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Research article

A simple method for solving matrix equations $AXB = D$ and $GXH = C$

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Abstract: A simple method to solve the common solution to the pair of linear matrix equations $AXB = D$ and $GXH = C$ is introduced. Some necessary and sufficient conditions for the existence of a common solution, and two expressions for the general common solution of the equation pair are provided by the proposed method. Subsequently, the results are applied to determine the solution of the matrix equation $AXB + GYH = D$ and the Hermitian solution of the matrix equation $AXB = D$.

Keywords: matrix equation; generalized inverse; common solution; Hermitian solution
Mathematics Subject Classification: 15A09, 15A24

1. Introduction

Throughout this paper, we denote the complex $m\times n$ matrix space by $\mathbb{C}^{m\times n}$, and denote the conjugate transpose, the inner inverse, the Moore-Penrose inverse, the range space and the null space of a complex matrix $A \in \mathbb{C}^{m\times n}$ by A^{H}, A^-, A^+, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. I_n represents the identity matrix of size n. $P_{\mathcal{L}}$ stands for the orthogonal projector on the subspace $\mathcal{L} \subset \mathbb{C}^n$. Furthermore, for a matrix $A \in \mathbb{C}^{m\times n}$, E_A and F_A stand for two idempotent matrices: $E_A = I_m - AA^-$, $F_A = I_n - A^-A$.

Finding a common solution to the pair of linear matrix equations

$$AXB = D, \quad GXH = C, \tag{1}$$

where $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}, G \in \mathbb{C}^{l\times n}, H \in \mathbb{C}^{p\times k}$ and $D \in \mathbb{C}^{m\times q}$, $C \in \mathbb{C}^{l\times k}$, has been studied by many authors. Woude [1, 2] studied the problem in the context of noninteracting control by measurement feedback with or without internal stability. Mitra [3, 4] has provided the necessary and sufficient conditions for the existence of a common solution, and the general common solution of the equation pair (1). Conditions for the existence of a common solution to the equations of (1) have also been studied by Shinozaki and Sibuya [5], von der Woude [6] and Navarra et al. [7]. Also, Özgüler and Akar [8] gave a condition for the solvability of (1) over a principle domain. Wang [9] studied the system (1) over arbitrary regular rings with identity. Dajić [10] considered it in associative ring with unit. Recently, the generalizations of (1) were considered in [11–15].

The one that is closely related to the equations of (1) is the following matrix equation:

$$AXB + GYH = D, \tag{2}$$

where $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}, G \in \mathbb{C}^{m\times l}, H \in \mathbb{C}^{k\times q}$ and $D \in \mathbb{C}^{m\times q}$. The solvability conditions and general solutions have been derived in [16–18] by using generalized inverses, the generalized singular value decomposition (GSVD) and the canonical correlation decomposition (CCD) of the matrices, respectively. Also, Peng and Peng [19] provided a finite iterative method for solving the matrix equation (2). Özgüler [20] discussed the solvability of the linear matrix equation (2) over an arbitrary principal ideal domain. Huang and Zeng [21] discussed the solvability of Eq (2) over any simple Artinian ring. Some generalizations of (2) and solving some constrained solutions of (2) were discussed in [22–26].

In this note, a simple method to solve the common solution of the equation pair (1) is introduced. The necessary and sufficient conditions for their solvability as well as two expressions for the general solution are provided by the proposed method. The results are given in terms of generalized inverses and orthogonal projectors, which are of the concise expressions compared with the existing methods. Subsequently, the results are applied to determine the solution of the matrix equation (2) and the Hermitian solution of the matrix equation $AXB = D$. The given numerical example validates the accuracy of the results.

2. Some lemmas

Lemma 1. *[27] Let $M \in \mathbb{C}^{m\times n}, N \in \mathbb{C}^{p\times q}, P \in \mathbb{C}^{m\times q}$. Then a necessary and sufficient condition for the matrix equation $MXN = P$ with respect to X is*

$$MM^{-}PN^{-}N = P,$$

or equivalently,

$$E_M P = 0,\ PF_N = 0. \tag{3}$$

In this case, the general solution can be written in the following parametric form

$$X = M^{-}PN^{-} + F_M V_1 + V_2 E_N, \tag{4}$$

where $V_1, V_2 \in \mathbb{C}^{n\times p}$ are arbitrary matrices.

Lemma 2. *[28] Let $A \in \mathbb{C}^{m\times k}, B \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{m\times n}$. Then the equation*

$$AX - YB = D \tag{5}$$

has a solution $X \in \mathbb{C}^{k\times n}, Y \in \mathbb{C}^{m\times l}$ if and only if $E_A D F_B = 0$. If this is the case, the general solution of Eq (5) has the form

$$X = A^{-}D + A^{-}ZB + F_A W, \tag{6}$$

$$Y = -E_A DB^{-} + Z - E_A ZBB^{-}, \tag{7}$$

where $W \in \mathbb{C}^{k\times n}, Z \in \mathbb{C}^{m\times l}$ are arbitrary matrices.

Lemma 3. *[27] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times n}$ and $C \in \mathbb{C}^{p\times q}$. Then the product $AB^{-}C$ does not depend on the choice of B^{-} if and only if $A = 0$, or $C = 0$, or $\mathcal{R}(A^{\mathrm{H}}) \subseteq \mathcal{R}(B^{\mathrm{H}})$ and $\mathcal{R}(C) \subseteq \mathcal{R}(B)$.*

Lemma 4. *[27] Let $A \in \mathbb{C}^{m\times k}$, $B \in \mathbb{C}^{l\times k}$, $C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times p}$. Then*

$$\begin{bmatrix} A \\ B \end{bmatrix}^{-} = [A^{-} - F_A(BF_A)^{-}BA^{-},\ F_A(BF_A)^{-}],$$

$$[C,\ D]^{-} = \begin{bmatrix} C^{-} - C^{-}D(E_CD)^{-}E_C \\ (E_CD)^{-}E_C \end{bmatrix}.$$

Lemma 5. *[29] Suppose that $P, Q \in \mathbb{C}^{p\times n}$. Then the matrix equation $PX = Q$ has a Hermitian solution $X \in \mathbb{C}^{n\times n}$ if and only if*

$$E_PQ = 0,\ QP^{\mathrm{H}} = PQ^{\mathrm{H}},$$

in which case, the general Hermitian solution is

$$X = P^{-}Q + F_P(P^{-}Q)^{\mathrm{H}} + F_PJF_P,$$

where $J \in \mathbb{C}^{n\times n}$ is an arbitrary Hermitian matrix.

Lemma 6. *[30] Assume that $A \in \mathbb{C}^{m\times n}$ and $\mathcal{T}$ is a subspace of $\mathbb{C}^{n}$. Let $\tilde{\mathcal{T}} = \mathcal{R}(P_{\mathcal{T}}A^{\mathrm{H}}) = P_{\mathcal{T}}\mathcal{R}(A^{\mathrm{H}})$, then*

$$\tilde{\mathcal{T}} = \mathcal{T} \cap (\mathcal{T} \cap \mathcal{N}(A))^{\perp},\ \tilde{\mathcal{T}}^{\perp} = \mathcal{T}^{\perp} \oplus (\mathcal{T} \cap \mathcal{N}(A)).$$

3. Main results

Theorem 1. *The pair of equations in (1) have a common solution X if and only if*

$$AA^{-}DB^{-}B = D,\ \ GG^{-}CH^{-}H = C,\ \ P_{\mathcal{T}}(A^{-}DB^{-} - G^{-}CH^{-})P_{\mathcal{S}} = 0, \tag{8}$$

where $\mathcal{T} = \mathcal{R}(A^{\mathrm{H}}) \cap \mathcal{R}(G^{\mathrm{H}})$, $\mathcal{S} = \mathcal{R}(B) \cap \mathcal{R}(H)$. In this case, the general common solution to the equations of (1) is given by

$$X = A^{-}DB^{-} + F_AL_1 + L_2E_B, \tag{9}$$

or equivalently,

$$X = G^{-}CH^{-} + F_GJ_1 + J_2E_H, \tag{10}$$

where

$$L_1 = (F_A - F_AF_GK^{-}A^{-}A)(\tilde{D} - Z_1E_B + Z_2E_H) + A^{-}AW_1 + F_AF_GF_KW_2, \tag{11}$$

$$L_2 = E_KA^{-}A\tilde{D}E_B - E_KA^{-}A\tilde{D}BB^{-}Q^{-}E_HE_B + Z_1 - E_KA^{-}AZ_1E_B + E_KA^{-}AZ_2E_QE_HE_B, \tag{12}$$

$$J_1 = -K^{-}A^{-}A\tilde{D} - K^{-}A^{-}A(-Z_1E_B + Z_2E_H) + F_KW_2, \tag{13}$$

$$J_2 = -E_KA^{-}A\tilde{D}BB^{-}Q^{-} + Z_2 - E_KA^{-}AZ_2QQ^{-}, \tag{14}$$

$K = A^{-}AF_G$, $Q = E_HBB^{-}$, $\tilde{D} = G^{-}CH^{-} - A^{-}DB^{-}$, and Z_1, Z_2, W_1, W_2 are arbitrary matrices.

Proof. By Lemma 1, if the first two conditions of (8) hold, then the general solutions of $AXB = D$ and $GXH = C$ are respectively given by Eqs (9) and (10). Now, we will find L_1, L_2, J_1 and J_2 such that $AXB = D, GXH = C$ has a common solution, namely,

$$A^{-}DB^{-} + F_A L_1 + L_2 E_B = G^{-}CH^{-} + F_G J_1 + J_2 E_H. \tag{15}$$

Obviously, Eq (15) can be equivalently written as

$$\tilde{A}\tilde{X} - \tilde{Y}\tilde{B} = \tilde{D}, \tag{16}$$

where

$$\tilde{A} = [F_A, -F_G],\ \tilde{B} = \begin{bmatrix} -E_B \\ E_H \end{bmatrix},\ \tilde{D} = G^{-}CH^{-} - A^{-}DB^{-},\ \tilde{X} = \begin{bmatrix} L_1 \\ J_1 \end{bmatrix},\ \tilde{Y} = [L_2, J_2].$$

According to Lemma 2, Eq (16) has a solution $(\tilde{X}, \tilde{Y})$ if and only if

$$E_{\tilde{A}}\tilde{D}F_{\tilde{B}} = 0. \tag{17}$$

By using Lemma 3, we have

$$\mathcal{R}(E_{\tilde{A}}) = \mathcal{R}(I - \tilde{A}\tilde{A}^{-}) = \mathcal{R}(I - \tilde{A}\tilde{A}^{+}) = \mathcal{N}(\tilde{A}^{\mathrm{H}}) = \mathcal{N}(F_A) \cap \mathcal{N}(F_G) = \mathcal{R}(A^{\mathrm{H}}) \cap \mathcal{R}(G^{\mathrm{H}}),$$

$$\mathcal{R}(F_{\tilde{B}}) = \mathcal{R}(I - \tilde{B}^{-}\tilde{B}) = \mathcal{N}(\tilde{B}) = \mathcal{N}(E_B) \cap \mathcal{N}(E_H) = \mathcal{R}(B) \cap \mathcal{R}(H).$$

Then, the relation of (17) is equivalent to

$$P_{\mathcal{T}}\tilde{D}P_{\mathcal{S}} = 0,$$

which is the third condition of (8). In which case, the general solution of Eq (16) is

$$\tilde{X} = \tilde{A}^{-}\tilde{D} + \tilde{A}^{-}Z\tilde{B} + F_{\tilde{A}}W, \tag{18}$$
$$\tilde{Y} = -E_{\tilde{A}}\tilde{D}\tilde{B}^{-} + Z - E_{\tilde{A}}Z\tilde{B}\tilde{B}^{-}. \tag{19}$$

By Lemma 4, we have

$$[F_A,\ -F_G]^{-} = \begin{bmatrix} F_A - F_A F_G K^{-}A^{-}A \\ -K^{-}A^{-}A \end{bmatrix}, \tag{20}$$

$$\begin{bmatrix} -E_B \\ E_H \end{bmatrix}^{-} = [-E_B + BB^{-}Q^{-}E_H E_B,\ BB^{-}Q^{-}]. \tag{21}$$

Inserting (20) and (21) into (18) and (19), we can get (11)–(14). □

At a first glance, the representation given by (10) is relatively simple comparing with that of (9). However, by careful inspection, we confirm that the equations of (9) and (10) are indeed the common solutions to the equations of (1).

Corollary 1. *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}, G \in \mathbb{C}^{m\times l}, H \in \mathbb{C}^{k\times q}$ and $D \in \mathbb{C}^{m\times q}$. Then the matrix equation* (2) *has a solution (X, Y) if and only if*

$$\begin{aligned} &A_1A_1^- E_A DH^- H = E_A D,\\ &GG^- DF_B B_1^- B_1 = DF_B,\\ &P_{\mathcal{T}_1}\left(A_1^- E_A DH^- - G^- DF_B B_1^-\right) P_{\mathcal{S}_1} = 0. \end{aligned} \tag{22}$$

If the above conditions are satisfied, the representation of the general solution to the equation of (2) *is*

$$X = A^-(D - GYH)B^- + F_A L_1 + L_2 E_B, \tag{23}$$

$$Y = G^- DF_B B_1^- + F_G W_2 + J_2 E_{B_1}, \tag{24}$$

where

$$A_1 = E_A G,\ B_1 = HF_B,\ \mathcal{T}_1 = \mathcal{R}(A_1^{\mathrm{H}}) = G^{\mathrm{H}}\mathcal{N}(A^{\mathrm{H}}),\ \mathcal{S}_1 = \mathcal{R}(B_1) = H\mathcal{N}(B),\ Q = E_{B_1}HH^-,$$

$$J_2 = -A_1^- A_1(G^- DF_B B_1^- - A_1^- E_A DH^-)HH^- Q^- + Z_2 - A_1^- A_1 Z_2 QQ^-,$$

and L_1, L_2, W_2, Z_2 are arbitrary matrices.

Proof. By Lemma 1, the matrix equation (2) with respect to X has a solution if and only if

$$E_A GYH = E_A D,\ \ GYHF_B = DF_B. \tag{25}$$

In which case, the general solution with respect to X is given by (23). Note that

$$\mathcal{R}(A_1^{\mathrm{H}}) = \mathcal{R}((E_A G)^{\mathrm{H}}) \subseteq \mathcal{R}(G^{\mathrm{H}}),\ \ \mathcal{R}(B_1) = \mathcal{R}(HF_B) \subseteq \mathcal{R}(H),\ A_1^- A_1 F_G = A_1^- E_A G F_G = 0.$$

Thus, by Theorem 1, we know that the equation of (25) have a common solution Y if and only if the conditions (22) are satisfied, and the general solution is given by (24). □

Corollary 2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times q}$ and $D \in \mathbb{C}^{m\times q}$. Then the matrix equation*

$$AXB = D \tag{26}$$

has a Hermitian solution X if and only if

$$\begin{aligned} &AA^- DB^- B = D,\\ &P_{\mathcal{T}_2}\left(A^- DB^- - (A^- DB^-)^{\mathrm{H}}\right) P_{\mathcal{T}_2} = 0, \end{aligned} \tag{27}$$

where $\mathcal{T}_2 = \mathcal{R}(A^{\mathrm{H}}) \cap \mathcal{R}(B)$. In which case, the general Hermitian solution of (26) *is*

$$X = \frac{1}{2}\left((B^{\mathrm{H}})^- D^{\mathrm{H}}(A^{\mathrm{H}})^- + A^- DB^-\right) + \frac{1}{2}(F_{B^{\mathrm{H}}} J_1 + J_2 E_{A^{\mathrm{H}}} + J_1^{\mathrm{H}} E_B + F_A J_2^{\mathrm{H}}), \tag{28}$$

where

$$J_1 = -K^- A^- A\left((B^{\mathrm{H}})^- D^{\mathrm{H}}(A^{\mathrm{H}})^- - A^- DB^-\right) - K^- A^- A(-Z_1 E_B + Z_2 E_{A^{\mathrm{H}}}) + F_K W_2,$$

$$J_2 = -E_K A^- A\left((B^{\mathrm{H}})^- D^{\mathrm{H}}(A^{\mathrm{H}})^- - A^- DB^-\right) BB^- Q^- + Z_2 - E_K A^- A Z_2 QQ^-,$$

$K = A^- AF_{B^{\mathrm{H}}}$, $Q = E_{A^{\mathrm{H}}} BB^-$, and Z_1, Z_2, W_2 are arbitrary matrices.

Proof. It is known that the equation of (26) has a Hermitian solution if and only if the following equations have a common solution

$$AXB = D, \quad B^{\mathrm{H}}XA^{\mathrm{H}} = D^{\mathrm{H}}. \tag{29}$$

According to Theorem 1, we can easily obtain the solvability conditions (27) of Eq (29). Notice that if X is a common solution of (29), then $\frac{1}{2}(X + X^{\mathrm{H}})$ is a Hermitian solution of (26). With this and Theorem 1, we can get (28). □

By using Corollary 2, we can solve the Hermitian solution of the matrix equation $AXB = D$ on a linear manifold with ease.

Corollary 3. *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times q}, D \in \mathbb{C}^{m\times q}$ and $P, Q \in \mathbb{C}^{p\times n}$. Then the matrix equation*

$$\begin{array}{c} AXB = D, \\ \text{s. t. } PX = Q, \ X^{\mathrm{H}} = X, \end{array} \tag{30}$$

has a solution X if and only if

$$E_P Q = 0, \ QP^{\mathrm{H}} = PQ^{\mathrm{H}}, \tag{31}$$

$$AF_P(AF_P)^-(D - AX_0B)(F_PB)^-F_PB = D - AX_0B, \tag{32}$$

$$P_{\mathcal{T}_3}\left((AF_P)^-(D - AX_0B)(F_PB)^- - ((AF_P)^-(D - AX_0B)(F_PB)^-)^{\mathrm{H}}\right)P_{\mathcal{T}_3} = 0, \tag{33}$$

where $X_0 = P^-Q + F_P(P^-Q)^{\mathrm{H}}$, $\mathcal{T}_3 = \mathcal{N}(P) \cap (\mathcal{N}(P) \cap \mathcal{R}(A))^\perp \cap (\mathcal{N}(P) \cap \mathcal{R}(B^{\mathrm{H}}))^\perp$.

Proof. By Lemma 5, we know that $PX = Q$ has a Hermitian solution X if and only if the conditions (31) hold. In which case, the general Hermitian solution of the equation $PX = Q$ is

$$X = P^-Q + F_P(P^-Q)^{\mathrm{H}} + F_PJF_P = X_0 + F_PJF_P, \tag{34}$$

where $J \in \mathbb{C}^{n\times n}$ is an arbitrary Hermitian matrix. Substituting (34) into the equation of (30) yields

$$AF_PJF_PB = D - AX_0B. \tag{35}$$

According to Corollary 2 and Lemma 6, we know that the equation of (35) has a Hermitian solution J if and only if the conditions (32) and (33) hold. □

By using Corollary 3, we can establish the solvability condition for the existence of a Hermitian solution of the matrix equation $AXB = D$ on a subspace.

Corollary 4. *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times q}, D \in \mathbb{C}^{m\times q}$, and let $\mathcal{L}$ be the subspace $\mathbb{C}^n$. Then the matrix equation*

$$\begin{array}{c} AXB = D, \\ \text{s. t. } \mathcal{R}(X) \subseteq \mathcal{L}, \ X^{\mathrm{H}} = X, \end{array} \tag{36}$$

has a solution X if and only if

$$\begin{array}{l} AP_{\mathcal{L}}(AP_{\mathcal{L}})^-D(P_{\mathcal{L}}B)^-P_{\mathcal{L}}B = D, \\ P_{\mathcal{T}_4}\left((AP_{\mathcal{L}})^-D(P_{\mathcal{L}}B)^- - ((AP_{\mathcal{L}})^-D(P_{\mathcal{L}}B)^-)^{\mathrm{H}}\right)P_{\mathcal{T}_4} = 0, \end{array} \tag{37}$$

where $\mathcal{T}_4 = \mathcal{L} \cap (\mathcal{L} \cap \mathcal{R}(A))^\perp \cap (\mathcal{L} \cap \mathcal{R}(B^{\mathrm{H}}))^\perp$.

Proof. It is evident that $\mathcal{R}(X) \subseteq \mathcal{L} \Leftrightarrow P_{L^\perp}X = 0$. By Corollary 3, we can easily achieve the solvability conditions (37) of Eq (36). □

4. A numerical example

Based on Theorem 1, we can describe an algorithm for obtaining a common solution to the pair of linear matrix equations (1).

Algorithm .

1) Input matrices A, B, C, D, G and H.

2) Compute F_A, F_G, E_B and E_H.

3) Compute $\tilde{A} = [F_A, -F_G]$, $\tilde{B} = \begin{bmatrix} -E_B \\ E_H \end{bmatrix}$ and $\tilde{D} = G^- C H^- - A^- D B^-$.

4) Compute $E_{\tilde{A}}$ and $F_{\tilde{B}}$.

5) If the conditions (8) and (17) are satisfied, go to 6); otherwise, the equations of (1) have no common solution, and stop.

6) Compute the matrices $K = A^- A F_G, Q = E_H B B^-$.

7) Compute L_1 and L_2 by (11) and (12), respectively.

8) Compute J_1 and J_2 by (13) and (14), respectively.

9) Compute X by (9) or by (10).

Example Let $k = 7, l = 8, m = 12, n = 10, p = 6$ and $q = 5$. The matrices A, B, C, D, G and H are given by

$$A = \begin{bmatrix} 3.3841 & -1.3291 & 2.3472 & 2.1069 & 1.2861 & 1.4575 & 1.8302 & 0.1195 & -2.9086 & -0.9130 \\ -3.8499 & 3.2224 & -3.5627 & 2.4056 & 1.9251 & 2.1999 & 0.5073 & 2.4268 & 2.6016 & 3.0440 \\ 2.0031 & 1.0308 & 1.4386 & -1.1009 & -0.2463 & -0.1047 & 0.3033 & 2.4811 & 2.7821 & -2.0057 \\ -0.3615 & -1.0511 & 1.8765 & 1.8875 & 1.3305 & 1.5464 & 1.0770 & -1.3507 & 0.5206 & 0.2354 \\ 0.8396 & 1.2427 & 0.5618 & -1.1523 & -0.4393 & -0.2103 & -0.1498 & 2.0070 & 2.3979 & -1.3476 \\ -1.3010 & 2.7000 & 1.1606 & 0.6390 & -0.7287 & 2.5401 & -1.0423 & 0.7335 & -2.8994 & -1.3105 \\ 3.4867 & -2.3471 & -2.9831 & 0.5334 & 2.2686 & -2.5837 & 2.7283 & 2.0678 & 3.6257 & 2.3488 \\ -2.5296 & 2.7764 & 1.3784 & 0.2023 & -0.1338 & 2.2178 & -0.7173 & 1.1172 & 2.5576 & -1.0291 \\ 1.9886 & 1.4513 & -2.7036 & 3.0171 & 1.6737 & 1.7751 & 1.5431 & 2.8624 & -5.1550 & 1.6403 \\ 1.8432 & 1.3214 & 2.3609 & 0.6419 & 0.6112 & 1.6995 & 0.9253 & 2.3621 & 1.4577 & -1.9330 \\ -2.3399 & -2.9004 & 2.7360 & 3.3460 & 2.5459 & 2.1536 & 1.7382 & -4.1331 & 2.1341 & 1.6190 \\ 2.0562 & 0.6674 & 2.8367 & 1.4244 & 1.3101 & 2.0604 & 1.5791 & 1.9793 & 1.8415 & -1.6444 \end{bmatrix},$$

$$B = \begin{bmatrix} 9.5013 & -4.5647 & 9.2181 & -4.1027 & 1.3889 \\ -2.3114 & 0.1850 & 7.3821 & 8.9365 & 2.0277 \\ 6.0684 & 8.2141 & 1.7627 & -0.5789 & 1.9872 \\ 4.8598 & 4.4470 & -4.0571 & 3.5287 & -6.0379 \\ 8.9130 & 6.1543 & 9.3547 & 8.1317 & 2.7219 \\ 7.6210 & 7.9194 & -9.1690 & 0.0986 & -1.9881 \end{bmatrix},$$

$$C = \begin{bmatrix} -4.4243 & 63.3495 & 57.1330 & -35.3497 & 94.3570 & 30.4116 & 103.5431 \\ 12.5767 & 11.5448 & 24.3666 & 34.9444 & 52.8061 & -16.8209 & -35.0123 \\ 4.7270 & 49.6253 & -53.1311 & -175.8609 & -55.6022 & 34.1111 & 192.2423 \\ 63.6385 & 124.8464 & 42.6792 & -98.0054 & 161.4329 & -21.1148 & 134.9706 \\ 28.7349 & 6.6986 & 40.6452 & 62.0110 & 72.0806 & -31.7856 & -59.6399 \\ 108.1412 & 100.7630 & 18.3534 & -145.4211 & 92.5247 & -38.9258 & 182.3775 \\ -7.2849 & 15.9978 & 7.7943 & -30.7668 & 5.4565 & 19.1774 & 55.3453 \\ -4.2597 & 65.4605 & -6.1941 & -71.6393 & 53.7233 & 12.3832 & 75.6285 \end{bmatrix},$$

$$D = 10^3 \times \begin{bmatrix} 1.7187 & 1.0608 & 0.8913 & 0.7105 & 0.3342 \\ -0.1399 & -0.1220 & -0.0108 & -0.0419 & -0.4514 \\ -0.5142 & 0.0006 & 0.0553 & 0.4558 & 0.0061 \\ 0.6801 & -0.1355 & 0.7746 & 0.0739 & 0.1986 \\ -0.6771 & -0.1321 & -0.1314 & 0.2385 & -0.0573 \\ 0.1313 & 0.4285 & 0.1771 & 0.2455 & 0.2599 \\ 0.5760 & 0.3211 & -0.0988 & 0.2740 & -0.5275 \\ -0.8144 & -0.5500 & 0.2302 & 0.0471 & 0.0762 \\ 1.9424 & 1.8729 & 0.2088 & 0.7696 & -0.1201 \\ 0.2437 & 0.3519 & 0.5792 & 0.6914 & 0.1667 \\ 0.9052 & -0.8601 & 1.2965 & -0.3310 & 0.2678 \\ 0.5841 & 0.3483 & 0.8712 & 0.7461 & 0.2035 \end{bmatrix},$$

$$G = \begin{bmatrix} 1.3652 & -0.6478 & 5.7981 & 4.6110 & 8.7437 & 2.1396 & 4.3992 & 6.0720 & 0.1286 & 0.1635 \\ 0.1176 & 9.8833 & 7.6037 & -5.6783 & 0.1501 & 6.4349 & 9.3338 & -6.2989 & -3.8397 & 1.9007 \\ 8.9390 & 5.8279 & -5.2982 & 7.9421 & 7.6795 & 3.2004 & -6.8333 & 3.7048 & 6.8312 & -5.8692 \\ 1.9914 & 4.2350 & 6.4053 & 0.5918 & 9.7084 & 9.6010 & 2.1256 & 5.7515 & -0.9284 & 0.5758 \\ 2.9872 & 5.1551 & 2.0907 & -6.0287 & -9.9008 & 7.2663 & 8.3924 & 4.5142 & 0.3534 & 3.6757 \\ 6.6144 & 3.3395 & 3.7982 & 0.5027 & 7.8886 & 4.1195 & -6.2878 & 0.4390 & 6.1240 & 6.3145 \\ 2.8441 & 4.3291 & -7.8333 & 4.1537 & 4.3866 & 7.4457 & 1.3377 & 0.2719 & -6.0854 & -7.1763 \\ 4.6922 & 2.2595 & 6.8085 & 3.0500 & 4.9831 & 2.6795 & 2.0713 & 3.1269 & 0.1576 & -6.9267 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.0704 & -0.3871 & 0.2970 & 0.4060 & -0.2410 & 0.0407 & -0.0698 \\ 0.2088 & 0.2624 & -0.3917 & -0.5369 & -0.0169 & -0.2079 & 0.1527 \\ -0.5101 & 0.3010 & 0.0086 & 0.1479 & 0.4768 & 0.2358 & -0.2671 \\ 0.2095 & -0.1370 & 0.2497 & -0.0462 & -0.1308 & 0.0378 & 0.4432 \\ -0.1319 & 0.1712 & 0.3884 & -0.1150 & 0.2857 & 0.3117 & 0.6693 \\ 0.5654 & 0.1514 & 0.2992 & 0.1829 & 0.5863 & -0.4149 & -0.1353 \end{bmatrix}.$$

It is easy to verify that the conditions (8) and (17) hold ($\|AA^{-}DB^{-}B - D\| = 3.3524e - 012$, $\|GG^{-}CH^{-}H - C\| = 3.5742e - 013$, and $\|P_{\mathcal{T}}(A^{-}DB^{-} - G^{-}CH^{-})P_{\mathcal{S}}\| = \|E_{\tilde{A}}\tilde{D}F_{\tilde{B}}\| = 4.5887e - 014$). According to Algorithm 1, by choosing $Z_1 = 0, Z_2 = 0, W_1 = 0$ and $W_2 = 0$, we can obtain a common solution X by (9) or by (10) as follows (In fact, the difference of the solutions computed by (9) and (10) is $5.1843e-14$):

$$X=\begin{bmatrix} -0.7593 & 7.1006 & 0.8776 & 4.1065 & 8.0839 & 7.7379 \\ -7.1008 & 0.2667 & 3.9975 & 3.4272 & -1.4408 & -9.0241 \\ 0.3669 & 3.8912 & 1.2076 & -8.4299 & 7.1447 & 5.6330 \\ 12.1231 & -0.7975 & 4.0895 & 0.9762 & 6.3728 & -7.7365 \\ -5.3094 & 7.0758 & 1.5034 & 9.2270 & 4.3845 & 1.1587 \\ 4.4918 & 6.4102 & -3.2166 & -0.4456 & 4.0534 & 9.7406 \\ 10.4318 & -6.4101 & 3.4834 & 1.7671 & -3.4256 & -3.1548 \\ -7.4411 & 3.1080 & 9.6018 & 6.3035 & 2.4320 & 2.5455 \\ 2.0221 & 2.1309 & -9.1917 & 2.7910 & -5.4906 & -9.2646 \\ 0.4320 & -2.6945 & 4.2894 & 7.2900 & 3.9208 & 7.6539 \end{bmatrix}.$$

Also, the absolute errors are estimated by

$$||AXB-D||=3.4544e-12, \quad ||GXH-C||=6.1471e-13,$$

which implies that X is a common solution to the matrix equations of (1).

5. Conclusions

In this paper, by choosing suitable parameter matrices L_1, L_2, J_1 and J_2 in the equations of (9) and (10), we have derived the necessary and sufficient conditions for the existence of a solution and two explicit representations of the general common solution to the pair of linear matrix equations (1) by means of the inner inverses and orthogonal projectors. In particular, our representation of the general common solution to the equations of (1) is in terms of only the coefficient and right-hand side matrices of the pair of matrix equations and some arbitrary matrices. Subsequently, the results are applied to determine the solvability conditions and the general common solution to the matrix equation (2) and the general Hermitian solution to $AXB = D$. Also, the results are applied to determine the solvability conditions for the matrix equation $AXB = D$ under some constraints (see Corollaries 3 and 4).

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Conflict of interest

The authors declare no conflict of interest.

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