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Research article

Global existence and finite time blow-up for a class of fractional p-Laplacian Kirchhoff type equations with logarithmic nonlinearity

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Abstract: In this paper, we study the initial-boundary value problem for a class of fractional p-Laplacian Kirchhoff diffusion equation with logarithmic nonlinearity. For both subcritical and critical states, by means of the Galerkin approximations , the potential well theory and the Nehari manifold, we prove the global existence and finite time blow-up of the weak solutions. Further, we give the growth rate of the weak solutions and study ground-state solution of the corresponding steady-state problem.

Keywords: Kirchhoff type; p-Laplacian; fractional; global existence; blow-up; ground-state solution
Mathematics Subject Classification: 39A13, 34B18, 34A08

1. Introduction

In this paper, we study the global existence and blow-up for the following nonlinear fractional p-Laplacian Kirchhoff diffusion equation.

$$\begin{cases} u_t + M([u]_{s,p}^p)(-\Delta)_p^s u = |u|^{\alpha p-2}u\ln|u|, & \text{in } \Omega\times(0,T), \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $1 < p < N/s$ (N is space latitude and satisfies $N \geq 1$), $1 \leq \alpha < p_s^*/p$, $p_s^* = \frac{Np}{N-sp}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $[u]_{s,p}$ is Gagliardo seminorm of u, $(-\Delta)_p^s$ is the fractional p-Laplacian operator and satisfies

$$(-\Delta)_p^s u(x) = 2\lim_{\beta\to 0^+}\int_{\mathbb{R}^N\setminus B_\beta(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}dy,$$

where $u(x) \in C^\infty$ and $u(x)$ has compact support in Ω, $B_\beta(x) \subset \mathbb{R}^N$ is the set of spheres with x as the center and β as the radius. Let $M(t) = a + bt^{\alpha-1}(t \geq 1)$, where a and b are constants and satisfy $a \geq 0$,

$b > 0$, in this paper we let $a = 0$, $b = 1$, $t = [u]^p_{s,p}$, i.e. $M([u]^p_{s,p}) = [u]^{\alpha p-p}_{s,p}$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, and use $\|\cdot\|_r$ to denote the norm of Lebesgue space $L^r(\Omega)$, where $r \in (0, +\infty)$. In view of the idea from [1], we define linear space $W^{s,p}(\Omega)$ as a fractional Sobolev space and discuss in the fractional Sobolev space $W^{s,p}_0(\Omega)$,

$$W^{s,p}_0(\Omega) = \{u \in W^{s,p}(\Omega) \mid u(t) = 0 \text{ a.e.in } \mathbb{R}^N\backslash\Omega\}.$$

The norm of space $W^{s,p}(\Omega)$ is defined as the following equation

$$\|u\|_{W^{s,p}} = \Big(\|u\|^p_p + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy\Big)^{\frac{1}{p}}.$$

According to the research result from [2], it can be concluded that

$$[u]_{s,p} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy\Big)^{\frac{1}{p}}.$$

Kirchhoff-type problems have a wide range of applications in mathematics and physics. In recent years, many scholars have devoted themselves to the study of Kirchhoff-type problems and have achieved many results [3–6]. Moreover, readers who are interested in this knowledge can refer to [7–15]. The research on the fractional p-Laplacian operator equation also achieved some results, such as [16–21]. Detailed knowledge about fractional-order related theories and differential inequalities can refer to [22–25] and the parabolic equation with logarithmic nonlinearity can refer to [26–28].

Until now, many scholars have studied the fractional Laplacian problem. In [1], the authors studied the fractional Kirchhoff-type problem.

$$\begin{cases} u_t + [u]^{2(\theta-1)}_s(-\Delta)^s u = |u|^{q-2}u \ln|u|, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $[u]_s$ is the Gagliardo seminorm of u, let

$$[u]_s = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy\Big)^{\frac{1}{2}},$$

where $s \in (0,1)$, $q \in (2\theta, 2^*)$, $\theta \in [1, N/(N-2s))$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary. They use the variational principle, Nehari manifold and potential well method to study the finite time blow-up of the solution of the problem (1.2) and the sufficient conditions for the global existence.

In [2], the authors studied the following initial value problem:

$$M([u]^p_{s,p})(-\Delta)^s_p u = h(x)|u|^{\theta p-2}u \ln|u| + \lambda|u|^{q-2}u, \qquad x \in \Omega, \tag{1.3}$$

where $h(x)$ is a sign-changing function on Ω, λ is a positive parameter, $M([u]^p_{s,p}) = [u]^{p(\theta-1)}_{s,p}$. By the method of the mountain pass lemma and Nehari manifold, the existence of the minimum energy solutions are obtained. For the steady-state equation, it is worth emphasizing that for the local minimum solution, Liu and Liao and Pan used a brand-new method in [29], and also obtained some properties of the corresponding equation.

In [3], the authors discuss the following equations

$$u_t + M([u]_s^2)\mathcal{L}_K u = |u|^{p-2}u \ln |u|, \qquad (x,t) \in \Omega \times (0,+\infty), \tag{1.4}$$

where $\mathcal{L}_K$ is a nonlocal integro-differential operator. They used the Galerkin approximation method and the potential well to prove the existence of a global weak solution with subcritical and critical states. According to the differential inequality, the blow-up solution of the equation is given. At the same time, the lower bound of the solution of problem (1.4) and the existence of the ground state solution of the corresponding steady-state problem was discussed. More details for fractional Laplacian equations with logarithmic nonlinearity can refer to [30–32].

Inspired by the above references, we study problem (1.1). Compared with problem (1.2), we consider the case of $1 < p < N/s$ and discuss the global existence and finite time blow-up of the solutions. If $p = 2$ and $1 < \alpha < N/(N-2s)$, we can turn problem (1.1) into problem (1.2). Compared with problem (1.3) and (1.4), we study the global existence and finite time blow-up of the solutions for fractional p-Laplacian Kirchhoff type equation with logarthmic nonlinearity. By the methods of the variational principle and Nehari manifold, as well as combining with the relevant theories and properties of the fractional Sobolev space definition, we consider both $E(u_0) < h$ and $E(u_0) = h$ cases and prove the global existence and the finite time blow-up of the solution for problem (1.1), and discuss the growth rate of the weak solutions and study ground-state solution of the corresponding steady-state problem.

The rest of the paper is organized as follws. In Section 2, we give some related definitions and lemmas. In Section 3, we prove the global existence and the finite time blow-up of the solution with subcritical state $E(u_0) < h$ of the problem (1.1), at the same time, we give the growth rate of the solutions and study ground-state solution of the corresponding steady-state problem. In Section 4, we prove the global existence and the finite time blow-up of the solution with critical state $E(u_0) = h$ of the problem (1.1).

2. Preliminaries and Lemmas

In this section, we give some related definitions and Lemmas needed to prove the conclusions later. First of all, we define

$$r(\varrho) = \Big(\frac{e\varrho}{(e-1)D_*^{\alpha p+\varrho}}\Big)^{\frac{1}{\varrho}}, \tag{2.1}$$

where D_* is the best embedding constant for $W_0^{s,p}(\Omega) \hookrightarrow L^{\alpha p+\varrho}(\Omega), \varrho \in (0, p_s^* - \alpha p)$.
The energy functional is

$$E(u) = \frac{1}{\alpha p}[u]_{s,p}^{\alpha p} - \frac{1}{\alpha p}\int_\Omega |u|^{\alpha p}\ln|u|dx + \frac{1}{(\alpha p)^2}\int_\Omega |u|^{\alpha p}dx. \tag{2.2}$$

The Nehari functional is

$$I(u) = [u]_{s,p}^{\alpha p} - \int_\Omega |u|^{\alpha p}\ln|u|dx. \tag{2.3}$$

And then, we define some sets as follows:

$$W := \{u(x) \in W_0^{s,p}(\Omega) \mid I(u(x)) > 0,\ E(u(x)) < h\} \cup \{0\}, \tag{2.4}$$

$$V := \{u(x) \in W_0^{s,p}(\Omega) \mid I(u(x)) < 0,\ E(u(x)) < h\}, \tag{2.5}$$

the mountain pass level

$$h := \inf_{u(x)\in\mathcal{N}} E(u(x)), \tag{2.6}$$

the Nehari manifold

$$\mathcal{N} = \{u(x) \in W_0^{s,p}(\Omega) \mid I(u(x)) = 0,\ u(x) \neq 0\}. \tag{2.7}$$

Further, define the positive set and the negative sets as follows

$$\mathcal{N}_+ = \{u(x) \in W_0^{s,p}(\Omega) \mid I(u(x)) > 0\}. \tag{2.8}$$

$$\mathcal{N}_- = \{u(x) \in W_0^{s,p}(\Omega) \mid I(u(x)) < 0\}. \tag{2.9}$$

From (2.2) and (2.3), we have

$$E(u) = \frac{1}{\alpha p} I(u) + \frac{1}{(\alpha p)^2} \int_\Omega |u|^{\alpha p} dx, \tag{2.10}$$

In order to facilitate the proof of the main results, next we give the necessary Lemmas and definitions.

Lemma 2.1 Let $u \in W_0^{s,p}(\Omega)$ and $[u]_{s,p} \neq 0, \rho \in (0, +\infty)$, then:

(i) $\lim\limits_{\rho\to 0^+} E(\rho u(x)) = 0,\ \lim\limits_{\rho\to+\infty} E(\rho u(x)) = -\infty$;

(ii) There is an unique value ρ^*, such that

$$\frac{d}{d\rho} E(\rho u(x))|_{\rho=\rho^*} = 0;$$

(iii) $E(\rho u(x))$ is increasing on $(0, \rho^*)$, decreasing on $(\rho^*, +\infty)$ and takes the maximum at ρ^*;

(iv) $I(\rho u(x)) > 0$ for all $0 < \rho < \rho^*$, $I(\rho u(x)) < 0$ for all $\rho > \rho^*$ and $I(\rho^* u(x)) = 0$.

Proof. (i) According to the known definition of $E(u(x))$ on (2.2), i.e.

$$E(u(x)) = \frac{1}{\alpha p}[u(x)]_{s,p}^{\alpha p} - \frac{1}{\alpha p} \int_\Omega |u(x)|^{\alpha p} \ln |u(x)| dx + \frac{1}{(\alpha p)^2} \int_\Omega |u(x)|^{\alpha p} dx,$$

we have

$$\begin{aligned} E(\rho u(x)) =& \frac{1}{\alpha p}\rho^{\alpha p}[u(x)]_{s,p}^{\alpha p} - \frac{1}{\alpha p}\rho^{\alpha p} \int_\Omega |u(x)|^{\alpha p} \ln |u(x)| dx \\ &- \frac{1}{\alpha p}\rho^{\alpha p} \int_\Omega |u(x)|^{\alpha p} \ln |\rho| dx + \frac{1}{(\alpha p)^2}\rho^{\alpha p} \int_\Omega |u(x)|^{\alpha p} dx \\ =& \rho^{\alpha p}\Big(\frac{1}{\alpha p}[u(x)]_{s,p}^{\alpha p} - \frac{1}{\alpha p} \int_\Omega |u(x)|^{\alpha p} \ln |u(x)| dx \\ &- \frac{1}{\alpha p} \int_\Omega |u(x)|^{\alpha p} \ln \rho + \frac{1}{(\alpha p)^2} \int_\Omega |u(x)|^{\alpha p} dx\Big), \end{aligned}$$

we can conclude that (i) holds.

(ii) Through simple calculations, we have

$$\frac{d}{d\rho}E(\rho u(x)) = \rho^{\alpha p-1}\Big([u(x)]_{s,p}^{\alpha p} - \int_\Omega |u(x)|^{\alpha p}\ln|u(x)|dx - \ln\rho\int_\Omega |u(x)|^{\alpha p}dx\Big), \tag{2.11}$$

therefore, (ii) holds.

(iii) By the result in (ii),we have

$$\frac{d}{d\rho}E(\rho u(x)) = \rho^{\alpha p-1}\Big([u(x)]_{s,p}^{\alpha p} - \int_\Omega |u(x)|^{\alpha p}\ln|u(x)|dx - \ln\rho\int_\Omega |u(x)|^{\alpha p}dx\Big) > 0,\ 0<\rho<\rho^*,$$

and

$$\frac{d}{d\rho}E(\rho u(x)) = \rho^{\alpha p-1}\Big([u(x)]_{s,p}^{\alpha p} - \int_\Omega |u(x)|^{\alpha p}\ln|u(x)|dx - \ln\rho\int_\Omega |u(x)|^{\alpha p}dx\Big) < 0,\ \rho^*<\rho<+\infty,$$

thus, (iii) holds.

(iv) According to the (2.11) and

$$I(\rho u(x)) = \rho^{\alpha p}[u(x)]_{s,p}^{\alpha p} - \rho^{\alpha p}\int_\Omega |u(x)|^{\alpha p}\ln|u(x)|dx - \rho^{\alpha p}\int_\Omega |u(x)|^{\alpha p}\ln\rho dx,$$

thus

$$I(\rho u(x)) = \rho\frac{d}{d\rho}E(\rho u(x)),$$

then, $I(\rho u(x)) > 0$ for all $0<\rho<\rho^*$, $I(\rho u(x)) < 0$ for all $\rho>\rho^*$ and $I(\rho u(x)) = 0$, (iv) holds.

Lemma 2.2 [1-2] If $\varrho \in (0, p_s^* - \alpha p)$, then:

$$e\varrho\ln t \le t^\varrho, \quad 1\le t<+\infty.$$

Proof. Set $g(t) = \ln t - \frac{t^\varrho}{e\varrho}$, $t\in[1,+\infty)$. By a simple derivative calculation of the function, we get $g^{'}(t) = \frac{1}{t} - \frac{1}{e\varrho}t^{\varrho-1}$ and let $g^{'}(t) = 0$, then $t^* = e^{\frac{1}{\varrho}}$. Obviously t^* is the maximum point of function $g(t)$, thus $g(t)\le g(t^*) = 0$ for all $t\in[1,+\infty)$. This proves the above inequality.

Lemma 2.3 If $\varrho\in(0,p_s^*-\alpha p)$. Let $u\in W_0^{s,p}(\Omega)$ and $[u]_{s,p}\neq 0$, we have

(i) If $0<[u]_{s,p}<r(\varrho)$, then $I(u)>0$;

(ii)If $I(u)\le 0$, then $[u]_{s,p}\ge r(\varrho)$.

Proof. By Lemma 2.2 and the definition of Nehari function $I(u)$, we have

$$\begin{aligned}
I(u) =&[u]_{s,p}^{\alpha p} - \int_\Omega |u(x)|^{\alpha p}\ln|u(x)|dx\\
>&[u]_{s,p}^{\alpha p} - \frac{1}{\varrho}\int_{\Omega(|u|<1)} |u(x)|^{\alpha p}\ln|u|^\varrho dx - \int_{\Omega(|u|\ge 1)}|u(x)|^{\alpha p}\ln|u|dx\\
\ge&[u]_{s,p}^{\alpha p} - \frac{1}{\varrho}\parallel u\parallel_{\alpha p+\varrho}^{\alpha p+\varrho} - \frac{1}{e\varrho}\parallel u\parallel_{\alpha p+\varrho}^{\alpha p+\varrho}\\
\ge&[u]_{s,p}^{\alpha p} - \frac{1}{\varrho}D_*^{\alpha p+\varrho}[u]_{s,p}^{\alpha p+\varrho} - \frac{1}{e\varrho}D_*^{\alpha p+\varrho}[u]_{s,p}^{\alpha p+\varrho}\\
=&[u]_{s,p}^{\alpha p}(1 - \frac{e-1}{e\varrho}D_*^{\alpha p+\varrho}[u]_{s,p}^{\varrho}),
\end{aligned}$$

we can get

$$I(u) > [u]_{s,p}^{\alpha p}(1 - \frac{e-1}{e\varrho} D_*^{\alpha p+\varrho}[u]_{s,p}^{\varrho}).$$

If $0 < [u]_{s,p} < r(\varrho)$, then

$$1 - \frac{e-1}{e\varrho} D_*^{\alpha p+\varrho}[u]_{s,p}^{\varrho} > 0,$$

thus (i) holds.
Similarly, assume $I(u) \leq 0$, i.e.

$$1 - \frac{e-1}{e\varrho} D_*^{\alpha p+\varrho}[u]_{s,p}^{\varrho} \leq 0,$$

thus (ii) holds.
Lemma 2.4 [2] If $s \in (0,1)$,$1 < p < N/s$ holds, then the functionals $E(u)$ and $I(u)$ are well-defined and continuous on $W_0^{s,p}(\Omega)$. Moreover, $E(u) \in C^1(W_0^{s,p}(\Omega), \mathbb{R})$, and $I(u) = \langle E'(u), u\rangle$ for all $u \in W_0^{s,p}(\Omega)$.
Proof. Since

$$\int_\Omega |u|^{\alpha p} \ln |u| dx \leq \frac{1}{\varrho} \|u\|_{\alpha p+\varrho}^{\alpha p+\varrho} \leq \frac{1}{\varrho} D_*^{\alpha p+\varrho}[u]_{s,p}^{\alpha p+\varrho},$$

where $1 \leq \alpha < p_s^*/p$, then we can claim that $E(u)$ and $I(u)$ are well-defined in $W_0^{s,p}(\Omega)$. Further, Similar to Lemma 2.3 in [2], one can prove the that $E(u) \in C^1(W_0^{s,p}(\Omega), \mathbb{R})$, and $I(u) = \langle E'(u), u\rangle$ for all $u \in W_0^{s,p}(\Omega)$.
Lemma 2.5 [33] Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$. Let $q \in [1, p^*)$, $\Omega \subseteq \mathbb{R}^N$ be a bounded extension domain for $W^{s,p}(\Omega)$ and ζ be a bounded subset of $L^p(\Omega)$. Suppose that

$$\sup_{f\in\zeta} \int_\Omega \int_\Omega \frac{|f(x) - f(x)|^p}{|x-y|^{N+sp}} dxdy < +\infty,$$

then ζ is pre-compact in $L^q(\Omega)$.
Lemma 2.6 Let $s \in (0,1)$, $1 < p < N/s$, $\alpha \in [1, p_s^*/p)$, $t \in [0,T)$, and assume that $u(x,t)$ is the solution of problem (1.1), then

$$\frac{d}{dt}\|u\|_2^2 = -2I(u).$$

Proof. By the definition of the weak solution, we have

$$\int_\Omega u_t u dx + [u]_{s,p}^{(\alpha-1)p} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))^2}{|x-y|^{N+sp}} dxdy = \int_\Omega u|u|^{\alpha p-2} u \ln |u| dx.$$

So

$$\begin{aligned}\frac{1}{2}\|u\|_2^2 &= -[u]_{s,p}^{\alpha p} + \int_\Omega |u|^{\alpha p} \ln |u| dx\\ &= -([u]_{s,p}^{\alpha p} - \int_\Omega |u|^{\alpha p} \ln |u| dx).\end{aligned}$$

Thus, we obtain

$$\frac{d}{dt}\|u\|_2^2 = -2I(u).$$

Definition 2.1 A function $u(x,t)$ is called a weak solution of problem (1.1), where $(x,t) \in \Omega \times [0,T)$. If $u \in L^\infty(0,T;W_0^{s,p}(\Omega))$ and $u_t \in L^2(0,T;L^2(\Omega))$, for all $v \in W_0^{s,p}(\Omega)$, $t \in (0,T)$, the following equation holds

$$\int_\Omega u_t v dx + [u]_{s,p}^{(\alpha-1)p} \iint_{R^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy = \int_\Omega v|u|^{\alpha p-2} u \ln |u| dx,$$

with $u(x,0) = u_0(x)$ in $W_0^{s,p}(\Omega)$. Moreover, for $0 \le t < T$, the following inequality holds

$$\int_0^t \|u_r\|_2^2 dr + E(u) \le E(u_0). \tag{2.12}$$

Definition 2.2 (Maximal existence time) Let $u(t)$ be a solution of problem (1.1), we define the maximal existence time T of $u(t)$ as follows:

(i) If $u(t)$ exists for $0 \le t < +\infty$, then $T = +\infty$.

(ii) If there exists a $t_0 \in (0,+\infty)$ such that $u(t)$ exists for $0 \le t < t_0$, but doesn't exist at $t = t_0$, then $T = t_0$.

Definition 2.3 (Finite time blow-up) Let $u(x)$ be weak solution of problem (1.1) , We call $u(t)$ a blow-up in finite time if the maximal existence time T is finite and

$$\lim_{t\to T^-} \int_0^T \|u\|_2^2 dr = +\infty.$$

3. Subcritical state ($E(u_0) < h$)

In this section, we use Galerkin approximation, potential well theory, Nehari manifold to study the problem (1.1) the global existence of the solution , blow-up in finite time of the solution with subcritical state $E(u_0) < h$ of the problem (1.1). In addition, we discuss the growth rate of the weak solution. Finally, we also discuss the ground state solution of the corresponding steady-state problem.

Theorem 3.1 Let $\alpha \in [1, p_s^*/p)$,$1 < p < N/s$. $u_0 \in W_0^{s,p}(\Omega)$, If $E(u_0) < h$ and $I(u_0) > 0$, then problem (1.1) admits a global weak solution $u(t) \in L^\infty(0,T;W_0^{s,p}(\Omega))$, $u_t \in L^2(0,T;L^2(\Omega))$ and $u(t) \in W$ for all $t \in [0,+\infty)$.

Proof. By the definition of the weak solution, we take a set of smooth functions $\eta = \eta_k(x)$ $(k = 1,2,\cdots)$ such that $\{\eta_k(x)\}_{k=1}^\infty$ is the basis function of $W_0^{s,p}(\Omega)$, and then construct a approximate solutions $u_m(t)$ of problem (1.1) as follows:

$$u_m = \sum_{k=1}^m d_{km}(t)\eta_k(x), \quad k = 1,2,\cdots,$$

i.e. $u_m : [0,T) \longmapsto W_0^{s,p}(\Omega)$, $u_m \in \text{span}\{\eta_1,\cdots\eta_m\}$, which satisfy

$$\begin{aligned}&\int_\Omega u_{mt}\eta_k dx + [u_m]_{s,p}^{\alpha p-p} \iint_{R^{2N}} \frac{|u_m(x)-u_m(y)|^{p-2}(u_m(x)-u_m(y))(\eta_k(x)-\eta_k(y))}{|x-y|^{N+sp}} dxdy \\ &= \int_\Omega \eta_k |u_m|^{\alpha p-2} u_m \ln |u_m| dx\end{aligned} \tag{3.1}$$

and

$$u_m(x,0) = \sum_{k=1}^{m} b_{km}\eta_k(x) \to u_0 \quad \text{in } W_0^{s,p}(\Omega), \tag{3.2}$$

where $b_{km} = (u_m(0), \eta_k)$ are given constants, $k = 1, 2, \cdots, m$. we use $(\cdot, \cdot)$ to represent the inner product in $L^2(\Omega)$.

Multiplying (3.1) by $d'_{km}(t)$, then summing k from 0 to m, and finally integrating t from 0 to t, we get

$$\int_0^t \|u_{mr}\|_2^2 dr + E(u_m) = E(u_m(0)) \quad 0 \leq t \leq T.$$

According to (3.2), we obtain $d_{km}(0) = b_{km}$. Noticing $E(u_0) < h$, $I(u_0) > 0$, we have

$$\lim_{m\to+\infty} (\int_0^t \|u_{mr}\|_2^2 dr + E(u_m)) = \lim_{m\to+\infty} E(u_m(0)) < h,$$

and

$$\lim_{m\to+\infty} I(u_m(0)) = I(u_0) > 0,$$

then for a sufficiently large m, we can obtain

$$\int_0^t \|u_{mr}\|_2^2 dr + E(u_m) = E(u_m(0)) < h, \tag{3.3}$$

and

$$I(u_m(0)) = I(u_0) > 0,$$

due to $W := \{u \in W_0^{s,p}(\Omega) \mid I(u) > 0,\ E(u) < h\} \cup \{0\}$, thus $u_m(x, 0) \in W$, this is true if m is big enough.

Let $m \to +\infty,\ 0 \leq t \leq T$, we prove $u_m(t) \in W$. Arguing by contraction, we suppose $u_m(t) \notin W$, for sufficiently large m, then exists a $t_0 \in (0, T)$ satisfying $u_m(t_0)$ is on the boundary of ∂W, we have

$$I(u_m(t_0)) = 0,\ u(t_0) \neq 0,$$

thus can get $u_m(t_0) \in \mathcal{N}$. And because $h = \inf_{u\in N} E(u)$ which yields $E(u_m(t_0)) \geq h$, this is paradoxically. Then $u_m(t_0) \in W$ for $m \to +\infty$ and $0 \leq t \leq T$.

By (3.3) and

$$E(u_m) = \frac{1}{\alpha p} I(u_m) + \frac{1}{(\alpha p)^2} \int_\Omega |u_m|^{\alpha p} dx,$$

we can get

$$\int_0^t \|u_{mr}\|_2^2 dr + \frac{1}{(\alpha p)^2} \|u_m\|_{\alpha p}^{\alpha p} < h.$$

For sufficiently large m, it yields

$$\frac{1}{(\alpha p)^2} \|u_m\|_{\alpha p}^{\alpha p} < h, \quad 0 \leq t \leq T, \tag{3.4}$$

$$\int_0^t \|u_{mr}\|_2^2 dr < h, \quad 0 \leq t \leq T, \tag{3.5}$$

$$\|u_m^{\alpha p-2}\|^q = \|u_m\|_{\alpha p}^{\alpha p} \leq C_*^{\alpha p}[u]_{s,p}^{\alpha p} < C_*^{\alpha p} r^{\alpha p}(\varrho), \quad 0 \leq t \leq T,\ q = \frac{\alpha p}{\alpha p - 2}, \tag{3.6}$$

where C_* is the best embedding constant for $W_0^{s,p}(\Omega) \hookrightarrow L^{\alpha p}(\Omega)$. Thus, we get $T = +\infty,\ u_m(t) \in W$ for sufficiently large m and $0 \leq t < +\infty$.

Next, we prove $u_m(t) \in L^\infty(0, T; W_0^{s,p}(\Omega))$ and $u_t \in L^2(0, T; L^2(\Omega))$. According to the (3.4) (3.5) and (3.6),we have $u_t \in L^2(0, T; L^2(\Omega))$, $u(t) \in L^\infty(0, T; W_0^{s,p}(\Omega))$. Based on the above proof, there exists a u and subsequence $\{u_n\}$ and $\{u_m\}$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(0, +\infty; W_0^{s,p}(\Omega)), \tag{3.7}$$

$$u_n \rightharpoonup u \quad \text{weakly star in } L^\infty(0, +\infty; W_0^{s,p}(\Omega)), \tag{3.8}$$

$$u_n^{\alpha p-2} \rightharpoonup u^{\alpha p-2} \quad \text{weakly star in } L^\infty(0, +\infty; L^q(\Omega)), \tag{3.9}$$

$$u_n \to u \quad \text{a.e. } \Omega, \tag{3.10}$$

$$u_{nt} \rightharpoonup u \quad \text{wealky in } L^2(0, +\infty; L^2(\Omega)). \tag{3.11}$$

By (3.7) and (3.11), we obtain

$$u_n \rightharpoonup u \quad \text{strongly in } L^2(0, +\infty; L^{\alpha p}(\Omega)), \tag{3.12}$$

then $|u|^{\alpha p-2} u_n \ln |u_n| \to |u|^{\alpha p-2} u \ln |u|$ a.e. in $\Omega \times (0, +\infty)$.
Since $u_n(t) \in W$ and

$$\begin{aligned}
&\int_\Omega \left| u_n(t)|u_n(t)|^{\alpha p-2} \ln |u_n(t)| \right|^{\alpha p} dx \\
&\leq \int_{\Omega(|u_n(x)| \leq 1)} \left| |u_n(t)|^{\alpha p-1} \ln |u_n(t)| \right|^{\alpha p} dx + \int_{\Omega(|u_n(x)|>1)} \left| |u_n(t)|^{\alpha p-1} \ln |u_n(t)| \right|^{\alpha p} dx \\
&\leq \frac{1}{(e\alpha p - e)^{\alpha p}} |\Omega| + (\frac{1}{e\varrho})^{\alpha p} \|u_n\|_{(\alpha p+\varrho-1)\alpha p}^{(\alpha p+\varrho-1)\alpha p} \\
&\leq \frac{1}{(e\alpha p - e)^{\alpha p}} |\Omega| + (\frac{S_*^{(\alpha p+\varrho-1)}}{e\varrho})^{\alpha p} [u_n]_{s,p}^{(\alpha p+\varrho-1)\alpha p} \\
&\leq \frac{1}{(e\alpha p - e)^{\alpha p}} |\Omega| + (\frac{S_*^{(\alpha p+\varrho-1)}}{e\varrho})^{\alpha p} r^{(\alpha p+\varrho-1)\alpha p}(\varrho),
\end{aligned}$$

where S_* is the imbedding constant for $W_0^{s,p}(\Omega) \hookrightarrow L^{(\alpha p+\varrho-1)\alpha p}(\Omega)$, which implies

$$|u|^{\alpha p-2} u_n \ln |u_n| \to |u|^{\alpha p-2} u \ln |u| \text{ weakly star in } L^\infty(0, +\infty;\ L^{\alpha p}(\Omega)). \tag{3.13}$$

In (3.1), we take a fixed k, let $m = n \to +\infty$, then there is

$$\begin{aligned}
&\int_\Omega u_t \eta_k dx + [u]_{s,p}^{\alpha p-p} \iint_{R^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\eta_k(x) - \eta_k(y))}{|x - y|^{N+sp}} dxdy \\
&= \int_\Omega \eta_k |u|^{\alpha p-2} u \ln |u| dx
\end{aligned}$$

and

$$\begin{aligned}&\int_{\Omega} u_t v dx + [u]_{s,p}^{\alpha p-p} \iint_{R^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy\\ &= \int_{\Omega} v|u|^{\alpha p-2} u \ln|u| dx,\end{aligned}$$

where $v \in W_0^{s,p}(\Omega),\ t \in (0,+\infty)$, and by (3.2) gives $u(x,0) = u_0(x)$ in $W_0^{s,p}(\Omega)$.

Finally, we prove the following inequality

$$\int_0^t \|u_r\|_2^2 dr + E(u) \leq E(u_0).$$

In order to achieve the above result, we first consider

$$\int_{\Omega} |u_m|^{\alpha p} \ln|u_m| dx - \int_{\Omega} |u|^{\alpha p} \ln|u| dx \leq 0.$$

Since $\varrho \ln s \leq s^{\varrho},\ s \in (0,+\infty),\ \varrho > 0$, we have

$$\begin{aligned}&\int_{\Omega} |u_m|^{\alpha p} \ln|u_m| dx - \int_{\Omega} |u|^{\alpha p} \ln|u| dx\\ &\leq \Big| \int_{\Omega} (|u_m|^{\alpha p} \ln|u_m| + uu_m|u_m|^{\alpha p-2} \ln|u_m| - uu_m|u_m|^{\alpha p-2} \ln|u_m| - |u|^{\alpha p} \ln|u|) dx \Big|\\ &\leq \int_{\Omega} \Big|(u_m - u)u_m|u_m|^{\alpha p-2} \ln|u_m|\Big| dx + \Big| \int_{\Omega} u(u_m|u_m|^{\alpha p-2} \ln|u_m| - u|u|^{\alpha p-2} \ln|u|) dx \Big|\\ &= \|u_m - u\|_{\alpha p} \frac{1}{\varrho} \|u_m\|_{\alpha p+\varrho-1}^{\alpha p+\varrho-1} + \Big| \int_{\Omega} u(u_m|u_m|^{\alpha p-2} - u|u|^{\alpha p-2} \ln|u|) dx \Big| \to 0\ (m \to +\infty).\end{aligned}$$

By the weak lower semicontinuity and the above inequality, we obtain

$$\begin{aligned}&E(u_0) + \frac{1}{\alpha p} \int_{\Omega} |u|^{\alpha p} \ln|u| dx\\ &= \liminf_{m\to+\infty} (E(u_m(0)) + \frac{1}{\alpha p} \int_{\Omega} |u_m|^{\alpha p} \ln|u_m| dx)\\ &= \liminf_{m\to+\infty} (E(u_m) + \frac{1}{\alpha p} \int_{\Omega} |u_m|^{\alpha p} \ln|u_m| dx + \int_0^t \|u_{mr}\|_2^2 dr)\\ &= \liminf_{m\to+\infty} (\frac{1}{\alpha p}[u_m]_{s,p}^{\alpha p} + \frac{1}{(\alpha p)^2} \|u_m\|_{\alpha p}^{\alpha p} + \int_0^t \|u_{mr}\|_2^2 dr)\\ &\geq \liminf_{m\to+\infty} \frac{1}{\alpha p}[u_m]_{s,p}^{\alpha p} + \liminf_{m\to+\infty} \frac{1}{(\alpha p)^2} \|u_m\|_{\alpha p}^{\alpha p} + \liminf_{m\to+\infty} \int_0^t \|u_{mr}\|_2^2 dr\\ &\geq \frac{1}{\alpha p}[u]_{s,p}^{\alpha p} + \frac{1}{(\alpha p)^2} \int_{\Omega} |u|^{\alpha p} dx + \int_0^t \|u_r\|_2^2 dr,\end{aligned}$$

which implies that the following inequality holds

$$\int_0^t \|u_r\|_2^2 dr + E(u) \leq E(u_0).$$

Theorem 3.2. Assume that $\alpha \in [1, p_s^*/p)$, $1 < p < N/s$ hold, and if $u_0 \in W_0^{s,p}(\Omega)$, $E(u(0)) < h$, $I(u(0)) < 0$, $u(t)$ is the weak solution of problem (1.1), then $u(t)$ blows up in a finite time T $(T > 0)$ and satisfies the following equation

$$\lim_{t\to T}\int_0^t \|u\|_2^2 dr = +\infty.$$

In addition, if $E(u_0) \le 0$, then the solution $u(t)$ of problem (1.1) also satisfies

$$\|u\|_2^2 \ge \Big(\frac{1}{M(t)}\Big)^{\frac{2}{\alpha p-2}},$$

where $M(t) = \|u_0\|_2^{\frac{2-\alpha p}{2}} - \frac{4}{\alpha p-2} t|\Omega|^{\frac{2-\alpha p}{2}}(\alpha p)^{-1}$.

Proof. Let $u(t)$ be weak solution of problem (1.1), and $E(u_0) < h$, $I(u_0) < 0$, we first define

$$Q(t) := \int_0^t \|u\|_2^2 dr,$$

then, we obtain

$$Q^{'}(t) = \|u\|_2^2,$$

and

$$Q^{''}(t) = 2(u, u_t) = -2I(u). \tag{3.14}$$

By (2.12), (3.14)and the following equation

$$E(u) = \frac{1}{\alpha p} I(u) + \frac{1}{(\alpha p)^2}\|u\|_{\alpha p}^{\alpha p},$$

we can get

$$Q^{''}(t) = -2I(u) \ge 2\alpha p \int_0^t \|u\|_2^2 dr + \frac{2}{\alpha p}\|u\|_{\alpha p}^{\alpha p} - 2\alpha p E(u_0). \tag{3.15}$$

Due to

$$\int_0^t (u_r, u) dr = \frac{1}{2}\int_0^t \frac{d}{dt}\|u\|_2^2 dr = \frac{1}{2}(\|u\|^2 - \|u_0\|^2),$$

square both sides of the above equation, we have

$$\Big(\int_0^t (u_r, u) dr\Big)^2 = \frac{1}{4}\Big((Q^{'}(t))^2 - 2\|u_0\|_2^2 Q^{'}(t) + \|u_0\|_2^4\Big). \tag{3.16}$$

By (3.15), (3.16) and the Schwartz's inequality, we can get

$$\begin{aligned} &Q^{''}(t)Q(t) - 2\alpha p\Big(\int_0^t (u_r, u) dr\Big)^2 \\ &\ge 2\alpha p \int_0^t \|u_r\|_2^2 dr \int_0^t \|u\|_2^2 - 2\alpha p\Big(\int_0^t (u_r, u) dr\Big)^2 + \frac{2}{\alpha p}\|u\|_{\alpha p}^{\alpha p} Q(t) - 2\alpha p E(u_0) Q(t), \end{aligned}$$

i,e.

$$
\begin{aligned}
&Q^{''}(t)Q(t) - \frac{\alpha p}{2}(Q^{'}(t))^2 \\
&\geq 2\alpha p\Big(\int_0^t \|u_r\|_2^2 dr \int_0^t \|u\|_2^2 dr - (\int_0^t (u_r, u)dr)^2\Big) + \frac{2}{\alpha p}\|u\|_{\alpha p}^{\alpha p} Q(t) \\
&- 2\alpha p E(u_0)Q(t) - \alpha p Q^{'}(t)\|u_0\|_2^2 + \frac{\alpha p}{2}\|u_0\|_2^4 \\
&\geq \frac{2}{\alpha p}\|u\|_{\alpha p}^{\alpha p}\, Q(t) - 2\alpha p E(u_0)Q(t) - \alpha p Q^{'}(t)\|u_0\|_2^2 + \frac{\alpha p}{2}\|u_0\|_2^4 \\
&\geq \frac{2}{\alpha p}\|u\|_{\alpha p}^{\alpha p}\, Q(t) - 2\alpha p E(u_0)Q(t) - \alpha p Q^{'}(t)\|u_0\|_2^2.
\end{aligned}
$$

Now, we discuss the two cases $E(u_0) \leq 0$, and $0 < E(u_0) < h$.

(a) If $E(u_0) \leq 0$, then

$$
Q^{''}(t)Q(t) - \frac{\alpha p}{2}(Q^{'}(t))^2 \geq \frac{2}{\alpha p}\|u\|_{\alpha p}^{\alpha p}\, Q(t) - \alpha p Q^{'}(t)\|u_0\|_2^2. \tag{3.17}
$$

Due to $I(u) < 0$ for all $t \in [0, T)$, so $Q^{''}(t) = -2I(u) > 0$, $Q^{'}(t) > Q^{'}(0) = \|u_0\|_2^2 > 0$. Then for all $\delta \in [0, t - t_0)$, where $t_0 \geq 0, t \geq t_0$ and $Q^{'}(t) > 0$, further we obtain

$$
Q(t) \geq \delta Q^{'}(t_0) > \delta Q^{'}(0). \tag{3.18}
$$

Moreover, we construct the following functional

$$
f(t) = \int_0^t \int_\Omega |u(x, r)|^2 dx dr + (T - t)\int_\Omega |u_0|^2 dx, \quad 0 \leq t < T.
$$

By simple calculation and Lemma 5, we have

$$
f^{'}(t) = \int_\Omega |u(x, t)|^2 dx - \int_\Omega |u_0|^2 dx,
$$

and

$$
f^{''}(t) = 2(u_t, u) = -2I(u).
$$

Since $I(u) < 0$, thus

$$
f^{''}(t) = 2(u_t, u) = -2I(u) > 0.
$$

By (2.10), (2.12) and Holder's inequality , we obtain

$$
\begin{aligned}
f^{''}(t) =& - 2\alpha p E(u(t)) + \frac{2}{\alpha p}\|u(t)\|_{\alpha p}^{\alpha p} \\
\geq& 2\alpha p \int_0^t \| u_r(r) \|_2^2\, dr + \frac{2}{\alpha p}\|u(t)\|_{\alpha p}^{\alpha p} - 2\alpha p E(u_0) \\
\geq& \frac{2}{\alpha p}\|u(t)\|_{\alpha p}^{\alpha p} \\
\geq& \frac{2}{\alpha p}|\Omega|^{\frac{2-\alpha p}{2}}\|u(t)\|_2^{\alpha p} \\
=& \frac{2}{\alpha p}|\Omega|^{\frac{2-\alpha p}{2}}(\gamma(t))^{\frac{\alpha p}{2}},
\end{aligned}
$$

where $\gamma(t) = f'(t) + \int_\Omega |u_0|^2 dx$, and $\gamma'(t) \geq 2(\alpha p)^{-1}|\Omega|^{\frac{2-\alpha p}{2}}(\gamma(t))^{\frac{\alpha p}{2}}$, by the differential inequality, we obtain

$$\|u\|_2^2 \geq \Big(\frac{1}{M(t)}\Big)^{\frac{2}{\alpha p-2}}, \tag{3.19}$$

where $M(t) = \|u_0\|_2^{\frac{2-\alpha p}{2}} - \frac{4}{\alpha p-2} t|\Omega|^{\frac{2-\alpha p}{2}}(\alpha p)^{-1}$. By (3.18) (3.19), for sufficiently large t, we have

$$Q''(t)Q(t) - \frac{\alpha p}{2}(Q'(t))^2 > 0.$$

(b) If $0 < E(u_0) < h$, by (3.18), $\alpha p > 2$ and Holder's inequality, then for a sufficiently large time t, we have

$$\begin{aligned}
&Q''(t)Q(t) - \frac{\alpha p}{2}(Q'(t))^2 \\
&\geq \frac{2}{\alpha p}\|u\|_{\alpha p}^{\alpha p} Q(t) - 2\alpha p E(u_0)Q(t) - \alpha p Q'(t)\|u_0\|_2^2 \\
&= (\frac{1}{\alpha p}\|u\|_{\alpha p}^{\alpha p} Q(t) - 2\alpha p E(u_0)Q(t)) + (\frac{1}{\alpha p}\|u\|_{\alpha p}^{\alpha p}Q(t) - \alpha p Q'(t)\|u_0\|_2^2) \\
&\geq \Big((\frac{1}{\alpha p}Q'(t) - 2\alpha p E(u_0))Q(t)\Big) + (\frac{1}{\alpha p}\|u\|_{\alpha p}^{\alpha p}Q(t) - \alpha p Q'(t)\|u_0\|_2^2) \\
&\geq \Big((\frac{1}{\alpha p}Q'(t) - 2\alpha p E(u_0))Q(t)\Big) + (\frac{1}{\alpha p}|\Omega|^{\frac{2-\alpha p}{2}}\|u(t)\|_2^{\alpha p}Q(t) - \alpha p Q'(t)\|u_0\|_2^2) \\
&> 0.
\end{aligned}$$

It implies for a sufficiently large time t, there exists a finite time $T > 0$ such that

$$\lim_{t\to T^-} Q(t) = +\infty.$$

Theorem 3.3 If $\alpha \in [1, p_s^*/p)$, $1 < p < N/s$ hold, then there exists a $v \in \mathcal{N}$ that satisfies the equation $E(v) = \lim_{u\in\mathcal{N}} E(u) = h$, and v is the ground state solution of the following steady-state problem

$$\begin{cases} M([u]_{s,p}^p)(-\Delta)_p^s u = |u|^{\alpha p-2}u\ln|u|, & x \in \Omega, \\ u = 0, & x \in R^N \setminus \Omega. \end{cases}$$

Proof. By the definition of h in (2.6), due to $u \in \mathcal{N}$, $I(u) = 0$, we have

$$h = \inf_{u\in\mathcal{N}} E(u) = \inf_{u\in\mathcal{N}}\{\frac{1}{\alpha p}I(u) + \frac{1}{(\alpha p)^2}\int_\Omega |u|^{\alpha p}dx\} = \inf_{u\in\mathcal{N}}\{\frac{1}{(\alpha p)^2}\|u\|_{\alpha p}^{\alpha p}\}.$$

Let $\{u_n\}$ be a minimizing sequence, where $\{u_n\} \subset \mathcal{N}$ and $\lim_{n\to+\infty} E(u_n) = h$, then

$$E(u_n) = \frac{1}{(\alpha p)^2}\|u_n\|_{\alpha p}^{\alpha p} = h.$$

Since $\alpha \in [1, p_s^*/p)$, $1 < p < N/s$, it implies that $\{u_n\}$ is bound in $W_0^{s,p}(\Omega)$.

According to the above discussion, there exists a subsequence of $\{u_n\}$. For the convenience, we still use $\{u_n\}$ to represent the subsequence, and further obtain

$$u_n \rightharpoonup v \quad \text{weakly in } W_0^{s,p}(\Omega),$$

$$u_n \rightharpoonup v \quad \text{strongly in } L^{\alpha p}(\Omega),$$

$$u_n \to v \quad \text{a.e. in } \Omega.$$

By Sobolev embedding theorem and logarithmic inequality, we have

$$\begin{aligned}\int_{\Omega^*} |u_n|^{\alpha p}\ln|u_n|dx &= -\int_{\Omega^*(|u_n|<1)} |u_n|^{\alpha p}\ln|u_n|dx + \int_{\Omega^*(|u_n|\geq 1)} |u_n|^{\alpha p}\ln|u_n|dx\\ &\leq \frac{1}{\alpha pe}|\Omega^*| + \frac{1}{\varrho e}\|u_n\|_{\alpha p+\varrho}^{\alpha p+\varrho}\\ &\leq \frac{1}{\alpha pe}|\Omega^*| + \frac{1}{\varrho e}D_*^{\alpha p+\varrho}[u_n]_{\alpha p+\varrho}^{\alpha p+\varrho},\end{aligned}$$

where $\Omega^* \subset \Omega$ is a measurable subset, thus we can get that $|u_n|^{\alpha p}\ln|u_n|$ is uniformly bound. By integral convergence theorem and $|u_n|^{\alpha p}\ln|u_n| \to |v|^{\alpha p}\ln|v|$ a.e. in Ω , we have

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{\alpha p}\ln|u_n|dx \to \int_\Omega |v|^{\alpha p}\ln|v|dx. \tag{3.20}$$

By Lemma 2.3(ii) and $\{u_n\} \subset \mathcal{N},\ I(u_n) = 0$, we can get

$$0 < r(\varrho) \leq [u_n]_{\alpha p} = (\int_\Omega |u_n|^{\alpha p}\ln|u_n|dx)^{\frac{1}{\alpha p}}.$$

In other words

$$0 < \int_\Omega |v|^{\alpha p}\ln|v|dx,$$

thus, we can conclude that $v \neq 0$.

According to (ii) of Lemma 2.1, there exists a $\rho^* > 0$ satifying $\rho^* v \in \mathcal{N},\ I(\rho^* v) = 0$, i.e.

$$I(\rho^* v) = (\rho^*)^{\alpha p}[v]_{s,p}^{\alpha p} - (\rho^*)^{\alpha p}\int_\Omega |v|^{\alpha p}\ln|v|dx - (\rho^*)^{\alpha p}\ln\rho^*\int_\Omega |v|^{\alpha p}dx = 0.$$

Let

$$b(\rho) := I(\rho v) = (\rho)^{\alpha p}[v]_{s,p}^{\alpha p} - (\rho)^{\alpha p}\int_\Omega |v|^{\alpha p}\ln|v|dx - (\rho)^{\alpha p}\ln\rho\int_\Omega |v|^{\alpha p}dx = 0,$$

by (iii) of Lemma 2.1 and taking $I(v) < 0$, $[u(x)]_{s,p}^{\alpha p} < \int_\Omega |u|^{\alpha p}\ln|u(x)|dx$, we can be sure that $0 < \rho^* < 1$.

Then by the definition of h , the weak lower semicontinuity and (3.20), we obtain

$$h = \lim_{n\to+\infty}\inf E(u_n) \geq E(v),$$

and

$$h = \lim_{n\to+\infty}\{\frac{1}{(\alpha p)^2}\|u_n\|_{\alpha p}^{\alpha p}\} \geq \lim_{n\to+\infty}\inf\|u_n\|_{\alpha p}^{\alpha p} \geq \frac{1}{(\alpha p)^2}\|v\|_{\alpha p}^{\alpha p}.$$

By simply calculating, we can get

$$E(\rho^* v) = (\frac{1}{\alpha p})^2 (\rho^*)^{\alpha p} \|v\|_{\alpha p}^{\alpha p} < (\frac{1}{\alpha p})^2 \|v\|_{\alpha p}^{\alpha p}.$$

This contradicts our definition of $E(\rho^* v) \geq h$, so we can conclude $I(v) = 0,\ h = E(v) = \inf_{u \in \mathcal{N}} E(u)$.

Next, we prove v is the ground state solution of the steady-state problem. From the above proof, we obtain $E(v) = h,\ I(v) = 0$. By the theory of Lagrange multipliers, there exists $\xi \in \mathbb{R}$ such that

$$E^{'}(v) - \xi I^{'}(v) = 0,$$

so we have

$$\xi \langle I^{'}(v), v \rangle = \langle E^{'}(v), v \rangle = I(v) > 0.$$

Since $I(v) = [v]_{\alpha p}^{\alpha p} - \int_\Omega |v|^{\alpha p} \ln |v| dx$, then there is

$$I^{'}(v) = \alpha p [v]_{\alpha p}^{\alpha p - 1} - \alpha p \int_\Omega |v|^{\alpha p - 1} \ln |v| dx - \int_\Omega |v|^{\alpha p - 1} dx.$$

Thus

$$\xi \langle I^{'}(v), v \rangle = \alpha p [v]^{\alpha p} - \alpha p \int_\Omega |v|^{\alpha p} \ln |v| dx - \int_\Omega |v|^{\alpha p} dx.$$

Note that $I(v) = 0,\ [v]_{s,p}^{\alpha p} = \int_\Omega |v|^{\alpha p} \ln |v| dx$, then

$$\xi \langle I^{'}(v), v \rangle = \alpha p [v]^{\alpha p} - \alpha p \int_\Omega |v|^{\alpha p} \ln |v| dx - \int_\Omega |v|^{\alpha p} dx = -\int_\Omega |v|^{\alpha p} dx < 0,$$

thus, $\xi = 0,\ E^{'}(v) = 0$. We get that v is the ground state solution of the steady-state problem.

4. Critical initial energy state ($E(u_0) = h$)

In this section, we will discuss the global existence and the finite time blow-up of weak solutions of problem (1.1) in the critical initial energy state ($E(u_0) = h$).

Theorem 4.1 Assume that $p \in (1, N/s),\ \alpha \in [1, p_s^*/p),\ u_0 \in W_0^{s,p}(\Omega)$ and $E(u_0) = h,\ I(u_0) \geq 0$ hold, then problem (1.1) admits a global weak solution $u(t) \in L^\infty(0, T; W_0^{s,p}(\Omega)),\ u_t \in L^2(0, T; L^2(\Omega))$ and $u(t) \in \overline{W}$ for all $t \in [0, +\infty)$.

Proof. We take a function ρ_m which satisfies $\rho_m > 0$ and $\lim_{n \to +\infty} \rho_m = 1$. Let $u(x, 0) = u_{0m}(x) = \rho_m u_0(x), x \in \Omega$, for the following equations

$$\begin{cases} u_t + M([u]_{s,p}^p)(-\Delta)_p^s u = |u|^{\alpha p - 2} u \ln |u|, & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_{0m}(x) = \rho_m u_0, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega \times (0, T). \end{cases} \tag{4.1}$$

Since $E(u_0) = h$, it implies $[u_0]_{s,p} \neq 0$. By Lemma (2.1) and $I(u_0) \geq 0, 0 < \rho_m < 1$, we obtain

$$[u_0]_{s,p}^{\alpha p} \geq \int_\Omega |u_0|^{\alpha p} \ln |u_0| dx,$$

and

$$(\rho)^{\alpha p} \ln \rho \int_\Omega |u_0|^{\alpha p} dx < 0.$$

By (iv) of lemma 2.1 ($\rho_m \frac{d}{d\rho_m} E(\rho_m u_0) = I(\rho_m u_0)$) and

$$\begin{aligned} I(\rho u_0) =& \rho^{\alpha p}[u_0]^{\alpha p}_{s,p} - \rho^{\alpha p} \int_\Omega |u_0|^{\alpha p} \ln |u_0| dx - \rho^{\alpha p} \ln \rho \int_\Omega |u_0|^{\alpha p} dx \\ >& \rho^{\alpha p}[u_0]^{\alpha p}_{s,p} - \rho^{\alpha p} \int_\Omega |u_0|^{\alpha p} \ln |u_0| dx \\ =& \rho^{\alpha p}([u_0]^{\alpha p}_{s,p} - \int_\Omega |u_0|^{\alpha p} \ln |u_0| dx) \\ \geq& 0, \end{aligned}$$

we have $I(\rho_m u_0) > 0$. (i.e. $\rho_m \frac{d}{d\rho_m} E(\rho_m u_0) = I(\rho_m u_0) > 0$). Then $E(\rho_m u(x))$ is monotonically increasing on $(0, \rho^*)$, we can get

$$E(u_{0m}) = E(\rho_m u_0) < E(u_0) = h.$$

From Theorem 3.1, it follows that for each m, the above problem (4.1) admits a global weak solution $u_m(t)$, where $u_m(t) \in L^\infty(0, +\infty; W^{s,p}_0(\Omega))$, $u_{mt} \in L^2(0, +\infty; L^2(\Omega))$, moreover $u_m(t) \in W$ for all $t \in [0, +\infty)$ and satisfying

$$\begin{aligned} \int_\Omega u_{mt} \varphi dx + [u_m]^{\alpha p - p}_{s,p} \iint_{R^{2N}} \frac{|u_m(x) - u_m(y)|^{p-2}(u_m(x) - u_m(x)(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy \\ = \int_\Omega \varphi |u_m|^{\alpha p - 2} u_m \ln |u_m| dx, \qquad (\forall\ \varphi \in W^{s,p}_0(\Omega)\ ,\ t > 0), \end{aligned}$$

in addition

$$\int_0^t \|u_{mr}\|_2^2 dr + E(u_m(t)) = E(u_{0m}) < h. \tag{4.2}$$

By (4.2) and the following equation

$$E(u_m(t)) = \frac{1}{\alpha p} I(u_m(t)) + \frac{1}{(\alpha p)^2} \int_\Omega |u_m(t)|^{\alpha p} dx,$$

we have

$$\int_0^t \|u_{mr}\|_2^2 dr + \frac{1}{(\alpha p)^2} \|u_m\|^{\alpha p}_{\alpha p} < h.$$

The remainder of the proof is similar to that in the proof of Theorem 3.1.

Theorem 4.2 Let $u(t)$ be the weak solution of problem (1.1), assume that $p \in (1, N/s)$, $\alpha \in [1, p^*_s/p)$, $u_0 \in W^{s,p}_0(\Omega)$ and $E(u_0) = h$, $I(u_0) < 0$ hold, then $u(t)$ blows up in a finite time T_*.

Proof. If $T_* = +\infty$, taking $Y(t) = \int_0^t \|u\|_2^2 dr$, we have

$$Y^{'}(t) = \|u\|_2^2,$$

and

$$Y^{''}(t) = 2(u, u_t) = -2I(u). \tag{4.3}$$

By (2.12), (4.3) and the following equation

$$E(u) = \frac{1}{\alpha p} I(u) + \frac{1}{(\alpha p)^2} \int_\Omega |u|^{\alpha p} dx,$$

we can get

$$Y^{''}(t) = -2I(u) \geq 2\alpha p \int_0^t \|u\|_2^2 dr + \frac{2}{\alpha p} \|u\|_{\alpha p}^{\alpha p} - 2\alpha ph. \tag{4.4}$$

Since

$$\int_0^t (u_r, u) dr = \frac{1}{2} \int_0^t \frac{d}{dt} \|u\|_2^2 dr = \frac{1}{2} (\|u\|_2^2 - \|u_0\|_2^2),$$

squaring both sides of the above equation, we have

$$\Big(\int_0^t (u_r, u) dr \Big)^2 = \frac{1}{4} \Big((Y^{'}(t))^2 - 2\|u_0\|_2^2 Y^{'}(t) + \|u_0\|_2^4 \Big). \tag{4.5}$$

By (4.4),(4.5) and the Schwartz's inequality, we can get

$$\begin{aligned}
&Y^{''}(t)Y(t) - 2\alpha p \Big(\int_0^t (u_r, u) dr \Big)^2 \\
&\geq 2\alpha p \int_0^t \|u_r\|_2^2 dr \int_0^t \|u\|_2^2 - 2\alpha p \Big(\int_0^t (u_r, u) dr \Big)^2 + \frac{2}{\alpha p} Y(t) \int_\Omega |u|^{\alpha p} dx - 2\theta ph Y(t),
\end{aligned}$$

i,e.

$$\begin{aligned}
&Y^{''}(t)Y(t) - \frac{\alpha p}{2} (Y^{'}(t))^2 \\
&\geq 2\alpha p \Big(\int_0^t \|u_r\|_2^2 dr \int_0^t \|u\|_2^2 dr - (\int_0^t (u_r, u) dr)^2 \Big) + \frac{2}{\alpha p} Y(t) \int_\Omega |u|^{\alpha p} dx \\
&- 2\alpha ph Y(t) - \alpha p Y^{'}(t) \|u_0\|_2^2 + \frac{\alpha p}{2} \|u_0\|_2^4 \\
&\geq \frac{2}{\alpha p} Y(t) \int_\Omega |u|^{\alpha p} dx - 2\alpha ph Y(t) - \alpha p Y^{'}(t) \|u_0\|_2^2 \\
&= (\frac{1}{\alpha p} Y(t) \int_\Omega |u|^{\alpha p} dx - 2\alpha ph Y(t)) + (\frac{1}{\alpha p} Y(t) \int_\Omega |u|^{\alpha p} dx - \alpha p Y^{'}(t) \|u_0\|_2^2)
\end{aligned}$$

Due to $I(u) < 0$ for all $t \in [0, T)$, so $Y''(t) = -2I(u) > 0$, $Y^{'}(t) > Y^{'}(0) = \|u_0\|_2^2 > 0$. For all $\delta \in [0, t - t_0)$, where $t_0 \geq 0, t \geq t_0$ and $Y^{'}(t) > 0$, we obtain

$$Y(t) \geq \delta Y^{'}(t_0) > \delta Y^{'}(0). \tag{4.6}$$

Taking $|u| > 1$ and noting that $\alpha p > 2$, we have

$$\frac{1}{\alpha p} \int_\Omega |u|^{\alpha p} dx - 2\alpha ph > \frac{1}{\alpha p} \int_\Omega |u|^2 dx - 2\alpha ph > \frac{1}{\alpha p} \int_\Omega |u_0|^2 dx - 2\alpha ph. \tag{4.7}$$

Moreover, by Holder's inequality, we have

$$\|u(t)\|_{\alpha p}^{\alpha p} \geq |\Omega|^{\frac{2-\alpha p}{2}} \|u(t)\|_2^{\alpha p}. \tag{4.8}$$

By (4.6), (4.7), (4.8), for sufficiently large t, it implies that

$$\frac{1}{\alpha p}Y(t)\int_{\Omega}|u|^{\alpha p}dx-2\alpha phY(t)>0,$$

and

$$\frac{1}{\alpha p}Y(t)\int_{\Omega}|u|^{\alpha p}dx-\alpha pY^{'}(t)\|u_0\|_2^2>0.$$

So

$$Y^{''}(t)Y(t)-\frac{\alpha p}{2}(Y^{'}(t))^2>0,$$

which gives

$$(Y^{-b}(t))^{''}=\frac{-b}{Y^{b+2}(t)}(Y(t)Y^{''}(t)-(b+1)(Y^{'}(t)^2))\leq 0,\quad b=\frac{\alpha p-2}{2}.$$

It implies for a sufficiently large time t, there exists a finite time T_* such that

$$\lim_{t\to T_*}Y^{-b}(t)=0,$$

and

$$\lim_{t\to T_*}Y(t)=+\infty.$$

This is contradictory to our assumption that $T_*=+\infty$.

Remark 1. Theorems 3.1 and 4.1 show that $E(u_0)\leq h$ and $I(u_0)>0$ holds, then problem (1.1) admits a global weak solution $u(t)\in L^{\infty}(0,T;W_0^{s,p}(\Omega)),\ u_t\in L^2(0,T;L^2(\Omega))$ and $u(t)\in W$ for all $t\in[0,+\infty)$.

Remark 2.One can see from Theorems 3.2 and 4.2 that blow-up of the weak solutions can always occur if $E(u_0)\leq h,\ I(u_0)<0$ holds.

5. Conclusions

In this work, we study the initial-boundary value problem for a class of fractional p-Laplace Kirchhoff diffusion equation with logarithmic nonlinearity. For both subcritical and critical states, by means of the Galerkin approximations , the potential well theory and the Nehari manifold, we prove the global existence and finite time blow-up of the weak solutions. Further, we give the growth rate of the weak solutions and study ground-state solution of the corresponding steady-state problem. Compared with problem (1.2), we consider the case of $1<p<N/s$ and discuss the global existence and finite time blow-up of the solutions. Compared with problem (1.3) and (1.4), we study the well-posedness of the solution for the fractional p-Laplacian Kirchhoff type evolution equation with logarithmic nonlinearity.

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Conflict of interest

The authors declare no conflict of interest.

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