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*Research article*

# Certain subclasses of spiral-like functions associated with $q$-analogue of Carlson-Shaffer operator

**Tamer M. Seoudy**[1,2,*] **and Amnah E. Shammaky**[3]

[1] Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt
[2] Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah, Saudi Arabia
[3] Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia

* **Correspondence:** Email: tms00@fayoum.edu.eg, tmsaman@uqu.edu.sa.

**Abstract:** In this paper, making use of the $q$-analogue of Carlson-Shaffer operator $L_q(a,c)$ we introduce a new subclass of spiral-like functions and discuss some subordination results and Fekete-Szego problem for this generalized function class. Further, some known and new results which follow as special cases of our results are also mentioned.

**Keywords:** convex functions; Fekete-Szego problem; Hadamard product; $q$-analogue of Carlson-Shaffer operator; spiral-like functions; starlike functions; subordinating factor sequence; univalent functions
**Mathematics Subject Classification:** 30C45, 30C50

## 1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{U}$. Also, let $\Omega$ be the class of all analytic functions $w$ in $\mathbb{U}$ that satisfy the conditions $w(0) = 0$ and $|w(z)| < 1\,(z \in \mathbb{U})$. If $f$ and $g$ are analytic in $\mathbb{U}$, we say that $f$ is subordinate to $g$, written as $f \prec g$ in $\mathbb{U}$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists $w \in \Omega$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence holds (see [4] and [11]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

A function $f \in \mathcal{A}$ is said to be in the class of $\gamma-$spiral-like functions of order $\lambda$ in $\mathbb{U}$, denoted by $\mathcal{S}^*(\gamma;\lambda)$ if

$$\Re\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} > \lambda\ \cos\gamma \quad \left(0 \leq \lambda < 1, |\gamma| < \frac{\pi}{2}; z \in \mathbb{U}\right). \tag{1.2}$$

The class $\mathcal{S}^*(\gamma;\lambda)$ was studied by Libera [10] and Keogh and Merkes [9]. Note that

1). $\mathcal{S}^*(\gamma;0) = \mathcal{S}^*(\gamma)$ is the class of spiral-like functions introduced by Špaček [17];
2). $\mathcal{S}^*(0;\lambda) = \mathcal{S}^*(\lambda)$ is the class of starlike functions of order $\lambda$;
3). $\mathcal{S}^*(0;0) = \mathcal{S}^*$ is the familiar class of starlike functions.

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.3}$$

we define the Hadamard product (or Convolution ) of $f$ and $g$ by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.4}$$

Also, for $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the Jackson's $q$-derivative operator or $q$-difference operator for a function $f \in \mathcal{A}$ is defined by (see [1–3, 6, 7, 15, 16])

$$D_q f(z) := \begin{cases} f'(0) & \text{if } z = 0, \\ \dfrac{f(z) - f(qz)}{(1-q)z} & \text{if } z \neq 0. \end{cases} \tag{1.5}$$

From (1.5), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q\, a_k z^{k-1}\ (z \neq 0), \tag{1.6}$$

where the $q$-integer number $[i]_q$ is defined by

$$[i]_q = \frac{1-q^i}{1-q} = 1 + q + q^2 + ... + q^{i-1}, \tag{1.7}$$

and

$$\lim_{q\to 1^-} D_q f(z) = \lim_{q\to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z), \tag{1.8}$$

for a function $f$ which is differentiable in a given subset of $\mathbb{C}$.

Next, in terms of the $q$-generalized Pochhammer symbol $\left([v]_q\right)_n$ given by

$$\left([v]_q\right)_n = [v]_q\ [v+1]_q\ [v+2]_q\ ...\ [v+n-1]_q\,, \tag{1.9}$$

we define the function $\phi_q(a, c; z)$ by

$$\phi_q(a, c; z) = z + \sum_{k=2}^{\infty} \frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}} z^k \quad \left(a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0, -1, -2, ...\}; z \in \mathbb{U}\right). \tag{1.10}$$

Corresponding to the function $\phi_q(a,c;z)$, we consider a linear operator $L_q(a,c) : \mathcal{A} \to \mathcal{A}$ which is defined by means of the following Hadamard product (or convolution):

$$L_q(a,c) f(z) = \phi_q(a,c;z) * f(z) = z + \sum_{k=2}^{\infty} \frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}} a_k z^k. \tag{1.11}$$

It is easily verified from (1.11) that

$$q^{a-1} z D_q \left( L_q(a,c) f(z) \right) = [a]_q \; L_q(a+1,c) f(z) - [a-1]_q L_q(a,c) f(z). \tag{1.12}$$

Moreover, for $f \in \mathcal{A}$, we observe that

1). $\lim_{q \to 1^-} L_q(a,c) f(z) = L(a,c) f(z)$, where $L(a,c)$ denotes the Carlson-Shaffer operator [5];
2). $L_q(\delta+1,1) f(z) = \mathcal{R}_q^{\delta} f(z) \, (\delta > 0)$, where $\mathcal{R}_q^{\delta}$ denotes the Ruscheweyh $q$-derivative of a function $f \in \mathcal{A}$ of order $\delta$ (see [8]);
3). $\lim_{q \to 1^-} L_q(\delta+1,1) f(z) = \mathcal{R}^{\delta} f(z) \, (\delta > 0)$, where $\mathcal{R}_q^{\delta}$ denotes the Ruscheweyh derivative of order $\delta$ (see [14]);
4). $L_q(a,a) f(z) = f(z)$ and $L_q(2,1) f(z) = z D_q f(z)$.

Making use of the $q$-analogue of Carlson-Shaffer operator $L_q(a,c)$, we introduce a new subclass of spiral-like functions.

**Definition 1.** For $0 \le t \le 1$, $0 \le \lambda < 1$ and $|\gamma| < \frac{\pi}{2}$, we let $\mathcal{S}_q^a(\gamma, \lambda, t)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic condition:

$$\Re \left\{ e^{i\gamma} \frac{z D_q \left( L_q(a,c) f(z) \right)}{(1-t) \; z + t \, L_q(a,c) f(z)} \right\} > \lambda \; \cos\gamma \tag{1.13}$$

$$\left( 0 \le t \le 1; 0 \le \lambda < 1; |\gamma| < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

We note that

1). For $t = 1$, $0 \le \lambda < 1$ and $|\gamma| < \frac{\pi}{2}$, we let $\mathcal{S}_q^a(\gamma, \lambda, 1) = \mathcal{S}_q^a(\gamma, \lambda)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic condition:

$$\Re \left\{ e^{i\gamma} \frac{z D_q \left( L_q(a,c) f(z) \right)}{L_q(a,c) f(z)} \right\} > \lambda \; \cos\gamma \tag{1.14}$$

$$\left( 0 \le \lambda < 1; |\gamma| < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

2). For $t = 0$, $0 \le \lambda < 1$ and $|\gamma| < \frac{\pi}{2}$, we let $\mathcal{S}_q^a(\gamma, \lambda, 0) = \mathcal{K}_q^a(\gamma, \lambda)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic condition:

$$\Re \left\{ e^{i\gamma} \; D_q \left( L_q(a,c) f(z) \right) \right\} > \lambda \; \cos\gamma \tag{1.15}$$

$$\left( 0 \le \lambda < 1; |\gamma| < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

The object of the present paper is to investigate the coefficient estimates and subordination properties for the class of functions $\mathcal{S}_q^a(\gamma, \lambda, t)$. Some interesting consequences of the results are also pointed out.

## 2. Membership characterizations

In this section, we obtain several sufficient conditions for a function $f \in \mathcal{A}$ to be in the class $\mathcal{S}_q^a(\gamma, \lambda, t)$.

**Theorem 1.** Let $f \in \mathcal{A}$ and let $\sigma$ be a real number with $0 \leq \sigma < 1$. If

$$\left| \frac{zD_q\left(L_q(a,c) f(z)\right)}{(1-t)\ z + t\, L_q(a,c) f(z)} - 1 \right| \leq 1 - \sigma \quad (z \in \mathbb{U}), \tag{2.1}$$

then $f \in \mathcal{S}_q^a(\gamma, \lambda, t)$ provided that

$$|\gamma| \leq \cos^{-1}\left(\frac{1-\sigma}{1-\lambda}\right) \tag{2.2}$$

*Proof.* From (2.1) it follows that

$$\frac{zD_q\left(L_q(a,c) f(z)\right)}{(1-t)\ z + t\, L_q(a,c) f(z)} = 1 + (1-\sigma)\, w(z),$$

where $w(z) \in \Omega$. We have

$$\begin{aligned} \Re\left\{ e^{i\gamma} \frac{zD_q\left(L_q(a,c) f(z)\right)}{(1-t)\ z + t\, L_q(a,c) f(z)} \right\} &= \Re\left\{ e^{i\gamma}\left[1 + (1-\sigma)\, w(z)\right] \right\} \\ &= \cos\gamma + (1-\sigma)\, \Re\left\{ e^{i\gamma} w(z) \right\} \\ &\geq \cos\gamma - (1-\sigma)\left| e^{i\gamma} w(z) \right| \\ &> \cos\gamma - (1-\sigma) \\ &\geq \lambda\ \cos\gamma \end{aligned}$$

provided that $|\gamma| \leq \cos^{-1}\left(\frac{1-\sigma}{1-\lambda}\right)$. Thus, the proof is completed. □

Putting $\sigma = 1 - (1-\lambda)\cos\gamma$ in Theorem 1, we obtain the following result.

**Corollary 1.** Let $f \in \mathcal{A}$. If

$$\left| \frac{zD_q\left(L_q(a,c) f(z)\right)}{(1-t)\ z + t\, L_q(a,c) f(z)} - 1 \right| \leq (1-\lambda)\cos\gamma \quad (z \in \mathbb{U}), \tag{2.3}$$

then $f \in \mathcal{S}_q^a(\gamma, \lambda, t)$.

In the following theorem, we obtain a sufficient condition for $f$ to be in $\mathcal{S}_q^a(\gamma, \lambda, t)$.

**Theorem 2.** A function $f(z)$ of the form (1.1) is in $\mathcal{S}_q^a(\gamma, \lambda, t)$ if

$$\sum_{k=2}^{\infty} \left\{ \left([k]_q - t\right)\sec\gamma + (1-\lambda)\, t \right\} \frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}} |a_k| \leq 1 - \lambda. \tag{2.4}$$

*Proof.* In virtue of Corollary 1, it suffices to show that the condition (2.3) is satisfied. We have

$$\left|\frac{zD_q\left(L_q(a,c)f(z)\right)}{(1-t)\;z+t\,L_q(a,c)f(z)}-1\right| = \left|\frac{\sum\limits_{k=2}^{\infty}\left\{[k]_q-t\right\}\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}\,a_k\,z^{k-1}}{1+t\sum\limits_{k=2}^{\infty}\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}a_k\,z^{k-1}}\right| < \frac{\sum\limits_{k=2}^{\infty}\left\{[k]_q-t\right\}\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}\,|a_k|}{1-\sum\limits_{k=2}^{\infty}t\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}\,|a_k|}.$$

The last expression is bounded above by $(1-\lambda)\cos\gamma$, if

$$\sum_{k=2}^{\infty}\left\{[k]_q-t\right\}\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}\,|a_k|\leq(1-\lambda)\cos\gamma\left\{1-\sum_{k=2}^{\infty}t\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}\,|a_k|\right\}$$

which is equivalent to

$$\sum_{k=2}^{\infty}\left\{\left([k]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}\,|a_k|\leq 1-\lambda.$$

This completes the proof of the Theorem 2. □

Putting $t=1$ in Theorem 2, we obtain the following corollary.

**Corollary 2.** A function $f(z)$ of the form (2.1) is in $\mathcal{S}_q^a(\gamma,\lambda)$ if

$$\sum_{k=2}^{\infty}\left\{\left([k]_q-1\right)\sec\gamma+1-\lambda\right\}\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}\,|a_k|\leq 1-\lambda. \tag{2.5}$$

Putting $t=0$ in Theorem 2, we obtain the following corollary.

**Corollary 3.** A function $f(z)$ of the form (2.1) is in $\mathcal{K}_q^a(\gamma,\lambda)$ if

$$\sum_{k=2}^{\infty}[k]_q\sec\gamma\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}\,|a_k|\leq 1-\lambda. \tag{2.6}$$

## 3. Subordination result

Before stating and proving our subordination result for the class $\mathcal{S}_q^a(\gamma,\lambda,t)$, we need the following definitions and a lemma due to Wilf [19].

**Definition 2** [19]**.** A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)=z+\sum\limits_{k=2}^{\infty}a_kz^k$ is regular, univalent and convex in $\mathbb{U}$, we have

$$\sum_{k=1}^{\infty}a_k\,b_k\prec f(z)\quad\left(a_1=1;z\in\mathbb{U}\right). \tag{3.1}$$

**Lemma 1** [19]**.** The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\Re\left\{1+2\sum_{k=1}^{\infty} b_k\, z^k\right\} > 0 \quad (z\in\mathbb{U})\,. \tag{3.2}$$

**Theorem 3.** Let $f \in \mathcal{S}_q^a(\gamma,\lambda,t)$ satisfy the coefficient inequality (2.4) with $a \ge c > 0$ and let $g(z)$ be any function in the usual class of convex functions $C$, then

$$\frac{\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{2\left[1-\lambda+\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}\right]}\,(f*g)(z) \prec g(z) \tag{3.3}$$

and

$$\Re\{f(z)\} > -\frac{1-\lambda+\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}. \tag{3.4}$$

The constant factor $\frac{\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)t\right\}\frac{[a]_q}{[c]_q}}{2\left[1-\lambda+\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)t\right\}\frac{[a]_q}{[c]_q}\right]}$ in (3.3) cannot be replaced by a larger number.

*Proof.* Let $f \in \mathcal{S}_q^a(\gamma,\lambda,t)$ satisfy the coefficient inequality (2.4) and suppose that

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in C.$$

Then, by Definition 2, the subordination (3.3) of our theorem will hold true if the sequence

$$\left\{\frac{\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{2\left[1-\lambda+\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}\right]}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $b_1 = 1$. In view of Lemma 1, it is equivalent to the inequality

$$\Re\left\{1+\sum_{k=1}^{\infty}\frac{\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{1-\lambda+\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}a_k\, z^k\right\} > 0 \quad (z\in\mathbb{U})\,. \tag{3.5}$$

By noting the fact that $\frac{\left\{\left([k]_q - t\right)\sec\gamma + (1-\lambda)t\right\}\left([a]_q\right)_{k-1}}{(1-\lambda)\left([c]_q\right)_{k-1}}$ is an increasing function for $k \ge 2$ and $a \ge c > 0$. In view of (2.4), when $|z| = r < 1$, we obtain

$$\Re\left\{1+\frac{\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{1-\lambda+\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}\sum_{k=1}^{\infty} a_k\, z^k\right\}$$

$$= \quad \Re\left\{1+\frac{\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{1-\lambda+\left\{\left([2]_q - t\right)\sec\gamma + (1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}z\right.$$

$$
+\frac{\sum_{k=2}^{\infty}\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}a_k\,z^k}{1-\lambda+\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}\Bigg\}
$$

$$
\begin{aligned}
\geq\ & 1-\frac{\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{1-\lambda+\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}r \\
& -\frac{\sum_{k=2}^{\infty}\left\{\left([k]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{\left([a]_q\right)_{k-1}}{\left([c]_q\right)_{k-1}}\left|a_k\right|\ r^k}{1-\lambda+\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}} \\
\geq\ & 1-\frac{\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{1-\lambda+\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}r \\
& -\frac{1-\lambda}{1-\lambda+\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}r \\
=\ & 1-r>0\quad(|z|=r<1)\,.
\end{aligned}
$$

This evidently proves the inequality (3.5) and hence also the subordination result (3.3) asserted by Theorem 3. The inequality (3.4) follows from (3.3) by taking

$$
g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k\in C.
$$

The sharpness of the multiplying factor in (3.3) can be established by considering a function

$$
F(z)=z-\frac{1-\lambda}{\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}z^2.
$$

Clearly $F\in\mathcal{S}_q^a(\gamma,\lambda,t)$ satisfy (2.4). Using (3.3) we infer that

$$
\frac{\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{2\left[1-\lambda+\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}\right]}F(z)\prec\frac{z}{1-z},
$$

and it follows that

$$
\min_{|z|\leq r}\left\{\frac{\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}}{2\left[1-\lambda+\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)\,t\right\}\frac{[a]_q}{[c]_q}\right]}\Re\{F(z)\}\right\}=-\frac{1}{2}.
$$

This shows that the constant $\frac{\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)t\right\}\frac{[a]_q}{[c]_q}}{2\left[1-\lambda+\left\{\left([2]_q-t\right)\sec\gamma+(1-\lambda)t\right\}\frac{[a]_q}{[c]_q}\right]}$ cannot be replaced by any larger one. □

For $t = 1$ in Theorem 3, we state the following corollary.
**Corollary 4.** Let $f \in \mathcal{S}_q^a(\gamma, \lambda)$ satisfy the coefficient inequality (2.5) with $a \geq c > 0$ and $g \in C$, then

$$\frac{\{q\sec\gamma + 1 - \lambda\}\frac{[a]_q}{[c]_q}}{2\left[1 - \lambda + \{q\sec\gamma + 1 - \lambda\}\frac{[a]_q}{[c]_q}\right]}(f * g)(z) \prec g(z) \tag{3.6}$$

and

$$\Re\{f(z)\} > -\frac{1 - \lambda + \{q\sec\gamma + 1 - \lambda\}\frac{[a]_q}{[c]_q}}{\{q\sec\gamma + 1 - \lambda\}\frac{[a]_q}{[c]_q}}. \tag{3.7}$$

The constant factor $\frac{\{q\sec\gamma+1-\lambda\}\frac{[a]_q}{[c]_q}}{2\left[1-\lambda+\{q\sec\gamma+(1-\lambda)\}\frac{[a]_q}{[c]_q}\right]}$ in (3.6) cannot be replaced by a larger number.

Taking $t = 0$ in Theorem 3, we state the next corollary.
**Corollary 5.** Let $f \in \mathcal{K}_q^a(\gamma, \lambda)$ satisfy the coefficient inequality (2.6) with $a \geq c > 0$ and $g \in C$, then

$$\frac{(1+q)\sec\gamma\frac{[a]_q}{[c]_q}}{2\left[1 - \lambda + (1+q)\sec\gamma\frac{[a]_q}{[c]_q}\right]}(f * g)(z) \prec g(z) \tag{3.8}$$

and

$$\Re\{f(z)\} > -\frac{1 - \lambda + (1+q)\sec\gamma\frac{[a]_q}{[c]_q}}{(1+q)\sec\gamma\frac{[a]_q}{[c]_q}}. \tag{3.9}$$

The constant factor $\frac{(1+q)\sec\gamma\frac{[a]_q}{[c]_q}}{2\left[1-\lambda+(1+q)\sec\gamma\frac{[a]_q}{[c]_q}\right]}$ in (3.8) cannot be replaced by a larger number.

## 4. Fekete-Szegö problem

The Fekete-Szegö problem consists in finding sharp upper bounds for the functional $\left|a_3 - \mu a_2^2\right|$ for various subclasses of $\mathcal{A}$ (see [13] and [18]). In order to obtain sharp upper-bounds for $\left|a_3 - \mu a_2^2\right|$ for the class $\mathcal{S}_q^a(\gamma, \lambda, t)$ the following lemma is required (see, e.g., [12, p.108]).
**Lemma 2.** Let the function $w \in \Omega$ be given by

$$w(z) = \sum_{k=1}^{\infty} w_k z^k \quad (z \in \mathbb{U}).$$

Then

$$|w_1| \leq 1, \quad |w_2| \leq 1 - |w_1|^2, \tag{4.1}$$

and

$$\left|w_2 - s\,w_1^2\right| \leq \max\{1, |s|\}, \tag{4.2}$$

for any complex number $s$. The functions $w(z) = z$ and $w(z) = z^2$or one of their rotations show that both inequalities (4.1) and (4.2) are sharp.

For the constants $\lambda$, $\gamma$ with $0 \le \lambda < 1$ and $|\gamma| < \frac{\pi}{2}$ denote

$$\mathcal{P}_{\lambda,\gamma}(z) = \frac{1 + e^{-i\gamma}\left(e^{-i\gamma} - 2\lambda\cos\gamma\right)z}{1-z} \quad (z \in \mathbb{U}). \tag{4.3}$$

The function $\mathcal{P}_{\lambda,\gamma}(z)$ maps the open unit disk $\mathbb{U}$ onto the half-plane $\mathcal{H}_{\lambda,\gamma} = \left\{w \in \mathbb{C} : \Re\left\{e^{i\gamma}w\right\} > \lambda\cos\gamma\right\}$. If

$$\mathcal{P}_{\lambda,\gamma}(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (z \in \mathbb{U}),$$

then it is easy to check that

$$p_k = 2e^{-i\gamma}(1-\lambda)\cos\gamma \quad (k \ge 1). \tag{4.4}$$

First we obtain sharp upper-bounds for the Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ with $\mu$ real parameter.
**Theorem 4.** Let $f \in \mathcal{S}_q^a(\gamma,\lambda,t)$ be given by (1.1) and let $\mu$ be a real number. Then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2-t\right)\left([a]_q\right)_2}\left[1 + \frac{2(1-\lambda)t}{1+q-t} - \mu\frac{2(1-\lambda)\left(1+q+q^2-t\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q-t)^2\left([a]_q\right)^2\left([c]_q\right)_2}\right] & (\mu \le \sigma_1) \\ \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2-t\right)\left([a]_q\right)_2} & (\sigma_1 \le \mu \le \sigma_2) \\ \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2-t\right)\left([a]_q\right)_2}\left[-1 - \frac{2(1-\lambda)t}{1+q-t} + \mu\frac{2(1-\lambda)\left(1+q+q^2-t\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q-t)^2\left([a]_q\right)^2\left([c]_q\right)_2}\right] & (\mu \ge \sigma_2) \end{cases} \tag{4.5}$$

where

$$\sigma_1 = \frac{t(1+q-t)\left([a]_q\right)^2\left([c]_q\right)_2}{\left(1+q+q^2-t\right)\left([c]_q\right)^2\left([a]_q\right)_2} \tag{4.6}$$

$$\sigma_2 = \frac{(1+q-t)(1+q-t\lambda)\left([a]_q\right)^2\left([c]_q\right)_2}{(1-\lambda)\left(1+q+q^2-t\right)\left([a]_q\right)_2\left([c]_q\right)^2} \tag{4.7}$$

and all estimates are sharp.

*Proof.* Suppose that $f \in \mathcal{S}_q^a(\gamma,\lambda,t)$ is given by (1.1). Then, from the definition of the class $\mathcal{S}_q^a(\gamma,\lambda,t)$, there exists $w \in \Omega$,

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + ...$$

such that

$$\frac{zD_q\left(L_q(a,c)f(z)\right)}{(1-t)\ z + t\, L_q(a,c)f(z)} = \mathcal{P}_{\lambda,\gamma}(w(z)) \quad (z \in \mathbb{U}). \tag{4.8}$$

We have

$$\frac{zD_q\left(L_q(a,c)f(z)\right)}{(1-t)\ z + t\, L_q(a,c)f(z)} = 1 + (1+q-t)\frac{[a]_q}{[c]_q}a_2 z$$

$$+ \left\{\left(1+q+q^2-t\right)\frac{\left([a]_q\right)_2}{\left([c]_q\right)_2}a_3 - t(1+q-t)\frac{\left([a]_q\right)^2}{\left([c]_q\right)^2}a_2^2\right\}z^2 + ... \tag{4.9}$$

Set

$$\mathcal{P}_{\lambda,\gamma}(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots .$$

From (4.4) we have

$$p_1 = p_2 = 2e^{-i\gamma}(1-\lambda)\cos\gamma.$$

Equating the coefficients of $z$ and $z^2$ on both sides of (4.8) and using (4.9), we obtain

$$a_2 = \frac{p_1\ [c]_q}{(1+q-t)\ [a]_q} w_1$$

and

$$a_3 = \frac{\left([c]_q\right)_2}{(1+q+q^2-t)\left([a]_q\right)_2}\left[p_1 w_2 + \left(p_2 + \frac{t}{(1+q-t)} p_1^2\right) w_1^2\right]$$

and thus we obtain

$$a_2 = \frac{2e^{-i\gamma}(1-\lambda)\cos\gamma\ [c]_q}{(1+q-t)\ [a]_q} w_1 \tag{4.10}$$

and

$$a_3 = \frac{2e^{-i\gamma}(1-\lambda)\cos\gamma\left([c]_q\right)_2}{(1+q+q^2-t)\left([a]_q\right)_2}\left[w_2 + \left(1 + \frac{2te^{-i\gamma}(1-\lambda)\cos\gamma}{1+q-t}\right) w_1^2\right]. \tag{4.11}$$

It follows

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{(1+q+q^2-t)\left([a]_q\right)_2}\left[|w_2| + \left|1 + \frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1+q-t}\left(t\ - \mu\frac{(1+q+q^2-t)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q-t)\ \left([a]_q\right)^2\left([c]_q\right)_2}\right)\right| |w_1|^2\right]$$

Making use of Lemma 2 we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{(1+q+q^2-t)\left([a]_q\right)_2}\left[1 + \left(\left|1 + \frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1+q-t}\left(t\ - \mu\frac{(1+q+q^2-t)\left([a]_q\right)_2\ \left([c]_q\right)^2}{\left([c]_q\right)_2(1+q-t)\ \left([a]_q\right)^2}\right)\right| - 1\right)|w_1|^2\right]$$

or

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{(1+q+q^2-t)\left([a]_q\right)_2}\left[1 + \left(\sqrt{1 + M(M+2)\cos^2\gamma} - 1\right)|w_1|^2\right], \tag{4.12}$$

where

$$M = \frac{2(1-\lambda)}{1+q-t}\left(t\ - \mu\frac{\left(1+q+q^2-t\right)\left([a]_q\right)_2\ \left([c]_q\right)^2}{\left([c]_q\right)_2(1+q-t)\ \left([a]_q\right)^2}\right). \tag{4.13}$$

Denote by

$$F(x,y) = \left[1 + \left(\sqrt{1 + M(M+2)x^2} - 1\right)y^2\right]$$

where $x = \cos\gamma$, $y = |w_1|$ and $(x,y) : [0,1] \times [0,1]$.

Simple calculation shows that the function $F(x,y)$ does not have a local maximum at any interior point of the open rectangle $(0,1) \times (0,1)$. Thus, the maximum must be attained at a boundary point. Since $F(x,0) = 1$, $F(0,y) = 1$ and $F(1,1) = |1+M|$, it follows that the maximal value of $F(x,y)$ may be $F(0,0) = 1$ or $F(1,1) = |1+M|$. Therefore, from (4.12) we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{(1+q+q^2-t)\left([a]_q\right)_2}\max\{1, |1+M|\}, \tag{4.14}$$

where M is given by (4.13). Consider first the case $|1 + M| \geq 1$. If $\mu \leq \sigma_1$, where $\sigma_1$ is given by (4.6), then $M \geq 0$ and from (4.14) we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2-t\right)\left([a]_q\right)_2}\left[1 + \frac{2(1-\lambda)t}{1+q-t} - \mu\frac{2(1-\lambda)\left(1+q+q^2-t\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q-t)^2\left([a]_q\right)^2\left([c]_q\right)_2}\right]$$

which is the first part of the inequality (4.5). If $\mu \geq \sigma_2$, where $\sigma_2$ is given by (4.7), then $M \leq -2$ and it follows from (4.14) that

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2-t\right)\left([a]_q\right)_2}\left[-1 - \frac{2(1-\lambda)t}{1+q-t} + \mu\frac{2(1-\lambda)\left(1+q+q^2-t\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q-t)^2\left([a]_q\right)^2\left([c]_q\right)_2}\right]$$

and this is the third part of (4.5).

Next, suppose $\sigma_1 \leq \mu \leq \sigma_2$. Then, $|1 + M| \leq 1$ and thus, from (4.14) we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2-t\right)\left([a]_q\right)_2}$$

which is the second part of the inequality (4.5). In view of Lemma 2, the results are sharp for $w(z) = z$ and $w(z) = z^2$ or one of their rotations. □

For $t = 1$ in Theorem 4, we state the following corollary.

**Corollary 6.** Let $f \in \mathcal{S}_q^a(\gamma, \lambda)$ be given by (1.1) and let $\mu$ be a real number. Then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{q(1+q)\left([a]_q\right)_2}\left[1 + \frac{2(1-\lambda)}{q} - \mu\frac{2(1-\lambda)(1+q)\left([a]_q\right)_2\left([c]_q\right)^2}{q\left([a]_q\right)^2\left([c]_q\right)_2}\right] & (\mu \leq \sigma_3) \\ \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{q(1+q)\left([a]_q\right)_2} & (\sigma_3 \leq \mu \leq \sigma_4) \\ \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{q(1+q)\left([a]_q\right)_2}\left[-1 - \frac{2(1-\lambda)}{q} + \mu\frac{2(1-\lambda)(1+q)\left([a]_q\right)_2\left([c]_q\right)^2}{q\left([a]_q\right)^2\left([c]_q\right)_2}\right] & (\mu \geq \sigma_4) \end{cases}$$

where

$$\sigma_3 = \frac{\left([a]_q\right)^2\left([c]_q\right)_2}{(1+q)\left([c]_q\right)^2\left([a]_q\right)_2}, \quad \sigma_4 = \frac{(1+q-\lambda)\left([a]_q\right)^2\left([c]_q\right)_2}{(1-\lambda)(1+q)\left([a]_q\right)_2\left([c]_q\right)^2}$$

and all estimates are sharp.

Taking $t = 0$ in Theorem 4, we state the next corollary.

**Corollary 7.** Let $f \in \mathcal{K}_q^a(\gamma, \lambda)$ be given by (1.1) and let $\mu$ be a real number. Then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2\right)\left([a]_q\right)_2}\left[1 - \mu\frac{2(1-\lambda)\left(1+q+q^2\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q)^2\left([a]_q\right)^2\left([c]_q\right)_2}\right] & (\mu \leq 0) \\ \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2\right)\left([a]_q\right)_2} & (0 \leq \mu \leq \sigma_5) \\ \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2\right)\left([a]_q\right)_2}\left[-1 + \mu\frac{2(1-\lambda)\left(1+q+q^2\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q)^2\left([a]_q\right)^2\left([c]_q\right)_2}\right] & (\mu \geq \sigma_5) \end{cases}$$

where

$$\sigma_5 = \frac{(1+q)^2\left([a]_q\right)^2\left([c]_q\right)_2}{(1-\lambda)\left(1+q+q^2\right)\left([a]_q\right)_2\left([c]_q\right)^2}$$

and all estimates are sharp.

We consider the Fekete-Szegö problem for the class $\mathcal{S}_q^a(\gamma,\lambda,t)$ with $\mu$ complex parameter.

**Theorem 5.** Let $f \in \mathcal{S}_q^a(\gamma,\lambda,t)$ be given by (1.1) and let $\mu$ be a complex number. Then,

$$\left|a_3-\mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2-t\right)\left([a]_q\right)_2} \max\left\{1, \left|\frac{2(1-\lambda)\cos\gamma}{1+q-t}\left(\mu\frac{\left(1+q+q^2-t\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q-t)\left([a]_q\right)^2\left([c]_q\right)_2}-t\right)-e^{i\gamma}\right|\right\}. \tag{4.15}$$

The result is sharp.

*Proof.* Assume that $f \in \mathcal{S}_q^a(\gamma,\lambda,t)$. Making use of (4.10) and (4.11) we obtain

$$\left|a_3-\mu a_2^2\right| = \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2-t\right)\left([a]_q\right)_2}\left|w_2-\left[\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1+q-t}\left(\mu\frac{\left(1+q+q^2-t\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q-t)\left([a]_q\right)^2\left([c]_q\right)_2}-t\right)-1\right]w_1^2\right|$$

The inequality (4.15) follows as an application of Lemma 2 with

$$s=\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1+q-t}\left(\mu\frac{\left(1+q+q^2-t\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q-t)\left([a]_q\right)^2\left([c]_q\right)_2}-t\right)-1.$$

□

For $t=1$ in Theorem 5, we state the following corollary.

**Corollary 8.** Let $f \in \mathcal{S}_q^a(\gamma,\lambda)$ be given by (1.1) and let $\mu$ be a complex number. Then,

$$\left|a_3-\mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{q(1+q)\left([a]_q\right)_2} \max\left\{1, \left|\frac{2(1-\lambda)\cos\gamma}{q}\left(\mu\frac{(1+q)\left([a]_q\right)_2\left([c]_q\right)^2}{\left([a]_q\right)^2\left([c]_q\right)_2}-1\right)-e^{i\gamma}\right|\right\}.$$

The result is sharp.

Taking $t=0$ in Theorem 5, we state the next corollary.

**Corollary 9.** Let $f \in \mathcal{K}_q^a(\gamma,\lambda)$ be given by (1.1) and let $\mu$ be a complex number. Then,

$$\left|a_3-\mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma\left([c]_q\right)_2}{\left(1+q+q^2\right)\left([a]_q\right)_2} \max\left\{1, \left|\mu\frac{2(1-\lambda)\cos\gamma\left(1+q+q^2\right)\left([a]_q\right)_2\left([c]_q\right)^2}{(1+q)^2\left([a]_q\right)^2\left([c]_q\right)_2}-e^{i\gamma}\right|\right\}.$$

The result is sharp.

## 5. Conclusion

Utilizing the concepts of quantum calculus, we defined new subclass of analytic functions associated with $q$-analogue of Carlson-Shaffer operator. For this subclass we investigated some useful results such as coefficient estimates, subordination properties and Fekete-Szegö problem. Their are some problems open for researchers such as distortion theorems, closure theorems, convolution propertiies and radii problems. Moreover, these results can be extended to multivalent functions and meromophic functions.

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## Conflict of interest

The authors declare that they have no competing interests.

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