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Research article

On the edge irregularity strength for some classes of plane graphs

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Abstract: Graph labeling is an assignment of (usually) positive integers to elements of a graph (vertices and/or edges) satisfying certain condition(s). In the last two decades, graph labeling research received much attention from researchers. This articles is about edge irregularity strength for some classes of plane graphs. Edge irregularity strength denoted by $es(G)$, was introduced by Ahmad et al. in 2014 as a modification of the well known irregularity strength by Chartrand in 1988. In this paper, the exact value of the edge irregularity strength for some clases of plane graphs is determined.

Keywords: irregular assignment; irregularity strength; edge irregularity strength; plane graphs
Mathematics Subject Classification: 05C78, 05C38

1. Introduction

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. *Graph labeling* is a mapping of elements of the graph, i.e. vertex and/or edges to a set of numbers (usually positive integers), called labels. If the domain is the vertex-set or the edge-set, the labeling is called *vertex labeling* or *edge labeling* respectively. Similarly if the domain is $V(G) \cup E(G)$, then the labeling is called *total labeling*. In 1988, Chartrand et al. [19] defined *irregular labeling* for a graph G as an assignment of labels from the set of natural numbers to the edges of G such that the sums of the labels assigned to the edges of each vertex are different. The minimum value of the largest label of an edge over all existing irregular labelings is known as the irregularity strength of G, and it is denoted by $s(G)$. The work of Chartrand et al. [19] opened a new horizon for graph theorists with a lot of research in this domain as confessed by the numerous articles investigating $s(G)$ for various families of graphs (see [7, 10, 18, 20, 21, 26, 27]).

In 2007, Baca et al. in [15] investigated two modifications of the irregularity strength of graphs, namely *total edge irregularity strength* denoted by $tes(G)$, and *total vertex irregularity strength* denoted by $tvs(G)$. Results on the total vertex irregularity strength and the total edge irregularity strength can be found in [1, 2, 4, 8, 11, 16, 23, 25, 27–30].

Motivated by the work of Chartrand et al. [19], Ahmad et al. in [3] introduced edge irregular k-labelings of graphs. A vertex k-labeling of graph G $\phi : V(G) \to \{1, 2, \cdots, k\}$ can be defined as an edge irregular k-labeling for G if for every two different edges e and f it is $w_\phi(e) \neq w_\phi(f)$, where the weight $w_\phi(e)$ of an edge $e = xy \in E(G)$ is defined as $w_\phi(xy) = \phi(x) + \phi(y)$. The minimum k for which the graph G has an edge irregular $k-$labeling is called the edge irregularity strength of G, denoted by es(G). In the same work [3], the authors proved a general lower bound of $es(G)$ and then determined exact values for several families of graphs such as paths, stars, double stars and Cartesian product of two paths. Over the last years, $es(G)$ has been investigated for different families of graphs including trees with the help of algorithmic solutions and amalgamated families of graph through corona product and Toeplitz graphs [5, 6, 9, 12–14, 31–35].

A lot of work has focused on graph labeling and that is evident from the recent survey by Gallian [22]. Still there is great potential of expansion in this area. That is why in this paper, we determine the exact value of edge irregularity strength for some classes of plane graphs. A planar graph is a graph that can be drawn on the plane in such a way that its edges do no intersect and only meet at their endpoints. A plane graph is a particular drawing of a planar graph on the Euclidean plane.

2. Main results

The following theorem establishes a general lower bound for the edge irregularity strength of a graph G (see [3]).

Theorem 1. *[3] Let $G = (V, E)$ be a simple graph with maximum degree Δ. Then*

$$es(G) \geq \max\left\{\left\lceil \frac{|E| + 1}{2} \right\rceil, \Delta\right\}.$$

We first discuss edge irregularity strength of plane graph C_n that is defined in [17] as follows: Let P_1, P_2 and P_3 be paths on vertices $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_{2n}$ and $c_1, c_2, \ldots, c_n$, respectively. Form the graph C_n from the disjoint union $P_1 \cup P_2 \cup P_3$ by adding the edges $\{a_i b_{2i-1},\ c_i b_{2i} : 1 \leq i \leq n\}$. Graph C_n is shown in Figure 1.

In the following theorem, we determine the exact value of the edge irregularity strength of C_n.

Theorem 2. *Let n be an integer with $n \geq 3$. Then $es(C_n) = 3n - 1$.*

Proof. Let C_n be the graph with vertex set $V(C_n) = \{a_i, c_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq 2n\}$ and edge set $E(C_n) = \{a_i a_{i+1}, c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{b_i b_{i+1} : 1 \leq i \leq 2n-1\} \cup \{a_i b_{2i-1}, c_i b_{2i} : 1 \leq i \leq n\}$. The maximum degree of C_n is $\Delta(C_n) = 3$. The order and size of graph C_n is $4n$ and $6n - 3$, respectively. From Theorem 1, we have $es(C_n) \geq \max\left\{\lceil \frac{6n-2}{2} \rceil, 3\right\} = \left\lceil \frac{6n-2}{2} \right\rceil = 3n - 1$. To prove the equality, it suffices to prove the existence of an optimal edge irregular $(3n-1)$-labeling.

We construct the labeling $\psi_1 : V(C_n) \to \{1, 2, 3, \ldots, 3n-1\}$ in the following way:

$\psi_1(a_i) = 2n - 1 + i$ for $1 \leq i \leq n$, $\psi_1(c_i) = i$ for $1 \leq i \leq n$,

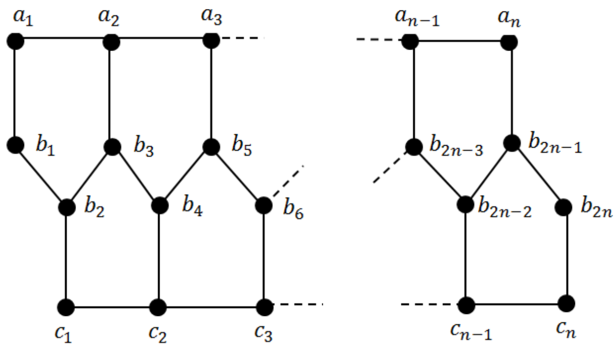

Figure 1. The plane graph C_n.

$$\psi_1(b_i) = \begin{cases} \frac{i}{2}, & \text{for } i \text{ even}, \ 1 \le i \le 2n \\ 2n + \frac{i-1}{2}, & \text{for } i \text{ odd}, \ 1 \le i \le 2n. \end{cases}$$

The edge weights are as follows:

$$w_{\psi_1}(a_i a_{i+1}) = 4n + 2i - 1, \ \text{for } 1 \le i \le n-1$$

$$w_{\psi_1}(b_i b_{i+1}) = 2n + i, \ \text{for } 1 \le i \le 2n-1$$

$$w_{\psi_1}(c_i c_{i+1}) = 2i + 1, \ \text{for } 1 \le i \le n-1$$

$$w_{\psi_1}(a_i b_{2i-1}) = 4n - 2 + 2i, \ \text{for } 1 \le i \le n$$

$$w_{\psi_1}(c_i b_{2i}) = 2i, \ \text{for } 1 \le i \le n$$

We can see that all vertex labels are at most $3n-1$. The edge weights under the labeling ψ_1 are distinct for all pairs of distinct edges and the labeling ψ_1 provides the upper bound on $es(C_n)$, i.e $es(C_n) \le 3n-1$. Combining with the lower bound, we conclude that $es(C_n) = 3n-1$. This completes the proof. □

Let A_n denotes the plane graph consisting of faces of length s where $s = 3, 4, 5$ and one external infinite face. Note that graph A_n is similar to C_n if one adds edges $b_{2i-1}b_{2i+1}$. The vertex set is $V(A_n) = \{a_i, b_i, c_i, d_i : 1 \le i \le n\}$ and the edge set is $E(A_n) = \{a_i a_{i+1}, c_i c_{i+1}, d_i d_{i+1}, b_i c_{i+1} : 1 \le i \le n-1\} \cup \{a_i b_i, b_i c_i, c_i d_i : 1 \le i \le n\}$. Moreover $|V(A_n)| = 4n$ and $|E(A_n)| = 7n-4$, see [11, 24]. The graph A_n is shown in Figure 2.

In the following theorem, we determine the exact value of the edge irregularity strength of A_n.

Theorem 3. *Let n be an integer with $n \ge 3$. Then $es(A_n) = \lceil \frac{7n-3}{2} \rceil$.*

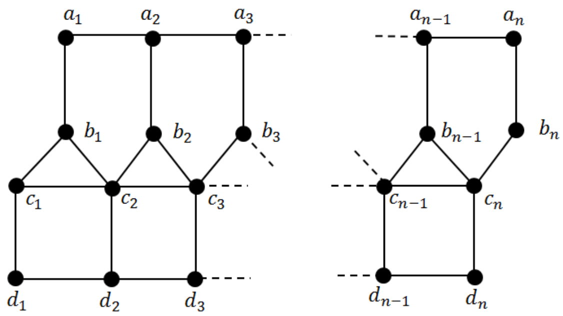

Figure 2. The plane graph A_n.

Proof. Let A_n be the graph with vertex set $V(A_n)$ and edge set $E(A_n)$ as defined previously. The maximum degree of A_n is $\Delta(A_n) = 5$. From Theorem 1, we have that $es(A_n) \geq \max\left\{\lceil\frac{7n-3}{2}\rceil, 5\right\} = \left\lceil\frac{7n-3}{2}\right\rceil$. To prove the equality, it suffices to prove the existence of an optimal edge irregular $\left\lceil\frac{7n-3}{2}\right\rceil$-labeling. Let $\psi_2 : V(A_n) \to \{1, 2, \ldots, \left\lceil\frac{7n-3}{2}\right\rceil\}$ such that

$$\psi_2(a_i) = \begin{cases} 1, & \text{for } i = 1 \\ \lfloor\frac{i-2}{2}\rfloor + 3i, & \text{for } 2 \leq i \leq n \end{cases}$$

$$\psi_2(b_i) = \begin{cases} 1, & \text{for } i = 1 \\ \lfloor\frac{i-1}{2}\rfloor + 3i - 1, & \text{for } 2 \leq i \leq n \end{cases}$$

$$\psi_2(c_i) = \begin{cases} 2, & \text{for } i = 1 \\ \lfloor\frac{i-1}{2}\rfloor + 3i - 2, & \text{for } 2 \leq i \leq n \end{cases}$$

$$\psi_2(d_i) = \begin{cases} 2, & \text{for } i = 1 \\ 3\lfloor\frac{i-2}{2}\rfloor + 2i + 2, & \text{for } 2 \leq i \leq n \end{cases}$$

The edge weights are as follows:

$$w_{\psi_2}(a_i a_{i+1}) = \begin{cases} 7, & \text{for } i = 1 \\ 7i + 1, & \text{for } 2 \leq i \leq n \end{cases}$$

$$w_{\psi_2}(c_i c_{i+1}) = \begin{cases} 6, & \text{for } i = 1 \\ 7i - 2, & \text{for } 2 \leq i \leq n - 1 \end{cases}$$

$$w_{\psi_2}(d_i d_{i+1}) = \begin{cases} 8, & \text{for } i = 1 \\ 7i, & \text{for } 2 \leq i \leq n - 1 \end{cases}$$

$$w_{\psi_2}(b_i c_{i+1}) = \begin{cases} 5, & \text{for } i = 1 \\ 7i - 1, & \text{for } 2 \leq i \leq n - 1 \end{cases}$$

$$w_{\psi_2}(a_ib_i) = \begin{cases} 2, & \text{for } i = 1 \\ 7i - 3, & \text{for } 2 \leq i \leq n \end{cases}$$

$$w_{\psi_2}(c_ib_i) = 2\lfloor\frac{i-1}{2}\rfloor + 6i - 3, \text{ for } 1 \leq i \leq n$$

$$w_{\psi_2}(c_id_i) = \begin{cases} 4, & \text{for } i = 1 \\ 2\left(\lfloor\frac{i-2}{2}\rfloor + 3i - 1\right), & \text{for } 2 \leq i \leq n \end{cases}$$

We can see that all vertex labels are at most $\lceil\frac{7n-3}{2}\rceil$. The edge weights under the labeling ψ_2 are distinct for all pairs of distinct edges and the labeling ψ_2 provides the upper bound on $es(A_n)$, i.e $es(A_n) \leq \lceil\frac{7n-3}{2}\rceil$. Combining with the lower bound, we conclude that $es(A_n) = \lceil\frac{7n-3}{2}\rceil$. This completes the proof. □

A quadrilateral snake Q_n is a plane graph obtained from a path $b_1, b_2, \ldots, b_n$ by adding new vertices $a_1, a_2, a_3 \ldots, a_{2(n-1)}$ and new edges $a_{2i-1}a_{2i}$, respectively and joining a_{2i-1} to $b - i$ and a_{2i} to b_{i+1}. The double quadrilateral snake $D(Q_n)$ is obtained from Q_n by adding new vertices $c_1, c_2, \ldots, c_{2(n-1)}$ and new edges $c_{2i-1}c_{2i}, c_{2i-1}b_i$ and $c_{2i}b_{i+1}$ for $1 \leq i \leq n-1$. It is clear that $|V(D(Q_n))| = 5n - 4$ and $|E(D(Q_n))| = 7(n-1)$. The graph $D(Q_n)$ is shown in Figure 3.

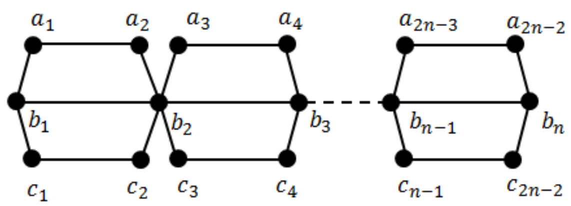

Figure 3. The plane graph $D(Q_n)$.

Theorem 4. *Let n be an integer with $n \geq 3$. Then $es(D(Q_n)) = \left\lceil\frac{7n-6}{2}\right\rceil$.*

Proof. Let $D(Q_n)$ be the plane graph with vertex set $V(D(Q_n)) = \{a_i, c_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq \frac{n}{2}+1\}$ and edge set $E(D(Q_n)) = \{a_{2i-1}a_{2i}, c_{2i-1}c_{2i} : 1 \leq i \leq \frac{n}{2}\} \cup \{b_ib_{i+1} : 1 \leq i \leq \frac{n}{2}\} \cup \{a_{2i-1}b_i : 1 \leq i \leq \frac{n}{2}+1\} \cup \{a_{2i}b_{i+1} : 1 \leq i \leq \frac{n}{2}\} \cup \{c_{2i}b_{i+1} : 1 \leq i \leq \frac{n}{2}\} \cup \{c_{2i-1}b_i : 1 \leq i \leq \frac{n}{2}+1\}$ where $n \geq 3$. The maximum degree of $D(Q_n)$ is $\Delta(D(Q_n)) = 6$. From Theorem 1, we have that $es(D(Q_n)) \geq \max\left\{\lceil\frac{7n-6}{2}\rceil, 6\right\} = \left\lceil\frac{7n-6}{2}\right\rceil$. To prove the equality, it suffices to prove the existence of an optimal edge irregular $\left\lceil\frac{7n-6}{2}\right\rceil$-labeling.

Let $\psi_3 : V(D(Q_n)) \to \{1, 2, \ldots, \left\lceil\frac{7n-6}{2}\right\rceil\}$ be the vertex labeling such that

$$\psi_3(a_i) = 2i - \lfloor\frac{i}{4}\rfloor - 1, \quad \text{for } 1 \leq i \leq 2n-2$$

$$\psi_3(b_i) = 3i + \left\lceil\frac{i-1}{2}\right\rceil - 2, \quad \text{for } 1 \leq i \leq n$$

$$\psi_3(c_i) = 2i - \lfloor\frac{i}{4}\rfloor, \quad \text{for } 1 \leq i \leq 2n-2$$

The edge weight are as follows:

$$w_{\psi_3}(a_{2i-1}a_{2i}) = 7i - 3, \quad \text{for } 1 \le i \le n-1,$$

$$w_{\psi_3}(a_{2i-1}b_i) = 7i - 5, \quad \text{for } 1 \le i \le n-1,$$

$$w_{\psi_3}(a_{2i}b_{i+1}) = 7i, \quad \text{for } 1 \le i \le n-1,$$

$$w_{\psi_3}(b_ib_{i+1}) = 7i - 2, \quad \text{for } 1 \le i \le n-1,$$

$$w_{\psi_4}(c_{2i-1}c_{2i}) = 7i - 1, \quad \text{for } 1 \le i \le n-1,$$

$$w_{\psi_3}(c_{2i-1}b_i) = 7i - 4, \quad \text{for } 1 \le i \le n-1,$$

$$w_{\psi_3}(c_{2i}b_{i+1}) = 7i + 1, \quad \text{for } 1 \le i \le n-1.$$

We can see that all vertex labels are at most $\lceil \frac{7n-6}{2} \rceil$. The edge weights under the labeling ψ_3 are distinct for all pairs of distinct edges and the labeling ψ_3 provides the upper bound on $es(D(Q_n))$, i.e. $es(D(Q_n)) \le \lceil \frac{7n-6}{2} \rceil$. Combining with the lower bound, we conclude that $es(D(Q_n)) = \lceil \frac{7n-6}{2} \rceil$. This completes the proof. □

3. Remarks and Conclusion

The problem studied in this paper is about edge irregularity strength of three classes of plane graphs. According to result for lower bound of $es(G)$ in Theorem 1 and all three upper bounds in Theorems 2, 3 and 4, we obtain the exact value for edge irregularity strength of these graphs.

The graphs considered in this paper are quite restricted. From our understanding, the $es(G)$ is indeed hard to compute for general graph G, or even for planar graphs. As families of planar graphs, we expect to study outerplanar graphs, planar graphs with bounded maximum degree, or with faces of small size, planar 2-trees, planar 3-trees, etc. They are common graph families in the literature of graph labeling and provide insight for other families of planar graphs that include them.

Presented graphs have bounded tree-width and path-width. Although this might be a mere observation, one can find a path decomposition such that the induced subgraphs in each partitions are identical (except possibly for the last one) and have $\lceil \frac{E}{n} \rceil$ edges. For example, for $D(Q_n)$ the path decomposition is defined by associating the 2-connected components to the nodes of the path. Two consecutive subgraphs have one vertex in common (cutvertex) and each subgraph has 7 edges (recall that $E(D(Q_n)) = 7n - 7$. One can compute the labeling for the first subgraph and propagate by linearly increasing the labels. If the edge irregularity is satisfied for the first two subgraphs, then it should hold for the entire graph. Such an approach (if correct) could potentially broaden the targeted graphs.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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