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Research article

Stability results of the functional equation deriving from quadratic function in random normed spaces

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Abstract: The authors introduce a finite variable quadratic functional equation and derive its solution. Also, authors examine its Hyers-Ulam stability in Random Normed space(RN-space) by means of two different methods.

Keywords: quadratic functional equation; Hyers-Ulam stability; fixed point theory; random normed spaces
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1. Introduction and preliminaries

A standard problem in the theory of functional equations is the subsequent subject:
When is it true that a function which approximately satisfies a functional equation must be close to an approximate solution of the functional equation?

We conclude that the functional equation is stable, if the problem satisfies the solution of the functional equation. In 1940, the stability problems of functional equations about group homomorphisms was introduced by Ulam [26]. In 1941, Hyers [10] gave is affirmative answer to Ulam's quaestion for additive groups (under the assumption that groups are Banach spaces). Rassias in [18] proved the generalized Hyers theorem for additive mappings.

The Rassias stability results provided manipulate during the last three decades in the growth of a generalization of the Hyers-Ulam stability conception. This novel method is called as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Also, in 1994, Rassias's generalization theorem was delivered by Gavruta [9] by replacing a general function $\epsilon\left(\|x\|^p + \|y\|^p\right)$ by $\varphi(x, y)$.

In 1982, J.M. Rassias [17] followed the modern approach of the Th.M.Rassias theorem in which he replaced the factor product of norms instead of sum of norms. Ravi et al., [21] investigated Gavruta's theorem for the unbounded Cauchy difference in the spirit of Rassias approach.

We mention here some quadratic functional equations papers concerning the stability (Ref. [7, 11, 16, 19, 20]). Recently, the stability of many functional equations in various spaces such as Banach spaces, fuzzy normed spaces and Random normed spaces have been broadly inspected by a numeral Mathematicians (Ref. [1–3, 6, 14, 15, 24]). In the consequence, we take on the usual terminology, notions and conventions of the theory of random normed spaces as in [4, 5, 12, 13, 22, 23].

All through this work, Δ^+ is the distribution functions spaces, i.e., the space of all mappings $V : R \cup \{-\infty, \infty\} \to [0, 1]$, such that F is left continuous and increasing on R, $V(0) = 0$ and $V(+\infty) = 1$. $D^+ \subset \Delta^+$ consisting of all functions $V \in \Delta^+$ for which $l^- V(+\infty) = 1$, where $l^-\phi(s)$ denotes $l^-\phi(s) = \lim_{t\to s^-} \phi(t)$. The space Δ^+ is partially ordered by the usual point wise ordering of functions, i.e., $V \leq W \iff V(t) \leq W(t)\ \forall t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ϵ_0 given by

$$\epsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. A Random Normed space (briefly, RN-space) is a triple (E, Ψ, Υ), where E is a vector space, Υ is a continuous $t-$norm and $\Psi : E \to D^+$ satisfying the following conditions:

1. $\Psi_s(t) = \varsigma_0(t)$ for all $t > 0$ if and only if $s = 0$;
2. $\Psi_{\alpha s}(t) = \Psi_s\left(\frac{t}{|\alpha|}\right)$ for all $s \in E,\ t \geq 0$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
3. $\Psi_{s_1+s_2}(t + u) \geq \Upsilon\left(\Psi_{s_1}(t), \Psi_{s_2}(u)\right)$ for all $s_1, s_2 \in E$ and $t, u \geq 0$.

Definition 1.2. Let (E, Ψ, Υ) be a RN-space.

(RN1) A sequence $\{s_m\}$ in E is said to be convergent to a point $s \in E$ if $\lim_{m\to\infty} \Psi_{s_m - s}(t) = 1,\ t > 0$.
(RN2) A sequence $\{s_m\}$ in E is called a Cauchy sequence if $\lim_{m\to\infty} \Psi_{s_m - s_l}(t) = 1,\ t > 0$.
(RN3) A RN-space (E, Ψ, Υ) is said to be complete if every Cauchy sequence in E is convergent.

Theorem 1.3. *[17] If (E, Ψ, Υ) is a RN-space and $\{s_m\}$ is a sequence in E such that $s_m \to s$, then $\lim_{m\to\infty} \Psi_{s_m}(t) = \Psi_s(t)$ almost everywhere.*

The authors introduce the finite variable quadratic functional equation is of the form

$$\sum_{a=1}^{m} \phi\left(-s_a + \sum_{b=1;b\neq a}^{m} s_b\right) = (m-4) \sum_{1\leq a<b\leq m} \phi\left(s_a + s_b\right) + \left(-m^2 + 6m - 4\right) \sum_{a=1}^{m} \phi(s_a) \tag{1.1}$$

where $m \geq 5$, and derive its solution. Also, we investigate its Hyers-Ulam stability in Random Normed space(RN-space) by using direct and fixed point methods.

2. General solution

Theorem 2.1. *If a mapping $\phi : E \to F$ satisfies the functional equation (1.1), then the mapping $\phi : E \to F$ is quadratic.*

Proof. Assume that the mapping $\phi : E \to F$ satisfies the functional equation (1.1). Replacing $(s_1, s_2, \cdots, s_m)$ by $(0, 0, \cdots, 0)$ in (1.1), we obtain $\phi(0) = 0$. Replacing $(s_1, s_2, \cdots, s_m)$ by $(s, 0, \cdots, 0)$ in (1.1), we have $\phi(-s) = \phi(s)$ for all $s \in E$. Therefore, the function ϕ is even. Next, replacing $(s_1, s_2, \cdots, s_m)$ by $(s, s, 0, \cdots, 0)$ in (1.1), we obtain

$$\phi(2s) = 2^2\phi(s) \tag{2.1}$$

for all $s \in E$. Replacing s by $2s$ in (2.1), we have

$$\phi(2^2 s) = 2^4\phi(s) \tag{2.2}$$

for all $s \in E$. Replacing s by $2s$ in (2.2), we get

$$\phi(2^3 s) = 2^6\phi(s) \tag{2.3}$$

for all $s \in E$. In general, for any positive integer n, we conclude that

$$\phi(2^n s) = 2^{2n}\phi(s) \tag{2.4}$$

for all $s \in E$. Now, replacing $(s_1, s_2, \cdots, s_m)$ by $(u, v, 0, \cdots, 0)$ in (1.1), we reach our desired results of ϕ. □

Remark 2.2. Let F be a linear space and a mapping $\phi : E \to F$ satisfies the functional equation (1.1), then the following results are true:

(1) $\phi(r^t s) = r^t\phi(s)$ for all $s \in E, r \in \mathbb{Q}$, t integers.
(2) $\phi(s) = s\phi(1)$ for all $s \in E$ if ϕ is continuous.

For our notational handiness, we define a mapping $\phi : E \to F$ by

$$D\phi(s_1, s_2, \cdots, s_m) = \sum_{a=1}^{m} \phi\left(-s_a + \sum_{b=1;b\neq a}^{m} s_b\right) - (m-4) \sum_{1\leq a<b\leq m} \phi\left(s_a + s_b\right) - \left(-m^2 + 6m - 4\right)\sum_{a=1}^{m} \phi(s_a)$$

for all $s_1, s_2, \cdots, s_m \in E$.

Throughout the paper, we consider E be a linear space and (E, Ψ, Υ) is a complete RN-space.

3. Result and discussion: Direct method

Theorem 3.1. *Let a mapping $\phi : E \to F$ for which there exists a mapping $\Phi : E^m \to D^+$ with*

$$\lim_{t\to\infty} T_{a=0}^{\infty}\left(\Phi_{2^{t+a}s_1,2^{t+a}s_2,\cdots,2^{t+a}s_m}\left(2^{2(t+a+1)}\epsilon\right)\right) = \lim_{t\to\infty}\Phi_{2^t s_1,2^t s_2,\cdots,2^t s_m}\left(2^{2t}\epsilon\right) = 1 \tag{3.1}$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$ such that the functional inequality with $\phi(0) = 0$ such that

$$\Psi_{D\phi(s_1,s_2,\cdots,s_m)}(\epsilon) \geq \Phi_{s_1,s_2,\cdots,s_m}(\epsilon) \tag{3.2}$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : E \to F$ satisfying the functional equation (1.1) with

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon) \geq T_{a=0}^{\infty}\left(\Phi_{2^{a+1}s,2^{a+1}s,0,\cdots,0}\left(2^{2(a+1)}\epsilon\right)\right), \tag{3.3}$$

for all $s \in E$ and all $\epsilon > 0$. The mapping $Q_2 : E \to F$ is defined by

$$\Psi_{Q_2(s)}(\epsilon) = \lim_{t\to\infty}\Psi_{\frac{\phi(2^t s)}{2^{2t}}}(\epsilon) \tag{3.4}$$

for all $s \in E$ and all $\epsilon > 0$.

Proof. Replacing $(s_1, s_2, \cdots, s_m)$ by $(s, s, 0, \cdots, 0)$ in (3.1), we obtain

$$\Psi_{2\phi(2s)-8\phi(s)}(\epsilon) \geq \Phi_{s,s,0,\cdots,0}(\epsilon) \tag{3.5}$$

for all $s \in E$ and all $\epsilon > 0$. It follows from (3.5) and (RN2), we get

$$\Psi_{\frac{\phi(2s)}{2^2}-\phi(s)}(\epsilon) \geq \Phi_{s,s,0,\cdots,0}\left(8\epsilon\right) \tag{3.6}$$

for all $s \in E$ and $\epsilon > 0$. Replacing s by $2^t s$ in (3.6), we have

$$\begin{aligned}\Psi_{\frac{\phi(2^{t+1}s)}{2^{2(t+1)}}-\frac{\phi(2^t s)}{2^{2t}}}(\epsilon) &\geq \Phi_{2^t s,2^t s,0,\cdots,0}\left(2^{2(t+1)}2\epsilon\right)\\ &\geq \Phi_{s,s,0,\cdots,0}\left(\frac{2^{2(t+1)}2\epsilon}{\alpha^t}\right)\end{aligned} \tag{3.7}$$

for all $s \in E$ and $\epsilon > 0$. Since

$$\frac{\phi(2^m s)}{2^{2m}} - \phi(s) = \sum_{t=0}^{m-1}\frac{\phi(2^{t+1}s)}{2^{2(t+1)}} - \frac{\phi(2^t s)}{2^{2t}} \tag{3.8}$$

for all $s \in E$ and $\epsilon > 0$. From (3.7) and (3.8), we get

$$\begin{aligned}\Psi_{\frac{\phi(2^m s)}{2^{2m}}-\phi(s)}\left(\sum_{t=0}^{m-1}\frac{\alpha^t\epsilon}{2^{2(t+1)}2}\right) &\geq \Phi_{s,s,0,\cdots,0}(\epsilon)\\ \Psi_{\frac{\phi(2^m s)}{2^{2m}}-\phi(s)}(\epsilon) &\geq \Phi_{s,s,0,\cdots,0}\left(\frac{\epsilon}{\sum_{t=0}^{m-1}\frac{\alpha^t}{2^{2(t+1)}2}}\right)\end{aligned} \tag{3.9}$$

for all $s \in E$ and $\epsilon > 0$. Replacing s by $2^l s$ in (3.9), we obtain

$$\Psi_{\frac{\phi(2^{m+l}s)}{2^{2(m+l)}} - \frac{\phi(2^m s)}{2^{2m}}}(\epsilon) \geq \Phi_{s,s,0,\cdots,0}\left(\frac{\epsilon}{\sum_{t=l}^{m+l-1} \frac{\alpha^t}{2^{2(t+1)}2}}\right) \tag{3.10}$$

for all $s \in E$ and $\epsilon > 0$. As $\Phi_{s,s,0,\cdots,0}\left(\frac{\epsilon}{\sum_{t=l}^{m+l-1} \frac{\alpha^t}{2^{2(t+1)}2}}\right) \to 1$ as $l, m \to \infty$, then $\left\{\frac{\phi(2^m s)}{2^{2m}}\right\}$ is a Cauchy sequence in (F, Ψ, Υ). Since (F, Ψ, Υ) is a complete RN-space, this sequence converges to some point $Q_2(s) \in F$. Fix $s \in E$ and put $l = 0$ in (3.10), we obtain

$$\Psi_{\frac{\phi(2^m s)}{2^{2m}} - \phi(s)}(\epsilon) \geq \Phi_{s,s,0,\cdots,0}\left(\frac{\epsilon}{\sum_{t=0}^{m-1} \frac{\alpha^t}{2^{2(t+1)}2}}\right) \tag{3.11}$$

and so, for every $\zeta > 0$, we get

$$\begin{aligned} \Psi_{Q_2(s)-\phi(s)}(\epsilon + \zeta) &\geq \Upsilon\left(\Psi_{Q_2(s)-\frac{\phi(2^m s)}{2^{2m}}}(\zeta), \Psi_{\frac{\phi(2^m s)}{2^{2m}} - \phi(s)}(\epsilon)\right) \\ &\geq \Upsilon\left(\Psi_{Q_2(s)-\frac{\phi(2^m s)}{2^{2m}}}(\zeta), \Phi_{s,s,0,\cdots,0}\left(\frac{\epsilon}{\sum_{t=0}^{m-1} \frac{\alpha^t}{2^{2(t+1)}2}}\right)\right) \end{aligned} \tag{3.12}$$

for all $s \in E$ and all $\epsilon, \zeta > 0$. Taking the limit $m \to \infty$ and using inequality (3.12), we have

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon + \zeta) \geq \Phi_{s,s,0,\cdots,0}\left(2(2^2 - \alpha)\epsilon\right) \tag{3.13}$$

for all $s \in E$ and all $\epsilon, \zeta > 0$. Since ζ was arbitrary, by taking $\zeta \to 0$ in (3.14), we get

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon) \geq \Phi_{s,s,0,\cdots,0}\left(2(2^2 - \alpha)\epsilon\right) \tag{3.14}$$

for all $s \in E$ and all $\epsilon > 0$. Replacing $(s_1, s_2, \cdots, s_m)$ by $(2^m s_1, 2^m s_2, \cdots, 2^m s_m)$ in (3.2), we obtain

$$\Psi_{D\phi(2^m s_1, 2^m s_2, \cdots, 2^m s_m)}(\epsilon) \geq \Phi_{2^m s_1, 2^m s_2, \cdots, 2^m s_m}\left(2^{2m}\epsilon\right) \tag{3.15}$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$. Since

$$\lim_{t\to\infty} T_{a=0}^{\infty}\left(\Phi_{2^{t+a}s_1, 2^{t+a}s_2, \cdots, 2^{t+a}s_m}\left(2^{2(t+a+1)}\epsilon\right)\right) = 1$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$. We conclude that Q_2 satisfies the functional equation (1.1). To prove the uniqueness of the quadratic mapping Q_2 . Assume that there exists a quadratic mapping $R_2 : E \to F$, which satisfies the inequality(3.14). Fix $s \in E$. Clearly, $Q_2(2^m s) = 2^{2m} Q_2(s)$ and $R_2(2^m s) = 2^{2m} R_2(s)$ for all $s \in E$. It follows from (3.14) that

$$\begin{aligned} \Psi_{Q_2(s)-R_2(s)}(\epsilon) &= \lim_{m\to\infty} \Psi_{\frac{Q_2(2^m s)}{2^{2m}} - \frac{R_2(2^m s)}{2^{2m}}}(\epsilon) \\ \Psi_{\frac{Q_2(2^m s)}{2^{2m}} - \frac{R_2(2^m s)}{2^{2m}}}(\epsilon) &\geq \min\left\{\Psi_{\frac{Q_2(2^m s)}{2^{2m}} - \frac{\phi(2^m s)}{2^{2m}}}\left(\frac{\epsilon}{2}\right), \Psi_{\frac{\phi(2^m s)}{2^{2m}} - \frac{R_2(2^m s)}{2^{2m}}}\left(\frac{\epsilon}{2}\right)\right\} \\ &\geq \Phi_{2^m s, 2^m s, 0, \cdots, 0}\left(2^{2m} 2(2^2 - \alpha)\epsilon\right) \\ &\geq \Phi_{s,s,0,\cdots,0}\left(\frac{2^{2m} 2(2^2 - \alpha)\epsilon}{\alpha^m}\right) \end{aligned} \tag{3.16}$$

for all $s \in E$ and all $\epsilon > 0$. Since, $\lim_{m\to\infty}\left(\frac{2^{2m}2(2^2-\alpha)\epsilon}{\alpha^m}\right) = \infty$, we get $\lim_{m\to\infty} \Phi_{s,s,0,\cdots,0}\left(\frac{2^{2m}2(2^2-\alpha)\epsilon}{\alpha^m}\right) = 1$. Therefore, it follows that $\Psi_{Q_2(s)-R_2(s)}(\epsilon) = 1$ for all $s \in E$ and all $\epsilon > 0$. And so $Q_2(s) = R_2(s)$. This completes the proof. □

Theorem 3.2. *Let a ampping $\phi : E \to F$ for which there exists a mapping $\Phi : E^m \to D^+$ with the condition*

$$\lim_{t\to\infty} T_{a=0}^{\infty}\left(\Phi_{\frac{s_1}{2^{t+a}},\frac{s_2}{2^{t+a}},\cdots,\frac{s_m}{2^{t+a}}}\left(\frac{\epsilon}{2^{2(t+a+1)}}\right)\right) = \lim_{t\to\infty}\Phi_{\frac{s_1}{2^t},\frac{s_2}{2^t},\cdots,\frac{s_m}{2^t}}\left(\frac{\epsilon}{2^{2t}}\right) = 1 \tag{3.17}$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$ such that the functional inequality with $\phi(0) = 0$ such that

$$\Psi_{D\phi(s_1,s_2,\cdots,s_m)}(\epsilon) \geq \Phi_{s_1,s_2,\cdots,s_m}(\epsilon) \tag{3.18}$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : E \to F$ satisfying the functional equation (1.1) and

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon) \geq T_{a=0}^{\infty}\left(\Phi_{\frac{s}{2^{a+1}},\frac{s}{2^{a+1}},0,\cdots,0}\left(\frac{\epsilon}{2^{2(a+1)}}\right)\right),\ \forall s \in E, \epsilon > 0. \tag{3.19}$$

The mapping $Q_2(s)$ is defined by

$$\Psi_{Q_2(s)}(\epsilon) = \lim_{t\to\infty}\Psi_{2^{2t}\phi\left(\frac{s}{2^t}\right)}(\epsilon) \tag{3.20}$$

for all $s \in E$ and all $\epsilon > 0$.

Corollary 3.3. *Let ς be positive real numbers. If $\phi : E \to F$ be a quadratic function which satisfies*

$$\Psi_{D\phi(s_1,s_2,\cdots,s_m)} \geq \Phi_{\varsigma}(\epsilon)$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : E \to F$ such that

$$\Psi_{Q_2(s)-\phi(s)}(\varsigma) \geq \Phi_{\frac{\epsilon}{2|2^2-1|}}(\epsilon)$$

for all $s \in E$ and $\epsilon > 0$.

Proof. If $s_1, s_2, \cdots, s_m = \varsigma$, then the proof is true from Theorem 3.1 and 3.2 by taking $\alpha = 2^0$. □

Corollary 3.4. *Let ς and θ be nonnegative real numbers with $\theta \in (0, 2) \cup (2, +\infty)$. If a quadratic mapping $\phi : E \to F$ satisfies*

$$\Psi_{D\phi(s_1,s_2,\cdots,s_m)} \geq \Phi_{\varsigma\sum_{i=1}^m \|s_i\|^\theta}(\epsilon)$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : E \to F$ such that

$$\Psi_{Q_2(s)-\phi(s)}(\varsigma) \geq \Phi_{\frac{\epsilon\|s\|^\theta}{|2^2-2^\theta|}}(\epsilon)$$

for all $s \in E$ and $\epsilon > 0$.

Proof. If $s_1, s_2, \cdots, s_m = \varsigma\sum_{i=1}^m \|s_i\|^\theta$, then the proof is true from Theorem 3.1 and 3.2 by taking $\alpha = 2^\theta$. □

Corollary 3.5. *Let ς and θ be nonnegative real numbers with $m\theta \in (0, 2) \cup (2, +\infty)$. If a quadratic mapping $\phi : E \to F$ satisfies*

$$\Psi_{D\phi(s_1,s_2,\cdots,s_m)} \geq \Phi_{\varsigma\left(\sum_{i=1}^m \|s_i\|^{m\theta}+\prod_{i=1}^m \|s_i\|^\theta\right)}(\epsilon)$$

for all $s_1, s_2, \cdots, s_m \in E$ *and all* $\epsilon > 0$. *Then there exists a unique quadratic mapping* $Q_2 : E \to F$ *such that*

$$\Psi_{Q_2(s)-\phi(s)}(\varsigma) \geq \Phi_{\frac{\epsilon \|s\|^{m\theta}}{|2^2 - 2^{m\theta}|}}(\epsilon)$$

for all $s \in E$ *and* $\epsilon > 0$.

Proof. If $s_1, s_2, \cdots, s_m = \varsigma \left(\sum_{i=1}^{m} \|s_i\|^{m\theta} + \prod_{i=1}^{m} \|s_i\|^{\theta} \right)$, then the proof is true from Theorem 3.1 and 3.2 by taking $\alpha = 2^{m\theta}$. □

4. Result and discussion: Fixed point method

Theorem 4.1. *If a mapping* $\phi : E \to F$ *for which there exists a function* $\Phi : E^m \to D^+$ *with*

$$\lim_{t \to \infty} \Phi_{\zeta_a^t s_1, \zeta_a^t s_2, \cdots, \zeta_a^t s_m} \left(\zeta_a^{2t} \epsilon \right) = 1 \tag{4.1}$$

for all $s_1, s_2, \cdots, s_m \in E$ *and all* $\epsilon > 0$ *and where* $\zeta_a = \begin{cases} 2 \ \textit{if}\ a = 0; \\ \frac{1}{2} \ \textit{if}\ a = 1; \end{cases}$ *satisfying the inequality*

$$\Psi_{D\phi(s_1, s_2, \cdots, s_m)}(\epsilon) \geq \Phi_{s_1, s_2, \cdots, s_m}(\epsilon) \tag{4.2}$$

for all $s_1, s_2, \cdots, s_m \in E$ *and all* $\epsilon > 0$. *If there exists* $L = L(a)$ *such that the function* $s \to \tau(s, \epsilon) = \Phi_{\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0}(2\epsilon)$ *has the property, that*

$$\tau(s, \epsilon) \leq L \frac{1}{\zeta_a^2} \tau(\zeta_a s, \epsilon), \quad \forall s \in E, \epsilon > 0. \tag{4.3}$$

Then there exists a unqiue quadratic mapping $Q_2 : E \to F$ *satisfies the functional equation (1.1) and satisfies*

$$\Psi_{Q_2(s)-\phi(s)} \left(\frac{L^{1-a}}{1-L} \epsilon \right) \geq \tau(s, \epsilon) \tag{4.4}$$

for all $s \in E$ *and all* $\epsilon > 0$.

Proof. Consider a general metric ρ on Δ such that $\rho(n_1, n_2) = \inf \left\{ \nu \in (0, \infty) / \Psi_{n_1(s)-n_2(s)}(\nu\epsilon) \geq \tau(s, \epsilon), s \in E, \epsilon > 0 \right\}$. It is easy to view that (Δ, ρ) is complete. Let us define a mapping $\Upsilon : \Delta \to \Delta$ by $\Upsilon n_1(s) = \frac{1}{\zeta_a^2} n_1(\zeta_a s)$, for all $s \in E$. Now for $n_1, n_2 \in \Delta$, we have $\rho(n_1, n_2) \leq \nu$.

$$\begin{aligned} \Rightarrow \Psi_{(n_1(s)-n_2(s))}(\nu\epsilon) &\geq \tau(s, \epsilon) \\ \Rightarrow \Psi_{\Upsilon n_1(s) - \Upsilon n_2(s)} \left(\frac{\nu\epsilon}{\zeta_a^2} \right) &\geq \tau(s, \epsilon) \\ \Rightarrow \rho(\Upsilon n_1(s) - \Upsilon n_2(s)) &\leq \nu L \\ \Rightarrow \rho(\Upsilon n_1, \Upsilon n_2) &\leq L \rho(n_1, n_2) \end{aligned} \tag{4.5}$$

for all $n_1, n_2 \in \Delta$. Therefore, υ is strictly contractive mapping on Δ with Lipschitz constant L. If follows from (3.5) that

$$\Psi_{2\phi(2s)-8\phi(s)}(\epsilon) \geq \Phi_{s,s,0,\cdots,0}(\epsilon) \tag{4.6}$$

for all $s \in E$ and all $\epsilon > 0$. It follows from (4.6) that

$$\Psi_{\frac{\phi(2s)}{2^4}-\phi(s)}(\epsilon) \geq \Phi_{s,s,0,\cdots,0}\,(8\epsilon) \tag{4.7}$$

for all $s \in E$ and all $\epsilon > 0$. Using (4.3) for $a = 0$, it reduces to

$$\Psi_{\frac{\phi(2s)}{2^4}-\phi(s)}(\epsilon) \geq L\tau(s, \epsilon)$$

for all $s \in E$ and all $\epsilon > 0$. Hence, we obtain

$$\rho\left(\Psi_{\Upsilon\phi(s)-\phi(s)}\right) \geq L = L^{1-a} < \infty \tag{4.8}$$

for all $s \in E$. Replacing s by $\frac{s}{2}$ in (4.7), we have

$$\Psi_{\frac{\phi(s)}{2^4}-\phi\left(\frac{s}{2}\right)}(\epsilon) \geq \Phi_{\frac{s}{2},\frac{s}{2},0,\cdots,0}\,(8\epsilon) \tag{4.9}$$

for all $s \in E$ and all $\epsilon > 0$. Using (4.3) for $a = 1$, it reduces to

$$\Psi_{\frac{\phi(s)}{2^4}-\phi\left(\frac{s}{2}\right)}(\epsilon) \geq \tau(s, \epsilon)$$

for all $s \in E$ and all $\epsilon > 0$. Hence, we arrive

$$\rho\left(\Psi_{\Upsilon\phi(s)-\phi(s)}\right) \geq L = L^{1-a} < \infty \tag{4.10}$$

for all $s \in E$. From (4.8) and (4.10), we can conclude

$$\rho\left(\Psi_{\Upsilon\phi(s)-\phi(s)}\right) \geq \infty \tag{4.11}$$

for all $s \in E$. In order to prove $Q_2 : E \to F$ satisfies the functional equation (1.1), the remaining proof is similar as in Theorem 3.1. As the function Q_2 is unique fixed point of Υ in $\Omega = \{\phi \in \Delta/\rho(\phi, Q_2) < \infty\}$. Finally, Q_2 is an unique function such that

$$\Psi_{Q_2(s)-\phi(s)}\left(\frac{L^{1-a}}{1-L}\epsilon\right) \geq \tau(s, \epsilon)$$

for all $s \in E$ and all $\epsilon > 0$. This completes the proof.

□

Corollary 4.2. *Let ς and θ be positive real numbers. If a quadratic mapping $\phi : E \to F$ satisfies*

$$\Psi_{D\phi(s_1,s_2,\cdots,s_m)} \geq \begin{cases} \Phi_{\varsigma}(\epsilon) \\ \Phi_{\varsigma\sum_{i=1}^{m}\|s_i\|^{\theta}}(\epsilon) \\ \Phi_{\varsigma\left(\sum_{i=1}^{m}\|s_i\|^{m\theta}+\prod_{i=1}^{m}\|s_i\|^{\theta}\right)}(\epsilon) \end{cases}$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : E \to F$ such that

$$\Psi_{Q_2(s)-\phi(s)}(\varsigma) \geq \begin{cases} \Phi_{\frac{\epsilon}{2|2^2-1|}}(\epsilon) \\ \Phi_{\frac{\epsilon\|s\|^{\theta}}{|2^2-2^{\theta}|}}(\epsilon);\ 0 < \theta < 2 \text{ or } \theta > 2, \\ \Phi_{\frac{\epsilon\|s\|^{m\theta}}{|2^2-2^{m\theta}|}}(\epsilon);\ 0 < \theta < \frac{2}{m} \text{ or } \theta > \frac{2}{m}, \end{cases}$$

for all $s \in E$ and $\epsilon > 0$.

Proof. Suppose

$$\Psi_{D\phi(s_1,s_2,\cdots,s_m)} \geq \begin{cases} \Phi_{\varsigma}(\epsilon) \\ \Phi_{\varsigma \sum_{i=1}^{m} \|s_i\|^{\theta}}(\epsilon) \\ \Phi_{\varsigma\left(\sum_{i=1}^{m} \|s_i\|^{m\theta} + \prod_{i=1}^{m} \|s_i\|^{\theta}\right)}(\epsilon) \end{cases}$$

for all $s_1, s_2, \cdots, s_m \in E$ and all $\epsilon > 0$. Then

$$\begin{aligned} \Phi_{\zeta_a^t s_1, \zeta_a^t s_2, \cdots, \zeta_a^t s_m}\left(\zeta_a^{2t}\epsilon\right) &= \begin{cases} \Phi_{\varsigma\zeta_a^{2t}}(\epsilon) \\ \Phi_{\varsigma \sum_{i=1}^{m} \|s_i\|^{\theta}\zeta_a^{(2-\theta)t}}(\epsilon) \\ \Phi_{\varsigma\left(\sum_{i=1}^{m} \|s_i\|^{m\theta}\zeta_a^{(2-\theta)t} + \prod_{i=1}^{m} \|s_i\|^{\theta}\zeta_a^{(2-m\theta)t}\right)}(\epsilon) \end{cases} \\ &= \begin{cases} \to 1 \text{ as } t \to \infty, \\ \to 1 \text{ as } t \to \infty, \\ \to 1 \text{ as } t \to \infty. \end{cases} \end{aligned}$$

But, we have $\tau(s, \epsilon) = \Phi_{\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0}(2\epsilon)$ has the property $L\frac{1}{\zeta_a^2}\tau(\zeta_a s, \epsilon)$ for all $s \in E$ and all $\epsilon > 0$. Now,

$$\begin{aligned} \tau(s, \epsilon) &= \begin{cases} \Phi_{\frac{\varsigma}{2}}(\epsilon) \\ \Phi_{\frac{2\varsigma\|s\|^{\theta}}{2^{\theta}2}}(\epsilon) \\ \Phi_{\frac{2\varsigma\|s\|^{m\theta}}{2^{m\theta}2}}(\epsilon) \end{cases} \\ L\frac{1}{\zeta_a^2}\tau(\zeta_a s, \epsilon) &= \begin{cases} \Phi_{\zeta_a^{-2}\tau(s)}(\epsilon) \\ \Phi_{\zeta_a^{\theta-2}\tau(s)}(\epsilon) \\ \Phi_{\zeta_a^{m\theta-2}\tau(s)}(\epsilon) \end{cases} \end{aligned}$$

By using Theorem 4.1, we prove the following cases:
Case-1: $L = 2^{-2}$ if $a = 0$

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon) \geq L\frac{1}{\zeta_a^2}\tau(\zeta_a s, \epsilon) \geq \Phi_{\frac{\varsigma}{2(2^2-1)}}(\epsilon)$$

Case-2: $L = 2^2$ if $a = 1$

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon) \geq L\frac{1}{\zeta_a^2}\tau(\zeta_a s, \epsilon) \geq \Phi_{\frac{\varsigma}{2(1-2^2)}}(\epsilon)$$

Case-3: $L = 2^{\theta-2}$ for $\theta < 2$ if $a = 0$

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon) \geq L\frac{1}{\zeta_a^2}\tau(\zeta_a s, \epsilon) \geq \Phi_{\frac{\varsigma\|s\|^{\theta}}{(2^2-2^{\theta})}}(\epsilon)$$

Case-4: $L = 2^{2-\theta}$ for $\theta > 2$ if $a = 1$

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon) \geq L\frac{1}{\zeta_a^2}\tau(\zeta_a s, \epsilon) \geq \Phi_{\frac{\varsigma\|s\|^{\theta}}{(2^{\theta}-2^2)}}(\epsilon)$$

Case-5: $L = 2^{m\theta-2}$ for $\theta < \frac{2}{m}$ if $a = 0$

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon) \geq L\frac{1}{\zeta_a^2}\tau(\zeta_a s, \epsilon) \geq \Phi_{\frac{\varsigma\|s\|^{m\theta}}{(2^2-2^{m\theta})}}(\epsilon)$$

Case-6: $L = 2^{2-m\theta}$ for $\theta > \frac{2}{m}$ if $a = 1$

$$\Psi_{Q_2(s)-\phi(s)}(\epsilon) \geq L\frac{1}{\zeta_a^2}\tau(\zeta_a s, \epsilon) \geq \Phi_{\frac{\varsigma\|s\|^{m\theta}}{(2^{m\theta}-2^2)}}(\epsilon)$$

Hence the proof is complete.

□

Counter example

Next, we show the following counter example replaced by the well-known counter example of Gajda [8] to the functional equation(1.1):

Example 4.3. Let a mapping $\phi : E \to F$ defined by

$$\phi(s) = \sum_{l=0}^{+\infty} \frac{\xi(2^m s)}{2^{2m}}$$

where

$$\xi(s) = \begin{cases} \psi s^2, & -1 < s < 1 \\ \psi, & \text{otherwise}, \end{cases} \tag{4.12}$$

where ψ is a constant, then the mapping $\phi : E \to F$ satisfies the inequality

$$|D\phi(s_1, s_2, \cdots, s_m)| \leq \left(-m^2 + 7m - 7\right)\frac{64}{3}\psi\left(\sum_{j=1}^{m} |s_j|^2\right), \tag{4.13}$$

for all $s_1, s_2, \cdots, s_l \in E$, but there does not exist a quadratic mapping $Q_2 : E \to F$ with a constant ε such that

$$|\phi(s) - Q_2(s)| \leq \varepsilon|s|^2 \tag{4.14}$$

for all $s \in E$.

Proof. It is easy to notice that ϕ is bounded by $\frac{4}{3}\psi$ on E. If $\sum_{j=1}^{l} |s_j|^2 \geq \frac{1}{2^2}$ or 0, then the left side of (4.13) is less than $\left(-m^2 + 7m - 7\right)\frac{4}{3}\psi$, and thus (4.13) is true. Assume that $0 < \sum_{j=1}^{m} |s_j|^2 < \frac{1}{2^2}$. Then there exists an integer l such that

$$\frac{1}{2^{2(l+2)}} \leq \sum_{j=1}^{m} |s_j|^2 < \frac{1}{2^{2(l+1)}}. \tag{4.15}$$

So that $2^{2l}|s_1| < \frac{1}{2^2}, 2^{2l}|s_2| < \frac{1}{2^2}, \cdots, 2^{2l}|s_m| < \frac{1}{2^2}$ and $2^m s_1, 2^m s_2, \cdots, 2^m s_m \in (-1,1)$ for all $m = 0, 1, 2, \cdots, l-1$. So, for $m = 0, 1, \cdots, l-1$

$$\sum_{a=1}^{m} \xi\left(2^m\left(-s_a + \sum_{b=1;b\neq a}^{m} s_b\right)\right) - (m-4)\sum_{1\leq a<b\leq m} \phi\left(2^m\left(s_a + s_b\right)\right) - \left(-m^2 + 6m - 4\right)\sum_{a=1}^{m} \phi(2^m s_a) = 0.$$

By the definition of ϕ, we obtain

$$\begin{aligned} |D\phi(s_1, s_2, \cdots, s_m)| &\leq \sum_{j=l}^{+\infty} \frac{1}{2^{2j}} |\xi(2^j s_1, 2^j s_2, \cdots, 2^j s_m)| \\ &\leq \sum_{j=l}^{+\infty} \frac{1}{2^{2j}} \left(-m^2 + 7m - 7\right)\psi \\ &\leq \left(-m^2 + 7m - 7\right)\frac{2^{2(1-l)}}{3}\psi. \end{aligned}$$

It follows from (4.15) that

$$|D\phi(s_1, s_2, \cdots, s_l)| \leq \left(-m^2 + 7m - 7\right)\frac{64}{3}\psi\left(\sum_{j=1}^{m} |s_j|^2\right), \tag{4.16}$$

for all $s_1, s_2, \cdots, s_m \in E$. Thus the function ϕ satisfies the inequality (4.13) for all $s_1, s_2, \cdots, s_l \in E$. We propose that there exists an quadratic mapping $Q_2 : E \to F$ with a constant $\varepsilon > 0$ satisfying the inequality (4.14). Since the function ϕ is bounded and continuous for all s in E, Q_2 is bounded on evry open interval containing the origin and continuous at the origin. By Remark 2.2, Q_2 must have the form $Q_2(v) = \gamma s^2$ for all $s \in E$. Thus we have

$$|\phi(s)| \leq (\varepsilon + |\gamma|)\,|s|^2$$

for all $s \in E$. However, we can select a non-negative integer l and $l\psi > \varepsilon + |\gamma|$. If $s \in \left(0, 2^{-l}\right)$, then $2^m s \in (0,1)$ for all $m = 0, 1, \cdots, l-1$ and for this s, we obtain

$$\phi(s) = \sum_{m=0}^{+\infty} \frac{\xi(2^m s)}{2^{2m}} \geq \sum_{m=0}^{l-1} \frac{\psi(2^m s)}{2^{2m}} = l\psi s > (\varepsilon + |\gamma|)|s|^2,$$

which is contradictory. □

5. Conclusions

We have introduced the generalized quadratic functional equation (1.1) and have obtained its general solution. Mainly, we have investigated Hyers-Ulam stability of the generalized quadratic functional equation (1.1) in Random Normed spaces by using direct and fixed point methods. Furthermore, we proved the counter example for the non-stability to the functional equation (1.1).

Conflict of interest

The authors declare that they have no conflict of interest.

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References

1. A. M. Alanazi, G. Muhiuddin, K. Tamilvanan, E. N. Alenze, A. Ebaid, K. Loganathan, Fuzzy stability results of finite variable additive functional equation: Direct and fixed point methods, *Mathematics*, **8** (2020), 1050. doi:10.3390/math8071050.
2. M. Arunkumar, S. Karthikeyan, S. Ramamoorthi, Generalized Ulam-Hyers stability of N-dimensional cubic functional equation in FNS and RNS:various methods, *Middle-East J. Sci. Res.,* **24** (2016), 386–404.
3. H. A. Kenary, RNS-approximately nonlinear additive functional equations, *J. Math. Extension*, **6** (2012), 11–20.
4. E. Baktash, Y. J. Cho, M. Jalili, R. Saadati, S. M. Vaezpour, On the stability of cubic mappings and quadratic mappings in random normed spaces, *J. Inequalities Appl.*, (2008), Article ID 902187, 11, doi:10.1155/2008/902187.
5. S. S. Chang, Y. J. Cho, S. M. Kang, *Nonlinear operator theory in probabilistic metric spaces*, Nova Science Publishers, Inc. New York, (2001).
6. S. S. Jin, Y. H. Lee, On the stability of the functional equation deriving from quadratic and additive function in random normed spaces via fixed point method, *J. Chungcheong Math. Soc.*, **25** (2012), 51–63.
7. K. Jun, Y. Lee, On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality, *Math. Inequal. Appl.*, **4** (2001), 93–118.
8. Z. Gajda, On stability of additive mappings, *Intern. J. Math. Math. Sci.*, **14** (1991), 431–434.
9. P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), 431–436.
10. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA.*, **27** (1941), 222–224.
11. Pl. Kannappan, Quadratic functional equation and inner product spaces, *Results Math.*, **27** (1995), 368–372.
12. D. Mihet, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, *J. Math. Anal. Appl.*, **343** (2008), 567–572.
13. D. Mihet, The fixed point method for fuzzy stability of the Jensen functional equation, *Fuzzy Sets Syst.*, doi:10.1016/j.fss.2008.06.014.

14. C. Park, K. Tamilvanan, G. Balasubramanian, B. Noori, A. Najati, On a functional equation that has the quadratic-multiplicative property, *Open Math.*, **18** (2020), 837–845.
15. C. Park, K. Tamilvanan, B. Noori, M. B. Moghimi, A. Najati, *Fuzzy normed spaces and stability of a generalized quadratic functional equation*, AIMS Math., **5** (2020), 7161—7174.
16. C. Park, Generalized quadratic mappings in several variables, *Nonlinear Anal. TMA*, **57** (2004), 713–722.
17. J. M. Rassias, On approximately of approximatelylinear mappings by linear mappings, *J. Funct. Anal. USA.*, **46** (1982), 126–130.
18. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300.
19. Th. M. Rassias, On the stability of the quadratic functional equation and its applications, *Studia Univ. Babes-Bolyai*, (1998), 89–124.
20. Th. M. Rassias, K. Shibata, Variational problem of some quadratic functionals in complex analysis, *J. Math. Anal. Appl.*, **228** (1998), 234–253.
21. K. Ravi, M. Arunkumar, J. M. Rassias, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *Int. J. Math. Sci. Autumn*, **3** (2008), 36–47.
22. B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, Elsevier, North Holand, New York, 1983.
23. A. N. Sherstnev, On the notion of a random normed space, *Dokl. Akad. Nauk SSSR*, **149** (1963), 280–283.
24. K. Tamilvanan, J. R. Lee, C. Park, Hyers-Ulam stability of a finite variable mixed type quadratic-additive functional equation in quasi-Banach spaces, *AIMS Math.*, **5** (2020), 5993–6005.
25. K. Tamilvanan, J. R. Lee, C. Park, Ulam stability of a functional equation deriving from quadratic and additive mappings in random normed spaces, *AIMS Math.*, **6** (2020), 908-–924.
26. S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John Wiley and Sons, 1964.

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