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Research article

Asymptotic expansion of a finite sum involving harmonic numbers

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Abstract: In the paper, we obtain asymptotic expansion of the finite sum of some sequences $S_n = \sum_{k=1}^{n} \left(n^2 + k\right)^{-1}$ by using the Euler's standard one of the harmonic numbers.

Keywords: asymptotic expansion; finite sum of some sequences; harmonic numbers
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1. Introduction

In [1, p.458], Graham, Knuth and Patashnik proposed a problem: Find the asymptotic value of the sum

$$S_n = \sum_{i=1}^{n} \frac{1}{n^2+i} = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} \tag{1.1}$$

with an absolute error of $O(n^{-7})$. The following answer

$$S_n = \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{4n^4} - \frac{2}{15n^5} + \frac{1}{12n^6} + O\left(\frac{1}{n^7}\right) \tag{1.2}$$

appeared in [1, p.459] and modestly be interpreted as probably correct.

In this paper we obtained an asymptotic expression of S_n with absolute error $O\left(1/n^{4l+1}\right)$ ($l \in \mathbb{N}$) by Euler's standard one of the harmonic numbers, and solved the above problem as a special case.

2. Main result and its proof

Theorem 2.1. Let l a given natural number and B_{2w} be an even-indexed Bernoulli number. Then as $n \to \infty$,

$$S_n = \sum_{i=1}^{n} \frac{1}{n^2+i} = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n}$$

$$= \frac{1}{n} - \frac{1}{2n^2} + \sum_{\substack{k=3\\ k\neq 4m\\ j=[k/4]}}^{4l-1} \frac{A_k}{n^k} + \sum_{\substack{k=4i\\ 1\le i\le l-1\\ j=i-1}} \frac{A_k}{n^k} + O\left(\frac{1}{n^{4l+1}}\right), \tag{2.1}$$

where

$$A_k = \frac{(-1)^{k-1}}{k} + \frac{(-1)^{k-2}}{2} + \sum_{w=1}^{j} \frac{B_{2w}}{2w} \frac{(-1)^{k-(2w+1)}}{(2w-1)!} \frac{(k-(2w+1))!}{(k-4w)!} \tag{2.2}$$

for $k \geq 3$.

Proof. Let γ be the Euler's constant and $P_{2l+1}(x)$ be a periodic Bernoulli polynomial. Then from Equation (7) , Inequalities (8) and (9) in [2]:

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{l} \frac{B_{2k}}{2k} \frac{1}{n^{2k}} + \int_n^\infty \frac{P_{2l+1}(x)}{x^{2l+1}} dx,$$

$$|P_{2l+1}(x)| \leq \frac{2(2l+1)!}{(2\pi)^{2l+1}} \sum_{r=1}^{\infty} \frac{1}{r^{2l+1}},$$

$$\left| \int_n^\infty \frac{P_{2l+1}(x)}{x^{2l+1}} dx \right| \leq \frac{4}{n} \sqrt{\frac{l}{\pi}} \left(\frac{l}{n\pi e}\right)^{2l},$$

one can obtain the Euler's standard asymptotic expansion of the harmonic numbers: as $n \to \infty$,

$$H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{l} \frac{B_{2k}}{2k} \frac{1}{n^{2k}} + O\left(\frac{1}{n^{2l+1}}\right). \tag{2.3}$$

By (2.3) we obtain

$$\begin{aligned} S_n &= H_{n^2+n} - H_{n^2} = \ln\left(n^2+n\right) - \ln n^2 + \frac{1}{2\left(n^2+n\right)} - \sum_{i=1}^{l} \frac{B_{2i}}{2i} \frac{1}{(n^2+n)^{2i}} - \sum_{i=l+1}^{\infty} \frac{B_{2i}}{2i} \frac{1}{(n^2+n)^{2i}} \\ &\quad - \left(\frac{1}{2n^2} - \sum_{i=1}^{l} \frac{B_{2i}}{2i} \frac{1}{n^{4i}}\right) + \sum_{i=l+1}^{\infty} \frac{B_{2i}}{2i} \frac{1}{n^{4i}} + O\left(\frac{1}{n^{4l+2}}\right) \\ &= \ln\left(1+\frac{1}{n}\right) + \frac{1}{2\left(n^2+n\right)} - \sum_{i=1}^{l} \frac{B_{2i}}{2i} \frac{1}{(n^2+n)^{2i}} - \sum_{i=l+1}^{\infty} \frac{B_{2i}}{2i} \frac{1}{(n^2+n)^{2i}} \\ &\quad - \frac{1}{2n^2} + \sum_{i=1}^{l} \frac{B_{2i}}{2i} \frac{1}{n^{4i}} + \sum_{i=l+1}^{\infty} \frac{B_{2i}}{2i} \frac{1}{n^{4i}} + O\left(\frac{1}{n^{4l+2}}\right). \end{aligned} \tag{2.4}$$

Since

$$(a+t)^{-n} = \frac{(-1)^{n-1}\left[(a+t)^{-1}\right]^{(n-1)}}{(n-1)!},\ a \in \mathbb{R}, \tag{2.5}$$

and for $-1 < t < 1$,

$$
\begin{aligned}
[\ln(1+t)]' &= (1+t)^{-1} = \sum_{k=0}^{\infty} (-1)^k t^k, \\
\ln(1+t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} t^{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k,
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{1}{n^2+n} &= \frac{1}{n^2}\frac{1}{1+\frac{1}{n}} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{n^{k+2}}, \\
\frac{1}{(n^2+n)^{2i}} &= \frac{1}{n^{4i}}\frac{1}{\left(1+\frac{1}{n}\right)^{2i}} = -\frac{1}{(2i-1)!} \sum_{k=2i-1}^{\infty} \frac{(-1)^k k!}{(k-(2i-1))!} \frac{1}{n^{k+2i+1}},
\end{aligned}
$$

and

$$
\begin{aligned}
S_n &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{1}{n^k} + \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{1}{n^{k+2}} \\
&\quad + \sum_{i=1}^{l} \frac{B_{2i}}{(2i)!} \sum_{k=2i-1}^{\infty} \frac{(-1)^k k!}{(k-(2i-1))!} \frac{1}{n^{k+2i+1}} + \sum_{i=l+1}^{\infty} \frac{B_{2i}}{(2i)!} \sum_{k=2i-1}^{\infty} \frac{(-1)^k k!}{(k-(2i-1))!} \frac{1}{n^{k+2i+1}} \\
&\quad - \frac{1}{2n^2} + \sum_{i=1}^{l} \frac{B_{2i}}{2i} \frac{1}{n^{4i}} + \sum_{i=l+1}^{\infty} \frac{B_{2i}}{2i} \frac{1}{n^{4i}} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{1}{n^k} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{k+2}} + \left[\sum_{i=1}^{l-1} \frac{B_{2i}}{(2i)!} \sum_{k=2i-1}^{\infty} \frac{(-1)^k k!}{(k-(2i-1))!} \frac{1}{n^{k+2i+1}} - \frac{B_{2l}}{2l} \frac{1}{n^{4l}} \right] \\
&\quad - \frac{1}{2n^2} + \left[\sum_{i=1}^{l-1} \frac{B_{2i}}{2i} \frac{1}{n^{4i}} + \frac{B_{2l}}{2l} \frac{1}{n^{4l}} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{1}{n^k} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{k+2}} + \sum_{i=1}^{l-1} \frac{B_{2i}}{(2i)!} \sum_{k=2i-1}^{\infty} \frac{(-1)^k k!}{(k-(2i-1))!} \frac{1}{n^{k+2i+1}} \\
&\quad - \frac{1}{2n^2} + \sum_{i=1}^{l-1} \frac{B_{2i}}{2i} \frac{1}{n^{4i}} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{1}{n^k} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{k+2}} + \frac{B_2}{2!} \sum_{k=1}^{\infty} \frac{(-1)^k k!}{(k-1)!} \frac{1}{n^{k+3}} + \frac{B_4}{4!} \sum_{k=3}^{\infty} \frac{(-1)^k k!}{(k-3)!} \frac{1}{n^{k+5}} \\
&\quad + \frac{B_6}{6!} \sum_{k=5}^{\infty} \frac{(-1)^k k!}{(k-5)!} \frac{1}{n^{k+7}} + \cdots + \frac{B_{2l-2}}{(2l-2)!} \sum_{k=2l-3}^{\infty} \frac{(-1)^k k!}{(k-(2l-3))!} \frac{1}{n^{k+2l-1}} \\
&\quad - \frac{1}{2n^2} + \frac{B_2}{2} \frac{1}{n^4} + \frac{B_4}{4} \frac{1}{n^8} + \frac{B_6}{6} \frac{1}{n^{12}} + \cdots + \frac{B_{2l-2}}{2l-2} \frac{1}{n^{4l-4}}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{(-1)^0}{1}\frac{1}{n} + \frac{(-1)^1}{2}\frac{1}{n^2} + \sum_{k=3}^{\infty}\frac{(-1)^{k-1}}{k}\frac{1}{n^k} + \frac{1}{2}\frac{(-1)^0}{n^2} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{n^{k+2}} + \frac{B_2}{2!}\sum_{k=1}^{\infty}\frac{(-1)^k k!}{(k-1)!}\frac{1}{n^{k+3}} \\
& + \frac{B_4}{4!}\sum_{k=3}^{\infty}\frac{(-1)^k k!}{(k-3)!}\frac{1}{n^{k+5}} + \frac{B_6}{6!}\sum_{k=5}^{\infty}\frac{(-1)^k k!}{(k-5)!}\frac{1}{n^{k+7}} + \cdots + \frac{B_{2l-2}}{(2l-2)!}\sum_{k=2l-3}^{\infty}\frac{(-1)^k k!}{(k-(2l-3))!}\frac{1}{n^{k+2l-1}} \\
& - \frac{1}{2n^2} + \frac{B_2}{2}\frac{1}{n^4} + \frac{B_4}{4}\frac{1}{n^8} + \frac{B_6}{6}\frac{1}{n^{12}} + \cdots + \frac{B_{2l-2}}{2l-2}\frac{1}{n^{4l-4}}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{(-1)^0}{1}\frac{1}{n} + \frac{(-1)^1}{2}\frac{1}{n^2} + \frac{1}{2}\frac{(-1)^0}{n^2} - \frac{1}{2n^2} \\
& + \sum_{k=3}^{\infty}\frac{(-1)^{k-1}}{k}\frac{1}{n^k} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{n^{k+2}} + \frac{B_2}{2!}\sum_{k=1}^{\infty}\frac{(-1)^k k!}{(k-1)!}\frac{1}{n^{k+3}} + \frac{B_4}{4!}\sum_{k=3}^{\infty}\frac{(-1)^k k!}{(k-3)!}\frac{1}{n^{k+5}} \\
& + \frac{B_6}{6!}\sum_{k=5}^{\infty}\frac{(-1)^k k!}{(k-5)!}\frac{1}{n^{k+7}} + \cdots + \frac{B_{2l-2}}{(2l-2)!}\sum_{k=2l-3}^{\infty}\frac{(-1)^k k!}{(k-(2l-3))!}\frac{1}{n^{k+2l-1}} \\
& + \frac{B_2}{2}\frac{1}{n^4} + \frac{B_4}{4}\frac{1}{n^8} + \frac{B_6}{6}\frac{1}{n^{12}} + \cdots + \frac{B_{2l-2}}{2l-2}\frac{1}{n^{4l-4}}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{n} - \frac{1}{2}\frac{1}{n^2} \quad & + \left[\sum_{k=3}^{\infty}\frac{(-1)^{k-1}}{k}\frac{1}{n^k} + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\frac{1}{n^{k+2}}\right] + \frac{B_2}{2!}\sum_{k=1}^{\infty}(-1)^k\frac{k!}{(k-1)!}\frac{1}{n^{k+3}} \\
& + \frac{B_4}{4!}\sum_{k=3}^{\infty}(-1)^k\frac{k!}{(k-3)!}\frac{1}{n^{k+5}} + \frac{B_6}{6!}\sum_{k=5}^{\infty}(-1)^k\frac{k!}{(k-5)!}\frac{1}{n^{k+7}} + \cdots \\
& + \frac{B_{2l-2}}{(2l-2)!}\sum_{k=2l-3}^{\infty}(-1)^k\frac{k!}{(k-(2l-3))!}\frac{1}{n^{k+2l-1}} \\
& + \frac{B_2}{2}\frac{1}{n^4} + \frac{B_4}{4}\frac{1}{n^8} + \frac{B_6}{6}\frac{1}{n^{12}} + \cdots + \frac{B_{2l-2}}{2l-2}\frac{1}{n^{4l-4}}.
\end{aligned}
$$

Similarly, from the brackets above we can extract the cubic term of $1/n$:

$$
\begin{aligned}
S_n = \frac{1}{n} - \frac{1}{2}\frac{1}{n^2} \quad & - \frac{1}{6n^3} + \left[\sum_{k=4}^{\infty}\frac{(-1)^{k-1}}{k}\frac{1}{n^k} + \frac{1}{2}\sum_{k=2}^{\infty}\frac{(-1)^k}{n^{k+2}} + \frac{B_2}{2!}\sum_{k=1}^{\infty}\frac{(-1)^k k!}{(k-1)!}\frac{1}{n^{k+3}}\right] \\
& + \frac{B_4}{4!}\sum_{k=3}^{\infty}\frac{(-1)^k k!}{(k-3)!}\frac{1}{n^{k+5}} + \frac{B_6}{6!}\sum_{k=5}^{\infty}\frac{(-1)^k k!}{(k-5)!}\frac{1}{n^{k+7}} + \cdots \\
& + \frac{B_{2l-2}}{(2l-2)!}\sum_{k=2l-3}^{\infty}\frac{(-1)^k k!}{(k-(2l-3))!}\frac{1}{n^{k+2l-1}} \\
& + \frac{B_2}{2}\frac{1}{n^4} + \frac{B_4}{4}\frac{1}{n^8} + \frac{B_6}{6}\frac{1}{n^{12}} + \cdots + \frac{B_{2l-2}}{2l-2}\frac{1}{n^{4l-4}}.
\end{aligned}
$$

Carrying out this idea and according to k is a multiple of 4 or not, we can classify the coefficients of $\left(1/n^k\right)$ as follows:

$$
S_n = \frac{1}{n} - \frac{1}{2n^2} + \sum_{\substack{k=3 \\ k\neq 4m \\ j=[k/4]}}^{4l-1}\left[\frac{(-1)^{k-1}}{k} + \frac{(-1)^{k-2}}{2} + \sum_{w=1}^{j}\frac{B_{2w}}{2w}\frac{(-1)^{k-(2w+1)}}{(2w-1)!}\frac{(k-(2w+1))!}{(k-4w)!}\right]\frac{1}{n^k}
$$

$$+\sum_{\substack{k=4i\\1\le i\le l-1\\j=i-1}}\left[\frac{(-1)^{k-1}}{k}+\frac{(-1)^{k-2}}{2}+\sum_{w=1}^{2j-2}\frac{B_{2w}}{2w}\frac{(-1)^{k-(2w+1)}}{(2w-1)!}\frac{(k-(2w+1))!}{(k-4w)!}\right]\frac{1}{n^k}+O\left(\frac{1}{n^{4l+1}}\right),$$

which is equivalent to (2.1) . □

3. Calculation of A_k and error analysis

By (2.1) and (2.2) we can obtain many concrete conclusions. Letting $k = 3, 4, 5, \cdots, 13, \cdots$ in (2.2) gives

(1) when $k = 3$, we have

$$A_3=\frac{(-1)^2}{3}+\frac{1}{2}(-1)=-\frac{1}{6};$$

(2) when $k = 4$, Then $i = 1$ and we have

$$A_4=\frac{(-1)^3}{4}+\frac{(-1)^2}{2}=\frac{1}{4};$$

(3) when $k = 5$, then $j = 1$ and

$$A_5=\frac{(-1)^4}{5}+\frac{(-1)^3}{2}+\frac{\frac{1}{6}}{2}\frac{(-1)^2}{1!}\frac{2!}{1!}=-\frac{2}{15};$$

(4) when $k = 6$, then $j = 1$ and

$$A_6=\frac{(-1)^5}{6}+\frac{(-1)^4}{2}+\frac{\frac{1}{6}}{2}\frac{(-1)^3}{1!}\frac{3!}{2!}=\frac{1}{12};$$

(5) when $k = 7$, then $j = 1$ and

$$A_7=\frac{(-1)^6}{7}+\frac{(-1)^5}{2}+\frac{\frac{1}{6}}{2}\frac{(-1)^4}{1!}\frac{4!}{3!}=-\frac{1}{42};$$

(6) when $k = 8$, then $i = 2$, $j = 1$ and

$$A_8=\frac{(-1)^7}{8}+\frac{(-1)^6}{2}+\frac{\frac{1}{6}}{2}\frac{(-1)^5}{1!}\frac{5!}{4!}=-\frac{1}{24};$$

(7) when $k = 9$, then $j = 2$ and

$$A_9=\frac{(-1)^8}{9}+\frac{(-1)^7}{2}+\frac{\frac{1}{6}}{2}\frac{(-1)^6}{1!}\frac{6!}{5!}+\frac{\left(-\frac{1}{30}\right)}{4}\frac{(-1)^4}{3!}\frac{4!}{1!}=\frac{7}{90};$$

(8) when $k = 10$, then $j = 2$ and

$$A_{10}=\frac{(-1)^9}{10}+\frac{(-1)^8}{2}+\frac{\frac{1}{6}}{2}\frac{(-1)^7}{1!}\frac{7!}{6!}+\frac{\left(-\frac{1}{30}\right)}{4}\frac{(-1)^5}{3!}\frac{5!}{2!}=-\frac{1}{10};$$

(9) when $k = 11$, then $j = 2$ and

$$A_{11} = \frac{(-1)^{10}}{11} + \frac{(-1)^9}{2} + \frac{\frac{1}{6}}{2}\frac{(-1)^8}{1!}\frac{8!}{7!} + \frac{\left(-\frac{1}{30}\right)}{4}\frac{(-1)^6}{3!}\frac{6!}{3!} = \frac{1}{11};$$

(10) when $k = 12$, then $i = 3$, $j = 2$ and

$$A_{12} = \frac{(-1)^{11}}{12} + \frac{(-1)^{10}}{2} + \frac{\frac{1}{6}}{2}\frac{(-1)^9}{1!}\frac{9!}{8!} + \frac{\left(-\frac{1}{30}\right)}{4}\frac{(-1)^7}{3!}\frac{7!}{4!} = -\frac{1}{24};$$

(11) when $k = 13$, then $i = 3$, $j = 2$ and

$$A_{13} = \frac{(-1)^{12}}{13} + \frac{(-1)^{11}}{2} + \frac{\frac{1}{6}}{2}\frac{(-1)^{10}}{1!}\frac{10!}{9!} + \frac{\left(-\frac{1}{30}\right)}{4}\frac{(-1)^8}{3!}\frac{8!}{5!} + + \frac{\frac{1}{42}}{6}\frac{(-1)^6}{5!}\frac{6!}{1!} = -\frac{89}{2730};$$

and so on. From this we can get the following corollary.

Corollary 3.1. Let

$$S_n = \sum_{i=1}^{n} \frac{1}{n^2+i} = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n}.$$

Then as $n \to \infty$,

$$\begin{aligned} S_n \quad = \quad & \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{4n^4} - \frac{2}{15n^5} + \frac{1}{12n^6} - \frac{1}{42n^7} - \frac{1}{24n^8} + \frac{7}{90n^9} \\ & - \frac{1}{10n^{10}} + \frac{1}{11n^{11}} - \frac{1}{24n^{12}} - \frac{89}{2730n^{13}} + O\left(\frac{1}{n^{14}}\right). \end{aligned} \tag{3.1}$$

As we can see, (3.1) affirms (1.2). Noting down

$$T_{n,k} = \sum_{i=1}^{k} \frac{A_i}{n^i},$$

from Theorem 2.1 we have that as $n \to \infty$,

$$S_n = \sum_{i=1}^{n} \frac{1}{n^2+i} = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} = T_{n,k} + O\left(\frac{1}{n^{k+1}}\right), \ k \geq 3.$$

The following is the error analysis table of S_n and $T_{n,k}$:

n	5	10	20	30
$S_n - T_{n,6}$	-3.8017×10^{-7}	-2.7290×10^{-9}	-2.0086×10^{-11}	-1.1484×10^{-12}
$S_n - T_{n,8}$	3.1262×10^{-8}	6.8643×10^{-11}	1.4258×10^{-13}	3.7872×10^{-15}
$S_n - T_{n,10}$	1.6799×10^{-9}	8.6526×10^{-13}	4.3339×10^{-16}	5.0517×10^{-18}
$S_n - T_{n,13}$	1.5455×10^{-11}	1.096×10^{-15}	7.2370×10^{-20}	2.5462×10^{-22}

,

which shows that $T_{n,k}$ has a good approximation to S_n.

4. Remarks

One of the reviewers pointed out that the key formula (2.4) of this paper can be obtained by the other two methods. The following two notes benefit from the reviewer's reminder. I hope the three methods provided in this paper will be helpful for further research in related fields.

Remark 4.1. Without referring to Euler-Maclaurin summation formula for the harmonic numbers H_n, we can obtain (2.4) when takinging $f(t) = 1/(n^2 + t)$ into the Euler-Maclaurin summation formula. In fact, by using the Euler-Maclaurin summation formula (see [3, p.45])

$$\sum_{k=0}^{n} f(k) = \int_0^n f(t)dt + \frac{1}{2}\left[f(0) + f(n)\right] + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right] + R_m,$$

where

$$R_m = \frac{nB_{2m+2}}{(2m+2)!} f^{(2m+2)}(\theta n),\ 0 < \theta < 1,\ n, m \in \mathbb{N}$$

and (2.5) , the key formula (2.4) follows from the relations

$$\begin{aligned} \int_0^n f(t)dt &= \ln(n^2 + n) - \ln n^2 = \ln\left(1 + \frac{1}{n}\right), \\ \frac{1}{2}\left[f(0) + f(n)\right] - \frac{1}{n^2} &= \frac{1}{2\left(n^2 + n\right)} - \frac{1}{2n^2} \end{aligned}$$

and

$$S_n = \sum_{k=0}^{n} f(k) - \frac{1}{n^2}.$$

Remark 4.2. The sum S_n can be written in the form

$$S_n = \psi(n^2 + n + 1) - \psi(n^2 + 1),\ \psi(z) = \frac{d}{dz} \ln \Gamma(z),$$

where the gamma function $\Gamma(z)$ is denoted by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \text{ (see [3, } p\text{.221])}$$

From known asymptotic expansion of the psi function $\psi(z)$:

$$\psi(z) \sim \ln z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)z^{2k}},\ \text{(see [3, } p\text{.224])}$$

and a property of $\psi(z)$:

$$\psi(z + 1) = \psi(z) + \frac{1}{z},\ \text{(see [3, } p\text{.224])}$$

the formula (2.4) follows.

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Conflict of interest

The author declares no conflict of interest in this paper.

References

1. R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete mathematics: A foundation for computer science*, 2 Eds., Amsterdam: Addison-Wesley Publishing Company, 1994.
2. D. E. Knuth, Euler's constant to 1271 places, *Math. Comput.*, **16** (1962), 275–280.
3. A. Jeffrey, *Handbook of mathematical formulas and integrals*, 3 Eds., Elsevier Academic Press, 2004.