\lambda^2\right)x(t)=f(t,x(t),\mathcal{D}_1^\alpha[x](t)+\epsilon h_\epsilon(t),$$

for $t\in(1,e)$ and $h_\epsilon\in C$, with boundary conditions (1.1)*-b, where $\epsilon<0$. Then $\|x-x_\epsilon\|_E=O(\epsilon)$.*

Proof. In accordance with Lemma 3.4, we have

$$\phi_{x_\epsilon}(t)=\int_1^t g(t-s)\mathcal{J}_1^\alpha[f_{x_\epsilon}+\epsilon h_\epsilon](s)\,\mathrm{d}s$$

and

$$\begin{aligned}\left|\phi_{x_\epsilon}(t)-\phi_x(t)\right| &= \left|\int_1^t g(t-s)\mathcal{J}_1^\alpha[f_{x_\epsilon}+\epsilon h_\epsilon](s)\,\mathrm{d}s-\int_1^t g(t-s)\mathcal{J}_1^\alpha[f_x](s)\,\mathrm{d}s\right| \\ &\le \left|\int_1^t g(t-s)\mathcal{J}_1^\alpha[f_{x_\epsilon}-f_x](s)\,\mathrm{d}s\right|+\epsilon\left|\int_1^t g(t-s)\mathcal{J}_1^\alpha[h_\epsilon](s)\,\mathrm{d}s\right| \\ &\le \frac{1}{\Gamma(1+\alpha)}\left(\left\|f_{x_\epsilon}-f_x\right\|+\epsilon\|h_\epsilon\|\right)\int_1^t(\ln s)^\alpha\,\mathrm{d}s \\ &\le \frac{N\|x-x_\epsilon\|_E+\epsilon\|h_\epsilon\|}{\Gamma(1+\alpha)}\rho_\alpha(t).\end{aligned}\tag{6.8}$$

and

$$\begin{aligned}\left|H_x(\xi,\beta)-H_{x_\epsilon}(\xi,\beta)\right| &\le \frac{1}{|\Delta|}\left[\beta\left|\phi_{x_\epsilon}(\xi)-\phi_x(\xi)\right|+\left|\phi_{x_\epsilon}(e)-\phi_x(e)\right|\right] \\ &\le \frac{1}{|\Delta|}\frac{N\|x-x_\epsilon\|_E+\epsilon\|h_\epsilon\|}{\Gamma(1+\alpha)}\left[\beta\rho_\alpha(\xi)+\rho_\alpha(e)\right].\end{aligned}\tag{6.9}$$

From (6.8) and (6.9), we derive

$$|x(t)-x_\epsilon(t)|\le\frac{\rho(t)}{\Gamma(1+\alpha)}\left(N\|x-x_\epsilon\|_E+\epsilon\|h_\epsilon\|\right).$$

On the other hand,

$$\begin{aligned}&\left|\mathcal{D}_1^\alpha[\phi_{x_\epsilon}](t)-\mathcal{D}_1^\alpha[\phi_x](t)\right| \\ &\quad=\left|\int_1^t\mathcal{D}_1^\alpha[g(t-1)](s)\left(\mathcal{J}_1^\alpha[f_{x_\epsilon}+\epsilon h_\epsilon]-\mathcal{J}_1^\alpha[f_x]\right)(t-s+1)\,\mathrm{d}s\right| \\ &\quad\le\left|\int_1^t\mathcal{D}_1^\alpha[g(t-1)](s)\mathcal{J}_1^\alpha[f_{x_\epsilon}-f_x](t-s+1)\,\mathrm{d}s\right| \\ &\qquad+\epsilon\left|\int_1^t\mathcal{D}_1^\alpha g(t-1)(s)\mathcal{J}_1^\alpha[h_\epsilon](t-s+1)\,\mathrm{d}s\right| \\ &\quad\le\left(N\|x-x_\epsilon\|_E+\epsilon\|h_\epsilon\|\right)\int_1^t\left|\mathcal{D}_1^\alpha[g(t-1)](s)\right|\mathcal{J}_1^\alpha[1](t-s+1)\,\mathrm{d}s\end{aligned}$$

and

$$\left|\mathcal{D}_1^\alpha\left[\phi_{x_\epsilon}\right](t)-\mathcal{D}_1^\alpha\left[\phi_x\right](t)\right|\leq\frac{R_{11}(t)}{\Gamma(1+\alpha)}\left(N\|x-x_\epsilon\|_E+\epsilon\|h_\epsilon\|\right),$$

where R_{11} is given by (4.9). Hence, we obtain

$$\begin{aligned}\left|\mathcal{D}_1^\alpha\left[x_\epsilon\right](t)\right. &\left.-\mathcal{D}_1^\alpha\left[x\right](t)\right|\leq\frac{R_{11}(t)}{\lambda\Gamma(1+\alpha)}\left(N\|x-x_\epsilon\|_E+\epsilon\|h_\epsilon\|\right)\\&+\frac{1}{|\Delta|}\frac{N\|x-x_\epsilon\|_E+\epsilon\|h_\epsilon\|}{\Gamma(1+\alpha)}\left[\beta\rho_\alpha(\xi)+\rho_\alpha(e)\right]\left|\mathcal{D}_1^\alpha\left[g(t-1)\right]\right|,\end{aligned}$$

and

$$\begin{aligned}&|x(t)-x_\epsilon(t)|+\left|\mathcal{D}_1^\alpha\left[x_\epsilon\right](t)-\mathcal{D}_1^\alpha\left[x\right](t)\right|\\&\leq\frac{N}{\Gamma(1+\alpha)}\left(\rho(t)+\frac{\left[\beta\rho_\alpha(\xi)+\rho_\alpha(e)\right]\left|\mathcal{D}_1^\alpha\left[g(t-1)\right]\right|}{|\Delta|}+\frac{R_{11}(t)}{\lambda}\right)\|x-x_\epsilon\|_E\\&\quad+\frac{\epsilon\|h_\epsilon\|}{\Gamma(1+\alpha)}\left(\rho(t)+\frac{\left[\beta\rho_\alpha(\xi)+\rho_\alpha(e)\right]\left|\mathcal{D}_1^\alpha\left[g(t-1)\right]\right|}{|\Delta|}+\frac{R_{11}(t)}{\lambda}\right).\end{aligned}$$

Consequently

$$\|x-x_\epsilon\|_E\leq\epsilon\frac{NQ}{1-NQ}\|h\|_*,$$

where Q is given by (3.18) and $\|h\|_*=\sup_{0<\epsilon}\|h_\epsilon\|$. It is obvious that $\|x-x_\epsilon\|_E=O(\epsilon)$. □

Let us introduce small perturbation in the boundary conditions of (1.1) such that

$$x(1)=0=\mathrm{D}^2x(1),\qquad x(e)=\beta x(\xi)+\epsilon, \tag{6.10}$$

for $\xi\in(1,e]$.

Theorem 6.3. *Assume the conditions of Theorem 4.1 hold. Let $x(t)$, $x_\epsilon(t)$ be respective solutions, of the problems* (1.1) *and the boundary conditions* (1.1)*-a with* (6.10)*. Then*

$$\|x-x_\epsilon\|_E=O(\epsilon).$$

Proof. Similar arguments as in the proof of Lemma 3.4, may lead to the solution of equations (1.1)-a and (6.10) that has the following form

$$\begin{aligned}x_\epsilon(t)=&\frac{1}{\lambda}\int_1^t\sin\lambda(t-s)\mathcal{J}_1^\alpha\left[f_{x_\epsilon}\right]\mathrm{d}s\\&+\frac{\beta}{\Delta}\sin\lambda(t-1)\int_1^\xi\sin\lambda(\xi-s)\mathcal{J}_1^\alpha\left[f_{x_\epsilon}\right]\mathrm{d}s\\&-\frac{1}{\Delta}\sin\lambda(t-1)\int_1^e\sin\lambda(e-s)\mathcal{J}_1^\alpha\left[f_{x_\epsilon}\right]\mathrm{d}s+\epsilon\frac{\lambda\sin\lambda(t-1)}{\Delta\cos\lambda}.\end{aligned}$$

Therefore

$$x_\epsilon(t)=\frac{1}{\lambda}\phi_{x_\epsilon}(t)+H_{x_\epsilon}(\xi,\beta)g(t-1)+\epsilon\frac{\lambda\sin\lambda(t-1)}{\Delta\cos\lambda},$$

and $\Delta\cos\lambda \neq 0$, where

$$\phi_{x_\epsilon}(t) = \int_1^t g(t-s)\mathcal{J}_1^\alpha[f_{x_\epsilon}](s)\,\mathrm{d}s$$

and $H_{x_\epsilon}(\xi,\beta) = \frac{1}{\Delta}(\beta\phi_{x_\epsilon}(\xi) + \phi_{x_\epsilon}(e))$. As before, we find that

$$|x(t) - x_\epsilon(t)| \leq \frac{N\rho(t)}{\Gamma(1+\alpha)}\|x - x_\epsilon\|_E + \frac{\epsilon\lambda}{|\Delta|}\left|\frac{\sin\lambda(t-1)}{\cos\lambda}\right|,$$

and

$$\begin{aligned}\left|\mathcal{D}_1^\alpha[x_\epsilon](t) - \mathcal{D}_1^\alpha[x](t)\right| \leq{} & \frac{R_{11}(t)}{\lambda\Gamma(1+\alpha)}N\|x - x_\epsilon\|_E + \epsilon\left|\frac{\lambda\mathcal{D}_1^\alpha\sin\lambda(t-1)}{\Delta\cos\lambda}\right| \\ & + \frac{1}{|\Delta|}\frac{N\|x - x_\epsilon\|_E}{\Gamma(1+\alpha)}[\beta\rho_\alpha(\xi) + \rho_\alpha(e)]\left|\mathcal{D}_1^\alpha[g(t-1)]\right|.\end{aligned}$$

Hence

$$\begin{aligned}|x(t) &- x_\epsilon(t)| + \left|\mathcal{D}_1^\alpha[x_\epsilon](t) - \mathcal{D}_1^\alpha[x](t)\right| \\ &\leq \frac{N}{\Gamma(1+\alpha)}\left[\rho(t) + \frac{[\beta\rho_\alpha(\xi) + \rho_\alpha(e)]\left|\mathcal{D}_1^\alpha[g(t-1)]\right|}{|\Delta|} + \frac{R_{11}(t)}{\lambda}\right]\|x - x_\epsilon\|_E \\ &\quad + \frac{\epsilon\lambda}{|\Delta\cos\lambda|}\left[|\sin\lambda(t-1)| + \frac{\lambda}{\Gamma(2-\alpha)}t(\ln t)^{1-\alpha}\right].\end{aligned}$$

Consequently

$$\|x - x_\epsilon\|_E \leq \frac{\epsilon\lambda}{|\Delta\cos\lambda|(1-NQ)}\left[1 + \frac{\lambda e^{2-\alpha}}{\Gamma(2-\alpha)}\right].$$

It is obvious that $\|x - x_\epsilon\|_E = O(\epsilon)$. □

7. Illustrative examples and applications

In this section, we present some examples to illustrate the validity and applicability of the main results.

Example 7.1. *Consider problem* (1.1) *with*

$$f(t,x,\tilde{x}) = \frac{1/6}{1 + |x| + |\tilde{x}|}. \tag{7.1}$$

Then f fulfills the Lipschitz condition (H1) such that $\lambda = 2$, $\beta = 2$, $\xi = \frac{3}{2}$,

$$N = \max\{N_1, N_2\} = \frac{1}{6}.$$

In Table 1, we show values of α, t and Q. Thus $NQ < 1$. Hence, by Theorem 4.1, the problem (1.1) *with* (7.1) *has a unique solution on* $[1, e]$.

Table 1. Some values of α, t and Q.

α	1	1	1	0.75	0.25	0.5	0.5	0.5
t	1	2	e	e	e	e	2	1
Q	1.38	1.35	2.94	3.58	4.58	4.15	1.90	0.20

Example 7.2. *Consider problem* (1.1) *with*

$$f(t, x, \tilde{x}) = \frac{5}{8}(\sin x + \cos x) + \tilde{x} \tag{7.2}$$

or

$$f(t, x, \tilde{x}) = \frac{1}{4}(\sin x + \cos x) + \frac{7}{6}\tilde{x}. \tag{7.3}$$

Then f fulfills the Lipschitz condition (H1), where $\lambda = 4$, $\beta = 2$, $\xi = \frac{3}{2}$, $N = \max\{N_1, N_2\} = \frac{5}{4}$ or $\frac{7}{6}$. In Table 2, we show values of α, t, Q, $\frac{5}{4}Q$ and $\frac{7}{6}Q$. Thus, the condition (4.1) *holds. Again, taking $N = \max\{N_1, N_2\} = \frac{5}{4}$ or $\frac{7}{6}$, we have $NQ < 1$. Note that all the assumptions of the Theorem 4.3 holds. Therefore problem* (1.1) *has a unique solution on E.*

Table 2. Some values of α, t, Q, $\frac{5}{4}Q$ and $\frac{7}{6}Q$.

α	1.000	1.000	0.75	0.50	0.25
t	1.030	1.002	1.03	1.03	1.03
Q	0.860	0.780	0.57	0.44	0.39
$5Q/4$	$1.008 > 1$	0.980	0.72	0.56	0.49
$7Q/6$	$0.940 < 1$	0.910	0.67	0.52	0.46

Example 7.3. *Consider problem* (1.1) *with*

$$f(t, x, \tilde{x}) = d_1(t)\sin[d_2(t)(x + \tilde{x})] + d_3(t)\cos[d_4(t)(x + \tilde{x})], \tag{7.4}$$

for $d_i \in C[1, e]$ with $i = 1, 2, 3, 4$, that fulfils (H1) with

$$N_1 = N_2 = |d_1(t)d_2(t)| + |d_3(t)d_4(t)| = 1,$$

for example $d_1(t) = d_3(t) = \frac{1}{4}$, $d_2(t) = d_4(t) = 2$. Thus, we can put $\beta = 2$, $\xi = \frac{3}{2}$, $N = 1$, $L_0 = \frac{1}{4}$. In Table 3, one can find some values of α, λ, t, Q and r, where r and Q are as defined in Theorem 4.3.

Table 3. Some values of α, λ, t, Q and r.

α	1.00	1.00	0.750	0.750	0.500	0.500	0.25	0.250
λ	1.35	7.76	7.760	9.600	7.760	9.600	7.80	9.600
t	1.00	1.50	1.500	1.000	1.500	1.000	2.00	1.000
Q	0.89	0.78	0.250	0.120	0.270	0.150	0.79	0.180
$r \geq$	2.03	0.89	0.084	0.035	0.093	0.045	0.94	0.054

Hence, by Theorem 4.3, problem (1.1) *with* (7.4) *has a unique solution on* B_r.

Example 7.4. *(Illustrative example on stability) Consider the FLE problem* (1.1) *with Hadamard fractional derivatives involving nonlocal boundary conditions:*

$$\begin{cases} \mathcal{D}_1^{\frac{3}{8}}\left(\mathrm{D}^2+\pi^2\right)x(t) = f\left(t, x(t), \mathcal{D}_1^{\frac{3}{8}}[x](t)\right), \quad t \in \left(1, \frac{15}{4}\right), \\ \mathrm{D}^2 x(1) = x(1) = 0, \\ x(\frac{15}{4}) = 2x(\frac{11}{5}), \end{cases} \tag{7.5}$$

with

$$f(t, x, \tilde{x}) = \frac{x^2}{5(t+1)^2(|x|+3)} + \frac{|x|}{5(t+1)^2} + \frac{\sin\tilde{x}}{4(t+3)^2} - 2, \tag{7.6}$$

where $\alpha = \frac{3}{8}$, $\lambda = \pi$, $\beta = 2$ *and* $\xi = \frac{11}{5}$. *It is obvious that*

$$\sin\pi\left(\frac{15}{4}-1\right) = 0.7071 \neq -1.1756 = 2\sin\pi\left(\frac{11}{5}-1\right),$$

So

$$\Delta = \lambda\left(\sin\lambda\,(e-1) - \beta\sin\lambda\,(\xi-1)\right) = 5.9146 \neq 0.$$

See the Figure 1. Also, we have

$$|f(t, x_1, \tilde{x}_1) - f(t, x_2, \tilde{x}_2)| \leq \frac{16}{135}|x_1 - x_2| + \frac{1}{49}|\tilde{x}_1 - \tilde{x}_2|,$$

for each $t \in \left[1, \frac{15}{4}\right]$ *and all* $x_i, \tilde{x}_i \in \mathbb{R}$, *here* $N_1 = \frac{16}{135}$ *and* $N_2 = \frac{1}{49}$. *Put* $\varphi(t) = \frac{t^2}{t^2+1}$ *and*

$$N = \max\left\{\frac{16}{135}, \frac{1}{49}\right\} = \frac{16}{135}.$$

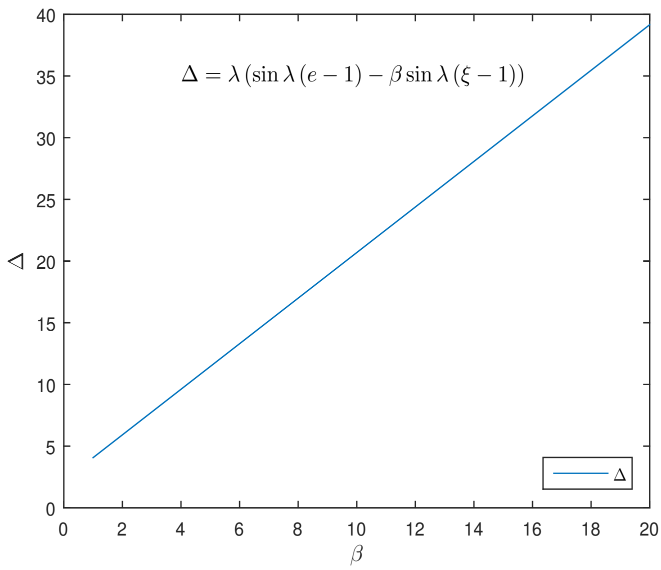


Figure 1. Numerical results of Δ where $\beta = 1, \ldots, 20$ in Example 7.4.

By using Eq (2.11)*, we obtain*

$$\rho_\alpha(t) \le \frac{1}{\frac{3}{8}+1}\left(\frac{15}{4}\right)^{\frac{3}{8}+1} = 4.4770$$

and using MatLab program,

$$\rho_\alpha\left(\frac{15}{4}\right) = \int_1^{\frac{15}{4}} (\ln s)^\alpha \, \mathrm{d}s = 2.4371,$$

$$\rho_\alpha\left(\frac{11}{5}\right) = \int_1^{\frac{11}{5}} (\ln s)^\alpha \, \mathrm{d}s = 0.8463.$$

Also, by applying Eq (3.17)*, we obtain*

$$M_\rho = \frac{1}{\pi} + \frac{3}{|5.9146|} = 0.8255,$$

Then we get

$$\begin{aligned} c_\varphi \ge{}& \frac{c\rho_\alpha(e)}{\lambda\Gamma(1+\alpha)} + \frac{c\beta\rho_\alpha(\xi)}{|\Delta|\Gamma(1+\alpha)}\left[1 + \frac{N\rho_\alpha(e)}{\Gamma(1+\alpha)-NM_\rho}\right] \\ &+ \frac{c\rho_\alpha(e)}{\Gamma(1+\alpha)}\left[\frac{1}{|\Delta|} + \frac{1}{\lambda}\frac{N\rho_\alpha(t)}{\Gamma(1+\alpha)-NM_\rho} + \frac{1}{|\Delta|}\frac{N\rho_\alpha(e)}{\Gamma(1+\alpha)-NM_\rho}\right]. \end{aligned}$$

Then the assumptions of Theorem 5.5 are satisfied. Then, problem (1.1) *is Ulam-Hyers stable and generalized Ulam-Hyers stable.*

Example 7.5. *(Illustrative example on solution dependence) Consider the FLE problem* (1.1) *with Hadamard fractional derivatives involving nonlocal boundary conditions:*

$$\begin{cases} \mathcal{D}_1^{\frac{7}{9}}\left(\mathrm{D}^2 + \left(\frac{4\pi}{3}\right)^2\right)x(t) = f\left(t, x(t), \mathcal{D}_1^{\frac{7}{9}}[x](t)\right), \quad t \in \left(1, \frac{12}{5}\right), \\ \mathrm{D}^2 x(1) = x(1) = 0, \\ x(\frac{12}{5}) = \frac{7}{3}x(\frac{9}{4}), \end{cases} \tag{7.7}$$

and

$$\mathcal{D}_1^{\frac{7}{9}-\epsilon}\left(\mathrm{D}^2 + \left(\frac{4\pi}{3}\right)^2\right)x(t) = f\left(t, x(t), \mathcal{D}_1^{\frac{7}{9}}[x](t)\right), \tag{7.8}$$

for $t \in (0,1)$ *and* $\epsilon > 0$ *with*

$$\begin{aligned} f(t,x,\tilde{x}) ={}& \frac{|x+4|(\sin^2(\pi t)+14)}{5+t} + \frac{(\tilde{x}^2+4)(\sin^2(3\pi t)+6)}{(t+2)(|\tilde{x}+1|)} \\ &+ \frac{|\tilde{x}|(\cos^2(3\pi t)+2)}{(t+0.5)^2} + \frac{13}{4}, \end{aligned} \tag{7.9}$$

where $\alpha = \frac{7}{9}$, $\lambda = \frac{4\pi}{3}$, $\beta = \frac{7}{3}$, $\xi = \frac{9}{4}$ *and* $0 < \frac{7}{9} - \epsilon < \alpha < 1$. *It is obvious that*

$$\sin \pi \left(\frac{15}{4} - 1 \right) = 0.7071 \neq -1.1756 = 2 \sin \pi \left(\frac{11}{5} - 1 \right),$$

So

$$\Delta = \lambda (\sin \lambda (e - 1) - \beta \sin \lambda (\xi - 1)) = 6.7606 \neq 0.$$

See Figure 2. Also, we have

$$|f(t, x_1, \tilde{x}_1) - f(t, x_2, \tilde{x}_2)| \leq \frac{4}{121} |x_1 - x_2| + \frac{2}{75} |\tilde{x}_1 - \tilde{x}_2|,$$

for each $t \in \left[1, \frac{12}{5}\right]$ *and all* $x_i, \tilde{x}_i \in \mathbb{R}$, *here* $N_1 = \frac{4}{121}$ *and* $N_2 = \frac{2}{75}$. *Put* $\varphi(t) = \frac{t^2}{t^2+1}$,

$$N = \max \left\{ \frac{4}{121}, \frac{2}{75} \right\}.$$

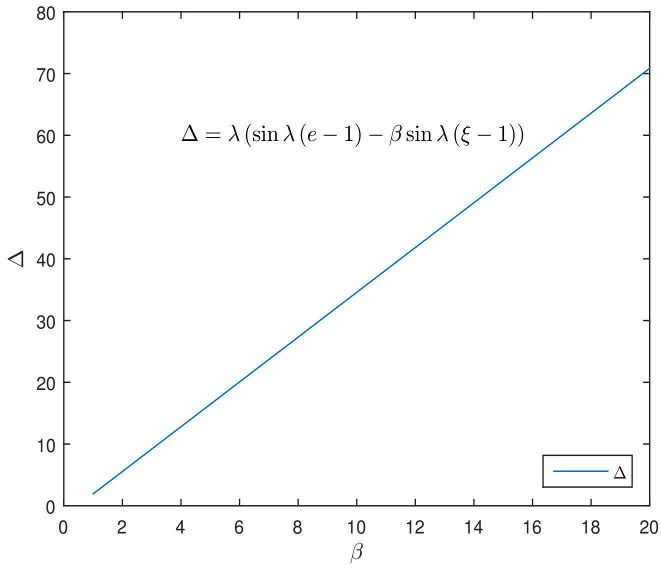


Figure 2. Numerical results of Δ where $\beta = 1, \ldots, 20$ in Example 7.5.

By using Eq (2.11), *we obtain*

$$\rho_\alpha(t) \leq \frac{1}{\frac{7}{9} + 1} \left(\frac{12}{5} \right)^{\frac{7}{9}+1} = 2.6671$$

and

$$\rho_\alpha \left(\frac{12}{5} \right) = \int_1^{\frac{12}{5}} (\ln s)^\alpha \, ds = 0.7953,$$

$$\rho_\alpha\left(\frac{9}{4}\right) = \int_1^{\frac{9}{4}} (\ln s)^\alpha \, \mathrm{d}s = 0.6639.$$

Also, by applying Eq (3.17)*, we obtain*

$$M_\rho = \frac{1}{\pi} + \frac{3}{|6.7606|} = 0.7620,$$

Then for $\epsilon = 0.1$*, by using MatLab program, we get*

$$\varrho_\epsilon(e) = \varrho_\epsilon\left(\frac{12}{5}\right) = \int_1^t \left|\frac{(\ln s)^{\frac{7}{9}-\epsilon}}{\Gamma\left(1+\frac{7}{9}-\epsilon\right)} - \frac{(\ln s)^\alpha}{\Gamma(1+\alpha)}\right| \mathrm{d}s = 0.1886,$$

$$\varrho_\epsilon(\xi) = \varrho_\epsilon\left(\frac{9}{4}\right) = \int_1^t \left|\frac{(\ln s)^{\frac{7}{9}-\epsilon}}{\Gamma\left(1+\frac{7}{9}-\epsilon\right)} - \frac{(\ln s)^\alpha}{\Gamma(1+\alpha)}\right| \mathrm{d}s = 0.1761$$

and

$$\varrho(t) = \frac{3}{4\pi}\varrho_\epsilon(t) + \frac{1}{|6.7606|}\left[\frac{7 \times 0.1761}{3} + \varrho_\epsilon(0.1886)\right].$$

Then the assumptions of Theorem 6.1 are satisfied. In addition to, by applying Eqs (6.6) *and* (6.7)*, we can calculate* $C_{22}(t)$*,* $C_{33}(t)$ *and* k_ϵ*.*

Table 4. Numerical results of $\mathcal{D}_1^\alpha\left(\mathrm{D}^2 + \lambda^2\right)x(t) = f(t, x(t), \mathcal{D}_1^\alpha[x](t))$ in Example 7.5 for $\alpha = \frac{1}{7}, \frac{1}{2}, \frac{7}{9}$, here $x(t) = \ln(t)$ and $(a) = \mathcal{D}_1^\alpha\left(\mathrm{D}^2 + \lambda^2\right)x(t)$, $(b) = f\left(t, x(t), \mathcal{D}_1^\alpha[x](t)\right)$.

			$\alpha = \frac{1}{7}$		$\alpha = \frac{1}{2}$		$\alpha = \frac{7}{9}$	
n	t_N	$x(t_N)$	(a)	(b)	(a)	(b)	(a)	(b)
0	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	1.0296	0.0292	−0.6481	20.4865	−2.0421	20.3037	−5.7442	20.6579
2	1.0601	0.0584	0.3210	20.6028	1.2297	20.4569	2.3563	20.9298
3	1.0915	0.0875	1.1548	20.8158	3.1716	20.6972	5.8459	21.1952
4	1.1238	0.1167	1.9214	21.0045	4.6113	20.9107	7.9468	21.4028
5	1.1571	0.1459	2.6447	21.0604	5.7818	21.0022	9.4115	21.4832
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	1.7926	0.5836	11.5237	19.1220	14.8965	19.3122	16.5953	19.4690
21	1.8456	0.6128	12.0495	19.0026	15.3004	19.1885	16.8158	19.3276
22	1.9003	0.642	12.5704	18.5841	15.6925	18.7749	17.0254	18.8997
23	1.9566	0.6712	13.0867	18.0950	16.0739	18.2893	17.2254	18.3996
24	2.0145	0.7004	13.5986	17.8788	16.4453	18.0647	17.4166	18.1580
25	2.0742	0.7296	14.1063	18.0394	16.8076	18.2061	17.5999	18.2808
26	2.1356	0.7587	14.6102	18.2910	17.1613	18.4376	17.7761	18.4949
27	2.1988	0.7879	15.1104	18.2508	17.5070	18.3843	17.9457	18.4272
28	2.2639	0.8171	15.6071	17.9090	17.8453	18.0343	18.1094	18.0647
29	2.3310	0.8463	16.1005	17.6773	18.1766	17.7921	18.2675	17.8109
30	2.4000	0.8755	16.5908	17.8398	18.5014	17.9399	18.4206	17.9478

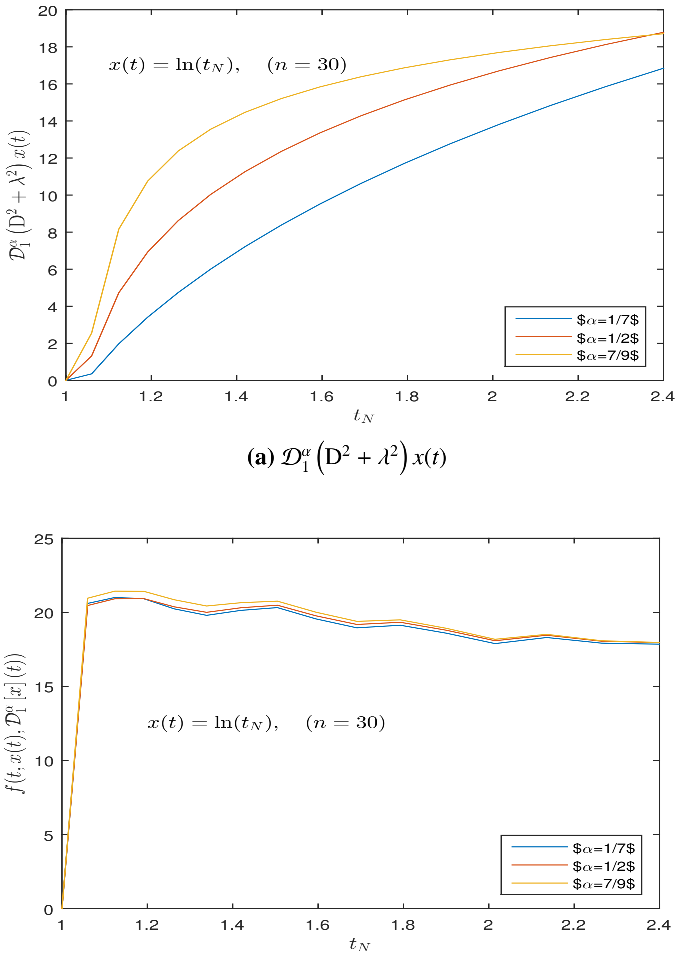
Now, we describ discretization method and use Theorem 2.3 for this example. Fix $n \geq 1$ *for* $N \in \{1, \cdots, n\}$*, define*

$$t_N = a\exp(\Delta T) = \exp(\Delta T),$$

with

$$\Delta T = \frac{1}{n} \ln \frac{b}{a} = \frac{1}{n} \ln \frac{12}{5}$$

and $[a, b] = [1, \frac{12}{5}]$.



(a) $\mathcal{D}_1^\alpha\left(\mathrm{D}^2 + \lambda^2\right)x(t)$

(b) $f\left(t, x(t), \mathcal{D}_1^\alpha\{x\}(t)\right)$

Figure 3. Numerical results of $\mathcal{D}_1^\alpha\left(\mathrm{D}^2 + \lambda^2\right)x(t)$ and $f\left(t, x(t), \mathcal{D}_1^\alpha[x](t)\right)$ where $x(t) = \ln t$ and $\alpha = \frac{1}{7}, \frac{1}{2}, \frac{7}{9}$ in Example 7.5, respectively.

Also,

$$(\tau_k^\alpha) = k^{1-\alpha} - (k-1)^{1-\alpha} = k^{1-\frac{7}{9}} - (k-1)^{1-\frac{7}{9}} = k^{\frac{2}{9}} - (k-1)^{\frac{2}{9}}$$

and

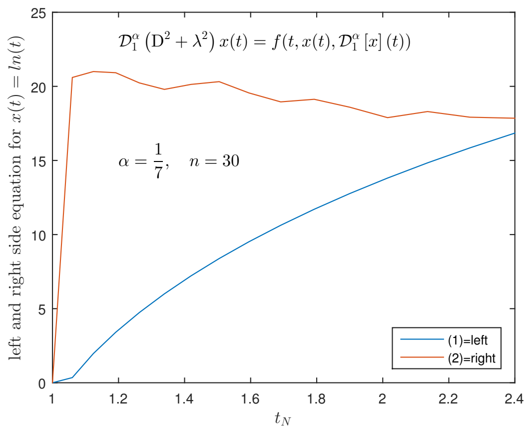
$$\zeta = \frac{(\Delta T)^{1-\alpha}}{a[1 - \exp(-\Delta T)]\Gamma(2-\alpha)} = \frac{\left(\frac{1}{n}\ln\frac{12}{5}\right)^{\frac{2}{9}}}{\left[1 - \exp\left(\frac{-1}{n}\ln\frac{12}{5}\right)\right]\Gamma\left(\frac{2}{9}\right)}.$$

Thus, for $f\left(t, x(t), \mathcal{D}_1^{\frac{7}{9}}[x](t)\right)$*, we get*

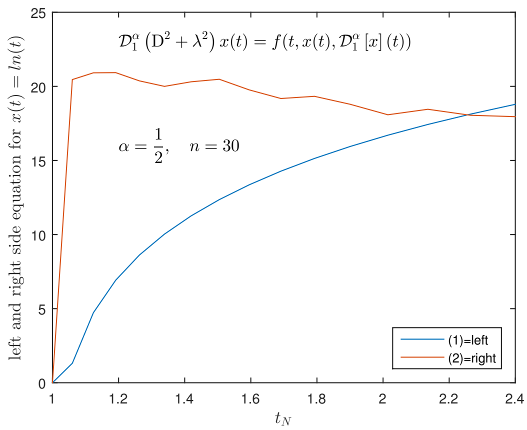
$$\mathcal{D}_1^{\frac{7}{9}}[x](t_N) = \tilde{\mathcal{D}}_1^{\frac{7}{9}}[x](t_N) + O\left(\frac{1}{n}\ln\frac{12}{5}\right),$$

where

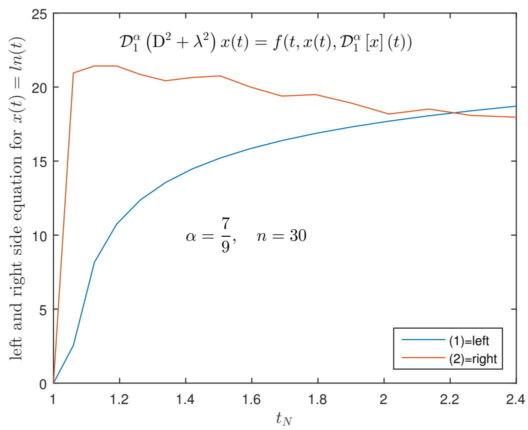
$$\begin{aligned}\tilde{\mathcal{D}}_1^{\frac{7}{9}}[x](t_N) &= \frac{x(a)}{\Gamma(1-\alpha)}\left(\ln\frac{t_N}{a}\right)^{-\alpha} + \zeta\sum_{k=1}^{N}(\tau_{N-k+1}^{\alpha})\frac{x(t_k)-x(t_{k-1})}{\exp(k\Delta T)}.t_k \\ &= \frac{x(1)}{\Gamma(1-\frac{7}{9})}(\ln t_N)^{-\frac{7}{9}} + \zeta\sum_{k=1}^{N}\left(\tau_{N-k+1}^{\frac{7}{9}}\right)\frac{x(t_k)-x(t_{k-1})}{\exp\left(\frac{k}{n}\ln\frac{12}{5}\right)}.t_k \\ &= \frac{x(1)}{\Gamma\left(\frac{2}{9}\right)}(\ln t_N)^{-\frac{7}{9}} + \zeta\sum_{k=1}^{N}\left(\tau_{N-k+1}^{\frac{7}{9}}\right)\frac{x(t_k)-x(t_{k-1})}{\exp\left(\frac{k}{n}\ln\frac{12}{5}\right)}.t_k,\end{aligned}$$



(a) $\alpha = \frac{1}{7}$



(b) $\alpha = \frac{1}{2}$



(c) $\alpha = \frac{7}{9}$

Figure 4. Numerical results of $(a) = \mathcal{D}_1^{\alpha}\left(\mathrm{D}^2 + \lambda^2\right)x(t)$ and $(b) = f(t, x(t), \mathcal{D}_1^{\alpha}[x](t))$ where $x(t) = \ln t$ in Example 7.5, respectively.

8. Conclusions

The Langevin equation has been proposed to describe dynamical processes in a fractal medium in which the fractal and memory properties with a dissipative memory kernel are incorporated. However, it has been realized that the classical Langevin equation failed to describe the complex systems. Thus, the consideration of LDE in frame of fractional derivatives becomes compulsory. As a result of this interest, several results have been revealed and different versions of LDE have been under study. In this paper, we have presented some results dealing with the existence and uniqueness of solutions for boundary value problem of nonlinear Langevin equation involving Hadamard fractional order. As a first step, the boundary value problem is transformed to a fixed point problem by applying the tools of Hadamard fractional calculus. Based on this, the existence results are established by means of the Schaefer's fixed point theorem and Banach contraction principle.

We claim that the results of this paper is new and generalize some earlier results. For instance, by taking $\alpha = 1$ in the results of this paper which can be considered a special case of a simple Jerk Chaotic circuit equation see [33]. The paper presented a discuss on the Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stabilities of the solution of the FLD using the generalization for the Gronwall inequality. We present an example to demonstrate the consistency to the theoretical findings. We also analyze the continuous dependence of solutions all on its right side function, initial value condition and the fractional order for FDE. Using these results, the properties of the solution process can be discussed through numerical simulation. We hope to consider this problem in a future work.

Availability of data and material

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

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Supplementary

Algorithm 1: The proposed method for the FLE problem with Hadamard fractional derivatives involving nonlocal boundary conditions (7.7) in Example 7.5 which we use the conditions of Theorem 2.3 there in.

```
function [ParamMatrix]= discretization_method3(alpha, a, e, lambda, n, x_t)
[xalpha, yalpha]=size(alpha);
DeltaT=log(e/a)/n;
ParamMatrix(1, 1) = 0;
ParamMatrix(1, 2) = a;
for j=3:2+7*yalpha
ParamMatrix(1, j) = 0;
end;

column=3;
j=1;
while j≤yalpha
for N=1:n
ParamMatrix(N+1, 1) = N;
tN=a*exp(N*DeltaT);
ParamMatrix(N+1, 2) = tN;
end;
for N=1:n
zeta = round(DeltaT^(1-alpha(j))/(a * (1- exp((-1)*DeltaT) )*gamma(2-alpha(j))), 6);
ParamMatrix(N+1, column) = zeta;
ParamMatrix(N+1, column+5) =round(eval(subs(x_t, ParamMatrix(N+1, 2))), 6);
s=0;
k=1;
while k≤N
taukalpha= (N-k+1)^(1-alpha(j)) - (N-k)^(1-alpha(j));
y2=eval(subs(x_t, ParamMatrix(k+1, 2)));
y1=eval(subs(x_t, ParamMatrix(k, 2)));
s = s + taukalpha*(y2-y1)*ParamMatrix(k+1, 2)/exp(k*DeltaT);
```

```
k=k+1;
end;
A=eval(subs(x_t, a))* (log(ParamMatrix(N+1, 2)/a))^((-1)*alpha(j));
HadamardD_xtN= round(A + zeta*s, 6);
ParamMatrix(N+1, column+1) = HadamardD_xtN;
end;
for N=1:n
s=0;
k=1;
while k≤N
taukalpha= (N-k+1)^(1-alpha(j)) - (N-k)^(1-alpha(j));
y2=lambda^2* eval(subs(x_t, ParamMatrix(k+1, 2)));
y1=lambda^2* eval(subs(x_t, ParamMatrix(k, 2)));
s = s + taukalpha*(y2-y1)*ParamMatrix(k+1, 2)/exp(k*DeltaT);
k=k+1;
end;
A=eval(subs(x_t, a))* (log(ParamMatrix(N+1, 2)/a))^((-1)*alpha(j));
HadamardD_xtN = round(A + zeta*s, 6);
ParamMatrix(N+1, column+2) = HadamardD_xtN;
end;
for N=1:n
s=0;
k=1;
while k≤N
taukalpha= (N-k+1)^(1-alpha(j)) - (N-k)^(1-alpha(j));
y2=eval(subs(diff(x_t,2), ParamMatrix(k+1, 2)));
y1=eval(subs(diff(x_t,2), ParamMatrix(k, 2)));
s = s + taukalpha*(y2-y1)*ParamMatrix(k+1, 2)/exp(k*DeltaT);
k=k+1;
end;
A=eval(subs(diff(x_t,2), a))* (log(ParamMatrix(N+1, 2)/a))^((-1)*alpha(j));
HadamardD_xtN= round(A + zeta*s, 5);
ParamMatrix(N+1, column+3) = HadamardD_xtN;
ParamMatrix(N+1, column+4) = ParamMatrix(N+1, column+2)+ParamMatrix(N+1, column+3);
ParamMatrix(N+1, column+6) = abs(ParamMatrix(N+1, column+5) +4)*((sin(pi*ParamMatrix(N+1, 2) ))^2
end;
j=j+1;
column=column+7;
end;
end
```