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Research article

Filter with its applications in fuzzy soft topological spaces

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Abstract: In this paper, we introduce the convergence of a fuzzy soft filter with the help of the Q-neighborhoods and study the relations between fuzzy soft nets and fuzzy soft filters. In addition, we use fuzzy soft filters to characterize some basic concepts of a fuzzy soft topological space, such as open sets, closure, T_2 separation and continuity.

Keywords: fuzzy soft set; fuzzy soft topology; fuzzy soft net; fuzzy soft filter; convergence
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1. Introduction

In 2001, Maji et al. [14] combined fuzzy sets [26] with soft sets [15] and proposed the concept of fuzzy soft sets. After that, the fuzzy soft set was applied to group theory, decision making, medical diagnosis and other fields (see [1,3,7,8,12,17,23–25,27]). Meanwhile, the theory of fuzzy soft set has been developed rapidly. Especially, the research on fuzzy soft topology has made a lot of achievements (see [2,6,9–11,13,16,18–21]).

Noting the contribution of the point approach in fuzzy topology, Roy and Samanta [18] defined a fuzzy soft point in a fuzzy soft topological space. In 2018, Ibedou and Abbas [10] redefined this concept. Recently, Gao and Wu [8] studied the properties of fuzzy soft point introduced in [10] deeply, and pointed that the fuzzy soft point given in [10] was more effective than that given in [18]. They also gave the definitions of a fuzzy soft net consisting of fuzzy soft points and its convergence. On these bases, they characterized the continuity of fuzzy soft mappings by the net approach. In 2014, Cetkin and Aygun [4] proposed the concept of fuzzy soft filters. Besides, Izzettin Demir et al. [5] investigated the convergence theory of fuzzy soft filters by using the technique of neighborhoods.

Moreover, they used the fuzzy soft filter convergence to characterize closure, continuity, product space and T_2 separation. Finally, they defined the notion of a fuzzy soft filter base and a fuzzy soft ultrafilter and obtained a few results analogous to the ones that held for fuzzy ultrafilters.

It is well known that Q-neighborhoods method has more merits than neighborhoods method. In Section 3, this paper redefines the concept of fuzzy soft filters convergence with the help of the Q-neighborhoods. In Section 4, the fuzzy soft filters are used to characterize some basic concepts of a fuzzy soft topological space, such as open sets, closure, T_2 separation and continuity. Finally, a brief conclusion is given in Section 5.

2. Preliminaries

Throughout this paper, U refers to an initial universe, and E is the set of all parameters for U. In this case, U is also denoted by (U, E). I^U is the set of all fuzzy subsets over U, where $I = [0, 1]$. The elements $\overline{0}, \overline{1} \in I^U$ respectively refer to the functions $\overline{0}(x) = 0$ and $\overline{1}(x) = 1$ for all $x \in U$. For an element $A \in I^U$, if there exists an $x \in U$ such that $A(x) = \lambda > 0$ and $A(y) = 0$, $\forall y \in U \backslash \{x\}$, then A is called a fuzzy point over U and is denoted by x_λ, meanwhile, x and λ are called the support and height of x_λ, respectively. The set of all fuzzy points over U is denoted by $FP(U)$.

The definitions in this section are sourced from the existing literature [8,10,18,19].

Definition 2.1. Let $A \subseteq E$. A mapping $F_A : E \to I^U$, is called a fuzzy soft set over (U, E), where $F_A(e) = \overline{0}$ if $e \in E \setminus A$ and $F_A(e) \neq \overline{0}$ if $e \in A$.

The set of all fuzzy soft sets over (U, E) is denoted by $FS(U, E)$.

The fuzzy soft set $F_\phi \in FS(U, E)$ is called the null fuzzy soft set and is denoted by $\tilde{\Phi}$. Here, $F_\phi(e) = \overline{0}$ for every $e \in E$.

For $F_E \in FS(U, E)$, if $F_E(e) = \overline{1}$ for all $e \in E$, then F_E is called the absolute fuzzy soft set and is denoted by $\tilde{E}$.

Let $F_A, F_B \in FS(U, E)$. If $F_A(e) \subseteq F_B(e)$ for all $e \in E$, then F_A is said to be a fuzzy soft subset of F_B and is denoted by $F_A \tilde{\subseteq} F_B$ or $F_B \tilde{\supseteq} F_A$. If $F_A \tilde{\subseteq} F_B$ and $F_B \tilde{\subseteq} F_A$, then F_A and F_B are said to be equivalent, denoted by $F_A = F_B$.

Remark 2.1. If $F_A \tilde{\subseteq} F_B$, then $A \subseteq B$.

Definition 2.2. Let $F_A, F_B \in FS(U, E)$.

(1) The complement of F_A, denoted by F_A^c, is then defined as

$$F_A^c(e) = \begin{cases} \overline{1} - F_A(e) & \text{for } e \in A, \\ \overline{1}, & \text{otherwise.} \end{cases}$$

(2) The union of F_A and F_B is also a fuzzy soft set F_C defined by $F_C(e) = F_A(e) \bigcup F_B(e)$ for all $e \in E$, where $C = A \bigcup B$, and is denoted by $F_C = F_A \tilde{\bigcup} F_B$.

(3) The intersection of F_A and F_B is also a fuzzy soft set F_C defined by $F_C(e) = F_A(e) \bigcap F_B(e)$ for all $e \in E$, where $C = A \bigcap B$, and is denoted by $F_C = F_A \tilde{\bigcap} F_B$.

Similarly, the union (intersection) of a family of fuzzy soft sets may be defined as

$\{F_{C_\alpha}:\alpha\in\Lambda\}$ and denoted by $\tilde{\bigcup}_{\alpha\in\Lambda}F_{C_\alpha}$ $(\tilde{\bigcap}_{\alpha\in\Lambda}F_{C_\alpha})$, where Λ is an arbitrary index set.

Remark 2.2. It is therefore clear that:

(1) $\tilde{\Phi}^c=\tilde{E},\ \tilde{E}^c=\tilde{\Phi}$;

(2) $(\tilde{\bigcup}_{\alpha\in\Lambda}F_{A_\alpha})^c=\tilde{\bigcap}_{\alpha\in\Lambda}F_{A_\alpha}^c$, $(\tilde{\bigcap}_{\alpha\in\Lambda}F_{A_\alpha})^c=\tilde{\bigcup}_{\alpha\in\Lambda}F_{A_\alpha}^c$.

Definition 2.3. A fuzzy soft topology τ over (U,E) is a family of fuzzy soft sets over (U,E) satisfying the following properties;

(1) $\tilde{\Phi},\ \tilde{E}\in\tau$;

(2) if $F_A,\ F_B\in\tau$, then $F_A\tilde{\bigcap}F_B\in\tau$;

(3) if $F_{A_\alpha}\in\tau$ for all $\alpha\in\Lambda$ (an index set), then $\tilde{\bigcup}_{\alpha\in\Lambda}F_{A_\alpha}\in\tau$.

If τ is a fuzzy soft topology over (U,E), the triple (U,E,τ) is said to be a fuzzy soft topological space. Each element of τ is called an open set. If F_A^c is an open set, then F_A is called a closed set.

Definition 2.4. Let (U,E,τ) be a fuzzy soft topological space, $F_A\in FS(U,E)$.

(1) The intersection of all closed sets $F_B\tilde{\supseteq}F_A$ is called the closure of F_A and is denoted by $\overline{F_A}$.

(2) The union of all open subsets of F_A over (U,E,τ) is called the interior of F_A and is denoted by $\operatorname{int}F_A$.

Definition 2.5. A mapping $\xi:E\to I^U$ is called a fuzzy soft point over (U,E) if there is an $e\in E$ such that $\xi(e)=x_\lambda\in FP(U)$, and $\xi(a)=\overline{0}$ when $a\in E\setminus\{e\}$.

In this case, ξ is also denoted by $P_e^{x_\lambda}$, and e is called its parameter support. The set of all fuzzy soft points over (U,E) is denoted by $FSP(U,E)$.

A fuzzy soft point is called a point if no confusion arises.

For $P_e^{x_\lambda}$, $P_f^{y_\beta}\in FSP(U,E)$, it is said that $P_e^{x_\lambda}$ is equal to $P_f^{y_\beta}$, denoted by $P_e^{x_\lambda}=P_f^{y_\beta}$, if and only if $x=y$, $\lambda=\beta$ and $e=f$.

Definition 2.6. A point $P_e^{x_\lambda}$ is said to be quasi-coincident with $F_A\in FS(U,E)$, which is denoted by $P_e^{x_\lambda}\tilde{\in}F_A$, if $\lambda+F_A(e)(x)>1$ for some $x\in U$.

Definition 2.7. A fuzzy soft set F_A is said to be quasi-coincident with F_B, which is denoted by F_AqF_B, if $F_A(e)(x)+F_B(e)(x)>1$ for some $x\in U$ and $e\in A\bigcap B$.

On the contrary, a fuzzy soft set F_A is said to be not quasi-coincident with F_B, which is denoted by $F_A\overline{q}F_B$, if $F_A(e)(x)+F_B(e)(x)\leq 1$ for all $x\in U$ and $e\in A\bigcap B$.

Definition 2.8. Let $\xi\in FSP(U,E)$ and $F_A,\ F_B\in FS(U,E)$ on a fuzzy soft topological space (U,E,τ).

(1) F_A is said to be a neighborhood of ξ if there exists $F_B\in\tau$ such that $\xi\in F_B\tilde{\subseteq}F_A$.

(2) F_A is called a Q-neighborhood of ξ if there exists $F_B \in \tau$ such that $\xi \tilde{\in} F_B \tilde{\subseteq} F_A$.

The set of all Q-neighborhoods of ξ is denoted by $\mathrm{A}(\xi)$.

Remark 2.3. It is clear that $\mathrm{A}(\xi)$ is a directed set with the partial order "$\tilde{\subseteq}$".

Definition 2.9. Let Δ be a directed set with the partial order "$\prec$". If $S(\delta) \in FSP(U,E)$ for any $\delta \in \Delta$, then $\{S(\delta), \delta \in \Delta\}$ is said to be a fuzzy soft net over (U,E), and is denoted by S for simplicity.

In particular, if $\{S(\delta), \delta \in \Delta\}$ is a fuzzy soft net over (U,E), and there exists an $F_A \in FS(U,E)$ such that $S(\delta) \in F_A$ for any $\delta \in \Delta$, then $\{S(\delta), \delta \in \Delta\}$ is said to be a fuzzy soft net in F_A.

A fuzzy soft net is called a net for simplicity if no confusion arises.

Definition 2.10. Let $F_A \in FS(U,E)$ and $S = \{S(\delta), \delta \in \Delta\}$ be a net over (U,E). If there exists $\delta_0 \in \Delta$ such that $S(\delta) \tilde{\in} F_A$ whenever $\delta_0 \prec \delta$, then S is said to be eventually quasi-coincident with F_A. If for each $\delta \in \Delta$ there exists $\delta_0 \in \Delta$ with $\delta \prec \delta_0$ such that $S(\delta_0) \tilde{\in} F_A$, then S is said to be frequently quasi-coincident with F_A.

Definition 2.11. A net $\{S(\delta), \delta \in \Delta\}$ over (U,E,τ) is said to be convergent to a point ξ if S is eventually quasi-coincident with each Q-neighborhood of ξ. In this case, ξ is called the limit of S and is denoted by $\lim S(\delta)$.

3. Convergence of fuzzy soft nets and fuzzy soft filters

In this section, we introduce the convergence of a fuzzy soft filter by using the Q-neighborhoods and study the relations between fuzzy soft nets and fuzzy soft filters.

Definition 3.1 [4]. A fuzzy soft filter Φ on (U,E,τ) is a nonempty collection of subsets of $FS(U,E)$ with the following properties:

(FSF1) $\tilde{\Phi} \notin \mathrm{F}$,

(FSF2) If $F_A, F_B \in \mathrm{F}$, then $F_A \tilde{\bigcap} F_B \in \mathrm{F}$,

(FSF3) If $F_A \in \mathrm{F}$ and $F_A \tilde{\subseteq} F_B$, then $F_B \in \mathrm{F}$.

If F_1 and F_2 are two fuzzy soft filters on (U,E,τ), we say that F_1 is finer than F_2 (or F_2 is coarser than F_1) if and only if $\mathrm{F}_1 \supseteq \mathrm{F}_2$.

The set of all fuzzy soft filters over (U,E) is denoted by $FSF(U,E)$.

Example 3.1. Let $\xi \in FSP(U,E)$, $\mathrm{F} = \{F_A \in FS(U,E) | \xi \tilde{\in} F_A\}$ be a fuzzy soft filter. $\mathrm{A}(\xi)$ is also a fuzzy soft filter and called the Q-neighborhood filter of ξ.

Definition 3.2. Let F be a fuzzy soft filter in a fuzzy soft topological space (U,E,τ). F is said to be convergent to the fuzzy soft point ξ, denoted by $\lim \mathrm{F} \equiv \xi$, if $\mathrm{A}(\xi) \subseteq \mathrm{F}$.

Now, we investigate the relations between fuzzy soft nets and fuzzy soft filters.

First, we show that a fuzzy soft filter may generate a fuzzy soft net. In fact, let $\mathrm{F} = \{F_{A_i}, i \in I\} \in FSF(U,E)$ and

$$S(i) \tilde{\in} F_{A_i} \text{ for all } i \in I. \quad (3.1)$$

The index set I forms a directed set under the relation "$\prec$", where $i \prec i'$ if and only if $F_{A_i} \tilde{\supseteq} F_{A_{i'}}$ for any $i, i' \in I$. Then $S_{\mathrm{F}} = \{S(i), i \in I\}$ is a fuzzy soft net.

Next, we show that a fuzzy soft net may generate a fuzzy soft filter. Suppose that $S = \{S(\delta), \delta \in \Delta\}$ is a fuzzy soft net in a fuzzy soft topological space (U,E,τ). Let

$$\mathrm{F}_S = \{F_A \in FS(U,E), \ S \text{ is eventually quasi-coincident with } F_A\}. \quad (3.2)$$

It is easy to see that $\mathrm{F}_S \in FSF(U,E)$.

Definition 3.3. (1) Let $\mathrm{F} = \{F_{A_\delta}, \delta \in \Delta\} \in FSF(U,E)$, and let $S(i)$ be defined as (3.1). Then $S_{\mathrm{F}} = \{S(i), i \in I\}$ is said to be the fuzzy soft net generated by F.
(2) Let $S = \{S(\delta), \delta \in \Delta\}$ be a fuzzy soft net in a fuzzy soft topological space (U,E,τ), and F_S be defined as (3.2). Then F_S is said to be the fuzzy soft filter generated by S.

In the rest of this paper, S_{F} and F_S represent the fuzzy soft net generated by F and the fuzzy soft filter generated by S respectively.

Theorem 3.1. Let (U,E,τ) be a fuzzy soft topological space and $\xi \in FSP(U,E)$. $\mathrm{F} = \{F_{A_i}, i \in I\} \in FSF(U,E)$, $S = \{S(\delta), \delta \in \Delta\}$ is a fuzzy soft net on (U,E). Then:

(1) F converges to ξ if and only if S_{F} converges to ξ.

(2) S converges to ξ if and only if F_S converges to ξ.

Proof. (1) (Necessity) Since F converges to ξ, then $\mathrm{A}(\xi) \subseteq \mathrm{F}$. For each $F_A \in \mathrm{A}(\xi)$, there exists $i \in I$ such that $F_{A_i} = F_A$. Therefore, $F_{A_i} \tilde{\supseteq} F_{A_{i'}}$ when $i \prec i' \in I$. So, $S(i') \tilde{\in} F_A$ when $i \prec i' \in I$. Thus, S_{F} converges to ξ.

(Sufficiency) Suppose F does not converge to ξ. Then, there exists an $F_A \in \mathrm{A}(\xi)$ such that $F_A \notin \mathrm{F}$. Therefore, $F_{A_i} \tilde{\not\subseteq} F_A$ for each $i \in I$. That is, there exist $x_i \in U$ and $e_i \in E$ such that $F_{A_i}(e_i)(x_i) > F_A(e_i)(x_i)$ for each $i \in I$. Take $\lambda_i \in (0,1)$ such that $F_{A_i}(e_i)(x_i) > 1 - \lambda_i > F_A(e_i)(x_i)$. Let $S(i) = P_{e_i}^{(x_i)_{\lambda_i}}$, then $S(i) \tilde{\in} F_{A_i}$ and $S(i) \tilde{\notin} F_A$. Therefore, the fuzzy soft net $S_{\mathrm{F}} = \{S(i), i \in I\}$ does not converge to ξ, which is a contradiction. Thus, F converges to ξ.

(2) (Necessity) Since S converges to ξ, then for each $F_A \in \mathrm{A}(\xi)$, S is eventually quasi-coincident with F_A. That is $F_A \in \mathrm{F}_S$. Therefore, F_S converges to ξ.

(Sufficiency) If F_S converges to ξ, then $\mathrm{A}(\xi) \subseteq \mathrm{F}$. That is, for each $F_A \in \mathrm{A}(\xi)$, S is eventually quasi-coincident with F_A. Therefore, S converges to ξ. □

4. Applications of fuzzy soft filters

In this section, we use fuzzy soft filters to describe open sets, closure, T_2 separation and continuity in fuzzy soft topological spaces. First, we give a lemma. Its proof is easy.

Lemma 4.1. Let (U,E,τ) be a fuzzy soft topological space. If for any $\xi \tilde{\in} F_A \in FS(U,E)$, there exists an open set $F_B \in \mathcal{A}(\xi)$ such that $F_B \tilde{\subseteq} F_A$, $F_C = \tilde{\bigcup}\{F_B | \xi \tilde{\in} F_A\}$, then $F_C = F_A$.

Theorem 4.1. Let (U,E,τ) be a fuzzy soft topological space. $F_A \in FS(U,E)$ is open if and only if $F_A \in \mathcal{F}$ for any fuzzy soft point $\xi \tilde{\in} F_A$ and $\mathcal{F} \in FSF(U,E)$ with $\lim \mathcal{F} \equiv \xi$.

Proof. (Necessity) Since F_A is open and $\xi \tilde{\in} F_A$, then $F_A \in \mathcal{A}(\xi)$. So $F_A \in \mathcal{F}$ follows from $\lim \mathcal{F} \equiv \xi$.

(Sufficiency) Let $\xi \tilde{\in} F_A$ arbitrarily. Since $\lim \mathcal{A}(\xi) = \xi$, then $F_A \in \mathcal{A}(\xi)$. That is, there exists an open set $F_B \in \mathcal{A}(\xi)$, such that $F_B \tilde{\subseteq} F_A$. From Lemma 4.1, $F_A = \tilde{\bigcup}\{F_B | \xi \tilde{\in} F_A\}$ is open. □

Remark 4.1. In [8], it is proved that $P_e^{x_\lambda} \tilde{\in} F_A$ if and only if $P_e^{x_\lambda} \notin F_A^c$. Additionally, if $P_e^{x_\lambda} \tilde{\in} F_A$, then there exists $0 < \mu < \lambda$ such that $P_e^{x_{1-\mu}} \in F_A$.

Theorem 4.2. Let (U,E,τ) be a fuzzy soft topological space. A fuzzy soft point $\xi \in \overline{F_A}$ if and only if there exists $\mathcal{F} \in FSF(U,E)$ such that $\lim \mathcal{F} \equiv \xi$ and $F_B \, q \, F_A$ for any $F_B \in \mathcal{F}$.

Proof. (Necessity) Let $\xi \in \overline{F_A}$, $\mathcal{F} = \mathcal{A}(\xi)$, then $\lim \mathcal{F} \equiv \xi$. For any $F_B \in \mathcal{A}(\xi)$, we suppose that F_B is open without loss of generality. To complete the proof, it is sufficient to show that $F_B q F_A$. In fact, if $F_B \bar{q} F_A$, then $F_A(d)(y) + F_B(d)(y) \le 1$ for any $y \in U$ and $d \in E$. Hence, $F_A \tilde{\subseteq} F_B^c$. Noting that F_B^c is closed, one gets $\overline{F_A} \tilde{\subseteq} F_B^c$. Therefore, $\xi \in F_B^c$. It follows from Remark 4.1 that $\xi \tilde{\notin} F_B$, which conflicts with $F_B \in \mathcal{A}(\xi)$. Thus, $F_B q F_A$.

(Sufficiency) Suppose that $\xi \notin \overline{F_A}$, then $\xi \tilde{\in} \overline{F_A}^c$ from Remark 4.1. Therefore, $\overline{F_A}^c \in \mathcal{A}(\xi)$. Since $\lim \mathcal{F} \equiv \xi$, then $\mathcal{A}(\xi) \subseteq \mathcal{F}$. Therefore, $\overline{F_A}^c \in \mathcal{F}$, and hence $\overline{F_A}^c q F_A$. That is, there exist $y \in U$ and $d \in E$ such that $\overline{F_A}^c(d)(y) + F_A(d)(y) > 1$. Equivalently, $\overline{F_A}(d)(y) < F_A(d)(y)$, which is a contradiction. Thus, $\xi \in \overline{F_A}$. □

Definition 4.1. Let (U,E,τ) be a fuzzy soft topological space. If for any two different points ξ and ζ, there exist $F_A \in \mathcal{A}(\xi)$ and $F_B \in \mathcal{A}(\zeta)$ such that $F_A \tilde{\bigcap} F_B = \tilde{\Phi}$, then (U,E,τ) is said to be T_2 separated.

Definition 4.2 [5]**.** A collection $\mathcal{B}$ of subsets of $FS(U,E)$ is called a base for a fuzzy soft filter on (U,E,τ) if the following two conditions are satisfied:

(B1) $\mathcal{B} \neq \Phi$ and $\tilde{\Phi} \notin \mathcal{B}$,

(B2) If F_A, $F_B \in \mathcal{B}$, then there is a $F_C \in \mathcal{B}$ such that $F_C \tilde{\subseteq} F_A \tilde{\bigcap} F_B$.

One readily sees that if $\mathcal{B}$ is a base for a fuzzy soft filter on (U,E,τ), the collection

$$\mathcal{F}_{\mathcal{B}} = \{F_A \in FS(U,E) : \text{there exists a } F_C \in \mathcal{B} \text{ such that } F_C \tilde{\subseteq} F_A\}$$

is a fuzzy soft filter on (U,E,τ). We say that the fuzzy soft filter $\mathcal{F}_{\mathcal{B}}$ is generated by $\mathcal{B}$.

Theorem 4.3. A fuzzy soft topological space (U,E,τ) is T_2 separated if and only if for any $\mathcal{F} \in FSF(U,E)$ does not converge to two different points at the same time.

Proof. (Necessity) Let $\mathcal{F} \in FSF(U,E)$ with $\lim \mathcal{F} = \xi \in FSP(U,E)$. For any fuzzy soft point $\zeta \neq \xi$, since (U,E,τ) be T_2 separated, there exist $F_A \in \mathcal{A}(\xi)$ and $F_B \in \mathcal{A}(\zeta)$ such that $F_A \tilde{\bigcap} F_B = \tilde{\Phi}$. From $F_A \in \mathcal{F}$, one knows that $F_B \notin \mathcal{F}$. Therefore, $\mathcal{F}$ does not converge to ζ.

(Sufficiency) Suppose (U,E,τ) is not T_2 separated. Then there exist two different points ξ and ζ such that $F_A \tilde{\bigcap} F_B \neq \tilde{\Phi}$ for any $F_A \in \mathcal{A}(\xi)$ and $F_B \in \mathcal{A}(\zeta)$. It is easy to see that $\mathcal{B} = \{F_A \tilde{\bigcap} F_B \,|\, F_A \in \mathcal{A}(\xi),\ F_B \in \mathcal{A}(\zeta)\}$ is a fuzzy soft filter base, and the fuzzy soft filter generated by $\mathcal{B}$ converges to ξ and ζ at the same time, which contradicts with the condition. Thus (U,E,τ) is T_2 separated. □

The following definition originates from [3].

Definition 4.3. Let $\varphi : U_1 \to U_2$ and $\psi : E_1 \to E_2$ be two functions. Then, the pair (φ,ψ) is called a fuzzy soft mapping from (U_1,E_1) to (U_2,E_2).

(1) Let $F_A \in FS(U_1,E_1)$. Then, the image of F_A under (φ,ψ) is the fuzzy soft set over (U_2,E_2) defined by $(\varphi,\psi)(F_A)$, where

$$(\varphi,\psi)(F_A)(k)(y) = \begin{cases} \bigvee\limits_{\varphi(x)=y}\ \bigvee\limits_{\psi(e)=k} F_A(e)(x), & \text{if } \varphi^{-1}(y) \neq \phi \\ \bar{0}, & \text{otherwise} \end{cases},\quad \forall k \in \psi(E_1),\ \ \forall y \in U_2.$$

(2) Let $F_B \in FS(U_2,E_2)$. Then, the pre-image of F_B under (φ,ψ) is the fuzzy soft set over (U_1,E_1) defined by $(\varphi,\psi)^{-1}(F_B)$, where

$$(\varphi,\psi)^{-1}(F_B)(e)(x) = F_B(\psi(e))(\varphi(x)),\ \ \forall e \in \psi^{-1}(E_2),\ \ \forall x \in U_1.$$

If both φ and ψ are injective (surjective), then the fuzzy soft mapping (φ,ψ) is said to be injective (surjective).

The composition of two fuzzy soft mappings (φ,ψ) from (U_1,E_1) to (U_2,E_2) and (φ',ψ') from (U_2,E_2) to (U_3,E_3) is defined as $(\varphi' \circ \varphi, \psi' \circ \psi)$ from (U_1,E_1) to (U_3,E_3).

Definition 4.4. Let (U_1,E_1,τ_1) and (U_2,E_2,τ_2) be two fuzzy soft topological spaces, $\xi \in FSP(U_1,E_1)$. A fuzzy soft mapping (φ,ψ): $(U_1,E_1,\tau_1) \to (U_2,E_2,\tau_2)$ is said to be fuzzy soft continuous at ξ if for any $F_B \in \mathcal{A}((\varphi,\psi)(\xi))$, there exists $F_A \in \mathcal{A}(\xi)$, such that $(\varphi,\psi)(F_A) \tilde{\subseteq} F_B$.

A fuzzy soft mapping (φ,ψ): $(U_1,E_1,\tau_1) \to (U_2,E_2,\tau_2)$ is said to be fuzzy soft continuous if (φ,ψ) is fuzzy soft continuous at each fuzzy soft point of (U_1,E_1,τ_1).

Remark 4.2. Theorem 6 in [8] implies that the fuzzy soft continuity of a fuzzy soft mapping in this paper is equivalent to that in [22].

Lemma 4.2 [22]. Let (U_1,E_1,τ_1) and (U_2,E_2,τ_2) be two fuzzy soft topological spaces and (φ,ψ): $(U_1,E_1,\tau_1)\rightarrow(U_2,E_2,\tau_2)$ is a fuzzy soft mapping. Then (φ,ψ) is fuzzy soft continuous if and only if $(\varphi,\psi)(\overline{F_A})\,\tilde{\subseteq}\,\overline{(\varphi,\psi)(F_A)}$, $\forall F_A\in FS(U_1,E_1)$.

Lemma 4.3. Let (U_1,E_1,τ_1) and (U_2,E_2,τ_2) be two fuzzy soft topological spaces and (φ,ψ): $(U_1,E_1,\tau_1)\rightarrow(U_2,E_2,\tau_2)$ is surjective, $\mathcal{F}\in FSF(U_1,E_1)$, then $(\varphi,\psi)(\mathcal{F})\in FSF(U_2,E_2)$.

Proof. (FSF1) and (FSF2) are obvious.

(FSF3) Let $F_A\in\mathcal{F}$ and $(\varphi,\psi)(F_A)\,\tilde{\subseteq}\,F_B\in FS(U_2,E_2)$. Since $F_A\,\tilde{\subseteq}\,(\varphi,\psi)^{-1}(\varphi,\psi)\,(F_A)\,\tilde{\subseteq}\,(\varphi,\psi)^{-1}$ (F_B), then $F_{A_1}=(\varphi,\psi)^{-1}(F_B)\in\mathcal{F}$. Noting that (φ,ψ) is surjective, one gets that $F_B=(\varphi,\psi)$ $(\varphi,\psi)^{-1}(F_B)=(\varphi,\psi)\,(F_{A_1})\in(\varphi,\psi)(\mathcal{F})$.

Therefore, $(\varphi,\psi)(\mathcal{F})\in FSF(U_2,E_2)$. □

Theorem 4.4. Let (U_1,E_1,τ_1) and (U_2,E_2,τ_2) be two fuzzy soft topological spaces, and (φ,ψ): $(U_1,E_1,\tau_1)\rightarrow(U_2,E_2,\tau_2)$ be surjective. Then, (φ,ψ) is fuzzy soft continuous if and only if $\lim(\varphi,\psi)(\mathcal{F})\equiv(\varphi,\psi)(\xi)$ for any $\mathcal{F}\in FSF(U_1,E_1)$ with $\lim\mathcal{F}\equiv\xi\in FSP(U_1,E_1)$.

Proof. (Necessity) Since (φ,ψ) is fuzzy soft continuous, by Definition 4.4, for any $F_B\in\mathcal{A}\left((\varphi,\psi)(\xi)\right)$, there exists $F_A\in\mathcal{A}(\xi)$, such that $(\varphi,\psi)(F_A)\,\tilde{\subseteq}\,F_B$. Let $\mathcal{F}\in FSF(U_1,E_1)$ with $\lim\mathcal{F}\equiv\xi$. Then $F_A\in\mathcal{F}$, and $(\varphi,\psi)(F_A)\in(\varphi,\psi)(\mathcal{F})$. Therefore, $F_B\in(\varphi,\psi)(\mathcal{F})$. So $\lim(\varphi,\psi)(\mathcal{F})\equiv(\varphi,\psi)(\xi)$.

(Sufficiency) To complete the proof, we shall show that $(\varphi,\psi)(\overline{F_A})\,\tilde{\subseteq}\,\overline{(\varphi,\psi)(F_A)}$ for any $F_A\in FS(U_1,E_1)$. Take $\zeta\in(\varphi,\psi)(\overline{F_A})$ arbitrarily. Then there is $\xi\in\overline{F_A}$ such that $\zeta=(\varphi,\psi)\left(\xi\right)$. By Theorem 4.2, there exists $\mathcal{F}\in FSF(U_1,E_1)$ such that $\lim\mathcal{F}\equiv\xi$ and $F_B\,q\,F_A$ for any $F_B\in\mathcal{F}$. From Lemma 4.3, $(\varphi,\psi)(\mathcal{F})\in FSF(U_2,E_2)$, and $\lim(\varphi,\psi)(\mathcal{F})\equiv\zeta$.

For any $F_C\in(\varphi,\psi)(\mathcal{F})$, there is $F_B\in\mathcal{F}$ such that $F_C=(\varphi,\psi)(F_B)$. Owing to $F_B\,q\,F_A$ and Theorem 5 in [8], we have $\left((\varphi,\psi)(F_B)\right)q\left((\varphi,\psi)(F_A)\right)$. That is, $F_C q\left((\varphi,\psi)(F_A)\right)$. It follows from Theorem 4.2 that $\zeta\in\overline{(\varphi,\psi)(F_A)}$. Recalling the arbitrariness of $\zeta\in(\varphi,\psi)(\overline{F_A})$, we have $(\varphi,\psi)(\overline{F_A})\,\tilde{\subseteq}\,\overline{(\varphi,\psi)(F_A)}$. □

5. Conclusion

In this paper, the convergence of a fuzzy soft filter is redefined by the Q-neighborhoods, some important properties of fuzzy soft topological spaces are characterized by the fuzzy soft filter. The obtained results demonstrate that the methods proposed in this paper are very useful and will provide powerful research tools for further research in this field.

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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