Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(3): 2486–2509.
DOI:10.3934/math.2021151
Received: 22 October 2020
Accepted: 21 December 2020
Published: 23 December 2020

Research article

Qualitative analysis of fractional relaxation equation and coupled system with Ψ–Caputo fractional derivative in Banach spaces

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Abstract: The aim of the reported results in this manuscript is to handle the existence, uniqueness, extremal solutions, and Ulam–Hyers stability of solutions for a class of Ψ-Caputo fractional relaxation differential equations and a coupled system of Ψ-Caputo fractional relaxation differential equations in Banach spaces. The obtained results are derived by different methods of nonlinear analysis like the method of upper and lower solutions along with monotone iterative technique, Banach contraction principle, and Mönch's fixed point theorem concerted with the measures of noncompactness. Furthermore, the Ulam–Hyers stability of the proposed system is studied. Finally, two examples are presented to illustrate our theoretical findings. Our acquired results are recent in the frame of a Ψ-Caputo derivative with initial conditions in Banach spaces via the monotone iterative technique. As a results, we aim to fill this gap in the literature and contribute to enriching this academic area.

Keywords: Ψ–Caputo fractional derivative; extremal solutions; monotone iterative technique; upper and lower solutions; relaxation; Ulam–Hyers stability; Banach spaces
Mathematics Subject Classification: 26A33, 34A08

1. Introduction

It is widely recognized that fractional differential equations (FDEs) become one of the most important research topics in numerous different disciplines such as mathematics, physics, biology, chemistry, finance, economics, and engineering, etc, see [1–4]. The elementary knowledge of fractional calculus can be found in the books of senior scholars [5, 6]. While some notable developments on this topic can be found in the monographs of several mathematicians [7–9].

In (2017), Almeida [10], investigated a novel fractional derivative called Ψ–Caputo fractional derivative, which was already mentioned in Kilbas et al., book [5] under the concept of fractional integrals and derivatives of a function with respect to another function and which was extended by several famous scientists [5, 11–14]. Consequently several papers on Ψ–Caputo FDEs with different techniques are available, we refer few of them in [15–17]. Other important findings on the existence, uniqueness, and stability of solutions dealing with various definitions of well known fractional derivatives can be found in the articles [18–30].

On the other hand, the monotone iterative technique combined with the method of upper and lower solutions have been used by several researchers, both for the scalar case and the abstract case with the end goal to establish the existence and uniqueness of extremal solutions for a class of nonlinear ordinary and FDEs (see, for instance, [31–40]) and the references therein. Motivated by the papers mentioned above, our goal is to extend the results of the recent paper [34] to the abstract framework. To our knowledge, no contributions exist, concerning the existence and uniqueness of extremal solution for a class of nonlinear FDEs in the frame of Ψ–Caputo derivative with initial conditions in Banach spaces via the monotone iterative technique. As a results, we aim to fill this gap in the literature and contribute to enriching this academic area. So, in this paper, we study the existence and uniqueness of extremal solution for the following Ψ-Caputo FDE in an ordered Banach space $\mathbb{Y}$:

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}\mathfrak{u}(\xi) + \mathrm{r}\mathfrak{u}(\xi) = \mathfrak{f}(\xi, \mathfrak{u}(\xi)), \quad \xi \in \mathcal{I} := [a, d], \\ \mathfrak{u}(a) = \mathfrak{u}_a, \end{cases} \tag{1.1}$$

where ${}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}$ is the Ψ-Caputo fractional derivative such that $0 < \varsigma \leq 1, \mathfrak{f} : \mathcal{I} \times \mathbb{Y} \longrightarrow \mathbb{Y}$ is a function fulfillments some suppositions that will be mentioned later, $\mathrm{r} > 0$ and $\mathfrak{u}_a \in \mathbb{Y}$.

Next, we continue the results obtained in our recently published work in [23] to prove other properties such as the existence and uniqueness of solutions as well as the Ulam–Hyers (UH) stability results for the following Ψ-Caputo fractional relaxation differential system (Ψ-Caputo FRDS):

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\varsigma_1;\Psi}\mathfrak{u}(\xi) + \mathrm{r}_1\mathfrak{u}(\xi) = \mathbb{G}_1(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)), \\ {}^{c}\mathbb{D}_{a^+}^{\varsigma_2;\Psi}\mathfrak{v}(\xi) + \mathrm{r}_2\mathfrak{v}(\xi) = \mathbb{G}_2(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)), \end{cases} \quad \xi \in \mathcal{I}, \tag{1.2}$$

with the initial conditions

$$\begin{cases} \mathfrak{u}(a) = \mu_1, \\ \mathfrak{v}(a) = \mu_2, \end{cases} \tag{1.3}$$

where $\varsigma_i \in (0, 1], \mathrm{r}_i > 0, \mathbb{G}_i : \mathcal{I} \times \aleph \times \aleph \longrightarrow \aleph, i = 1, 2$ are functions fulfillments some suppositions that will be mentioned later, $\aleph$ is a Banach space with norm $\|\cdot\|$ and $\mu_1, \mu_2 \in \aleph$.

We organize the present work as follows: In Sect. 2, we recall basic concepts and results that will be called in the proof of our results. In Sect. 3, we apply the the monotone iterative technique in the presence of upper and lower solutions method to establish the existence and uniqueness of extremal solutions for the given problem (1.1). Whereas Sect. 4 is devoted to the existence and uniqueness of solutions to the coupled system (1.2)–(1.3). Moreover, Sect. 5, contains the UH stability of the proposed system (1.2)–(1.3). Also, two examples to illustrate the effectiveness of the feasibility of our abstract results are provided in Sect. 6. the work is terminated by some concluding remarks in Sect. 7.

2. Background materials

In this portion, we provide some fundamental concepts on the cones in a Banach space and Kuratowski's measure of noncompactness (KMN) as well as some facts about fractional calculus theory.

All over this part, we suppose that $(\mathbb{Y}, \|\cdot\|, \leq)$ is a partially ordered Banach space whose positive cone $\mathcal{K} = \{y \in \mathbb{Y} \mid y \geq \theta\}$ (θ is the zero element of $\mathbb{Y}$). Note that every cone $\mathcal{K}$ in $\mathbb{Y}$ defines a partial ordering in $\mathbb{Y}$ given by

$$z_1 \leq z_2 \quad \text{if and only if} \quad z_2 - z_1 \in \mathcal{K}.$$

Definition 2.1 ([44]). Let $\mathbb{Y}$ be an ordered Banach space with zero element θ. A cone $\mathcal{K} \subset \mathbb{Y}$ is a normal if $\exists\ \nu > 0$ such that

$$\theta \leq z_1 \leq z_2 \Rightarrow \|z_1\| \leq \nu\|z_2\|, \ \forall z_1, z_2 \in \mathbb{Y}.$$

where ν is the normal constant of $\mathcal{K}$, which is the smallest positive number fulfilling the above condition.

For any $z_1, z_2 \in \mathbb{Y}, z_1 \leq z_2$, The segment $[z_1, z_2]$ is a set in $\mathbb{Y}$ defined by

$$[z_1, z_2] = \{z \in \mathbb{Y} : z_1 \leq z \leq z_2\}.$$

Definition 2.2 ([44]). An operator $\mathcal{T} : \mathbb{Y} \longrightarrow \mathbb{Y}$ is said to be increasing if

$$x \leq y \Rightarrow \mathcal{T}x \leq \mathcal{T}y.$$

Let now $\mathcal{I} := [a, d]$ $(0 < a < d < \infty)$ be a finite interval and $\Psi : \mathcal{I} \to \mathbb{R}$ be an increasing function with $\Psi'(\xi) \neq 0$, for all $\xi \in \mathcal{I}$, and let $C(\mathcal{I}, \mathbb{Y})$ be the Banach space of all continuous functions $\mathfrak{u}$ from $\mathcal{I}$ into $\mathbb{Y}$ with the norm

$$\|\mathfrak{u}\|_\infty = \sup_{\xi \in \mathcal{I}} \|\mathfrak{u}(\xi)\|.$$

Plainly, $C(\mathcal{I}, \mathbb{Y})$ is an ordered Banach space whose partial ordering $\leq$ reduced by a positive cone $\mathcal{K}_C = \{\mathfrak{u} \in C(\mathcal{I}, \mathbb{Y}) : \mathfrak{u}(\xi) \geq \theta, \xi \in \mathcal{I}\}$ which is also normal with the same normal constant ν. For more details on cone theory, see [44].

A measurable function $\mathfrak{u} : \mathcal{I} \to \mathbb{Y}$ is Bochner integrable if and only if $\|\mathfrak{u}\|$ is Lebesgue integrable.

By $L^1(\mathcal{I}, \mathbb{Y})$ we denote the space of Bochner-integrable functions $\mathfrak{u} : \mathcal{I} \to \mathbb{Y}$, with the norm

$$\|\mathfrak{u}\|_1 = \int_a^d \|\mathfrak{u}(\xi)\| \mathrm{d}\xi.$$

Next, we define the KMN and grant some of its significant properties.

Definition 2.3 ([41]). The KMN $\Upsilon(\cdot)$ defined on bounded set $\mathbb{Q}$ of Banach space $\mathbb{Y}$ is

$$\Upsilon(\mathbb{Q}) := \inf\{\sigma > 0 \ : \ \mathbb{Q} = \cup_{k=1}^{n} \mathbb{Q}_k \text{ and } \operatorname{diam}(\mathbb{Q}_k) \leq \sigma \ \text{ for } \ k = 1, 2, \cdots, n\}.$$

The following properties about the KMN are well known.

Lemma 2.4 ([41]). *Let $\mathbb{Y}$ be a Banach space and $\mathbb{Q}_1, \mathbb{Q}_2 \subset \mathbb{Y}$ be bounded. The following properties are satisfied :*

(1) $\Upsilon(\mathbb{Q}_1) \leq \Upsilon(\mathbb{Q}_2)$ *if* $\mathbb{Q}_1 \subset \mathbb{Q}_2$*;*

(2) $\Upsilon(\mathbb{Q}) = \Upsilon(\overline{\mathbb{Q}}) = \Upsilon(conv\ \mathbb{Q})$*, where conv* $\mathbb{Q}$ *means the convex hull of* $\mathbb{Q}$*;*

(3) $\Upsilon(\mathbb{Q}) = 0$ *if and only if* $\overline{\mathbb{Q}}$ *is compact, where* $\overline{\mathbb{Q}}$ *means the closure hull of* $\mathbb{Q}$*;*

(4) $\Upsilon(\sigma\mathbb{Q}) = |\sigma|\Upsilon(\mathbb{Q})$*, where* $\sigma \in \mathbb{R}$*;*

(5) $\Upsilon(\mathbb{Q}_1 \cup \mathbb{Q}_2) = \max\{\Upsilon(\mathbb{Q}_1), \Upsilon(\mathbb{Q}_2)\}$*;*

(6) $\Upsilon(\mathbb{Q}_1 + \mathbb{Q}_1) \leq \Upsilon(\mathbb{Q}_1) + \Upsilon(\mathbb{Q}_2)$*, where* $\mathbb{Q}_1 + \mathbb{Q}_2 = \{p \mid p = q_1 + q_2, q_1 \in \mathbb{Q}_1, q_2 \in \mathbb{Q}_2\}$*;*

(7) $\Upsilon(\mathbb{Q} + q) = \Upsilon(\mathbb{Q})$*, for any* $q \in \mathbb{Y}$*;*

(8) *If the map* $\mathcal{T} : \mathrm{dom}(\mathcal{T}) \subset \mathbb{Y} \to \mathbb{Y}$ *is Lipschitz continuous with constant* k*, then* $\Upsilon(\mathcal{T}(\mathbb{Q})) \leq \mathrm{k}\Upsilon(\mathbb{Q})$ *for any bounded subset* $\mathbb{Q} \subset \mathrm{dom}(\mathcal{T})$.

The next lemmas are a prerequisite in our analysis.

Lemma 2.5 ([45]). *Let Λ be a bounded and equicontinuous subset of $C(\mathcal{I}, \mathbb{Y})$. Then the function $\xi \to \Upsilon(\Lambda(\xi))$ it has the property of continuity on $\mathcal{I}$, with*

$$\Upsilon_C(\Lambda) = \max_{\xi \in \mathcal{I}} \Upsilon(\Lambda(\xi)),$$

and

$$\Upsilon\left(\int_{\mathcal{I}} \Lambda(\ell)d\ell\right) \leq \int_{\mathcal{I}} \Upsilon(\Lambda(\ell))d\ell,$$

where $\Lambda(\ell) = \{b(\ell) : b \in \Lambda\}, \ell \in \mathcal{I}$.

Lemma 2.6 ([42]). *Let Λ is a bounded subset of Banach space $\mathbb{Y}$. Then for each ε, there is a sequence $\{y_n\}_{n=1}^{\infty} \subset \Lambda$, such that*

$$\Upsilon(\Lambda) \leq 2\Upsilon(\{y_n\}_{n=1}^{\infty}) + \varepsilon.$$

We say $\Lambda \subset L^1(\mathcal{I}, \mathbb{Y})$ is an uniformly integrable if there exists $v \in L^1(\mathcal{I}, \mathbb{R}_+)$ comply with

$$\|\mathrm{u}(\xi)\| \leq v(\xi), \text{ for all } \mathrm{u} \in \Lambda \text{ and a.e. } \xi \in \mathcal{I}.$$

Lemma 2.7 ([46]). *If $\{y_n\}_{n=1}^{\infty} \subset L^1(\mathcal{I}, \mathbb{Y})$ is uniformly integrable, then $\xi \mapsto \Upsilon(\{y_n(\xi)\}_{n=1}^{\infty})$ is measurable, and*

$$\Upsilon\left(\left\{\int_a^{\xi} y_n(\ell)\mathrm{d}\ell\right\}_{n=1}^{\infty}\right) \leq 2\int_a^{\xi} \Upsilon(\{y_n(\ell)\}_{n=1}^{\infty})\mathrm{d}\ell.$$

A fixed point technique advantageous to our aims is the following.

Theorem 2.8 ([47]). *Let $\mathbb{Y}$ a Banach space and Λ be a closed, bounded and convex subset of $\mathbb{Y}$ such that $\theta \in \Lambda$, and let $\mathcal{N}$ be a continuous map of Λ into itself. If the implication*

$$\mathfrak{Q} = \overline{conv}\mathcal{N}(\mathfrak{Q}), or\ \mathfrak{Q} = \mathcal{N}(\mathfrak{Q}) \cup \{\theta\} \Rightarrow \Upsilon(\mathfrak{Q}) = 0, \tag{2.1}$$

holds for every subset $\mathfrak{Q} \subset \Lambda$, then $\mathcal{N}$ has a fixed point.

Now, we supply some properties and results regarding the Ψ-fractional calculus as follows.

Definition 2.9 ([5, 10]). For $\varsigma > 0$ and $\xi \in \mathcal{I}$, the Ψ–Riemann-Liouville fractional integral of order ς is given by

$$\mathbb{I}_{a^+}^{\varsigma;\Psi} z(\xi) = \frac{1}{\Gamma(\varsigma)} \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} z(\ell) \mathrm{d}\ell, \tag{2.2}$$

where $z\colon \mathcal{I} \longrightarrow \mathbb{R}$ is an integrable function and $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(\varsigma) = \int_0^{+\infty} \xi^{\varsigma-1} e^{-\xi} \mathrm{d}\xi, \quad \varsigma > 0.$$

Definition 2.10 ([10]). Let $n \in \mathbb{N}$ and $\Psi, z \in C^n(\mathcal{I}, \mathbb{R})$ be two functions. Then the Ψ–Riemann–Liouville fractional derivative of a function z of order ς is given by

$$\begin{aligned} \mathbb{D}_{a^+}^{\varsigma;\Psi} z(\xi) &= \left(\frac{1}{\Psi'(\xi)} \frac{d}{dt} \right)^n \mathbb{I}_{a^+}^{n-\varsigma;\Psi} z(\xi) \\ &= \frac{1}{\Gamma(n-\varsigma)} \left(\frac{1}{\Psi'(\xi)} \frac{d}{dt} \right)^n \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{n-\varsigma-1} z(\ell) \mathrm{d}\ell, \end{aligned}$$

where $n = [\varsigma] + 1$.

Definition 2.11 ([10]). Let $n \in \mathbb{N}$ and $\Psi, z \in C^n(\mathcal{I}, \mathbb{R})$ be two functions. The Ψ-Caputo fractional derivative of z of order ς is defined by

$${}^c\mathbb{D}_{a^+}^{\varsigma;\Psi} z(\xi) = \mathbb{I}_{a^+}^{n-\varsigma;\Psi} \left(\frac{1}{\Psi'(\xi)} \frac{d}{dt} \right)^n z(\xi),$$

where $n = [\varsigma] + 1$ for $\nu \notin \mathbb{N}$, $n = \varsigma$ for $\varsigma \in \mathbb{N}$.
For the sake of brevity, let us take

$$z_{\Psi}^{[n]}(\xi) = \left(\frac{1}{\Psi'(\xi)} \frac{d}{d\xi} \right)^n z(\xi).$$

From the definition, it is clear that

$${}^c\mathbb{D}_{a^+}^{\varsigma;\Psi} z(\xi) = \begin{cases} \int_a^{\xi} \frac{\Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{n-\varsigma-1}}{\Gamma(n-\varsigma)} z_{\Psi}^{[n]}(\ell) \mathrm{d}\ell & , \quad \text{if } \varsigma \notin \mathbb{N}, \\ z_{\Psi}^{[n]}(\xi) & , \quad \text{if } \varsigma \in \mathbb{N}. \end{cases}$$

In fact, since the fractional integrals of a function z with respect to another function Ψ are generated by iterating the local integral $I_{a^+}^{1;\Psi} z(\xi) = \int_a^{\xi} z(s)\Psi'(s)ds$, then the fractional derivative of a function z with respect to another function Ψ in the sense of Riemann-Liouville and in the case of $\varsigma = n$ is natural number will be reduced to the local fractional operator $z_{\Psi}^{[n]}(\xi)$. Then, the Caputo type local behavior is accordingly follows via Remark 1 in [13]. For example, if $\varsigma = 1$ then $[1] + 1 = 2$ and

$${}_aD^{1,\Psi} z(\xi) = [\frac{d\xi}{\Psi'(\xi)}]^2 \int_a^{\xi} (\xi - s)^{2-1-1} z(s)\Psi'(s)ds = z_{\Psi}^{[1]}(\xi) = \frac{z'(\xi)}{\Psi'(\xi)}.$$

Then, on the light of (16) in Remark 1 in [13], we have

$${}_a^CD^{1,\Psi} z(\xi) = \frac{1}{\Psi'(\xi)} \frac{d}{d\xi}[z(\xi) - \xi(a)] = \frac{z'(\xi)}{\Psi'(\xi)}.$$

Lemma 2.12 ([10, 15]). *Let $\varsigma, \beta > 0$, and $z \in C(\mathcal{I}, \mathbb{R})$. Then for each $\xi \in \mathcal{I}$ we have*

(1) ${}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}\mathbb{I}_{a^+}^{\varsigma;\Psi}z(\xi) = z(\xi)$,

(2) $\mathbb{I}_{a^+}^{\varsigma;\Psi}{}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}z(\xi) = z(\xi) - z(a), \quad 0 < \varsigma \leq 1$,

(3) $\mathbb{I}_{a^+}^{\varsigma;\Psi}(\Psi(\xi) - \Psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\varsigma+\beta)}(\Psi(\xi) - \Psi(a))^{\varsigma+\beta-1}$,

(4) ${}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}(\Psi(\xi) - \Psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\varsigma)}(\Psi(\xi) - \Psi(a))^{\beta-\varsigma-1}$,

(5) ${}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}(\Psi(\xi) - \Psi(a))^{k} = 0$, *for all* $k \in \{0, \ldots, n-1\}, n \in \mathbb{N}$.

Definition 2.13 ([43]). The Mittag–Leffler functions (MLFs) of one and two parameters are defined respectively as

$$\mathbb{M}_{\varsigma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\varsigma k + 1)}, \quad z \in \mathbb{R}, \ \varsigma > 0.$$

and

$$\mathbb{M}_{\varsigma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\varsigma k + \beta)}, \ \varsigma, \beta > 0 \text{ and } z \in \mathbb{R}. \tag{2.3}$$

It is obvious that $\mathbb{M}_{1,1}(z) = \mathbb{M}_1(z) = e^z$.

Some essential properties of the MLFs are listed in the following Lemma.

Lemma 2.14 ([48]). *Let $\varsigma \in (0, 1)$ and $x \in \mathbb{R}$. Then the following properties are satisfied:*

(1) *$\mathbb{M}_{\varsigma}$ and $\mathbb{M}_{\varsigma,\varsigma}$ are nonnegative,*

(2) $\mathbb{M}_{\varsigma}(x) \leq 1, \mathbb{M}_{\varsigma,\varsigma}(x) \leq \frac{1}{\Gamma(\varsigma)}$, *for any $x < 0$.*

The following lemma is a generalization of Gronwall's inequality.

Theorem 2.15 ([26]). *Assume u, v be two integrable functions and w continuous, with domain $\mathcal{I}$. Let $\Psi \in C^1(\mathcal{I}, \mathbb{R}_+)$ an increasing function such that $\Psi'(\xi) \neq 0, \forall \xi \in \mathcal{I}$. Suppose that*

(1) *u and v are non-negative,*

(2) *w is non-decreasing and non-negative.*

If

$$u(\xi) \leq v(\xi) + w(\xi) \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} u(\ell) \mathrm{d}\ell, \ \xi \in \mathcal{I}.$$

Then

$$u(\xi) \leq v(\xi) + \int_a^{\xi} \sum_{n=0}^{\infty} \frac{(w(\xi)\Gamma(\varsigma))^n}{\Gamma(n\varsigma)} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{n\varsigma-1} v(\ell) \mathrm{d}\ell, \ \xi \in \mathcal{I}.$$

Remark 2.16. Notice that, for an abstract function $z : \mathcal{I} \longrightarrow \mathbb{Y}$, the integrals which show in the preceding definitions are taken in Bochner's frame (see [49]).

The next lemma has an important role in demonstrating our main results.

Lemma 2.17. *Let $\omega, \lambda > 0$. Then for all $\xi \in [a, d]$ we have*

$$\mathbb{I}_{a^+}^{\omega;\Psi} e^{\lambda(\Psi(\xi)-\Psi(a))} \leq \frac{e^{\lambda(\Psi(\xi)-\Psi(a))}}{\lambda^{\omega}}.$$

Proof. From equation (2.2), we have

$$\mathbb{I}_{a^+}^{\omega;\Psi} e^{\lambda(\Psi(\xi)-\Psi(a))} = \frac{1}{\Gamma(\omega)} \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\omega-1} e^{\lambda(\Psi(\ell)-\Psi(a))} d\ell.$$

Using the change of variables $\mathrm{y} = \Psi(\xi) - \Psi(\ell)$ we get

$$\mathbb{I}_{a^+}^{\omega;\Psi} e^{\lambda(\Psi(\xi)-\Psi(a))} = \frac{e^{\lambda(\Psi(\xi)-\Psi(a))}}{\Gamma(\omega)} \int_0^{\Psi(\xi)-\Psi(a)} \mathrm{y}^{\omega-1} e^{-\lambda \mathrm{y}} d\mathrm{y}.$$

Using now the change of variables $\mathrm{v} = \lambda \mathrm{y}$ in the above equation we get

$$\begin{aligned} \mathbb{I}_{a^+}^{\omega;\Psi} e^{\lambda(\Psi(\xi)-\Psi(a))} &= \frac{e^{\lambda(\Psi(\xi)-\Psi(a))}}{\Gamma(\omega)\lambda^{\omega}} \int_0^{\lambda(\Psi(\xi)-\Psi(a))} \mathrm{v}^{\omega-1} e^{-\mathrm{v}} d\mathrm{v} \\ &\leq \frac{e^{\lambda(\Psi(\xi)-\Psi(a))}}{\Gamma(\omega)\lambda^{\omega}} \int_0^{\infty} \mathrm{v}^{\omega-1} e^{-\mathrm{v}} d\mathrm{v} \\ &= \frac{e^{\lambda(\Psi(\xi)-\Psi(a))}}{\lambda^{\omega}}. \end{aligned}$$

This completes the proof. □

Remark 2.18 ([27])**.** On the space $C(\mathcal{I}, \mathbb{Y})$ we define a Bielecki type norm $\|\cdot\|_{\mathfrak{B}}$ as below

$$\|z\|_{\mathfrak{B}} := \sup_{\xi \in \mathcal{I}} \frac{\|z(\xi)\|}{e^{\lambda(\Psi(\xi)-\Psi(a))}}, \quad \lambda > 0. \tag{2.4}$$

Consequently, we have the following proprieties

1. $(C(\mathcal{I}, \mathbb{Y}), \|\cdot\|_{\mathfrak{B}})$ is a Banach space.
2. The norms $\|\cdot\|_{\mathfrak{B}}$ and $\|\cdot\|_{\infty}$ are equivalent on $C(\mathcal{I}, \mathbb{Y})$, where $\|\cdot\|_{\infty}$ denotes the Chebyshev norm on $C(\mathcal{I}, \mathbb{Y})$, i.e;
$$\iota_1 \|\cdot\|_{\mathfrak{B}} \leq \|\cdot\|_{\infty} \leq \iota_2 \|\cdot\|_{\mathfrak{B}},$$
where
$$\iota_1 = 1, \quad \iota_2 = e^{\lambda(\Psi(d)-\Psi(a))}.$$

3. Cauchy problem of Ψ-Caputo FDE (1.1)

In this section, we apply the well-known MIT together with the method of UP and LO solutions and the theory of measure of noncompactness to investigate the existence and uniqueness of extremal solutions for the Cauchy problem (1.1) in an ordered Banach space $\mathbb{Y}$.

Before we give our main results, let us defining what we mean by a solution of Ψ-Caputo FDE (1.1).

Definition 3.1. A function $\mathfrak{u} \in C(\mathcal{I}, \mathbb{Y})$ be a solution of problem (1.1) such that ${}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}\mathfrak{u}$ exists and is continuous on $\mathcal{I}$, if $\mathfrak{u}$ satisfies the equation ${}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}\mathfrak{u}(\xi) + \mathrm{r}\mathfrak{u}(\xi) = \mathfrak{f}(\xi, \mathfrak{u}(\xi))$, for each $\xi \in \mathcal{I}$ and the condition $\mathfrak{u}(a) = \mathfrak{u}_a$.

Now, we present the definition of lower and upper solutions of Ψ-Caputo FDE (1.1).

Definition 3.2. A function $\mathfrak{u} \in C(\mathcal{I}, \mathbb{Y})$ is called a lower solution of the Ψ-Caputo FDE (1.1) if it satisfies the following inequalities

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}\mathfrak{u}(\xi) + \mathrm{r}\mathfrak{u}(\xi) \leq \mathfrak{f}(\xi, \mathfrak{u}(\xi)), \quad \xi \in (a, d], \\ \mathfrak{u}(a) \leq \mathfrak{u}_a. \end{cases} \tag{3.1}$$

If all inequalities of (3.1) are inverted, we say that $\mathfrak{u}$ is an upper solution of the Ψ-Caputo FDE (1.1).

The following key lemma is substantial to forward in demonstrating the main results.

Lemma 3.3. *[34, Lemma 4] Let $\varsigma \in (0, 1]$ be fixed, $\mathrm{r} \in \mathbb{R}$ and $h \in C(\mathcal{I}, \mathbb{R})$. Then, the linear initial value problem*

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi}\mathfrak{u}(\xi) + \mathrm{r}\mathfrak{u}(\xi) = h(\xi), \ \ \xi \in \mathcal{I} := [a, d], \\ \mathfrak{u}(a) = \mathfrak{u}_a, \end{cases} \tag{3.2}$$

has a unique solution is given by

$$\begin{aligned} \mathfrak{u}(\xi) =& \mathfrak{u}_a \mathbb{M}_{\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(a))^{\varsigma}) \\ &+ \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\nu-1} \mathbb{M}_{\varsigma,\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(\ell))^{\varsigma}) h(\ell) \mathrm{d}\ell. \end{aligned} \tag{3.3}$$

As a result of Lemma 3.3, the problem (1.1) can be converted to an integral equation which takes the following form

$$\begin{aligned} \mathfrak{u}(\xi) =& \mathfrak{u}_a \mathbb{M}_{\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(a))^{\varsigma}) + \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \\ &\times \mathbb{M}_{\varsigma,\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(\ell))^{\varsigma}) \mathfrak{f}(\ell, \mathfrak{u}(\ell)) \mathrm{d}\ell. \end{aligned} \tag{3.4}$$

Now, we are willing to give and prove our main findings.

Theorem 3.4. *Let $\mathbb{Y}$ be an ordered Banach space, whose positive cone $\mathcal{K}$ is normal with normal constant ν. Let the following assumpitions are fulfilled*

(H_1) *There exist $\mathfrak{u}_0, \mathfrak{y}_0 \in C(\mathcal{I}, \mathbb{Y})$ such that $\mathfrak{u}_0$ and $\mathfrak{y}_0$ are lower and upper solutions of the Ψ-Caputo FDE* (1.1) *respectively, with $\mathfrak{u}_0 \leq \mathfrak{y}_0$.*

(H_2) *The function $\mathfrak{f} : \mathcal{I} \times \mathbb{Y} \longrightarrow \mathbb{Y}$ be continuous.*

(H_3) *$\mathfrak{f}$ is increasing with respect to the second variable. i.e*

$$\mathfrak{f}(\xi, z_1) \leq \mathfrak{f}(\xi, z_2),$$

for any $\xi \in \mathcal{I}$, and $z_1, z_2 \in \mathbb{Y}$ with $\mathfrak{u}_0(\xi) \leq z_1 \leq z_2 \leq \mathfrak{y}_0(\xi)$.

(H_4) *There exists a constant* $\mathbb{L} > 0$*, such that for any* $\xi \in \mathcal{I}$ *and decreasing or increasing monotone sequence* $\{\mathfrak{u}_n(\xi)\} \subset [\mathfrak{u}_0(\xi), \mathfrak{y}_0(\xi)]$,

$$\Upsilon(\{\mathfrak{f}(\xi, \mathfrak{u}_n(\xi))\}) \le \mathbb{L}\Upsilon(\{\mathfrak{u}_n(\xi)\}).$$

Then the Ψ*-Caputo FDE* (1.1) *has minimal and maximal solutions that are between* $\mathfrak{u}_0$ *and* $\mathfrak{y}_0$ *which can be acquired by a monotone iterative procedure starting from* $\mathfrak{u}_0$ *and* $\mathfrak{y}_0$*, respectively.*

Proof. Transform the integral representation (3.4) of the problem (1.1) into a fixed point problem as follows:

$$\mathfrak{u} = \mathcal{T}\mathfrak{u}, \quad \mathfrak{u} \in C(\mathcal{I}, \mathbb{Y}),$$

where $\mathcal{T} : C(\mathcal{I}, \mathbb{Y}) \longrightarrow C(\mathcal{I}, \mathbb{Y})$ is defined by

$$\begin{aligned}\mathcal{T}\mathfrak{u}(\xi) =& \mathfrak{u}_a\mathbb{M}_\varsigma(-\mathrm{r}(\Psi(\xi) - \Psi(a))^\varsigma) + \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \\ & \times \mathbb{M}_{\varsigma,\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(\ell))^\varsigma)\mathfrak{f}(\ell, \mathfrak{u}(\ell))\mathrm{d}\ell. \end{aligned} \tag{3.5}$$

From the continuity of $\mathfrak{f}$, $\mathcal{T}$ is well defined. On the other side, for any $\mathfrak{u} \in \mathrm{D} = [\mathfrak{u}_0, \mathfrak{y}_0] = \{y \in C(\mathcal{I}, \mathbb{Y}) : \mathfrak{u}_0 \le y \le \mathfrak{y}_0\}$. and $\xi \in \mathcal{I}$, (H_3) implies

$$\mathfrak{f}(\xi, \mathfrak{u}_0(\xi)) \le \mathfrak{f}(\xi, \mathfrak{u}(\xi)) \le \mathfrak{f}(\xi, \mathfrak{y}_0(\xi)),$$

i.e.

$$\theta \le \mathfrak{f}(\xi, \mathfrak{u}(\xi)) - \mathfrak{f}(\xi, \mathfrak{u}_0(\xi)) \le \mathfrak{f}(\xi, \mathfrak{y}_0(\xi)) - \mathfrak{f}(\xi, \mathfrak{u}_0(\xi)).$$

Therefore, from the normality of $\mathcal{K}$ we can get

$$\|\mathfrak{f}(\xi, \mathfrak{u}(\xi)) - \mathfrak{f}(\xi, \mathfrak{u}_0(\xi))\| \le \nu\|\mathfrak{f}(\xi, \mathfrak{y}_0(\xi)) - \mathfrak{f}(\xi, \mathfrak{u}_0(\xi))\|.$$

Thus

$$\begin{aligned}\|\mathfrak{f}(\xi, \mathfrak{u}(\xi))\| &\le \|\mathfrak{f}(\xi, \mathfrak{u}(\xi)) - \mathfrak{f}(\xi, \mathfrak{u}_0(\xi))\| + \|\mathfrak{f}(\xi, \mathfrak{u}_0(\xi))\| \\ &\le \nu\|\mathfrak{f}(\xi, \mathfrak{y}_0(\xi)) - \mathfrak{f}(\xi, \mathfrak{u}_0(\xi))\| + \|\mathfrak{f}(\xi, \mathfrak{u}_0(\xi))\| \\ &:= c.\end{aligned}$$

Hence

$$\|\mathfrak{f}(\xi, \mathfrak{u}(\xi))\| \le c, \mathfrak{u} \in \mathrm{D}. \tag{3.6}$$

Now, we complete the proof by a series of steps.

Step 1: In this step, we will show the continuity of the operator $\mathcal{T}$ on D. To do this, let $\{\mathfrak{u}_n\}$ be a sequence in D such that $\mathfrak{u}_n \to \mathfrak{u}$ in D as $n \to \infty$. With ease, we find that $\mathfrak{f}(\ell, \mathfrak{u}_n(\ell)) \to \mathfrak{f}(\ell, \mathfrak{u}(\ell))$, as $n \to +\infty$, due to $\mathfrak{f}$ is a continuous. Also, from 3.6 we get the following inequality:

$$\frac{\Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1}}{\Gamma(\varsigma)}\|\mathfrak{f}(\ell, \mathfrak{u}_n(\ell)) - \mathfrak{f}(\ell, \mathfrak{u}(\ell))\| \le \frac{2c\Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1}}{\Gamma(\varsigma)}.$$

From fact that the function $\ell \mapsto 2c\Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1}$ is Lebesgue integrable over $[a, \xi]$ along with the Lebesgue dominated convergence theorem, we attain

$$\int_a^{\xi} \frac{\Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1}}{\Gamma(\varsigma)} \|\mathfrak{f}(\ell, \mathfrak{u}_n(\ell)) - \mathfrak{f}(\ell, \mathfrak{u}(\ell))\| \mathrm{d}\ell \to 0 \ \text{ as } \ n \to +\infty.$$

It follows that $\|\mathcal{T}\mathfrak{u}_n - \mathcal{T}\mathfrak{u}\| \to 0$ as $n \to +\infty$. Hence the operator $\mathcal{T}$ is continuous.

Step 2: We shall show that the operator $\mathcal{T}$ has the following two properties:

(P_1) $\mathcal{T}$ is an increasing operator in D;

(P_2) $\mathfrak{u}_0 \leq \mathcal{T}\mathfrak{u}_0, \mathcal{T}\mathfrak{y}_0 \leq \mathfrak{y}_0$.

To prove (P_1), let $z_1, z_2 \in$ D, such that $z_1 \leq z_2$. Then, from (H_3) we obtain

$$\begin{aligned}
\mathcal{T}z_1(\xi) =& \mathfrak{u}_a \mathbb{M}_{\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(a))^{\varsigma}) + \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \\
& \times \mathbb{M}_{\varsigma,\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(\ell))^{\varsigma}) \mathfrak{f}(\ell, z_1(\ell)) \mathrm{d}\ell \\
\leq & \mathfrak{u}_a \mathbb{M}_{\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(a))^{\varsigma}) + \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \\
& \times \mathbb{M}_{\varsigma,\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(\ell))^{\varsigma}) \mathfrak{f}(\ell, z_2(\ell)) \mathrm{d}\ell \\
=& \mathcal{T}z_2(\xi),
\end{aligned}$$

which implies that $\mathcal{T}z_1 \leq \mathcal{T}z_2$. Therefore, $\mathcal{T}$ is an increasing operator.

To prove (P_2), let $h(\xi) = {}^{c}\mathbb{D}_{a^+}^{\varsigma;\Psi} \mathfrak{u}_0(\xi) + \mathrm{r}\mathfrak{u}_0(\xi)$. Definition 3.2, implies $h(\xi) \leq \mathfrak{f}(\xi, \mathfrak{u}_0(\xi))$. By Lemma 3.3, we obtain

$$\begin{aligned}
\mathfrak{u}_0(\xi) =& \mathfrak{u}_a \mathbb{M}_{\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(a))^{\varsigma}) + \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \\
& \times \mathbb{M}_{\varsigma,\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(\ell))^{\varsigma}) h(\ell) \mathrm{d}\ell \\
\leq & \mathfrak{u}_a \mathbb{M}_{\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(a))^{\varsigma}) + \int_a^{\xi} \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \\
& \times \mathbb{M}_{\varsigma,\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(\ell))^{\varsigma}) \mathfrak{f}(\ell, \mathfrak{u}_0(\ell)) \mathrm{d}\ell \\
=& \mathcal{T}\mathfrak{u}_0(\xi).
\end{aligned}$$

Hence, $\mathfrak{u}_0 \leq \mathcal{T}\mathfrak{u}_0$. In the same way, we get $\mathcal{T}\mathfrak{y}_0 \leq \mathfrak{y}_0$. Therefore, for every $\mathfrak{u} \in$ D, we have

$$\mathfrak{u}_0 \leq \mathcal{T}\mathfrak{u}_0 \leq \mathcal{T}\mathfrak{u} \leq \mathcal{T}\mathfrak{y}_0 \leq \mathfrak{y}_0.$$

From the above arguments, we conclude that $\mathcal{T}$: D $\to$ D is a continuous increasing operator.

Step 3: $\mathbb{T}(\mathrm{D})$ is equicontinuous on $\mathcal{I}$. To do this, choosing, $\xi_1, \xi_2 \in \mathcal{I}$, with $\xi_1 \leq \xi_2$. By (3.6) and Lemma 2.14 we have

$$\begin{aligned}
&\|\mathcal{T}\mathfrak{u}(\xi_2) - \mathcal{T}\mathfrak{u}(\xi_1)\| \\
\leq& \|\mathfrak{u}_a\| \left|\mathbb{M}_\varsigma(-\mathrm{r}(\Psi(\xi_2) - \Psi(a))^\varsigma) - \mathbb{M}_\varsigma(-\mathrm{r}(\Psi(\xi_1) - \Psi(a))^\varsigma)\right| \\
&+ \int_a^{\xi_1} \frac{\Psi'(\ell)\left[(\Psi(\xi_1) - \Psi(\ell))^{\varsigma-1} - (\Psi(\xi_2) - \Psi(\ell))^{\varsigma-1}\right]}{\Gamma(\varsigma)} \|\mathfrak{f}(\ell, \mathfrak{u}(\ell))\| d\ell \\
&+ \int_{\xi_1}^{\xi_2} \frac{\Psi'(\ell)(\Psi(\xi_2) - \Psi(\ell))^{\varsigma-1}}{\Gamma(\varsigma)} \|\mathfrak{f}(\xi, \mathfrak{u}(\ell))\| \, d\ell \\
\leq& \|\mathfrak{u}_a\| \left|\mathbb{M}_\varsigma(-\mathrm{r}(\Psi(\xi_2) - \Psi(a))^\varsigma) - \mathbb{M}_\varsigma(-\mathrm{r}(\Psi(\xi_1) - \Psi(a))^\varsigma)\right| \\
&+ \frac{2c}{\Gamma(\varsigma+1)} (\Psi(\xi_2) - \Psi(\xi_1))^\varsigma.
\end{aligned}$$

Since the function $\mathbb{M}_\varsigma(-\mathrm{r}(\Psi(\xi) - \Psi(a))^\varsigma)$ is continuous on $\mathcal{I}$, the right-hand side of the previous inequality approaches to zero when $\xi_1 \to \xi_2$ independently of $\mathfrak{u} \in \mathrm{D}$. This implies that $\mathbb{T}(\mathrm{D})$ is equicontinuous on $\mathcal{I}$.

Now define two sequences $\{\mathfrak{u}_n\}$ and $\{\mathfrak{y}_n\}$ in D, by the iterative scheme

$$\mathfrak{u}_n = \mathcal{T}\mathfrak{u}_{n-1}, \quad \mathfrak{y}_n = \mathcal{T}\mathfrak{y}_{n-1}, \text{ for } n = 1, 2, \ldots \tag{3.7}$$

Then from the monotonicity of $\mathcal{T}$, we have

$$\mathfrak{u}_0 \leq u_1 \leq \cdots \leq \mathfrak{u}_n \leq \cdots \leq \mathfrak{y}_n \leq \cdots \leq v_1 \leq \mathfrak{y}_0. \tag{3.8}$$

Step 4: We show that $\{\mathfrak{u}_n\}$ and $\{\mathfrak{y}_n\}$ are convergent in $C(\mathcal{I}, \mathbb{Y})$.

Let $\Omega = \{\mathfrak{u}_n : n \in \mathbb{N}\}$ and $\Omega_0 = \{\mathfrak{u}_{n-1} : n \in \mathbb{N}\}$. By (3.8) and the normality of the positive cone $\mathcal{K}$, we get that Ω and Ω_0 are bounded. From $\Omega_0 = \Omega \cup \{\mathfrak{u}_0\}$, we have

$$\Upsilon(\Omega(\xi)) = \Upsilon(\Omega_0(\xi)), \text{ for all } \xi \in \mathcal{I}.$$

Let

$$\rho(\xi) = \Upsilon(\Omega(\xi)) = \Upsilon(\Omega_0(\xi)), \text{ for all } \xi \in \mathcal{I}.$$

Since $\Omega = \mathcal{T}\Omega_0$, we have

$$\Upsilon(\Omega(\xi)) = \Upsilon(\mathcal{T}\Omega_0(\xi)), \text{ for all } \xi \in \mathcal{I}.$$

Now, we will show that $\rho(\xi) \equiv 0$ on $\mathcal{I}$. By (H_4), Lemmas 2.7, 2.14 and the properties of Υ we obtain the following estimates:

$$\begin{aligned}
\rho(\xi) = \Upsilon(\Omega(\xi)) = \Upsilon(\mathcal{T}\Omega_0(\xi)) &\leq \Upsilon\left\{\int_a^\xi \frac{\Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1}}{\Gamma(\varsigma)} \mathfrak{f}(\ell, \mathfrak{u}_{n-1}(\ell)) d\ell\right\} \\
&\leq \frac{2}{\Gamma(\varsigma)} \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \Upsilon(\{\mathfrak{f}(\ell, \mathfrak{u}_{n-1}(\ell))\}) d\ell \\
&\leq \frac{2\mathbb{L}}{\Gamma(\varsigma)} \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \Upsilon(\{\mathfrak{u}_{n-1}(\ell)\}) d\ell
\end{aligned}$$

$$\leq \frac{2\mathbb{L}}{\Gamma(\varsigma)} \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1}\Omega_0(\ell)d\ell$$
$$= \frac{2\mathbb{L}}{\Gamma(\varsigma)} \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1}\rho(\ell)d\ell.$$

Hence by Lemma 2.15, we obtain $\rho(\xi) \equiv 0$ on $\mathcal{I}$. Since Ω is equi-continuous, we have from Lemma 2.5 that

$$\Upsilon(\Omega) = \Upsilon(\Omega_0) = \max_{\xi \in \mathcal{I}} \Upsilon(\Omega_0(\xi)) = 0. \tag{3.9}$$

So, $\{\mathfrak{u}_n\}$ are relatively compact in $C(\mathcal{I}, \mathbb{Y})$. Hence, $\{\mathfrak{u}_n\}$ has a convergent subsequence in $C(\mathcal{I}, \mathbb{Y})$. Combining this with the monotonicity and the normality of the cone $\mathcal{K}$, without difficulty we can prove that $\{\mathfrak{u}_n\}$ itself is convergent in $C(\mathcal{I}, \mathbb{Y})$, i.e., there exists $\underline{\mathfrak{u}} \in C(\mathcal{I}, \mathbb{Y})$ such that $\lim_{n\to\infty} \mathfrak{u}_n = \underline{\mathfrak{u}}$. Similarly, it can be proved that there exists $\bar{\mathfrak{y}} \in C(\mathcal{I}, \mathbb{Y})$ such that $\lim_{n\to\infty} \mathfrak{y}_n = \bar{\mathfrak{y}}$.

Using Lebesgue dominated convergence theorem, and letting $n \to \infty$ in, (3.7) we see that

$$\underline{\mathfrak{u}} = \mathcal{T}\underline{\mathfrak{u}}, \qquad \bar{\mathfrak{y}} = \mathcal{T}\bar{\mathfrak{y}}.$$

Therefore, $\underline{\mathfrak{u}}, \bar{\mathfrak{y}} \in C(\mathcal{I}, \mathbb{Y})$ are fixed points of $\mathcal{T}$.

Step 5: We show the minimal and maximal property of $\underline{\mathfrak{u}}, \bar{\mathfrak{y}}$. Suppose that z^* is a fixed point of $\mathcal{T}$ in D, then we have

$$\mathfrak{u}_0(\xi) \leq z^*(\xi) \leq \mathfrak{y}_0(\xi), \quad \xi \in \mathcal{I}.$$

By the monotonicity of $\mathcal{T}$, it is uncomplicated to find that

$$u_1(\xi) = (\mathcal{T}\mathfrak{u}_0)(\xi) \leq (\mathcal{T}z^*)(\xi) = z^*(\xi) \leq (\mathcal{T}\mathfrak{y}_0)(\xi) = v_1(\xi), \quad \xi \in \mathcal{I}.$$

Repeating the above arguments, we get

$$\mathfrak{u}_n \leq z^* \leq \mathfrak{y}_n, \quad n = 1, 2, \ldots. \tag{3.10}$$

Taking $n \to \infty$ in (3.10), we get $\underline{\mathfrak{u}} \leq z^* \leq \bar{\mathfrak{y}}$. Thus $\underline{\mathfrak{u}}, \bar{\mathfrak{y}}$ are the minimal and maximal fixed points of $\mathcal{T}$ in D, so, they also are the minimal and maximal solutions of problem (1.1) in D. Moreover, $\underline{\mathfrak{u}}$ and $\bar{\mathfrak{y}}$ can be obtained by the iterative procedure (3.7) beginning from $\mathfrak{u}_0$ and $\mathfrak{y}_0$, respectively. This finishes the proof. □

Our next theorem to prove the uniqueness of solution for the Ψ-Caputo FDE (1.1) by applying the monotone iterative technique.

Theorem 3.5. *Let $\mathbb{Y}$ be an ordered Banach space whose positive cone $\mathcal{K}$ is normal with normal constant ν. Suppose that (H_1)–(H_3) are fulfilled. Further, we suppose that:*

(H_5) *There exists a constant* $\mathrm{k} > 0$ *with*

$$\mathfrak{f}(\xi, z_2) - \mathfrak{f}(\xi, z_1) \leq \mathrm{k}(z_2 - z_1), \quad \textit{for any } \ \xi \in \mathcal{I},$$

and $\mathfrak{u}_0 \leq z_1 \leq z_2 \leq \mathfrak{y}_0$.

Then Ψ-Caputo FDE (1.1) *has a unique solution between* $\mathfrak{u}_0$ *and* $\mathfrak{y}_0$*, which can be acquired by the iterative procedure beginning from* $\mathfrak{u}_0$ *or* $\mathfrak{y}_0$.

Proof. Let $\{z_n\} \subset \mathrm{D}$ be an increasing monotone sequence, and $n, m \in \mathbb{N}$ with $n > m$. (H_2) and (H_5) imply

$$\theta \leq \mathfrak{f}(\xi, z_n) - \mathfrak{f}(\xi, z_m) \leq \mathrm{k}(z_n - z_m).$$

From the normality of positive cone $\mathcal{K}$, we obtain

$$\|\mathfrak{f}(\xi, z_n) - \mathfrak{f}(\xi, z_m)\| \leq \mathrm{k}\nu \|z_n - z_m\|.$$

So by Lemma 2.4, we get

$$\Upsilon(\{\mathfrak{f}(\xi, z_n)\}) \leq \mathrm{k}\nu \Upsilon(\{z_n\}).$$

Then condition (H_4) holds and from the Theorem 3.4, we realize that Ψ-Caputo FDE (1.1) has minimal and maximal solutions $\underline{\mathfrak{u}}$ and $\bar{\mathfrak{y}}$ in D. Next we prove that $\underline{\mathfrak{u}}(\xi) \equiv \bar{\mathfrak{y}}(\xi)$ in $\mathcal{I}$.

Thanks to Lemma 2.14 and (H_5), for each $\xi \in \mathcal{I}$, we obtain

$$\begin{aligned}
\theta \leq & \bar{\mathfrak{y}}(\xi) - \underline{\mathfrak{u}}(\xi) = \mathcal{T}\bar{\mathfrak{y}}(\xi) - \mathcal{T}\underline{\mathfrak{u}}(\xi) \\
= & \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \mathbb{M}_{\varsigma,\varsigma}(-\mathrm{r}(\Psi(\xi) - \Psi(\ell))^\varsigma) \\
& \times (\mathfrak{f}(\ell, \bar{\mathfrak{y}}(\ell)) - \mathfrak{f}(\ell, \underline{\mathfrak{u}}(\ell)))d\ell \\
\leq & \frac{\mathrm{k}}{\Gamma(\varsigma)} \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} (\bar{\mathfrak{y}}(\ell) - \underline{\mathfrak{u}}(\ell))d\ell.
\end{aligned}$$

By the normality of positive cone $\mathcal{K}$, it follows that

$$\left\|\bar{\mathfrak{y}}(\xi) - \underline{\mathfrak{u}}(\xi)\right\| \leq \frac{\mathrm{k}\nu}{\Gamma(\varsigma)} \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma-1} \|\bar{\mathfrak{y}}(\ell) - \underline{\mathfrak{u}}(\ell)\| d\ell. \tag{3.11}$$

By Lemma 2.15, we attain $\bar{\mathfrak{y}}(\xi) \equiv \underline{\mathfrak{u}}(\xi)$ on $\mathcal{I}$. Hence, $\bar{\mathfrak{y}} \equiv \underline{\mathfrak{u}}$ is the unique solution of the Ψ-Caputo FDE (1.1) in D. Thus, the proof is finished. □

4. Coupled systems of Ψ-Caputo FRDS (1.2)–(1.3)

In this portion, we plan to prove our main theoretical findings of the existence and uniqueness of solution for the Ψ-Caputo FRDS (1.2)–(1.3).

Here, we offer the definition and lemma of a solution for Ψ-Caputo FRDS (1.2)–(1.3).

Definition 4.1. By a solution of coupled systems of Ψ-Caputo FRDS (1.2)–(1.3), we mean a pair of continuous functions $(\mathfrak{u}, \mathfrak{v}) \in C(\mathcal{I}, \aleph) \times C(\mathcal{I}, \aleph)$ those satisfy equations (1.2) on $\mathcal{I}$, and conditions (1.3).

Now by using the basic concepts mentioned in [13, 34] we can easily derive the following lemma which is useful to prove our main results.

Lemma 4.2. *[13, 34] Let* $\varsigma_1, \varsigma_2 \in (0,1]$ *be fixed,* $\mathrm{r}_1, \mathrm{r}_2 > 0$ *and* $\mathbb{G}_1, \mathbb{G}_2 \in C(\mathcal{I} \times \aleph \times \aleph, \aleph)$*. Then the coupled systems of* Ψ*-Caputo FRDS* (1.2)–(1.3) *is equivalent to the following integral equations*

$$\begin{cases} \mathfrak{u}(\xi) = & \mu_1 \mathbb{M}_{\varsigma_1}(-\mathrm{r}_1(\Psi(\xi)-\Psi(a))^{\varsigma_1}) + \int_a^{\xi} \Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_1-1} \\ & \times \mathbb{M}_{\varsigma_1,\varsigma_1}(-\mathrm{r}_1(\Psi(\xi)-\Psi(\ell))^{\varsigma_1})\mathbb{G}_1(\ell, \mathfrak{u}(\ell), \mathfrak{v}(\ell))d\ell \\ \mathfrak{v}(\xi) = & \mu_2 \mathbb{M}_{\varsigma_2}(-\mathrm{r}_2(\Psi(\xi)-\Psi(a))^{\varsigma_2}) + \int_a^{\xi} \Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_2-1} \\ & \times \mathbb{M}_{\varsigma_2,\varsigma_2}(-\mathrm{r}_2(\Psi(\xi)-\Psi(\ell))^{\varsigma_2})\mathbb{G}_2(\ell, \mathfrak{u}(\ell), \mathfrak{v}(\ell))d\ell. \end{cases} \tag{4.1}$$

In order to establish our main results, we introduce the following assumptions.

(H_1') $\mathbb{G}_1, \mathbb{G}_2 : \mathcal{I} \times \aleph \times \aleph \longrightarrow \aleph$ are continuous functions.

(H_2') There exist constants $\mathbb{L}_i > 0,\ i = 1, 2$ such that

$$\|\mathbb{G}_i(\xi, \mathfrak{u}_1, \mathfrak{v}_1) - \mathbb{G}_i(\xi, \mathfrak{u}_2, \mathfrak{v}_2)\| \le \mathbb{L}_i(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \|\mathfrak{v}_1 - \mathfrak{v}_2\|),$$

for all $\xi \in \mathcal{I}$ and each $\mathfrak{u}_1, \mathfrak{v}_1, \mathfrak{u}_2, \mathfrak{v}_2 \in \aleph$.

(H_3') There exist real constants $\mathbb{K}_1, \mathbb{K}_2 > 0$ and a continuous non-decreasing function $\phi_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+,\ i = 1, 2$ such that

$$\|\mathbb{G}_i(\xi, \mathfrak{u}, \mathfrak{v})\| \le \mathbb{K}_i \phi_i(\|\mathfrak{u}\| + \|\mathfrak{v}\|), \text{for any } \xi \in \mathcal{I} \text{ and each } \mathfrak{u}, \mathfrak{v} \in \aleph.$$

(H_4') For each bounded set $\mathcal{H} \subset \aleph \times \aleph$, and each $\xi \in \mathcal{I}$, the following inequality holds

$$\Upsilon(\mathbb{G}_i(\xi, \mathcal{H})) \le \mathbb{K}_i \Upsilon(\mathcal{H}),\ i = 1, 2.$$

Our first theorem on the uniqueness relies on the fixed point theorem of Banach combined with the Bielecki norm.

Theorem 4.3. *If the assumptions* (H_1')–(H_2') *are true, then the coupled system of* Ψ*-Caputo FRDS* (1.2)–(1.3) *has a unique solution.*

Proof. Let $C(\mathcal{I}, \aleph)$ be a Banach space equipped with the Bielecki norm type $\|\cdot\|_{\mathfrak{B}}$ defined in (2.4). Consequently, the product space $\mathcal{E} := C(\mathcal{I}, \aleph) \times C(\mathcal{I}, \aleph)$ is a Banach space, endowed with the Bielecki norm

$$\|(\mathfrak{u}, \mathfrak{v})\|_{\mathcal{E},\mathfrak{B}} = \|\mathfrak{u}\|_{\mathfrak{B}} + \|\mathfrak{v}\|_{\mathfrak{B}}.$$

We define an operator $\mathbb{S} = (\mathbb{S}_1, \mathbb{S}_2) \colon \mathcal{E} \to \mathcal{E}$ by:

$$\mathbb{S}(\mathfrak{u}, \mathfrak{v}) = (\mathbb{S}_1(\mathfrak{u}, \mathfrak{v}), \mathbb{S}_2(\mathfrak{u}, \mathfrak{v})). \tag{4.2}$$

where

$$\begin{aligned} \mathbb{S}_i(\mathfrak{u}, \mathfrak{v})(\xi) =& \mu_i \mathbb{M}_{\varsigma_i}(-\mathrm{r}_i(\Psi(\xi)-\Psi(a))^{\varsigma_i}) + \int_a^{\xi} \Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_i-1} \\ & \times \mathbb{M}_{\varsigma_i,\varsigma_i}(-\mathrm{r}_i(\Psi(\xi)-\Psi(\ell))^{\varsigma_i})\mathbb{G}_i(\ell, \mathfrak{u}(\ell), \mathfrak{v}(\ell))d\ell. \end{aligned} \tag{4.3}$$

It should be noted that $\mathbb{S}$ is well-defined since both $\mathbb{G}_1$ and $\mathbb{G}_2$ are continuous. Now, we make use of the fixed point theorem of Banach to show that $\mathbb{S}$ has a unique fixed point. In this moment, we must show that $\mathbb{S}$ is a contraction mapping on $\mathcal{E}$ with respect to Bielecki's norm $\|\cdot\|_{\mathcal{E},\mathfrak{B}}$. Note that by definition of operator $\mathbb{S}$, for any $(\mathfrak{u}_1,\mathfrak{v}_1),(\mathfrak{u}_2,\mathfrak{v}_2)\in\mathcal{E}$ and $\xi\in\mathcal{I}$, using (H_2'), and Lemmas 2.14, 2.17, we can get

$$\begin{aligned}
&\left\|\mathbb{S}_i(\mathfrak{u}_1,\mathfrak{v}_1)(\xi)-\mathbb{S}_i(\mathfrak{u}_2,\mathfrak{v}_2)(\xi)\right\|\\
\leq&\mathbb{L}_i(\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathfrak{B}}+\|\mathfrak{v}_1-\mathfrak{v}_2\|_{\mathfrak{B}})\int_a^\xi\frac{\Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_i-1}}{\Gamma(\varsigma_i)}e^{\lambda(\Psi(\ell)-\Psi(a))}\mathrm{d}\ell\\
\leq&\frac{e^{\lambda(\Psi(\xi)-\Psi(a))}}{\lambda^{\varsigma_i}}\mathbb{L}_i(\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathfrak{B}}+\|\mathfrak{v}_1-\mathfrak{v}_2\|_{\mathfrak{B}}).
\end{aligned}$$

Hence

$$\left\|\mathbb{S}_i(\mathfrak{u}_1,\mathfrak{v}_1)-\mathbb{S}_i(\mathfrak{u}_2,\mathfrak{v}_2)\right\|_{\mathfrak{B}}\leq\frac{\mathbb{L}_i}{\lambda^{\varsigma_i}}(\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathfrak{B}}+\|\mathfrak{v}_1-\mathfrak{v}_2\|_{\mathfrak{B}}).$$

This implies that

$$\left\|\mathbb{S}(\mathfrak{u}_1,\mathfrak{v}_1)-\mathbb{S}(\mathfrak{u}_2,\mathfrak{v}_2)\right\|_{\mathcal{E},\mathfrak{B}}\leq\left[\frac{\mathbb{L}_1}{\lambda^{\varsigma_1}}+\frac{\mathbb{L}_2}{\lambda^{\varsigma_2}}\right]\|(\mathfrak{u}_1,\mathfrak{v}_1)-(\mathfrak{u}_2,\mathfrak{v}_2)\|_{\mathcal{E},\mathfrak{B}}.$$

We can choose $\lambda>0$ such that $\frac{\mathbb{L}_1}{\lambda^{\varsigma_1}}+\frac{\mathbb{L}_2}{\lambda^{\varsigma_2}}<1$, so the operator $\mathbb{S}$ is a contraction with respect to Bielecki's norm $\|\cdot\|_{\mathcal{E},\mathfrak{B}}$. Thus, an application of Banach's fixed point theorem shows that $\mathbb{S}$ has a unique fixed point. So the coupled system of Ψ-Caputo FRDS (1.2)–(1.3) has a unique solution in the space $\mathcal{E}$. This completes the proof. □

Theorem 4.4. *Let the hypotheses (H_1'), (H_3') and (H_4') be fulfilled. Then the coupled system of Ψ-Caputo FRDS* (1.2)–(1.3) *has at least one solution defined on $\mathcal{I}$*

Proof. In order to use the Theorem 2.8, we define a subset $\mathbb{B}_\delta$ of $\mathcal{E}$ by

$$\mathbb{B}_\delta=\left\{(\mathfrak{u},\mathfrak{v})\in\mathcal{E}:\|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{E},\infty}\leq\delta\right\},$$

with $\delta>0$, such that

$$\delta\geq\sum_{i=1}^{2}(\|\mu_i\|+\mathbb{P}_{\varsigma_i,\Psi}\mathbb{K}_i\phi_i(\delta)).$$

where

$$\mathbb{P}_{\varsigma_i,\Psi}=\frac{(\Psi(d)-\Psi(a))^{\varsigma_i}}{\Gamma(\varsigma_i+1)},\ i=1,2.$$

Notice that $\mathbb{B}_\delta$ is convex, closed and bounded subset of the Banach space $\mathcal{E}$. We shall prove that $\mathbb{S}$, satisfies all conditions of Theorem 2.8 in a two steps.

Step 1: we show that the operator $\mathbb{S}$ maps the set $\mathbb{B}_\delta$ into itself. Indeed, for any $(\mathfrak{u},\mathfrak{v})\in\mathbb{B}_\delta$ and for each $\xi \in \mathcal{I}$. By Lemma 2.14 together with assumption (H_3') we can get

$$\begin{aligned}\left\|\mathbb{S}_i(\mathfrak{u},\mathfrak{v})(\xi)\right\| &\leq \|\mu_i\| + \int_a^\xi \frac{\Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_i-1}}{\Gamma(\varsigma_i)}\|\mathbb{G}_i(\ell,\mathfrak{u}(\ell),\mathfrak{v}(\ell))\|\,\mathrm{d}\ell \\ &\leq \|\mu_i\| + \int_a^\xi \frac{\Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_i-1}}{\Gamma(\varsigma_i)}\mathbb{K}_i\phi_i(\|\mathfrak{u}(\ell)\|+\|\mathfrak{v}(\ell)\|)\,\mathrm{d}\ell \\ &\leq \|\mu_i\| + \mathbb{K}_i\phi_i(\delta)\int_a^\xi \frac{\Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_i-1}}{\Gamma(\varsigma_i)}\mathrm{d}\ell \\ &\leq \|\mu_i\| + \mathbb{K}_i\mathbb{P}_{\varsigma_i,\Psi}\phi_i(\delta), \quad i=1,2.\end{aligned}$$

Hence

$$\left\|\mathbb{S}(\mathfrak{u},\mathfrak{v})\right\|_{\mathcal{E},\infty} \leq \sum_{i=1}^{2}(\|\mu_i\| + \mathbb{K}_i\mathbb{P}_{\varsigma_i,\Psi}\phi_i(\delta)) \leq \delta.$$

This proves that $\mathbb{S}$ transforms the ball $\mathbb{B}_\delta$ into itself. Moreover, in view of assumptions (H_1'), (H_3') and by a similar deduction in Theorem 3.4, one can easily verify that $\mathbb{S}:\mathbb{B}_\delta \longrightarrow \mathbb{B}_\delta$ is continuous and $\mathbb{S}(\mathbb{B}_\delta)$ is equi-continuous on $\mathcal{I}$.

Step 2: Now we prove that the Mönch's condition holds. For this purpose, let $\Delta = \Delta_1 \cap \Delta_2$ and Δ_i be a subset of $\mathbb{B}_R$ such that $\Delta_i \subset \overline{conv}\left(\mathbb{S}_i(\Delta_i)\cup\{0\}\right), i=1,2$. Δ_i is bounded and equi-continuous, and therefore the function $\mathfrak{f}_i(\xi) = \Upsilon(\Delta_i(\xi))$ is continuous on $\mathcal{I}$. By the properties of the KMN, Lemma 2.5 and (H_4'), we have

$$\begin{aligned}\mathfrak{f}_1(\xi) &= \Upsilon(\Delta_1(\xi)) \leq \Upsilon(\overline{conv}(\mathcal{T}_1(\Delta_1)(\xi)\cup\{0\})) \leq \Upsilon(\mathcal{T}_1(\Delta_1)(\xi)) \\ &\leq \Upsilon\left\{\int_a^\xi \frac{\Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_1-1}}{\Gamma(\varsigma_1)}\mathbb{G}_1(\ell,\mathfrak{u}(\ell),\mathfrak{v}(\ell))\mathrm{d}\ell : (\mathfrak{u},\mathfrak{v})\in\Delta_1\right\} \\ &\leq \int_a^\xi \frac{\Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_1-1}}{\Gamma(\varsigma_1)}\Upsilon(\mathbb{G}_1(\ell,\Delta_1(\ell)))\,\mathrm{d}\ell \\ &\leq \frac{\mathbb{K}_1}{\Gamma(\varsigma_1)}\int_a^\xi \Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_1-1}\Upsilon(\Delta_1(\ell))\,\mathrm{d}\ell \\ &\leq \frac{\mathbb{K}_1}{\Gamma(\varsigma_1)}\int_a^\xi \Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_1-1}\mathfrak{f}_1(\ell)\mathrm{d}\ell.\end{aligned}$$

Hence by means of Lemma 2.15, we get $\mathfrak{f}_1(\xi) = \Upsilon(\Delta_1(\xi)) = 0$, for each $\xi \in \mathcal{I}$. Similarly, we have $\mathfrak{f}_2(\xi) = 0$. Hence $\Upsilon(\Delta(\xi)) \leq \Upsilon(\Delta_1(\xi)) = 0$ and $\Upsilon(\Delta(\xi)) \leq \Upsilon(\Delta_2(\xi)) = 0$, this shows that $\Delta(\xi)$ is relatively compact in $\aleph\times\aleph$. By Ascoli-Arzelá theorem, Δ is relatively compact in $\mathbb{B}_\delta$. Invoking Theorem 2.8 we deduce that $\mathcal{T}$ has a fixed point which is a solution of Ψ-Caputo FRDS (1.2)–(1.3). This finishes the proof. □

5. Stability analysis for the Ψ-Caputo FRDS (1.2)–(1.3)

In this part of the manuscript, we analyze the UH stability for the proposed Ψ-Caputo FRDS (1.2)–(1.3).

For some $\varepsilon_1, \varepsilon_2 > 0$, we consider the following inequalities:

$$\begin{cases} \left\| ({}^{c}\mathbb{D}_{a^+}^{\varsigma_1;\Psi}\tilde{\mathfrak{u}})(\xi) + \mathrm{r}_1\tilde{\mathfrak{u}}(\xi) - \mathbb{G}_1(\xi, \tilde{\mathfrak{u}}(\xi), \tilde{\mathfrak{v}}(\xi)) \right\| \le \varepsilon_1, \\ \left\| ({}^{c}\mathbb{D}_{a^+}^{\varsigma_2;\Psi}\tilde{\mathfrak{v}})(\xi) + \mathrm{r}_2\tilde{\mathfrak{v}}(\xi) - \mathbb{G}_2(\xi, \tilde{\mathfrak{u}}(\xi), \tilde{\mathfrak{v}}(\xi)) \right\| \le \varepsilon_2. \end{cases} \quad \xi \in \mathcal{I}. \tag{5.1}$$

Definition 5.1. The Ψ-Caputo FRDS (1.2)–(1.3) is UH stable with respect to the Bielecki's norm if there exists a positive real number c such that for each pair $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}_+ \times \mathbb{R}_+$ and for each solution $(\tilde{\mathfrak{u}}, \tilde{\mathfrak{v}}) \in \mathcal{E}$ of the inequalities (5.1), there exists a unique solution $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{E}$ of (1.2)–(1.3) with

$$\|(\tilde{\mathfrak{u}}, \tilde{\mathfrak{v}}) - (\mathfrak{u}, \mathfrak{v})\|_{\mathcal{E},\mathfrak{B}} \le c\varepsilon,$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$.

Remark 5.2. A function $(\tilde{\mathfrak{u}}, \tilde{\mathfrak{v}}) \in \mathcal{E}$ is a solution of the inequalities (5.1) if and only if there exist a functions $g_1, g_2 \in C(\mathcal{I}, \aleph)$ (which depend upon $\tilde{\mathfrak{u}}$ and $\tilde{\mathfrak{v}}$ respectively, such that

(i) $\|g_1(\xi)\| \le \varepsilon_1, \quad \|g_2(\xi)\| \le \varepsilon_2, \quad \xi \in \mathcal{I}$;

(ii) and

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\varsigma_1;\Psi}\tilde{\mathfrak{u}}(\xi) + \mathrm{r}_1\tilde{\mathfrak{u}}(\xi) = \mathbb{G}_1(\xi, \tilde{\mathfrak{u}}(\xi), \tilde{\mathfrak{v}}(\xi)) + g_1(\xi), \\ {}^{c}\mathbb{D}_{a^+}^{\varsigma_2;\Psi}\tilde{\mathfrak{v}}(\xi) + \mathrm{r}_2\tilde{\mathfrak{v}}(\xi) = \mathbb{G}_2(\xi, \tilde{\mathfrak{u}}(\xi), \tilde{\mathfrak{v}}(\xi)) + g_2(\xi). \end{cases} \quad \xi \in \mathcal{I},$$

.

Lemma 5.3. *Let* $(\tilde{\mathfrak{u}}, \tilde{\mathfrak{v}}) \in \mathcal{E}$ *be the solution of the inequalities* (5.1)*, then the following of the inequalities will be satisfied:*

$$\begin{cases} \left\| \tilde{\mathfrak{u}}(\xi) - \mathbb{S}_1(\tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})(\xi) \right\| & \le \varepsilon_1\mathbb{P}_{\varsigma_1,\Psi} \\ \left\| \tilde{\mathfrak{v}}(\xi) - \mathbb{S}_2(\tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})(\xi) \right\| & \le \varepsilon_2\mathbb{P}_{\varsigma_2,\Psi}, \end{cases}$$

where $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *are defined by* (4.3)*.*

Proof. By Remark 5.2 (ii), we have

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\varsigma_1;\Psi}\tilde{\mathfrak{u}}(\xi) + \mathrm{r}_1\tilde{\mathfrak{u}}(\xi) = \mathbb{G}_1(\xi, \tilde{\mathfrak{u}}(\xi), \tilde{\mathfrak{v}}(\xi)) + g_1(\xi), \\ {}^{c}\mathbb{D}_{a^+}^{\varsigma_2;\Psi}\tilde{\mathfrak{v}}(\xi) + \mathrm{r}_2\tilde{\mathfrak{v}}(\xi) = \mathbb{G}_2(\xi, \tilde{\mathfrak{u}}(\xi), \tilde{\mathfrak{v}}(\xi)) + g_2(\xi), \end{cases} \quad \xi \in \mathcal{I}, \tag{5.2}$$

with the following initial conditions

$$\begin{cases} \tilde{\mathfrak{u}}(a) = \mu_1, \\ \tilde{\mathfrak{v}}(a) = \mu_2. \end{cases} \tag{5.3}$$

Thanks to Lemma 3.3, the integral representation of (5.2)–(5.3) is expressed as

$$\begin{cases} \tilde{\mathfrak{u}}(\xi) = & \mu_1\mathbb{M}_{\varsigma_1}(-\mathrm{r}_1(\Psi(\xi) - \Psi(a))^{\varsigma_1}) + \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma_1-1} \\ & \times\mathbb{M}_{\varsigma_1,\varsigma_1}(-\mathrm{r}_1(\Psi(\xi) - \Psi(\ell))^{\varsigma_1})(\mathbb{G}_1(\ell, \tilde{\mathfrak{u}}(\ell), \tilde{\mathfrak{v}}(\ell)) + g_1(\ell))d\ell \\ \tilde{\mathfrak{v}}(\xi) = & \mu_2\mathbb{M}_{\varsigma_2}(-\mathrm{r}_2(\Psi(\xi) - \Psi(a))^{\varsigma_2}) + \int_a^\xi \Psi'(\ell)(\Psi(\xi) - \Psi(\ell))^{\varsigma_2-1} \\ & \times\mathbb{M}_{\varsigma_2,\varsigma_2}(-\mathrm{r}_2(\Psi(\xi) - \Psi(\ell))^{\varsigma_2})(\mathbb{G}_2(\ell, \tilde{\mathfrak{u}}(\ell), \tilde{\mathfrak{v}}(\ell)) + g_2(\ell))d\ell. \end{cases} \tag{5.4}$$

It follows from (5.4), together with Remark 5.2 (i), and Lemma 2.14 that

$$\begin{cases}\left\|\tilde{\mathfrak{u}}(\xi)-\mathbb{S}_1(\tilde{\mathfrak{u}},\tilde{\mathfrak{v}})(\xi)\right\| & \leq \int_a^\xi \frac{\Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_1-1}}{\Gamma(\varsigma_1)} g_1(\ell)\mathrm{d}\ell \leq \varepsilon_1\mathbb{P}_{\varsigma_1,\Psi} \\ \left\|\tilde{\mathfrak{v}}(\xi)-\mathbb{S}_2(\tilde{\mathfrak{u}},\tilde{\mathfrak{v}})(\xi)\right\| & \leq \int_a^\xi \frac{\Psi'(\ell)(\Psi(\xi)-\Psi(\ell))^{\varsigma_2-1}}{\Gamma(\varsigma_2)} g_2(\ell)\mathrm{d}\ell \leq \varepsilon_2\mathbb{P}_{\varsigma_2,\Psi}.\end{cases}$$

□

Theorem 5.4. *Let the assumptions* (H_1')–(H_2') *are satisfied. Then* Ψ*-Caputo FRDS* (1.2)–(1.3) *is UH stable with respect to the Bielecki's norm.*

Proof. Let $(\mathfrak{u},\mathfrak{v}) \in \mathcal{E}$ be the unique solution of Ψ-Caputo FRDS (1.2)–(1.3) and $(\tilde{\mathfrak{u}},\tilde{\mathfrak{v}})$ be any solution satisfying (5.1), then by (H_2') and Lemmas 2.17, 5.3 and we can get

$$\begin{aligned}\left\|\tilde{\mathfrak{u}}(\xi)-\mathfrak{u}(\xi)\right\| &\leq \left\|\tilde{\mathfrak{u}}(\xi)-\mathbb{S}_1(\tilde{\mathfrak{u}},\tilde{\mathfrak{v}})(\xi)\right\| + \left\|\mathbb{S}_1(\tilde{\mathfrak{u}},\tilde{\mathfrak{v}})(\xi)-\mathbb{S}_1(\mathfrak{u},\mathfrak{v})(\xi)\right\| \\ &\leq \varepsilon_1\mathbb{P}_{\varsigma_1,\Psi} + \frac{e^{\lambda(\Psi(\xi)-\Psi(a))}}{\lambda^{\varsigma_1}}\mathbb{L}_1\big(\|\tilde{\mathfrak{u}}-\mathfrak{u}\|_{\mathfrak{B}}+\|\tilde{\mathfrak{v}}-\mathfrak{v}\|_{\mathfrak{B}}\big).\end{aligned}$$

Hence we get

$$\|\tilde{\mathfrak{u}}-\mathfrak{u}\|_{\mathfrak{B}} \leq \varepsilon_1\mathbb{P}_{\varsigma_1,\Psi} + \frac{\mathbb{L}_1}{\lambda^{\varsigma_1}}\|(\tilde{\mathfrak{u}},\tilde{\mathfrak{v}})-(\mathfrak{u},\mathfrak{v})\|_{\mathcal{E},\mathfrak{B}}.$$

Similarly, we have

$$\|\tilde{\mathfrak{v}}-\mathfrak{v}\|_{\mathfrak{B}} \leq \varepsilon_2\mathbb{P}_{\varsigma_2,\Psi} + \frac{\mathbb{L}_2}{\lambda^{\varsigma_2}}\|(\tilde{\mathfrak{u}},\tilde{\mathfrak{v}})-(\mathfrak{u},\mathfrak{v})\|_{\mathcal{E},\mathfrak{B}}.$$

This leads to

$$\left[1-\left[\frac{\mathbb{L}_1}{\lambda^{\varsigma_1}}+\frac{\mathbb{L}_2}{\lambda^{\varsigma_2}}\right]\right]\|(\tilde{\mathfrak{u}},\tilde{\mathfrak{v}})-(\mathfrak{u},\mathfrak{v})\|_{\mathcal{E},\mathfrak{B}} \leq \varepsilon_1\mathbb{P}_{\varsigma_1,\Psi}+\varepsilon_2\mathbb{P}_{\varsigma_2,\Psi}, \tag{5.5}$$

Since we can choose $\lambda > 0$ such that $\frac{\mathbb{L}_1}{\lambda^{\varsigma_1}}+\frac{\mathbb{L}_2}{\lambda^{\varsigma_2}} < 1$. Therefore, (5.5) is equivalent to

$$\|(\tilde{\mathfrak{u}},\tilde{\mathfrak{v}})-(\mathfrak{u},\mathfrak{v})\|_{\mathcal{E},\mathfrak{B}} \leq \left[1-\left[\frac{\mathbb{L}_1}{\lambda^{\varsigma_1}}+\frac{\mathbb{L}_2}{\lambda^{\varsigma_2}}\right]\right]^{-1}\left(\mathbb{P}_{\varsigma_1,\Psi}+\mathbb{P}_{\varsigma_2,\Psi}\right)\varepsilon,$$

where $\varepsilon = \max\{\varepsilon_1,\varepsilon_2\}$.

Hence, the Ψ-Caputo FRDS (1.2)–(1.3) is UH stable with respect to Bielecki's norm $\|\cdot\|_{\mathfrak{B}}$. □

Remark 5.5. Importing the same logic as in Theorem 5.4. One can easily show that the Ψ-Caputo FRDS (1.2)–(1.3) is generalized HU, HU-Rassias and generalized HU-Rassias stable with respect to Bielecki's norm $\|\cdot\|_{\mathfrak{B}}$.

6. Examples

To illustrate our results, we provide two examples.

Let

$$\aleph = \ell^1 = \left\{z=(z_1,z_2,\ldots,z_j,\ldots),\ \sum_{j=1}^{\infty}|z_j|<\infty\right\},$$

be the Banach space with the norm $\|z\| = \sum_{j=1}^{\infty}|z_j|$.

Example 6.1. Consider the following Ψ-Caputo FRDS assumed in ℓ^1 :

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{0.5;\Psi}\mathfrak{u}(\xi) + 2\mathfrak{u}(\xi) = \mathbb{G}_1(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)), \quad \xi \in \mathcal{I} := [a, d]\,, \\ {}^{c}\mathbb{D}_{a^+}^{0.5;\Psi}\mathfrak{v}(\xi) + 3\mathfrak{v}(\xi) = \mathbb{G}_2(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)), \quad \xi \in \mathcal{I} := [a, d]\,, \\ \mathfrak{u}(0) = (0.5, 0.25, \dots, 0.5^n, \dots)\,, \\ \mathfrak{v}(0) = (1, 0, \dots, 0, \dots)\,. \end{cases} \tag{6.1}$$

In this case we take

$$\varsigma_1 = \varsigma_2 = 0.5, \mathrm{r}_1 = 2, \mathrm{r}_2 = 3,$$

and $\mathbb{G}_1, \mathbb{G}_2 : \mathcal{I} \times \ell^1 \times \ell^1 \longrightarrow \ell^1$ given by

$$\mathbb{G}_1(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)) = \left\{ \frac{1}{\xi + 1} \left(\frac{1}{2^k} + \frac{\mathfrak{u}_k(\xi) + \mathfrak{v}_k(\xi)}{\|\mathfrak{u}(\xi)\| + \|\mathfrak{v}(\xi)\| + 1} \right) \right\}_{k \geq 1},$$

$$\mathbb{G}_2(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)) = \left\{ \left(\frac{1}{k^3} + \sin(|\mathfrak{u}_k(\xi)| + |\mathfrak{v}_k(\xi)|) \right) e^{-\xi} \right\}_{k \geq 1}.$$

It is clear that condition (H_1') holds, and as

$$\|\mathbb{G}_i(\xi, \mathfrak{u}_1, \mathfrak{v}_1) - \mathbb{G}_i(\xi, \mathfrak{u}_2, \mathfrak{v}_2)| \leq (\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \|\mathfrak{v}_1 - \mathfrak{v}_2\|),\ i = 1, 2,$$

for all $\xi \in \mathcal{I}$ and each $\mathfrak{u}_1, \mathfrak{v}_1, \mathfrak{u}_2, \mathfrak{v}_2 \in \ell^1$. Hence condition (H_2') holds with $\mathbb{L}_1 = \mathbb{L}_2 = 1$. Moreover, if we choose, $\lambda > 4$, it follows that the mapping $\mathbb{S}$ is a contraction with respect to Bielecki's norm. Hence by Theorem 4.3 the coupled system (6.1) has a unique solution which belong to the space $C(\mathcal{I}, \ell^1) \times C(\mathcal{I}, \ell^1)$. Besides, Theorem 5.4 implies that the coupled system (6.1) is Ulam–Hyers stable with respect to the Bielecki's norm.

Let Now

$$\aleph = c_0 = \{z = (z_1, z_2, \dots, z_n, \dots) : z_n \to 0\ (n \to \infty)\}\,,$$

be the Banach space of real sequences converging to zero, endowed its usual norm

$$\|z\|_\infty = \sup_{n \geq 1} |z_n|.$$

Example 6.2. Consider the following Ψ-Caputo FRDS posed in c_0 :

$$\begin{cases} {}^{c}\mathbb{D}^{0.85}\mathfrak{u}(\xi) + 0.1\mathfrak{u}(\xi) = \mathbb{G}_1(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)), \quad \xi \in \mathcal{I} := [0, 1]\,, \\ {}^{c}\mathbb{D}^{0.75}\mathfrak{v}(\xi) + 0.1\mathfrak{v}(\xi) = \mathbb{G}_2(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)), \quad \xi \in \mathcal{I} := [0, 1]\,, \\ \mathfrak{u}(0) = (0, 0, \dots, 0, \dots)\,, \\ \mathfrak{v}(0) = (0, 0, \dots, 0, \dots)\,. \end{cases} \tag{6.2}$$

Notice that, the proposed problem is a special case of the Ψ-Caputo FRDS (1.2)–(1.3), where

$$\varsigma_1 = 0.85, \varsigma_2 = 0.75, \mathrm{r}_1 = 0.1, \mathrm{r}_2 = 0.3, a = 0, b = 1, \Psi(\xi) = \xi,$$

and $\mathbb{G}_1, \mathbb{G}_2 : \mathcal{I} \times c_0 \times c_0 \longrightarrow c_0$ given by

$$\mathbb{G}_1(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)) = \left\{ \frac{1}{e^{\xi} + 9} \left(\frac{1}{2^n} + \ln(1 + |\mathfrak{u}_n(\xi)| + |\mathfrak{v}_n(\xi)|) \right) \right\}_{n \geq 1},$$
$$\mathbb{G}_2(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)) = \left\{ \frac{\sin(\xi)}{\xi + 2} \left(\frac{1}{n^2} + \arctan(|\mathfrak{u}_n(\xi)| + |\mathfrak{v}_n(\xi)|) \right) \right\}_{n \geq 1}.$$

It is obvious that, assumption (H_1') of the Theorem 4.4 is satisfied. On the one side, for each $\xi \in \mathcal{I}$ we have

$$\begin{aligned} \|\mathbb{G}_1(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi))\|_\infty &\leq \left\| \frac{1}{e^{\xi} + 9} \left(\frac{1}{2^n} + (|\mathfrak{u}_n(\xi)| + |\mathfrak{v}_n(\xi)|) \right) \right\|_\infty \\ &\leq \frac{1}{10}(\|\mathfrak{u}(\xi)\| + \|\mathfrak{v}(\xi)\| + 1) \\ &= \mathbb{K}_1 \phi_1(\|\mathfrak{u}(\xi)\| + \|\mathfrak{v}(\xi)\|), \end{aligned}$$

and

$$\begin{aligned} \|\mathbb{G}_2(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi))\|_\infty &\leq \left\| \frac{1}{\xi + 2} \left(\frac{1}{n^2} + (|\mathfrak{u}_n(\xi)| + |\mathfrak{v}_n(\xi)|) \right) \right\|_\infty \\ &\leq \frac{1}{2}(\|\mathfrak{u}(\xi)\| + \|\mathfrak{v}(\xi)\| + 1) \\ &= \mathbb{K}_2 \phi_2(\|\mathfrak{u}(\xi)\| + \|\mathfrak{v}(\xi)\|). \end{aligned}$$

Thus, assumption (H_3') of the Theorem 4.4 is satisfied with $\mathbb{K}_1 = \frac{1}{10}, \mathbb{K}_2 = \frac{1}{2}$, and $\phi_1(w) = \phi_2(w) = 1 + w,\ w \in [0, \infty)$, On the other hand, for any bounded set $\mathcal{H} \subset c_0 \times c_0$, we have

$$\Upsilon(\mathbb{G}_i(\xi, \mathcal{H})) \leq \mathbb{K}_i \Upsilon(\mathcal{H}),\ i = 1, 2.$$

Hence (H_4') is satisfied. Consequently, Theorem 4.4 implies that Ψ-Caputo FRDS (6.2) has at least one solution $(\mathfrak{u}, \mathfrak{v}) \in C(\mathcal{I}, c_0) \times C(\mathcal{I}, c_0)$.

7. Conclusion

The existence and uniqueness theorems of solutions to two classes of Ψ-Caputo-type FDEs and FRDS in Banach spaces have been developed. For the mentioned theorems, the obtained results have been derived by different methods of nonlinear analysis like the method of upper and lower solutions along with the monotone iterative technique, Banach contraction principle, and Mönch's fixed point theorem concerted with the measures of noncompactness. Also, some convenient results about UH stability have been established by utilizing some results of nonlinear analysis. The acquired results have been justified by two pertinent examples. To the best of our knowledge, the current results are recent for FDEs and FRDS involving generalized Caputo fractional derivative. Moreover, these results proven in Banach spaces. Apart from this, the FDEs and FRDS for different values of Ψ includes the study of FDEs and FRDS involving the fractional derivative operators: standard Caputo, Caputo-Hadamard, Caputo-Katugampola, and many other operators.

Finally, we would like to point out that the use of fractional operators with different kernels, reflected by the use of the increasing function Ψ used in the power law, is important in modeling certain physical and engineering problems in which we have memory. This confirms the need of the non-locality nature when we deal with such models. Moreover, the dependency of the kernel on the function Ψ provides us with more possibilities or choices in fitting the real data of some models.

In the future, the above results and analysis can be extended to more sophisticated and applicable problems of FDEs and FRDS involving Ψ-Hilfer operator. It will be also of interest to discuss the implementation of certain conditions in such case studies using the monotone iterative technique.

Acknowledgments

We would like to thank the referees very much for their valuable comments and suggestions. Moreover, all authors have read and approved the final revised version. The author Thabet Abdeljawad would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Conflict of interest

The authors declare no conflict of interest.

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