*Mathematics*

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(3): 2590–2606.
DOI: 10.3934/math.2021157
Received: 11 September 2020
Accepted: 21 December 2020
Published: 25 December 2020

*Research article*

# Thickness of the subgroup intersection graph of a finite group

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**Abstract:** Let $G$ be a finite group. The intersection graph of subgroups of $G$ is a graph whose vertices are all non-trivial subgroups of $G$ and in which two distinct vertices $H$ and $K$ are adjacent if and only if $H \cap K \neq 1$. In this paper, we classify all finite abelian groups whose thickness and outerthickness of subgroup intersection graphs are 1 and 2, respectively. We also investigate the thickness and outerthickness of subgroup intersection graphs for some finite non-abelian groups.

**Keywords:** thickness; outerthickness; subgroup intersection graph; finite group
**Mathematics Subject Classification:** 05C25, 05C10, 20D60

## 1. Introduction

Associating graphs to algebraic structures and studying their algebraic properties employing the methods in graph theory has been an interesting topic for mathematicians in the last decades and continuously arousing people's wide attention. For instance, Cayley graphs of groups [14], power graphs of groups ([2, 15, 24]), zero-divisor graphs of semigroups [17], zero-divisor graphs of commutative rings [5], total graphs of rings [1], unit graphs of rings [6], zero-divisor graphs of modules [9]. Bosák initiated the study of the intersection graphs of semigroups in [10]. Later, Csákány and Pollák defined the intersection graph of subgroups of a finite group and studied the graphic properties in [16]. In the recent years, several interesting properties of the intersection graphs of subgroups of groups have been obtained in the literature. In 1975, Zelinka investigated the intersection graphs of subgroups of finite abelian groups [33]. Akbari, Heydari and Maghasedi obtained some properties of the intersection graphs of subgroups of solvable groups in [7]. Shen classified finite groups with disconnected intersection graphs of subgroups in [29]. Ma gave an upper bound for the diameter of intersection graphs of finite simple groups in [23]. Rajkumar and Devi defined the intersection graph of cyclic subgroups of groups and obtained the clique number of intersection graph of subgroups of some non-abelian groups [26, 27]. Recall that, for a finite group $G$, the intersection graph of subgroups of $G$, denoted by $\Gamma(G)$, is a graph with all non-trivial subgroups of $G$ as its vertices and two distinct vertices $H$ and $K$ are adjacent in $\Gamma(G)$ if and only if $H \cap K \neq 1$.

The thickness of a graph, as a topological invariant of a graph, was first defined by Tutte in [30]. The problem is the same as determining whether $K_9$ is biplanar or not, that is, a union of two planar graphs and it was showed in [11]. Tutte generalized the problem by defining the concept of the thickness of a graph. It is an important research object in topological graph theory, and it also has important applications to VLSI design and network design (see [4, 28]). Outerthickness seems to be studied first in Geller's unpublished manuscript, where it was shown that $\theta_0(K_7)$ is 3 by similar exhaustive search as in the case of the thickness of $K_9$. It is known that determining the thickness of a graph is NP-hard [22]. Until now, there are only a few results on the thickness of graphs. The attention has been focused on finding upper bounds of thickness and outerthickness. In 1964, Beineke, Frank and Moon obtained the thickness of the complete bipartite graph [13]. Beineke and Harary obtained the thickness of the complete graph ([3, 12]). Chartrand, Geller and Hedetniemi conjectured that planar graph has an edge partition into two outerplanar graphs in [21]. Guy and Nowwkowski obtained the outerthickness of the complete graph and complete bipartite graph in [19, 20]. In [18], Ding et al. proved that every planar graph has an edge partition into two outerplanar graphs.

For the planarity of intersection graph of subgroups of a finite group, Ahmadi and Taeri, Kayacan and Yaraneri independently classified all finite groups with planar intersection graphs in [8, 25], respectively. By their main theorem, it is easy to obtain the planarity of intersection graph of subgroups of a finite abelian group.

**Theorem 1.1.** [8, 25] *Let $G$ be a finite abelian group. Then $\Gamma(G)$ is planar if and only if $G$ is one of the following groups:* $\mathbb{Z}_{p^\alpha}(\alpha = 2, 3, 4, 5), \mathbb{Z}_{p^\alpha q}(\alpha = 1, 2), \mathbb{Z}_{pqr}, \mathbb{Z}_p \oplus \mathbb{Z}_p, \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}$, *where $p, q, r$ are distinct primes.*

Motivated by the above theorem, the aim of this paper is to classify finite abelian groups whose thickness and outerthickness of intersection graphs of subgroups are 1 and 2, respectively. We also investigate the thickness and outerthickness of intersection graphs of subgroups of some finite non-abelian groups in the last section.

## 2. Preliminaries

Let us recall some basic definitions and notations in graph theory. Let $\Gamma$ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, and all graphs in this paper are simple graphs, that are, undirect, no loops and no multiple edges. We use $K_n$ to denote the complete graph on $n$ vertices and $K_{m,n}$ to denote the complete bipartite graph with one partition consisting of $m$ vertices and another partition consisting of $n$ vertices. The degree of a vertex $v$ in $\Gamma$, denoted by $deg_\Gamma(v)$, is the number of vertices incident to $v$. If $deg_\Gamma(v) = k$, then the vertex is denoted by $v_k$. A subgraph $\Gamma'$ of $\Gamma$ is provided $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. A clique of a graph $\Gamma$ is a complete subgraph of $\Gamma$. A maximal clique of a graph is a clique such that adding any vertex into the clique cannot be a clique. The clique number of a graph $\Gamma$ is the number of vertices in a maximal clique in $\Gamma$ with maximum size and it is denoted by $\omega(\Gamma)$. A subgraph $H$ of $\Gamma$ is a spanning subgraph of $\Gamma$ if $V(H) = V(\Gamma)$. The union of graphs $\Gamma_1$ and $\Gamma_2$ is the graph $\Gamma_1 \cup \Gamma_2$ with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$. Let $\Gamma_1 + \Gamma_2$ be a graph obtained from the union $\Gamma_1 \cup \Gamma_2$ by adding edges between each vertex in $\Gamma_1$ and each vertex in $\Gamma_2$. If $\Gamma_2$ has exactly one vertex $v$, we shall use $\Gamma_1 + v$ instead of $\Gamma_1 + \Gamma_2$. A graph is planar if it can be drawn in a plane such that no two edges intersect expect at their end vertices. An outerplanar graph is a planar graph that can be embedded in the plane without crossings in such a way that all the vertices lie in the boundary of the unbounded face of the embedding. Recall that the thickness of a graph $\Gamma$, denoted by $\theta(\Gamma)$, is the minimum number of planar subgraphs whose union is $\Gamma$. Similarly, the outerthickness $\theta_0(\Gamma)$ is obtained where the planar subgraphs are replaced by outerplanar subgraphs in the previous definition. Obviously, the thickness of a planar graph is 1 and the outerthickness of an outerplanar graph is 1.

The following lemmas are obvious and are used frequently later.

**Lemma 2.1.** [32, Lemma 2.1] *If $\Gamma_1$ is a subgraph of $\Gamma_2$, then $\theta(\Gamma_1) \leq \theta(\Gamma_2)$ and $\theta_0(\Gamma_1) \leq \theta_0(\Gamma_2)$.*

**Lemma 2.2.** [12, 19, 3] (1) $\theta(K_n) = \begin{cases} \lfloor \frac{n+7}{6} \rfloor & n \neq 9, 10 \\ 3 & n = 9, 10 \end{cases}$.

(2) $\theta_0(K_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil & n \neq 7 \\ 3 & n = 7 \end{cases}$.

*where $\lceil x \rceil$ denotes the least integer greater than or equal to x, $\lfloor x \rfloor$ denotes the largest integer greater than or equal to x*

By Lemma 2.2, one can see that $\theta(K_n) \geq 3$ for $n \geq 9$, $\theta_0(K_n) \geq 3$ for $n \geq 7$.

From Euler's formula, a planar graph $\Gamma$ has at most $3|V| - 6$ edges and an outerpalnar graph $\Gamma$ has at most $2|V| - 3$ edges, furthermore, one can get the lower bound for the thickness and outerthickness of a graph as follows.

**Lemma 2.3.** [13] *If $\Gamma = (V, E)$ is a graph with $|V| = n$ and $|E| = m$, then $\theta(\Gamma) \geq \lceil \frac{m}{3n-6} \rceil$ and $\theta_0(\Gamma) \geq \lceil \frac{m}{2n-3} \rceil$.*

**Lemma 2.4.** *Let $\Gamma$ be a graph. If $\theta(\Gamma) = k$, then $\theta(\Gamma + v_k) = k$. In particular, if $\Gamma$ is a complete graph, then $\theta(\Gamma + v_t) = k (1 \leq t \leq 2k)$.*

*Proof.* If $\Gamma$ is a planar graph, then $\Gamma + v_1$ is clearly a planar graph. If $\Gamma$ is not a planar graph, then we may assume that $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_k\}$ is a planar decomposition of $\Gamma$, where each $\Gamma_i$ is a spanning subgraph of $\Gamma$, then $\Gamma_i + v_1$ are planar graphs. Hence $\{\Gamma_1 + v_1, \Gamma_2 + v_1, \cdots, \Gamma_k + v_1\}$ is a decomposition of $\Gamma + v_k$. Thus, $\theta(\Gamma + v_k) \leq k$. By Lemma 2.1, $\theta(\Gamma + v_k) \geq \theta(\Gamma) = k$ and thus $\theta(\Gamma + v_k) = k$. In particular, if $\Gamma$ is a complete graph, then $\theta(\Gamma + v_t) = k (1 \leq t \leq 2k)$. □

**Lemma 2.5.** [31] *A graph is outerplanar if and only if it has no subgraph homeomorphic to $K_4$ or $K_{2,3}$.*

## 3. Classification of finite abelian groups

The aim of this section is to classify finite abelian groups whose intersection graphs have thickness and outerthickness 1 and 2. We first consider the case of finite cyclic groups.

**Theorem 3.1.** *Let $G = \mathbb{Z}_n$ be a cyclic group of order $n$ and $\Gamma(G)$ is not an empty graph. Then the following statements are hold.*

(1) $\theta(\Gamma(G)) = 1$ *if and only if* $G$ *is one following groups:*

$$\mathbb{Z}_{p^\alpha}(\alpha = 2,3,4,5), \mathbb{Z}_{p_1 p_2^\alpha}(\alpha = 1,2), \mathbb{Z}_{p_1 p_2 p_3},$$

*where* $p_1$, $p_2$, $p_3$ *and* $p$ *are distinct primes.*

(2) $\theta(\Gamma(G)) = 2$ *if and only if* $G$ *is one following groups:*

$$\mathbb{Z}_{p^\alpha}(\alpha = 6,7,8,9), \mathbb{Z}_{p_1 p_2^\alpha}(\alpha = 3,4), \mathbb{Z}_{p_1^2 p_2^\alpha}(\alpha = 2,3), \mathbb{Z}_{p_1 p_2 p_3^2}, \mathbb{Z}_{p_1 p_2 p_3 p_4},$$

*where* $p_1$, $p_2$, $p_3$, $p_4$ *and* $p$ *are distinct primes.*

*Proof.* Suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$. Then we have $V(\Gamma(\mathbb{Z}_n)) = \prod_{i=1}^{s}(\alpha_i+1)-2$. We complete our proofs by discussing the number of prime divisors of $n$.

**Case 1**. $s \geq 5$. It is clear that

$$\langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_4\rangle, \langle p_1p_2\rangle, \langle p_1p_3\rangle, \langle p_1p_4\rangle, \langle p_1p_2p_3\rangle, \langle p_1p_2p_3p_4\rangle$$

are nine non-trivial subgroups of $G$. Note that any pair of these nine subgroups has non-trivial intersection. So they form a complete graph $K_9$, which is a subgraph of $\Gamma(G)$. Thus, $\omega(\Gamma(G)) \geq 9$. By Lemma 2.1, $\theta(\Gamma(G)) \geq \theta(K_9) = 3$.

**Case 2**. $s = 4$. If $\alpha_4 \geq 2$, then $G$ has at least nine non-trivial subgroups, namely,

$$\langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_4\rangle, \langle p_1p_2\rangle, \langle p_1p_3\rangle, \langle p_1p_4\rangle, \langle p_1p_2p_3\rangle, \langle p_1p_2p_3p_4\rangle.$$

Note that any pair of these nine subgroups has non-trivial intersection. So they form a complete graph $K_9$, which is a subgraph of $\Gamma(G)$. Thus, $\theta(\Gamma(G)) \geq \theta(K_9) = 3$ by Lemma 2.1. If $\alpha_4 = 1$, then $G \simeq \mathbb{Z}_{p_1p_2p_3p_4}$. So, $G$ has totally 14 non-trivial subgroups, namely,

$$\langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_4\rangle, \langle p_1p_2\rangle, \langle p_1p_3\rangle, \langle p_1p_4\rangle, \langle p_2p_3\rangle, \langle p_2p_4\rangle,$$
$$\langle p_3p_4\rangle, \langle p_1p_2p_3\rangle, \langle p_1p_2p_4\rangle, \langle p_1p_3p_4\rangle, \langle p_2p_3p_4\rangle.$$

Note that $\langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle$, $\langle p_1p_2\rangle$, $\langle p_1p_3\rangle$, $\langle p_2p_3\rangle$, $\langle p_1p_2p_3\rangle$ form $K_7$ as a subgraph of $\Gamma(G)$. So $\omega(\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})) \geq 7$ and $V(\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})) = 14$. By Lemma 2.1 and Lemma 2.2, $2 = \theta(K_7) \leq \theta(\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})) \leq \theta(K_{14}) = 3$. A planar decomposition of $\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})$ shown in Figure 1 deduces that $\theta(\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})) = 2$.

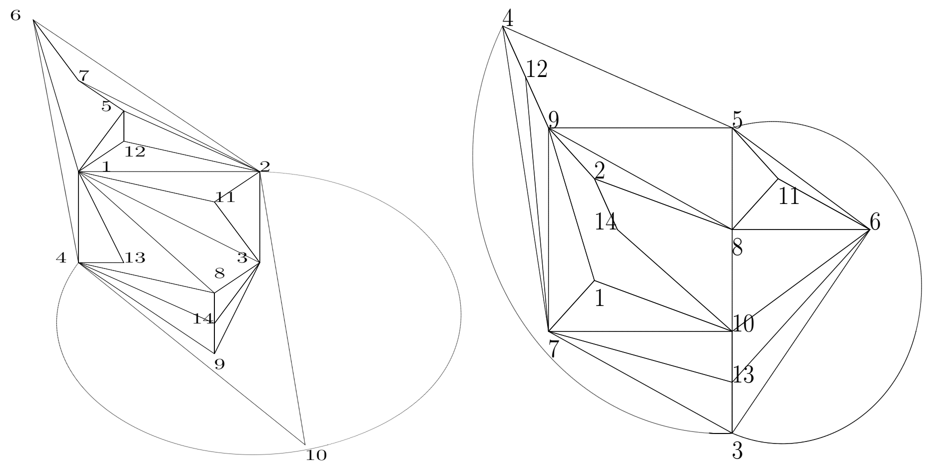


**Figure 1.** A planar decomposition of $\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})$.

**Case 3**. $s = 3$. If $\alpha_3 \geq 3$, then

$$\langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_3^2\rangle, \langle p_1p_2\rangle, \langle p_1p_3\rangle, \langle p_2p_3\rangle, \langle p_1p_2p_3\rangle, \langle p_1p_3^2\rangle$$

are nine non-trivial subgroups of $G$. Any pair of these nine subgroups has non-trivial intersection. So they form $K_9$, which is a subgraph of $\Gamma(G)$. By Lemma 2.1, $\theta(\Gamma(G)) \geq \theta(K_9) = 3$. If $\alpha_2 = \alpha_3 = 2$, then

$$\langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_3^2\rangle, \langle p_1p_2\rangle, \langle p_1p_3\rangle, \langle p_2p_3\rangle, \langle p_1p_2p_3\rangle, \langle p_1p_3^2\rangle$$

are nine non-trivial subgroups of $G$. Any pair of these nine subgroups has non-trivial intersection. They also form $K_9$ as a subgraph of $\Gamma(G)$. So $\theta(\Gamma(G)) \geq \theta(K_9) = 3$ by Lemma 2.1.

If $\alpha_1 = \alpha_2 = 1, \alpha_3 = 2$, then $G$ has totally ten non-trivial subgroups, namely,

$$\langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_3^2\rangle, \langle p_1p_2\rangle, \langle p_1p_3\rangle, \langle p_1p_3^2\rangle, \langle p_2p_3\rangle, \langle p_2p_3^2\rangle, \langle p_1p_2p_3\rangle.$$

Note that $\langle p_1\rangle$, $\langle p_2\rangle$, $\langle p_3\rangle$, $\langle p_1p_2\rangle$, $\langle p_1p_3\rangle$, $\langle p_2p_3\rangle$, $\langle p_1p_2p_3\rangle$ form $K_7$ as a subgraph of $\Gamma(G)$. Then $\omega(\Gamma(G)) \geq 7$ and $V(\Gamma(G)) = 10$. By Lemma 2.1 and Lemma 2.2, $2 = \theta(K_7) \leq \theta(\Gamma(G)) \leq \theta(K_{10}) = 3$. We can give a planar decomposition of $\Gamma(\mathbb{Z}_{p_1p_2p_3^2})$ as shown in Figure 2. Thus, $\theta(\Gamma(G) = 2$.

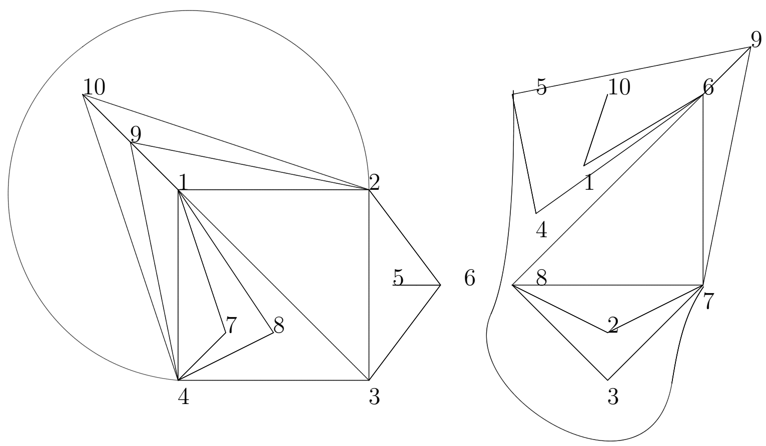


**Figure 2.** A planar decomposition of $\Gamma(\mathbb{Z}_{p_1p_2p_3^2})$.

If $\alpha_1 = \alpha_2 = \alpha_3 = 1$, by Theorem 1.1, $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ is a planar graph. So $\theta(\Gamma(\mathbb{Z}_{p_1p_2p_3})) = 1$.

**Case 4**. $s = 2$. In this case, it easy to see that $\Gamma(G) \cong K_{\alpha_1\alpha_2-1} + (K_{\alpha_1} \cup K_{\alpha_2})$. Then $\omega(\Gamma(\mathbb{Z}_{p_1^{\alpha_1}p_2^{\alpha_2}})) = \alpha_1\alpha_2 - 1 + \alpha_2$. If $\alpha_1\alpha_2 - 1 + \alpha_2 \geq 9$, then $\theta(\Gamma(G)) \geq \theta(K_9) = 3$. So $\alpha_1\alpha_2 - 1 + \alpha_2 \leq 8$ and implies that $\alpha_2 \leq 4$.

If $\alpha_2 = 4$, then $\alpha_1 = 1$. So, $\Gamma(G) \simeq K_3 + (K_4 \cup K_1)$. By Lemma 2.1 and Lemma 2.2, $2 = \theta(K_7) \leq \theta(\Gamma(\mathbb{Z}_{p_1p_2^4})) \leq \theta(K_8) = 2$. This deduces that $\theta(\Gamma(\mathbb{Z}_{p_1p_2^4})) = 2$.

If $\alpha_2 = 3$, then $\alpha_1 \leq 2$. If $\alpha_1 = 2$, then $\Gamma(G) \simeq K_5 + (K_3 \cup K_2)$, By Lemma 2.1 and Lemma 2.2, $2 = \theta(K_8) \leq \theta(\Gamma(\mathbb{Z}_{p_1^2p_2^3})) \leq \theta(K_{10}) = 3$. A planar decomposition of $\Gamma(\mathbb{Z}_{p_1^2p_2^3})$ as shown in Figure 3 deduces that $\theta(\Gamma(\mathbb{Z}_{p_1^2p_2^3})) = 2$. If $\alpha_1 = 1$, then $\Gamma(G) \simeq K_2 + (K_3 \cup K_1)$. By Lemma 2.1 and Lemma 2.2, $2 = \theta(K_5) \leq \theta(\Gamma(\mathbb{Z}_{p_1p_2^3})) \leq \theta(K_6) = 2$. So, $\theta(\Gamma(\mathbb{Z}_{p_1p_2^3})) = 2$.

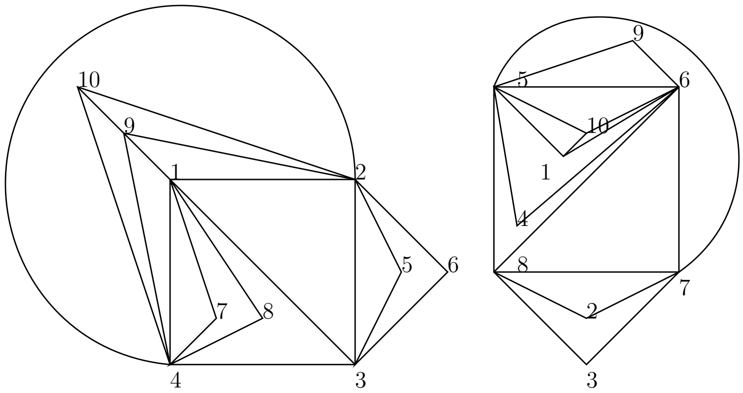


**Figure 3.** A planar decomposition of $\Gamma(\mathbb{Z}_{p_1^2p_2^3})$.

If $\alpha_2 = \alpha_1 = 2$, then $\Gamma(G) \simeq K_3 + (K_2 \cup K_2)$. By Lemma 2.1 and Lemma 2.2, $2 = \theta(K_5) \leq \theta(\Gamma(\mathbb{Z}_{p_1^2p_2^2})) \leq \theta(K_7) = 2$. Thus, $\theta(\Gamma(\mathbb{Z}_{p_1^2p_2^2})) = 2$.

If $\alpha_2 = 2$ and $\alpha_1 = 1$ or $\alpha_1 = \alpha_2 = 1$, then by Theorem 1.1, $\Gamma(\mathbb{Z}_{p_1p_2})$ and $\Gamma(\mathbb{Z}_{p_1p_2^2})$ are planar graphs. So, $\theta(\Gamma(\mathbb{Z}_{p_1p_2})) = \theta(\Gamma(\mathbb{Z}_{p_1p_2^2})) = 1$.

**Case 5**. $s = 1$. In this case, $\Gamma(\mathbb{Z}_{p^\alpha}) \simeq K_{\alpha-1}$. By Lemma 2.2, it deduces that if $\alpha = 2, 3, 4, 5$, then $\theta(\Gamma(\mathbb{Z}_{p^\alpha})) = 1$; and if $\alpha = 6, 7, 8, 9$, then $\theta(\Gamma(\mathbb{Z}_{p^\alpha})) = 2$.

This completes the proof. □

For the outerthickness of $\Gamma(G)$ of a cyclic group $G$, we have following theorem.

**Theorem 3.2.** *Let $G = \mathbb{Z}_n$ be a cyclic group of order $n$ and $\Gamma(G)$ is not an empty graph. Then the following statements are hold.*

(1) $\theta_0(\Gamma(G)) = 1$ *if and only if G is one following groups:* $\mathbb{Z}_{p^\alpha}(\alpha = 2, 3, 4)$, $\mathbb{Z}_{p_1p_2^\alpha}(\alpha = 1, 2)$, $\mathbb{Z}_{p_1p_2p_3}$, *where $p_1$, $p_2$, $p_3$ and $p$ are primes.*

(2) $\theta_0(\Gamma(G)) = 2$ *if and only if G is one following groups:* $\mathbb{Z}_{p^\alpha}(\alpha = 5, 6, 7)$, $\mathbb{Z}_{p_1^2p_2^2}$, $\mathbb{Z}_{p_1p_2^3}$, *where $p_1$, $p_2$ and $p$ are primes.*

*Proof.* Suppose that $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$. According to the proof of Theorem 3.1, if $s \geq 4$, then $\theta_0(\Gamma(G)) \geq 3$. We proceed with three cases.

**Case 1**. $s = 3$. Again according to the proof of Theorem 3.1, if $\theta_0(\Gamma(G)) \leq 2$, then $G \simeq \mathbb{Z}_{p_1p_2p_3}$. In this case, it is clear that $\Gamma(G)$ contains no subgraph homeomorphic to $K_4$ or $K_{2,3}$. By Lemma 2.5, $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ is an outerplanar graph. So, $\theta_0(\Gamma(\mathbb{Z}_{p_1p_2p_3})) = 1$.

**Case 2**. $s = 2$. In this case, $\Gamma(G) \simeq K_{\alpha_1\alpha_2-1} + (K_{\alpha_1} \cup K_{\alpha_2})$. Then $\omega(\Gamma(G)) = \alpha_1\alpha_2 - 1 + \alpha_2$. If $\alpha_1\alpha_2 - 1 + \alpha_2 \geq 7$, then $\theta_0(\Gamma(G)) \geq 3$. So $\alpha_1\alpha_2 - 1 + \alpha_2 \leq 6$. This deduces that $\alpha_2 \leq 3$.

If $\alpha_2 = 3$, then $\alpha_1 = 1$. So $\Gamma(G) \simeq K_2 + (K_3 \cup K_1)$. By Lemma 2.1 and Lemma 2.2, $2 = \theta_0(K_5) \leq \theta_0(\Gamma(\mathbb{Z}_{p_1p_2^3})) \leq \theta_0(K_6) = 2$. Hence $\theta_0(\Gamma(\mathbb{Z}_{p_1p_2^3})) = 2$.

If $\alpha_2 = 2, \alpha_1 = 2$, then $\Gamma(G) \simeq K_3 + (K_2 \cup K_2)$. By Lemma 2.1 and Lemma 2.2, $2 = \theta_0(K_5) \leq \theta_0(\Gamma(\mathbb{Z}_{p_1^2p_2^2})) \leq \theta_0(K_7) = 3$. An outerplanar decomposition of $\Gamma(\mathbb{Z}_{p_1^2p_2^2})$ as shown in Figure 4 deduces that $\theta_0(\Gamma(\mathbb{Z}_{p_1^2p_2^2})) = 2$.

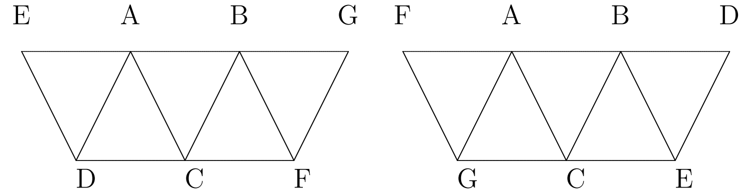


**Figure 4.** An outerplanar decomposition of $\Gamma(\mathbb{Z}_{p_1^2 p_2^2})$.

If $\alpha_2 = 2, \alpha_1 = 1$, then $\Gamma(G) \simeq K_1 + (K_2 \cup K_1)$. It is clear that $\Gamma(G)$ contains no subgraph homeomorphic to $K_4$ or $K_{2,3}$. By Lemma 2.5, $\Gamma(\mathbb{Z}_{p_1 p_2^2})$ is an outerplanar graph. Then, $\theta_0(\Gamma(\mathbb{Z}_{p_1 p_2^2})) = 1$.

If $\alpha_1 = \alpha_2 = 1$, then $\Gamma(G) \simeq K_2$. So, $\Gamma(\mathbb{Z}_{p_1 p_2})$ is an outerplanar graph and thus $\theta_0(\Gamma(\mathbb{Z}_{p_1 p_2})) = 1$.

**Case 3**. $s = 1$. In this case, $\Gamma(Z_{p^\alpha}) \simeq K_{\alpha-1}$. By Lemma 2.2, it deduces that if $\alpha = 2, 3, 4$, then $\theta_0(\Gamma(\mathbb{Z}_{p^\alpha})) = 1$; and if $\alpha = 5, 6, 7$, then $\theta_0(\Gamma(\mathbb{Z}_{p^\alpha})) = 2$. The proof is complete. □

Now, we consider the case of finite non-cyclic groups. Let $G$ be a finite Abelian group but not a cyclic group. Then by fundamental theorem of finite abelian group, we have $G \simeq G_1 \oplus G_2 \oplus \cdots \oplus G_s$, where $G_i (i = 1, 2, \cdots, s)$ is a cyclic group of prime power order.

**Lemma 3.3.** *Let $G \simeq G_1 \oplus G_2 \oplus \cdots \oplus G_s$, where $s \geq 2$ and $G_i (i = 1, 2, \cdots, s)$ is a cyclic group of prime power order. If $s \geq 4$, then $\theta(\Gamma(G)) \geq 3$.*

*Proof.* Set

$$H_1 = \langle (1, 0, 0, 0, \ldots, 0) \rangle, H_2 = \langle (1, 0, 0, 0, \ldots, 0), (0, 1, 0, 0, \ldots, 0) \rangle,$$

$$H_3 = \langle (1, 0, 0, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0) \rangle, H_4 = \langle (1, 0, 0, 0, \ldots, 0), (0, 0, 0, 1, \ldots, 0) \rangle,$$

$$H_5 = \langle (1, 0, 0, 0, \ldots, 0), (0, 1, 1, 0, \ldots, 0) \rangle, H_6 = \langle (1, 0, 0, 0, \ldots, 0), (0, 1, 0, 1, \ldots, 0) \rangle,$$

$$H_7 = \langle (1, 0, 0, 0, \ldots, 0), (0, 0, 1, 1, \ldots, 0) \rangle, H_8 = \langle (1, 0, 0, 0, \ldots, 0), (0, 1, 1, 1, \ldots, 0) \rangle,$$

$$H_9 = \langle (1, 0, 0, 0, \ldots, 0), (0, 1, 0, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0), (0, 1, 1, 0, \ldots, 0) \rangle.$$

Then they are nine non-trivial subgroups of $G$. Note that any pair of these nine subgroups has non-trivial intersection. So they form $K_9$, which is a subgraph of $\Gamma(G)$. Thus, $\theta(\Gamma(G)) \geq \theta(K_9) = 3$.

**Theorem 3.4.** *Let $G$ be a finite non-cyclic abelian group. Then the following statements are hold.*

(1) $\theta(\Gamma(G)) = 1$ *if and only if $G$ is one following groups: $\mathbb{Z}_2 \oplus \mathbb{Z}_{2p}$, $\mathbb{Z}_p \oplus \mathbb{Z}_p$, where $p$ is a prime.*

(2) $\theta(\Gamma(G)) = 2$ *if and only if $G$ is one following groups: $\mathbb{Z}_3 \oplus \mathbb{Z}_{3q}$, $\mathbb{Z}_5 \oplus \mathbb{Z}_{5q}$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_8$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $q$ is a prime.*

*Proof.* Let the number of distinct prime factors of order of $|G|$ be $s$. By Lemma 3.3, we know that if $\theta(\Gamma(G)) \leq 2$, then $s \leq 3$. By assumption that $G$ is not cyclic, we have three cases for our discussion.

**Case 1**. $G \simeq \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta} \oplus \mathbb{Z}_{p^\gamma}$, $p$ is a prime.

**Case 2**. $G \simeq \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta} \oplus \mathbb{Z}_{q^\gamma}$, $p, q$ are distinct primes.

**Case 3**. $G \simeq \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}$, $p$ is a prime.

For the **Case 1**, if there is at least one in $\{\alpha, \beta, \gamma\}$ greater than 1, then we may assume $\gamma \geq 2$. Set

$$A_1 = \langle(0,0,1),(0,1,0)\rangle, A_2 = \langle(0,0,1),(1,0,0)\rangle, A_3 = \langle(1,1,0),(1,1,1)\rangle,$$
$$A_4 = \langle(0,1,0),(1,0,0)\rangle, A_5 = \langle(0,1,0),(1,0,1)\rangle, A_6 = \langle(1,0,0),(0,1,1)\rangle,$$
$$A_7 = \langle(1,1,0),(0,1,1)\rangle, A_8 = \langle(0,0,p),(0,1,0)\rangle, A_9 = \langle(0,0,p),(1,0,0)\rangle.$$

It is easy to see that they are nine non-trivial subgroups of $G$. Note that any pair of these nine subgroups has non-trivial intersection. So they form $K_9$, which is a subgraph of $\Gamma(G)$. Thus, $\theta(\Gamma(G)) \geq \theta(K_9) = 3$.

Now, we consider the case of $\alpha = \beta = \gamma = 1$. If $p \geq 3$, we let

$$B_1 = \langle(0,0,1),(0,1,0)\rangle, B_2 = \langle(0,0,1),(1,0,0)\rangle, B_3 = \langle(0,0,1),(1,1,0)\rangle,$$
$$B_4 = \langle(0,0,1),(-1,1,0)\rangle, B_5 = \langle(0,1,0),(1,0,0)\rangle, B_6 = \langle(0,1,0),(1,0,1)\rangle,$$
$$B_7 = \langle(0,1,0),(-1,0,1)\rangle, B_8 = \langle(0,1,1),(1,0,0)\rangle, B_9 = \langle(0,1,1),(1,0,1)\rangle.$$

They also form $K_9$ and $\theta(\Gamma(G)) \geq \theta(K_9) = 3$.

If $p = 2$, then $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus Z_2$. $G$ has totally 14 non-trivial subgroups, namely,

$$C_1 = \langle(0,0,1),(0,1,0)\rangle, C_2 = \langle(0,0,1),(1,0,0)\rangle, C_3 = \langle(1,1,0),(1,1,1)\rangle,$$
$$C_4 = \langle(0,1,0),(1,0,0)\rangle, C_5 = \langle(0,1,0),(1,0,1)\rangle, C_6 = \langle(1,0,0),(0,1,1)\rangle,$$
$$C_7 = \langle(1,1,0),(0,1,1)\rangle, C_8 = \langle(0,0,1)\rangle, C_9 = \langle(1,0,0)\rangle, C_{10} = \langle(1,1,0)\rangle,$$
$$C_{11} = \langle(0,1,0)\rangle, C_{12} = \langle(1,0,1)\rangle, C_{13} = \langle(0,1,1)\rangle, C_{14} = \langle(1,1,1)\rangle.$$

It is easy to see that $\Gamma(G) = K_7 + 7v_3$. By Lemma 2.4, we get $\theta(\Gamma(G)) = \theta(K_7) = 2$.

For the **Case 2**, we may assume $\alpha \leq \beta$. If $\beta \geq 2$, we let

$$D_1 = \langle(0,p,0)\rangle, D_2 = \langle(0,1,0),(0,0,1)\rangle, D_3 = \langle(0,1,0),(1,0,0)\rangle,$$
$$D_4 = \langle(0,1,0)\rangle, D_5 = \langle(0,p,0),(0,0,1)\rangle, D_6 = \langle(0,p,0),(1,0,0)\rangle,$$
$$D_7 = \langle(1,1,0)\rangle, D_8 = \langle(0,p,0),(0,0,1),(1,0,0)\rangle, D_9 = \langle(0,p,0),(0,0,1),(1,1,0)\rangle.$$

Note that $D_i \cap D_j = D_1$ for all $i \neq j$. So, they form a $K_9$ and $\theta(\Gamma(G)) \geq \theta(K_9) = 3$.

If $\gamma \geq 2$, then we let

$$E_1 = \langle(0,0,q)\rangle, E_2 = \langle(0,0,1),(1,0,0)\rangle, E_3 = \langle(0,0,1),(1,1,0)\rangle,$$
$$E_4 = \langle(0,0,q),(0,1,0)\rangle, E_5 = \langle(0,0,q),(1,0,0)\rangle, E_6 = \langle(0,0,q),(1,1,0)\rangle,$$
$$E_7 = \langle(0,0,1)\rangle, E_8 = \langle(0,0,1),(0,1,0)\rangle, E_9 = \langle(0,0,q),(0,1,0),(1,0,0)\rangle.$$

Note that $E_i \cap E_j = E_1$ for all $i \neq j$. So, they form a $K_9$ and $\theta(\Gamma(G)) \geq \theta(K_9) = 3$.

Now, let $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_q$. Then all non-trivial subgroups of $G$ are:

$$I = \langle(0,0,1),(0,1,0)\rangle,$$

$$M_i = \langle(0,0,1),(1,i,0)\rangle, \quad \text{where} \quad 0 \le i \le p-1,$$

$$T = \langle(0,1,0),(1,0,0)\rangle, K = \langle(0,1,0)\rangle,$$

$$N_j = \langle(1,j,0)\rangle, \quad \text{where} \quad 0 \le j \le p-1.$$

It is not difficult to see that $\Gamma(G) = (K_{p+3} - e) + (p+1)v_2$. If $p \ge 7$, then $\theta(\Gamma(G)) \ge \theta(K_9) = 3$. So, $p \le 5$. By Lemma 2.2, if $p = 2$, then $\theta(\Gamma(G)) = 1$; and if $p = 3, 5$, then $\theta(\Gamma(G)) = 2$.

For the **Case 3**, $G \simeq \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}$. We may assume that $\beta \ge \alpha$. If $\alpha \ge 2$ and $\beta \ge 3$, then we set

$$F_1 = \langle(0,p^\alpha)\rangle, F_2 = \langle(0,p)\rangle, F_3 = \langle(0,1)\rangle, F_4 = \langle(0,1),(1,0)\rangle, F_5 = \langle(0,p),(1,0)\rangle,$$

$$F_6 = \langle(0,1),(p,0)\rangle, F_7 = \langle(p,1)\rangle, F_8 = \langle(0,p),(p,0)\rangle, F_9 = \langle(p,p)\rangle.$$

They are non-trivial subgroups of $G$. Also, $F_1 = F_i \cap F_j$ for all $i \ne j$. It follows that they form $K_9$ as a subgraph of $\Gamma(G)$. Thus, $\theta(\Gamma(G)) \ge \theta(K_9) = 3$.

If $\alpha = \beta = 2$, then $G \simeq \mathbb{Z}_{p^2} \oplus Z_{p^2}$. Now, $G$ has totally $p+1$ subgroups of order $p$, i.e., $\langle(0,p)\rangle, \langle(p,0)\rangle, \langle(p,p)\rangle, \cdots \langle(p^2-p,p)\rangle$; $G$ has totally $p^2+p+1$ subgroups of order $p^2$, i.e., $Q = \langle(0,p),(p,0)\rangle, \langle(1,0)\rangle, \langle(0,-1)\rangle, \langle(1,-1)\rangle, \cdots \langle(p^2-1,-1)\rangle$; $\langle(1,p)\rangle, \langle(1,2p)\rangle, \langle(1,3p)\rangle, \cdots \langle(1,p^2-p)\rangle$; $G$ has totally $p+1$ subgroups of order $p^3$, i.e., $\langle(0,1),(p,0)\rangle, \langle(0,p),(1,0)\rangle, \langle(0,p),(p,0),(1,1)\rangle, \langle(0,p),(p,0),(1,2)\rangle, \cdots \langle(0,p),(p,0),(1,p-1)\rangle$. It is not difficult to see that $\Gamma(\mathbb{Z}_{p^2} \oplus Z_{p^2}) \simeq K_{p+2} + (p+1)K_{p+1}$. So, $\omega(\Gamma(\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2})) = 2p+3$, $V(\Gamma(\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2})) = p^2+3p+3$. If $p \ge 3$, then $\omega(\Gamma(G)) \ge 9$ and $\theta(\Gamma(G)) \ge 3$. So $p = 2$. By Lemma 2.1 and Lemma 2.2, $2 = \theta(K_7) \le \theta(\Gamma(\mathbb{Z}_4 \oplus \mathbb{Z}_4) \le \theta(K_{13}) = 3$. On the other hand, a planar decomposition of $\Gamma(\mathbb{Z}_4 \oplus \mathbb{Z}_4)$ shown in Figure 5 gives $\theta(\Gamma(\mathbb{Z}_4 \oplus \mathbb{Z}_4)) = 2$.

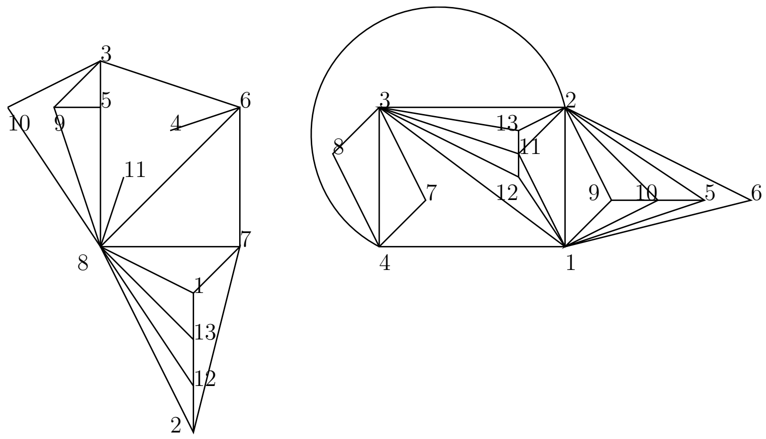


**Figure 5.** A planar decomposition of $\Gamma(Z_4 \oplus Z_4)$.

If $\alpha = 1$, then $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_{p^\beta}$. All subgroups of $G$ can be listed below.
$\langle(0,1)\rangle, \langle(1,1)\rangle, \cdots \langle(p-1,1)\rangle$,
$\langle(0,p)\rangle, \langle(1,p)\rangle, \cdots \langle(p-1,p)\rangle$,
$\langle(0,p^2)\rangle, \langle(1,p^2)\rangle, \cdots \langle(p-1,p^2)\rangle$,
$\cdots\cdots$
$\langle(0,p^{\beta-1})\rangle, \langle(1,p^{\beta-1})\rangle, \cdots \langle(p-1,p^{\beta-1})\rangle$.

It is clear that $\Gamma(G) \simeq K_{\beta-1} + (K_{p\beta-p+1} \cup pK_1)$. So $K_{\beta p+\beta-p}$ is a subgraph of $\Gamma(G)$. If $\beta \geq 4$, then $\omega(\Gamma(G)) \geq 9$ and $\theta(\Gamma(G)) \geq 3$. So, $\beta \leq 3$.

If $\beta = 3$, then $p = 2$ and $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_8$. So $\Gamma(G) \simeq K_2 + (K_5 \cup 2K_1)$ and $\theta(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_8) \geq \theta(K_7) = 2$. By Lemma 2.4, $\theta(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_8)) = \theta(K_7) = 2$.

If $\beta = 2$, then $p \leq 5$. If $p = 2$, then $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $\Gamma(G) \simeq K_1 + (K_3 \cup 2K_1)$. So, $\theta(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_4)) \geq \theta(K_4) = 1$. By Lemma 2.4, $\theta(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_4)) = \theta(K_4) = 1$. If $p = 3$, then $G \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_9$ and $\Gamma(G) \simeq K_1 + (K_4 \cup 3K_1)$. Thus, $\theta(\Gamma(\mathbb{Z}_3 \oplus \mathbb{Z}_9)) \geq \theta(K_5) = 2$. By Lemma 2.4, $\theta(\Gamma(\mathbb{Z}_3 \oplus \mathbb{Z}_9)) = \theta(K_5) = 2$. If $p = 5$, then $G \simeq \mathbb{Z}_5 \oplus \mathbb{Z}_{25}$ and $\Gamma(G) \simeq K_1 + (K_6 \cup 5K_1)$. Thus, $\theta(\Gamma(\mathbb{Z}_5 \oplus \mathbb{Z}_{25})) \geq \theta(K_7) = 2$. By Lemma 2.4, then $\theta(\Gamma(\mathbb{Z}_5 \oplus \mathbb{Z}_{25})) = \theta(K_7) = 2$.

If $\beta = 1$, then $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$. So, $\Gamma(G) \simeq (p+1)K_1$. By Theorem 1.1, $\theta(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p)) = 1$.

This completes our proof. □

**Theorem 3.5.** *Let G be a finite non-cyclic abelian group. Then the following statements are hold.*

(1) $\theta_0(\Gamma(G)) = 1$ *if and only if* $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$*, where p is a prime.*

(2) $\theta_0(\Gamma(G)) = 2$ *if and only if* $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}$ *or* $\mathbb{Z}_3 \oplus \mathbb{Z}_{3p}$*, where p is a prime.*

*Proof.* Suppose $G \simeq G_1 \oplus G_2 \oplus G_3 \oplus \cdots \oplus G_s$, where $s \geq 2$ and $G_i (i = 1, 2, 3, \cdots, s)$ are cyclic groups. According to the proof of Theorem 3.4, if $\theta_0(\Gamma(G)) \leq 2$, then $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$ or $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_q$ or $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$.

**Case 1**. $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$. In this case, $\Gamma(G) \simeq (p+1)K_1$. It is clear that $\Gamma(G)$ contains no subgraph homeomorphic to $K_4$ or $K_{2,3}$. By Lemma 2.5, $\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p)$ is an outerplanar graph, then $\theta_0(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p)) = 1$.

**Case 2**. $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_q$. Again by the proof of Theorem 3.4, we know that $\omega(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_q)) = p+2$. If $p \geq 5$, then $\theta_0(\Gamma(G)) \geq \theta_0(K_7) = 3$. So $p = 2, 3$. If $p = 2$, then $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_{2q}$. It is easy to see that $\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_{2q}) = (K_5 - e) + 3v_2$. By Lemma 2.1 and Lemma 2.2, $2 = \theta_0(K_4) \leq \theta_0(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_{2q})) \leq \theta_0(K_5) = 2$. Thus, $\theta_0(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_{2q})) = 2$. If $p = 3$, then $G \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_{3q}$. It is easy to see that $\Gamma(\mathbb{Z}_3 \oplus \mathbb{Z}_{3q}) = (K_6 - e) + 4v_2$. By Lemma 2.1 and Lemma2.2, $2 = \theta_0(K_5) \leq \theta_0(\Gamma(\mathbb{Z}_3 \oplus \mathbb{Z}_{3q})) \leq \theta_0(K_6) = 2$. So $\theta_0(\Gamma(\mathbb{Z}_3 \oplus \mathbb{Z}_{3q})) = 2$.

**Case 3**. $G \simeq \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$. In this case, $\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}) \simeq K_1 + (K_{p+1} + pK_1)$. Thus, $\omega(\Gamma(G)) = p+2$. If $\omega(\Gamma(G)) \geq 7$, then $\theta_0(G) \geq 3$. Thus, $p = 2, 3$. If $p = 2$, then $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $\Gamma(G) \simeq K_1 + (K_3 \cup 2K_1)$. By Lemma 2.4, $\theta_0(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_4)) = \theta_0(K_4) = 2$. If $p = 3$, then $G \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_9$ and $\Gamma(G) \simeq K_1 + (K_4 \cup 3K_1)$. By Lemma 2.4, $\theta_0(\Gamma(\mathbb{Z}_3 \oplus \mathbb{Z}_9)) = \theta_0(K_5) = 2$.

The proof is complete. □

Combine Theorems 3.1–3.5, we get the following theorem.

**Theorem 3.6.** *Let G be a finite abelian group and let* $p_1, p_2, p_3, p_4$ *and p be primes. Then the following statements are hold.*

(1) $\theta(\Gamma(G)) = 1$ *if and only if G is one following groups:*

$$\mathbb{Z}_{p^\alpha} (\alpha = 2, 3, 4, 5), \mathbb{Z}_{p_1 p_2^\alpha} (\alpha = 1, 2), \mathbb{Z}_{p_1 p_2 p_3}, \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}, \mathbb{Z}_p \oplus \mathbb{Z}_p.$$

(2) $\theta(\Gamma(G)) = 2$ *if and only if G is one following groups:*

$$\mathbb{Z}_{p^\alpha} (\alpha = 6, 7, 8, 9), \mathbb{Z}_{p_1 p_2^\alpha} (\alpha = 3, 4), \mathbb{Z}_{p_1^2 p_2^\alpha} (\alpha = 2, 3), \mathbb{Z}_{p_1 p_2 p_3^2}, \mathbb{Z}_{p_1 p_2 p_3 p_4},$$

$$\mathbb{Z}_3 \oplus \mathbb{Z}_{3p}, \mathbb{Z}_5 \oplus \mathbb{Z}_{5p}, \mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

(3) $\theta_0(\Gamma(G)) = 1$ *if and only if G is one following groups:*

$$\mathbb{Z}_{p^\alpha}(\alpha = 2, 3, 4), \mathbb{Z}_{p_1 p_2^\alpha}(\alpha = 1, 2), \mathbb{Z}_{p_1 p_2 p_3}, \mathbb{Z}_p \oplus \mathbb{Z}_p.$$

(4) $\theta_0(\Gamma(G)) = 2$ *if and only if G is one following groups:*

$$\mathbb{Z}_{p^\alpha}(\alpha = 5, 6, 7), \mathbb{Z}_{p_1^2 p_2^2}, \mathbb{Z}_{p_1 p_2^3}, \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}, \mathbb{Z}_3 \oplus \mathbb{Z}_{3p}.$$

## 4. Some finite non-abelian groups

In this section, we investigate the thickness and outerthickness of intersection graphs of subgroups of some finite non-abelian groups. For a positive integer $n$, $\tau(n)$ denotes the number of positive divisor of $n$; $\sigma(n)$ denotes the sum of all the positive divisors of $n$. $S_n$ and $A_n$ are denoted the symmetric group and alternating group of degree $n$ acting on $\{1, 2, 3, \cdots, n\}$, respectively.

For any integer $n \geq 3$, the dihedral group of order $2n$ is defined by

$$D_n = \{a, b | a^n = b^2 = 1, ab = ba^{-1}\}.$$

For any integer $n \geq 2$, the generalized quaternion group of order $4n$ is defined by

$$Q_n = \{a, b | a^{2n} = b^4 = 1, a^n = b^2, b^{-1}ab = a^{-1}\}.$$

For any prime $p$ and integer $\alpha \geq 3$, the modular group of order $p^\alpha$ is defined by

$$M_{p^\alpha} = \{a, b | a^{p^{\alpha-1}} = b^p = 1, bab^{-1} = a^{p^{\alpha-2}+1}\}.$$

For any integer $n \geq 4$, the quasi-dihedral group of order $2^n$ is defined by

$$QD_{2^n} = \{a, b | a^{2^{\alpha-1}} = b^2 = 1, bab^{-1} = a^{2^{\alpha-2}-1}\}.$$

**Lemma 4.1.** [27] *The following statements are hold.*

(1) *For $n \geq 3$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $V(\Gamma(D_n)) = \tau(n) + \sigma(n) - 2$ and $\omega(\Gamma(D_n)) = \sigma(n) - n - 1 + \prod_{i=1}^k \alpha_i$.*

(2) *Let $n \geq 2$ be an integer and $2n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_2, \ldots, p_k$ are distinct odd primes and $\alpha_i \geq 1$. Then $V(\Gamma(Q_n)) = \tau(2n) + \sigma(n) - 2$ and $\omega(\Gamma(Q_n)) = \sigma(n) - 1 + \alpha_1 \prod_{i=2}^k (\alpha_i + 1)$.*

(3) *For $\alpha \geq 4$, $V(\Gamma(QD_{2^\alpha})) = \alpha + 3(2^{\alpha-1} - 1)$ and $\omega(\Gamma(QD_{2^\alpha})) = \alpha + 2^{\alpha-1} - 3$.*

**Theorem 4.2.** *Let $G = S_n$. Then the following statements are hold.*

(1) *If $n = 3$, then $\theta(\Gamma(G)) = 1$.*

(2) *If $n \geq 4$, then $\theta(\Gamma(G)) \geq 3$.*

*Proof.* If $G = S_3$, then it is easy to see that $\Gamma(S_3) \simeq 4K_1$, which is a planar graph. So, $\theta(\Gamma(G)) = 1$. If $n \geq 4$, then we set

$S_1 = \{(1), (12), (34), (12)(34)\}$,

$S_2 = \{(1), (12), (13), (23), (123), (132)\}$,

$S_3 = \{(1), (12), (14), (24), (124), (142)\}$,

$S_4 = \{(1), (13), (14), (34), (134), (143)\}$,

$S_5 = \{(1), (23), (24), (34), (234), (243)\}$,

$S_6 = \{(1), (13), (24), (13)(24), (14)(23), (12)(34), (1234), (1432)\}$,
$S_7 = \{(1), (14), (23), (13)(24), (14)(23), (12)(34), (1243), (1342)\}$,
$S_8 = \{(1), (13), (24), (13)(24), (14)(23), (12)(34), (1324), (1423)\}$,
$S_9 = \{(1), (123), (132), (124), (142), (134), (143), (234), (243), (13)(24), (14)(23), (12)(34)\}$.

Then they are nine non-trivial subgroups of $S_n$ and form a complete graph $K_9$ as a subgraph of $\Gamma(S_n)$. Thus, $\theta(\Gamma(G)) \geq \theta(K_9) \geq 3$. □

**Theorem 4.3.** *Let $G = A_n$. Then the following statements are hold.*
(1) $\theta(\Gamma(G)) = 1$ *if and only if* $n = 4$.
(2) *If* $n \geq 5$, *then* $\theta(\Gamma(G)) \geq 3$.

*Proof.* If $G \simeq A_4$, then $\Gamma(A_4) \simeq 4K_1 \cup K_{1,3}$, which is a planar graph. So, $\theta(\Gamma(G)) = 1$. If $n \geq 5$, we let
$L_1 = \{(1), (12345), (13524), (14253), (15432), (12)(35), (13)(45), (14)(23), (15)(24), (25)(34)\}$,
$L_2 = \{(1), (12453), (13542), (14325), (15234), (12)(34), (13)(25), (14)(35), (15)(24), (23)(45)\}$,
$L_3 = \{(1), (12354), (13425), (14532), (15243), (12)(34), (13)(45), (14)(25), (15)(23), (24)(35)\}$,
$L_4 = \{(1), (12435), (13254), (14523), (15342), (12)(45), (13)(24), (14)(35), (15)(23), (25)(34)\}$,
$L_5 = \{(1), (12534), (13245), (14352), (15423), (12)(45), (13)(25), (14)(23), (15)(34), (24)(35)\}$,
$L_6 = \{(1), (12543), (13452), (14235), (15324), (12)(35), (13)(24), (14)(25), (15)(34), (23)(45)\}$,
$L_7 = \{(1), (123), (124), (134), (234), (132), (142), (143), (243), (13)(24), (14)(23), (12)(34)\}$,
$L_8 = \{(1), (123), (125), (135), (235), (132), (152), (153), (253), (12)(35), (13)(25), (15)(23)\}$,
$L_9 = \{(1), (124), (125), (145), (245), (142), (152), (154), (254), (12)(45), (14)(25), (15)(24)\}$.

Then they are nine non-trivial subgroups of $A_n$ and form a complete graph $K_9$ as a subgraph of $\Gamma(A_n)$. Thus, $\theta(\Gamma(G)) \geq \theta(K_9) \geq 3$. □

**Theorem 4.4.** *Let $G = D_n (n \geq 3)$ be the dihedral group of order $2n$. Then the following statements are hold.*
(1) $\theta(\Gamma(G)) = 1$ *if and only if* $G \simeq D_4$ *or* $D_p$, *where* $p$ *is an odd prime.*
(2) $\theta(\Gamma(G)) = 2$ *if and only if* $G \simeq D_6, D_9, D_{25}$ *or* $D_{10}$.

*Proof.* For $n \geq 3$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$. By Lemma 4.1, we know that $\omega(\Gamma(D_n)) = \sigma(n) - n - 1 + \prod_{i=1}^{k} \alpha_i$. If $k \geq 2$ and $\alpha_k \geq 2$, then $\omega(\Gamma(D_n)) \geq p_1 + p_k + p_1 p_k \geq 11$. Thus, if $\theta(\Gamma(G)) \leq 2$, then $n = p_1 p_2$ or $n = p^{\alpha}$.

If $n = p_1 p_2$, then by Lemma 4.1, $\omega(\Gamma(D_n)) = \sigma(n) - n - 1 + \prod_{i=1}^{k} \alpha_i = p_1 + p_2 + 1$. If $p_1 + p_2 + 1 \geq 9$, then $\theta(\Gamma(D_n)) \geq 3$. Thus, if $\theta(\Gamma(D_n)) \leq 2$, then $p_1 + p_2 + 1 \leq 8$. Thus, $p_1 = 2$, $p_2 = 3$ or $p_1 = 2$, $p_2 = 5$.

If $p_1 = 2$, $p_2 = 3$, then $\omega(\Gamma(D_6)) = 6$. So, $\theta(\Gamma(D_6)) \geq \theta(K_6) = 2$. Note that $deg_{\Gamma(D_6)}(\langle a^i b \rangle) = \tau(6) - 2 = 2$ and $V(\langle a^i b \rangle) = 6$. By Lemma 2.4, $\theta(\Gamma(D_6)) = \theta(\Gamma(D_6) - 6v_2)$. On the other hand, $V(\Gamma(D_6) - 6v_2) = \tau(6) - 1 + \sigma(6) - 6 - 1 = 8$, then $2 = \theta(K_6) \leq \theta(\Gamma(D_6) - 6v_2) \leq \theta(K_8) = 2$. Hence, $\theta(\Gamma(D_6)) = 2$.

If $p_1 = 2$ and $p_2 = 5$, then $\omega(\Gamma(D_{10})) = 8$. So, $\theta(\Gamma(D_{10})) \geq \theta(K_8) = 2$. Note that $deg_{\Gamma(D_{10})}(\langle a^i b \rangle) = \tau(10) - 2 = 2$ and $V(\langle a^i b \rangle) = n = 10$. By Lemma 2.4, $\theta(\Gamma(D_{10})) = \theta(\Gamma(D_{10}) - 10v_2)$. As $V(\Gamma(D_{10}) - 10v_2) = \tau(10) - 1 + \sigma(10) - 10 - 1 = 10$, we have $2 = \theta(K_8) \leq \theta(\Gamma(D_{10}) - 10v_2) \leq \theta(K_{10}) = 3$. On the other hand, a planar decomposition of $\Gamma(D_{10}) - 10v_2$ as shown in Figure 6 deduces that $\theta(\Gamma(D_{10})) = 2$.

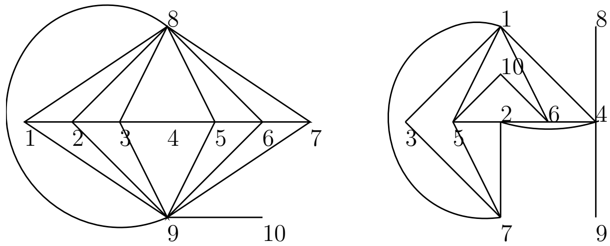


**Figure 6.** A planar decomposition of $\Gamma(D_{10}) - 10v_2$.

If $n = p^\alpha$, then by Lemma 4.1, $\omega(\Gamma(D_n)) = \frac{p^\alpha - p}{p-1} + \alpha$. If $\omega(\Gamma(D_n)) \geq 9$, then $\theta(\Gamma(D_n)) \geq 3$. So, $\alpha \leq 2$.

If $\alpha = 1$, then $\Gamma(D_p) \simeq (p+1)K_1$ and thus $\theta(\Gamma(D_p)) = 1$. If $\alpha = 2$, then $\omega(\Gamma(D_{p^2})) = p + 2$. Note that $deg_{\Gamma(D_{p^2})}(\langle a^i b\rangle) = \tau(p^2) - 2 = 1$, $V(\langle a^i b\rangle) = n = p^2$, and $V(\Gamma(D_{p^2})) = \tau(n) + \sigma(n) - 2 = 1 + p + p^2 + 3 - 2 = p^2 + p + 2$. So, $\Gamma(D_{p^2}) = K_{p+2} + p^2 v_1$. By Lemma 2.4, $\theta(\Gamma(D_{p^2})) = \theta(K_{p+2})$. Since $p + 2 \geq 9$ deduces $\theta(\Gamma(D_{p^2})) \geq 3$, we have $p + 2 \leq 8$, that is, $p = 2, 3, 5$. By Lemma 2.2, we get if $p = 2$, then $\theta(\Gamma(D_4)) = 1$; and if $p = 3, 5$, then $\theta(\Gamma(D_9)) = \theta(\Gamma(D_{25})) = 2$. □

**Theorem 4.5.** *Let $G = Q_n$. Then the following statements are hold.*

(1) $\theta(\Gamma(G)) = 1$ *if and only if* $G = Q_2$.

(2) $\theta(\Gamma(G)) = 2$ *if and only if* $G \simeq Q_3$ *or* $Q_5$.

*Proof.* Let $n \geq 2$ be an integer and $2n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_i$ are distinct primes and $\alpha_i \geq 1$. By Lemma 4.1, $\omega(\Gamma(Q_n)) = \sigma(n) - 1 + \alpha_1 \prod_{i=2}^{k}(\alpha_i + 1)$. If $k \geq 2$, then $\omega(\Gamma(Q_n)) \geq 1 + p_1 + p_2 + p_1 p_2 - 1 + 1 \times 2 \times 2 = 4 + p_1 + p_2 + p_1 p_2 > 4 + 2 + 3 + 2 \times 3 > 9$ and $\theta(\Gamma(Q_n)) > 3$. So, if $\theta(\Gamma(Q_n)) \leq 2$, then we have $n = p^\alpha$.

If $n = 2^\alpha$, $\omega(\Gamma(Q_{2^\alpha})) = \sigma(n) - 1 + \alpha_1 \prod_{i=2}^{k}(\alpha_i + 1) = 2^{\alpha+1} - 1 + \alpha$, $V(\Gamma(Q_{2^\alpha})) = \tau(2n) + \sigma(n) - 2 = 2^{\alpha+1} - 1 + \alpha$, then $\Gamma(Q_{2^\alpha})$ is a complete graph. If $2^{\alpha+1} - 1 + \alpha \geq 9$, then $\theta(\Gamma(Q_n)) \geq 3$. So, if $\theta(\Gamma(Q_n)) \leq 2$, then $2^{\alpha+1} - 1 + \alpha \leq 8$ and thus $\alpha = 1$. In this case, $\theta(\Gamma(Q_2)) = 1$.

If $n = p^\alpha$ and $p \neq 2$, then $\omega(\Gamma(Q_n)) = \sigma(n) - 1 + \alpha_1 \prod_{i=2}^{k}(\alpha_i + 1) = 1 + p + p^2 + \cdots + p^{\alpha-1} + p^\alpha + 1 \times (\alpha + 1) - 1 = \frac{p^{\alpha+1} - 1}{p-1} + \alpha \geq 9$, then $\theta(\Gamma(Q_n)) \geq 3$. So, if $\theta(\Gamma(Q_n)) \leq 2$, then $\frac{p^{\alpha+1}-1}{p-1} + \alpha \leq 8$, then $\alpha = 1$.

For $\alpha = 1$, $\omega(\Gamma(Q_n)) = \sigma(n) - 1 + \alpha_1 \prod_{i=2}^{k}(\alpha_i + 1) = p + 2$ and $V(\Gamma(Q_n)) = \tau(2n) + \sigma(n) - 2 = p + 3$. So, $\theta(K_{p+2}) \leq \theta(\Gamma(Q_n)) \leq \theta(K_{p+3})$. If $\omega(\Gamma(Q_n)) = p + 2 \geq 9$, then $\theta(\Gamma(Q_n)) \geq 3$. So, if $\theta(\Gamma(Q_n)) \leq 2$, then $p + 2 \leq 8$, then $p = 3, 5$.

By Lemma 2.1 and Lemma 2.2, $2 = \theta(K_5) \leq \theta(\Gamma(Q_3)) \leq \theta(K_6) = 2$ and $2 = \theta(K_7) \leq \theta(\Gamma(Q_5)) \leq \theta(K_8) = 2$, then $\theta(\Gamma(Q_3)) = \theta(\Gamma(Q_5)) = 2$. □

**Theorem 4.6.** *Let $G = QD_{2^\alpha}$. Then $\theta(\Gamma(G)) \geq 3$.*

*Proof.* For $\alpha \geq 4$, by Lemma 4.1, we have $\omega(\Gamma(QD_{2^\alpha})) = \alpha + 2^{\alpha-1} - 3 \geq 9$. Thus, $\theta(\Gamma(QD_{2^\alpha})) \geq \theta(K_9) = 3$. □

**Theorem 4.7.** *Let $G \simeq M_{p^\alpha}$. Then the following statements are hold.*

(1) $\theta(\Gamma(G)) = 1$ *if and only if* $G = M_8$.

(2) $\theta(\Gamma(G)) = 2$ *if and only if* $G \simeq M_{27}, M_{125}$ *or* $M_{16}$.

*Proof.* If $p^\alpha = 8$, then $\Gamma(M_8) \simeq \Gamma(D_4)$. So $\theta(\Gamma(M_8)) = 1$. If $p^\alpha \neq 8$, then $\Gamma(M_{p^\alpha}) \simeq \Gamma(Z_p \oplus Z_{p^{\alpha-1}})$. If $\theta(\Gamma(G)) \leq 2$, then by the proof of Theorem 3.5, we know $\alpha = 3, 4$. If $\alpha = 3$, then $\Gamma(M_{p^3}) \simeq$

$K_1 + (K_{p+1} \cup pK_1)$. By $\theta(\Gamma(G)) \le 2$, we get $p = 3, 5$. By Lemma 2.4, $\theta(\Gamma(M_{27})) = \theta(K_5) = 2$ and $\theta(\Gamma(M_{125})) = \theta(K_7) = 2$. If $\alpha = 4$, then $p = 2$. By Theorem 3.5, $\theta(\Gamma(M_{16})) = \theta(\Gamma(Z_2 \oplus Z_8)) = 2$. □

**Theorem 4.8.** *Let $G = S_n$. Then the following statements are hold.*
(1) *If $n = 3$, then $\theta_0(\Gamma(G)) = 1$.*
(2) *If $n \ge 4$, then $\theta_0(\Gamma(G)) \ge 3$.*

*Proof.* If $G = S_3$, then $\Gamma(S_3) \simeq 4K_1$, which is an outerplanar graph. So $\theta_0(\Gamma(S_3)) = 1$. By the proof of Theorem 4.2, if $n \ge 4$, then $\theta_0(\Gamma(G)) \ge \theta_0(\Gamma(K_9)) = 3$. □

**Theorem 4.9.** *Let $G \simeq A_n$. Then the following statements are hold.*
(1) *If $n = 4$, then $\theta_0(\Gamma(G)) = 1$.*
(2) *If $n \ge 5$, then $\theta_0(\Gamma(G)) \ge 3$.*

*Proof.* Note that $\Gamma(A_4) \simeq 4K_1 \cup K_{1,3}$. It is clear that $\Gamma(A_4)$ contains no subgraph homeomorphic to $K_4$ or $K_{2,3}$. By Lemma 2.5, $\Gamma(A_4)$ is an outerplanar graph. So $\theta_0(\Gamma(A_4)) = 1$. By the proof of Theorem 4.3, if $n \ge 5$, then $\theta_0(\Gamma(G)) \ge \theta_0(K_9) = 3$. □

**Theorem 4.10.** *Let $G \simeq D_n$. Then the following statements are hold.*
(1) $\theta_0(\Gamma(G)) = 1$ *if and only if $G \simeq D_p$, where $p$ is a prime.*
(2) $\theta_0(\Gamma(G)) = 2$ *if and only if $G$ is one following groups: $D_4$, $D_6$, $D_9$.*

*Proof.* For $n \ge 3$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_k$. According to the proof of Theorem 4.4, if $\theta_0(\Gamma(D_n)) \le 2$, then $n = p_1 p_2$ or $p^\alpha$.

If $n = p_1 p_2$, then $\omega(\Gamma(D_n)) = \sigma(n) - n - 1 + \prod_{i=1}^{k} \alpha_i = p_1 + p_2 + 1$. If $p_1 + p_2 + 1 \ge 7$, then $\theta_0(\Gamma(D_n)) \ge 3$. So, if $\theta_0(\Gamma(D_n)) \le 2$, then $p_1 + p_2 + 1 \le 6$, that is, $p_1 = 2$, $p_2 = 3$. As $\omega(\Gamma(D_6)) = 2 + 3 + 1 = 6$, we know $\theta_0(\Gamma(D_6)) \ge \theta_0(K_6) = 2$. An outerplanar decomposition of $\Gamma(D_6) - 6v_2$ as shown in Figure 7 gives $\theta_0(\Gamma(D_6)) = 2$.

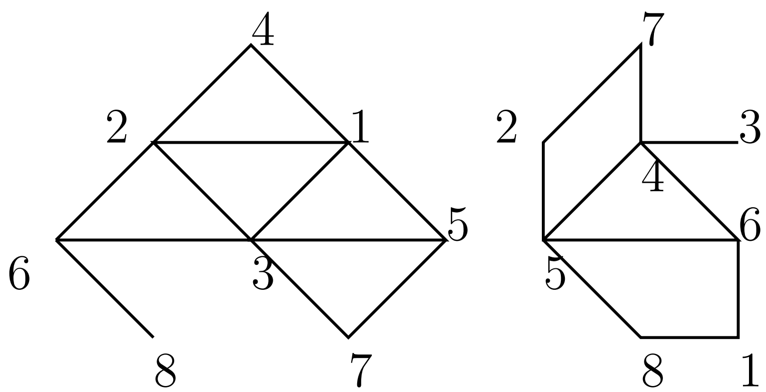


**Figure 7.** An outerplanar decomposition of $\Gamma(D_6 - 6v_2)$.

If $n = p^\alpha$, then $\omega(\Gamma(D_n)) = \sigma(n) - n - 1 + \prod_{i=1}^{k} \alpha_i = p + p^2 + \cdots + p^{\alpha-1} + \alpha = \frac{p^\alpha - p}{p-1} + \alpha \ge 7$, then $\theta_0(\Gamma(D_n)) \ge 3$. So, if $\theta_0(\Gamma(D_n)) \le 2$, then $\frac{p^\alpha - p}{p-1} + \alpha \le 6$, so $\alpha = 1, 2$.

If $\alpha = 1$, then $\Gamma(D_p) \simeq (p+1)K_1$. So, $\theta_0(\Gamma(D_p)) = 1$.

If $\alpha = 2$, $\Gamma(D_{p^2}) = K_{p+2} + p^2 v_1$. By Lemma 2.4, $\theta_0(\Gamma(D_{p^2})) = \theta_0(K_{p+2})$. As $p + 2 \ge 7$, then $\theta_0(\Gamma(D_{p^2})) \ge 3$. So, if $\theta_0(\Gamma(D_{p^2})) \le 2$, then $p + 2 \le 6$, then $p = 2, 3$. By Lemma 2.2, we get if $p = 2$, then $\theta_0(\Gamma(D_4)) = \theta_0(K_4) = 2$; and if $p = 3$, then $\theta_0(\Gamma(D_9)) = \theta_0(K_5) = 2$. □

**Theorem 4.11.** *Let $G \simeq Q_n$. Then the following statements are hold.*

(1) $\theta_0(\Gamma(G)) \geq 2$.

(2) $\theta_0(\Gamma(G)) = 2$ *if and only if* $G \simeq Q_2$ *or* $Q_3$.

*Proof.* According the proof of Theorem 4.5, if $\theta_0(\Gamma(Q_n)) \leq 2$, then $n = p^\alpha$.

If $n = 2^\alpha$, again by the proof of Theorem 4.5, we have $\Gamma(Q_{2^\alpha}) = K_{2^{\alpha+1}-1+\alpha}$. If $2^{\alpha+1} - 1 + \alpha \geq 7$, then $\theta_0(\Gamma(Q_n)) \geq 3$. So $\theta_0(\Gamma(Q_n)) \leq 2$, then $2^{\alpha+1} - 1 + \alpha \leq 6$, so $\alpha = 1$. In this case, $\theta_0(\Gamma(Q_2)) = 2$.

If $n = p^\alpha$ and $p \neq 2$, then $\omega(\Gamma(Q_n)) = \frac{p^{\alpha+1}-1}{p-1} + \alpha \geq 7$ deduces $\theta_0(\Gamma(Q_n)) \geq 3$. So, if $\theta_0(\Gamma(Q_n)) \leq 2$, then $\frac{p^{\alpha+1}-1}{p-1} + \alpha \leq 6$ and $\alpha = 1$.

In this case, $\omega(\Gamma(Q_n)) = \sigma(n) - 1 + \alpha_1 \prod_{i=2}^{k}(\alpha_i + 1) = p + 2$ and $V(\Gamma(Q_n)) = \tau(2n) + \sigma(n) - 2 = p + 3$. So, $\theta_0(K_{p+2}) \leq \theta_0(\Gamma(Q_n)) \leq \theta_0(K_{p+3})$. If $\omega(\Gamma(Q_n)) = p+2 \geq 7$, then $\theta_0(\Gamma(Q_n)) \geq 3$. So, if $\theta_0(\Gamma(Q_n)) \leq 2$, then $p + 2 \leq 6$, so $p = 3$. By Lemma 2.1 and Lemma 2.2, $2 = \theta_0(K_5) \leq \theta_0(\Gamma(Q_3)) \leq \theta_0(K_6) = 2$, then $\theta_0(\Gamma(Q_3)) = 2$. □

**Theorem 4.12.** *Let $G \simeq QD_{2^\alpha}$. Then $\theta_0(\Gamma(G) \geq 3$.*

*Proof.* For $\alpha \geq 4$, by Lemma 4.1, we have $\omega(\Gamma(QD_{2^\alpha})) = \alpha + 2^{\alpha-1} - 3 \geq 9$. Thus, $\theta_0(\Gamma(QD_{2^\alpha})) \geq \theta_0(K_9) = 3$. □

**Theorem 4.13.** *Let $G = M_{p^\alpha}$. Then the following statements are hold.*

(1) $\theta_0(\Gamma(G)) \geq 2$.

(2) $\theta_0(\Gamma(G)) = 2$ *if and only if* $G \simeq M_8$ *or* $M_{27}$.

*Proof.* If $p^\alpha = 8$, then $\Gamma(M_8) \simeq \Gamma(D_4)$. By Theorem 4.4, $\theta_0(\Gamma(M_8)) = 2$. If $p^\alpha \neq 8$, then $\Gamma(M_{p^\alpha}) \simeq \Gamma(Z_p \oplus Z_{p^{\alpha-1}})$. By the proof of Theorem 4.7, we get $\alpha = 3, p = 3$. In this case, we have $\theta_0(\Gamma(M_{27})) = 2$. □

## 5. Conclusions and further work

In this paper, we classify all finite abelian groups whose thickness and outerthickness of subgroup intersection graphs are 1 and 2, respectively. We also investigate the thickness and outerthickness of subgroup intersection graphs for some finite non-abelian groups. The method is finding a forbidden subgraph of $\Gamma(G)$, like $K_9$ and then determining the remaining cases. The key point is how to separate $\Gamma(G)$ into two planar graphs. Next, one can consider to determine all finite abelian groups $G$ whose $\Gamma(G)$ has thickness or outerthickness 3. For a general finite group $G$, how to determine the thickness and outerthickness of $\Gamma(G)$ is a challenge. Of course, investigating other properties of $\Gamma(G)$ is a valuable exploration.

## Acknowledgments

We would like to thank the referees for a careful reading of the paper and for their valuable comments. This work was supported by the National Natural Science Foundation of China (Grant No. 11661013, 11961050), the Guangxi Natural Science Foundation (Grant No. 2020GXNSFAA159103, 2020GXNSFAA159053) and High-level talents for scientific research of Beibu Gulf University

(2020KYQD07). The second author is supported by the Science and Technology Research Project of Jiangxi Education Department.

## Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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